# Dynamics and Asymptotic Profiles of Steady States of an Epidemic Model in Advective Environments *

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**Abstract**

We study the dynamics of a SIS epidemic model of reaction-diffusion-advection type. The persistence of infected and susceptible populations and the global stability of the disease free equilibrium are established when the basic reproduction number is greater than or less than or equal to one, respectively. We further consider the effects of diffusion and advection on asymptotic profiles of endemic equilibrium: When the advection rate is relatively large comparing to the diffusion rates of both populations, then two populations persist and concentrate at the downstream end. As the diffusion rate of the susceptible population tends to zero, the density of the infected population decays exponentially for positive advection rate but linearly when there is no advection. Our results suggest that advection can help speed up the elimination of disease.

**Keywords**: Disease dynamics; Advective environment; Spatial heterogeneity; Asymptotic profile; Steady state

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## 1 Introduction

It has been widely recognized that environmental heterogeneity and individual motility are significant factors that should be taken into account in studying disease dynamics. For the Susceptible-Infected-Susceptible (SIS) epidemic reaction-diffusion models, some recent works are able to capture the effect of the spatial heterogeneity of environment and movement of individuals on the persistence or extinction of diseases [2, 3, 7, 9, 10, 13, 20, 21, 22, 23, 28]. In [2] Allen et al. proposed a SIS epidemic reaction-diffusion model without advection. In [21] Peng and Liu discussed the global stability of the endemic equilibrium in some special cases for the model of Allen et al. The effects of large and small diffusion rates of the susceptible and infected populations on the persistence and extinction of the disease were considered in [20, 22]. Allen et al. also investigated a discrete SIS-model in [1]. Peng and Zhao [23] recently considered the same SIS reaction-diffusion model of Allen et al., but the rates of disease transmission and recovery are assumed to be spatially heterogeneous and temporally periodic. In [7, 28] the authors consider an SIS model with mass action infection mechanism. In [13] Li et al. provided qualitative analysis on an SIS reaction diffusion system with a linear source term. Ge et al. introduced a free boundary model for characterizing the spreading front of the disease in [9]. In these works the populations are assumed to adopt random diffusion in the habitats.

In some heterogeneous environments populations may assume biased or passive movement in certain directions [4, 19], e.g., due to the external environmental forces such as wind [6], water flow [12, 15, 16, 17, 18] and so on, which usually can be described by adding an advection term to the existing reaction-diffusion models. For the spatial epidemic model with advection in heterogeneous environment, it is of interest to understand how the diffusion and advection jointly affect the persistence or extinction of the infectious diseases. Such studies may have significant implications for predicting the patterns of disease occurrence and for designing optimal control strategies as well.

The following SIS reaction-diffusion-advection model in one dimensional domain was considered in [5]:

$$\begin{cases} \bar{S}_t = d_S\bar{S}_{xx} - q\bar{S}_x - \beta(x)\dfrac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} + \gamma(x)\bar{I}, & 0<x<L,\ t>0, \\ \bar{I}_t = d_I\bar{I}_{xx} - q\bar{I}_x + \beta(x)\dfrac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} - \gamma(x)\bar{I}, & 0<x<L,\ t>0, \\ d_S\bar{S}_x - q\bar{S} = d_I\bar{I}_x - q\bar{I} = 0, & x=0,L,\ t>0, \\ \bar{S}(x,0) = \bar{S}_0(x) \geq 0, \quad \bar{I}(x,0) = \bar{I}_0(x) \geq 0, & 0<x<L, \end{cases} \tag{1.1}$$

where $\bar{S}(x,t)$ and $\bar{I}(x,t)$ denote the density of susceptible and infected individuals at time $t$ and position $x$ in the interval $[0,L]$, respectively; the positive constants $d_S$ and $d_I$ are diffusion coefficients for the susceptible and infected populations; $q$ is the effective speed of the current (sometimes we call $q$ the advection rate); $L$ is the size of the habitat, and we call $x = 0$ the upstream end and $x = L$ the downstream end. The functions $\beta(x)$ and $\gamma(x)$ are assumed to be positive, Hölder continuous on $[0, L]$ and they represent the rates of disease transmission and recovery at $x$, respectively. Here both populations satisfy no-flux boundary conditions, which means that there is no population flux across the upstream and downstream ends, so that both susceptible and infected populations live in a closed environment. As the term $\bar{S}\bar{I}/(\bar{S}+\bar{I})$ is a Lipschitz continuous function of $\bar{S}$ and $\bar{I}$ in the open first quadrant, its definition can be extended to the closure of the first quadrant by setting it to be zero when either $\bar{S} = 0$ or $\bar{I} = 0$. Here we also assume that there is a positive number of infected individuals initially, that is,

$$\textbf{(A)} \qquad \int_0^L \bar{I}(x,0)\,\mathrm{d}x > 0, \ \text{with } \bar{S}(x,0) \geq 0 \text{ and } \bar{I}(x,0) \geq 0 \text{ for } x \in (0,L).$$

As discussed in [5], system (1.1) admits a unique classical solution $(\bar{S}(x,t), \bar{I}(x,t))$ which exists globally in time. Let

$$N := \int_0^L \left[\bar{S}(x,0) + \bar{I}(x,0)\right]\,\mathrm{d}x > 0 \tag{1.2}$$

be the total number of individuals in $(0, L)$ at $t = 0$. It turns out that the total population remains constant in time, i.e.,

$$\int_0^L [\bar{S}(x,t) + \bar{I}(x,t)]\,\mathrm{d}x \equiv N, \ \ t \geq 0. \tag{1.3}$$

From (1.3), we know that any solution $(\bar{S}(x,t), \bar{I}(x,t))$ satisfies $L^1$ space bound uniformly for $t \in [0, \infty)$. In fact, it can be concluded that for any fixed $q \geq 0$, $\|\bar{S}(\cdot,t)\|_{L^\infty((0,L))}$ and $\|\bar{I}(\cdot,t)\|_{L^\infty((0,L))}$ are also uniformly bounded in $[0,\infty)$ (see Proposition 2.2). Unless otherwise stated, it is always assumed that assumption (A) holds and the class of initial data $(\bar{S}_0, \bar{I}_0)$ satisfies (1.2) for some (common) fixed positive constant $N$. By adopting the same terminology as in [5], we say that $x$ is a *low-risk* site if the local disease transmission rate $\beta(x)$ is lower than the local disease recovery rate $\gamma(x)$. Similarly, $x$ is a *high-risk* site if $\beta(x) > \gamma(x)$.

The second half of this paper concerns the non-negative equilibrium solutions of (1.1), that is, the non-negative solutions of the following system:

$$\begin{cases} d_S S_{xx} - qS_x - \beta(x)\dfrac{SI}{S+I} + \gamma(x)I = 0, & 0 < x < L, \\ d_I I_{xx} - qI_x + \beta(x)\dfrac{SI}{S+I} - \gamma(x)I = 0, & 0 < x < L, \\ d_S S_x - qS = d_I I_x - qI = 0, & x = 0,\ L. \end{cases} \tag{1.4}$$

Here, $S(x)$ and $I(x)$ denote the density of susceptible and infected individuals at equilibrium, respectively, at $x \in [0, L]$. In view of (1.3), we impose the additional hypothesis

$$\int_0^L [S(x) + I(x)]\,\mathrm{d}x = N. \tag{1.5}$$

Since (1.5) is a population model, only solutions $(S(x), I(x))$ satisfying $S(x) \geq 0$ and $I(x) \geq 0$ on $[0, L]$ are of interest. A *disease-free equilibrium* (DFE) is a solution of (1.4)-(1.5) so that $I(x) = 0$ for every $x \in (0, L)$; An *endemic equilibrium* (EE) of (1.4)-(1.5) is a solution in which $I(x) > 0$ for some $x \in (0, L)$. We denote a DFE by $(\tilde{S}, 0)$ and an EE by $(S_e, I_e)$. By direct computations and condition (1.5), we get $\tilde{S}(x) = qNe^{(q/d_S)x}/d_S(e^{qL/d_S} - 1)$. Thus (1.4)-(1.5) has a unique *disease-free equilibrium*, which is spatially inhomogeneous.

According to the definition of the basic reproduction number in existing literatures [8, 26, 27], in [5] we introduced the basic reproduction number for model (1.1) as follows:

$$\mathcal{R}_0(d_I, q) = \sup_{\substack{\varphi \in H^1((0,L)) \\ \varphi \neq 0}} \left\{ \frac{\int_0^L \beta(x) e^{\frac{q}{d_I}x} \varphi^2 \, \mathrm{d}x}{d_I \int_0^L e^{\frac{q}{d_I}x} \varphi_x^2 \, \mathrm{d}x + \int_0^L \gamma(x) e^{\frac{q}{d_I}x} \varphi^2 \, \mathrm{d}x} \right\}. \tag{1.6}$$

From the definition of the basic reproduction number of (1.1), it can be seen that $\mathcal{R}_0$ is a smooth function of $d_I$ and $q$. It was shown in [5] that $\mathcal{R}_0$ is a threshold value for the stability of the disease-free equilibrium: Namely, if $\mathcal{R}_0 < 1$ then DFE is globally asymptotically stable, and if $\mathcal{R}_0 > 1$ then the DFE is unstable. Our first theorem improves this earlier result in [2] and [5] as follows:

**Theorem 1.1.** *The equation* (1.1) *generates a semiflow* $\Phi$ *in* $X = \{(S_0, I_0) \in C([0, L]; \mathbb{R}_+) : \int_0^L (S_0 + I_0) \, \mathrm{d}x = N\}$ *such that* $\Phi_t : X \to X$ *is compact for each* $t > 1$.

(a) *If* $\mathcal{R}_0 \leq 1$, *then the DFE* $(\tilde{S}, 0) = \left( \frac{qN}{d_S(1 - e^{-qL/d_S})} e^{-q(L-x)/d_S}, 0 \right)$ *is globally asymptotically stable among solutions with initial data in* $X$.

(b) *If* $\mathcal{R}_0 > 1$, *then the following hold.*

  (i) *There exists* $\epsilon_0 > 0$ *(independent of initial data) such that for any solution* $(\bar{S}, \bar{I})$ *of* (1.1) *such that* $I_0 \not\equiv 0$, *we have*

$$\liminf_{t \to \infty} \left[ \inf_{0 < x < L} \bar{I}(x, t) \right] \geq \epsilon_0.$$

  (ii) *There exists at least one EE, denoted as* $(S_e, I_e)$.

  (iii) *If, in addition,* $d_S = d_I = d > 0$, *then the EE is unique, and is globally asymptotically stable among solutions of* (1.1) *with initial data in* $X$ *satisfying* $I_0 \not\equiv 0$.

Previously, the global asymptotic stability of the DFE was proved under the stronger assumption $\mathcal{R}_0 < 1$ in [2] for the case $q = 0$; [23] for $q = 0$ with time-periodic coefficients; and in [5] for $q > 0$. A special case of Theorem 1.1, when $d_S = d_I$ and $q = 0$, was attempted in [21]. Our general argument in Section 2 proving that the DFE is globally asymptotically stable under the necessary and sufficient condition of $\mathcal{R}_0 \leq 1$, without additional assumptions, seems to be new. And we expect this argument to be applicable in other SIS type PDE models with precompact trajectories.

Theorem 1.1 in particular says that a necessary and sufficient condition for the existence of EE is that $\mathcal{R}_0 > 1$. Characterizing the dependence of $\mathcal{R}_0$ on parameters $d_S, d_I, q, \beta(\cdot), \gamma(\cdot)$ is an important and challenging problem. We refer the reader to [2] for the case $q = 0$ and to [5] for the case $q > 0$. In fact, by setting the test function $\varphi \equiv 1$ in the definition of $\mathcal{R}_0$ in (1.6), we have

$$\mathcal{R}_0 \geq \frac{\int_0^L e^{qx/d_I}\beta(x)\,\mathrm{d}x}{\int_0^L e^{qx/d_I}\gamma(x)\,\mathrm{d}x}. \tag{1.7}$$

And one can derive the following two sufficient conditions for the existence of EE which are relevant for our purposes in this paper.

- $\beta(L) > \gamma(L)$ and the ratio $q/d_I$ is sufficiently large;
- $\int_0^L \beta(x)\,\mathrm{d}x > \int_0^L \gamma(x)\,\mathrm{d}x$ and the ratio $q/d_I$ is sufficiently small.

Next, we proceed to study qualitative properties of EE of (1.1) under the constraint (1.5) when diffusion and advection rates $d_S$, $d_I$ and $q$ vary. First, we examine the case when $\beta(L) > \gamma(L) > 0$. To this end, define, for any $\eta \in [0, \infty)$, $\alpha^* = \alpha^*(\eta)$ to be the unique positive root of

$$\int_0^1 \left[\frac{1}{1 + \alpha^* z^{(1-\eta)}}\right]\mathrm{d}z = \frac{\gamma(L)}{\beta(L)}. \tag{1.8}$$

In particular, $\alpha^*(\eta) > 0$ for all $\eta \geq 0$, and $\alpha^*(1) = \frac{\beta(L)}{\gamma(L)} - 1$.

The following result describes the asymptotic profiles of EE when $q$ is relatively large with respect to $d_I$ and $d_S$.

**Theorem 1.2.** *Assume that $\beta(L) > \gamma(L)$. Then there exists some positive constant $C$, independent of $d_S, d_I$ and $q$, such that (1.1) has at least one EE whenever $q/d_I \geq C$.*

(i) ***(exponential decay)*** *For each $\eta_0 \geq 1$, there exist $C_1, C_2 > 0$ and $0 < \delta < 1$ such that for $q/d_S, q^2/d_S \geq C_1$ and $d_I/d_S \leq \eta_0$, any EE $(S_e(x), I_e(x))$ of (1.1) satisfies*

$$\left|S_e(x) - S_e(L)e^{-\frac{q}{d_S}(L-x)}\right| + \left|I_e(x) - I_e(L)e^{-\frac{q}{d_I}(L-x)}\right| \leq C_2 \min\left\{\frac{q}{d_S}, \frac{1}{q}\right\} e^{-\frac{\delta q}{d_S}(L-x)}. \tag{1.9}$$

*Moreover, as*

$$\frac{q}{d_S} \to \infty, \quad \frac{q^2}{d_S} \to \infty, \quad \frac{d_I}{d_S} \to \eta \in [0, \infty), \tag{1.10}$$

(ii) ***(concentration)*** *the limiting profiles of the populations are given by*

$$(S_e(\cdot),\ I_e(\cdot)) \to \left(\frac{N}{1+\eta\alpha^*}\delta(\cdot - L),\ \frac{\eta\alpha^* N}{1+\eta\alpha^*}\delta(\cdot - L)\right) \quad \textit{in distribution sense,} \tag{1.11}$$

*where* $\alpha^* = \alpha^*(\eta) > 0$ *is given in* (1.8)*, and* $\delta(\cdot - L)$ *is the Dirac measure at* $L$*;*

(iii) ***(downstream density)*** *the densities at the downstream end satisfy*

$$\frac{d_S}{q}(S_e(L), I_e(L)) \to \left(\frac{N}{1+\eta\alpha^*(\eta)}, \frac{\alpha^*(\eta)N}{1+\eta\alpha^*(\eta)}\right). \tag{1.12}$$

**Remark 1.** *For the sake of clarity, the above theorem is stated for the case when both* $q/d_S$ *and* $q/d_I$ *are large and comparable. In fact, our proofs remain applicable for the case* $q/d_I \to \infty$ *and* $q/d_S \to \xi \in [0,\infty)$*. This will be taken up in Subsection 3.1.*

Theorem 1.2, interpreted biologically, says that both susceptible and infected populations concentrate at the downstream end ($x = L$), provided that the advection rate is relatively large comparing to the diffusion rates of the populations. To further illustrate the result, we state two special cases as a corollary.

**Corollary 1.3.** *Assume* $\beta(L) > \gamma(L)$.

(i) *For each fixed* $d_S, d_I > 0$*, as* $q \to \infty$*, any EE of* (1.1) *satisfies*

$$(S_e(\cdot), I_e(\cdot)) \to \left(\frac{d_S N}{d_S + d_I\alpha^*}\delta(\cdot - L), \frac{d_I\alpha^* N}{d_S + d_I\alpha^*}\delta(\cdot - L)\right) \quad \textit{in distribution sense,}$$

*where* $\alpha^* = \alpha^*(d_S, d_I)$ *is uniquely determined by*

$$\int_0^1 \left[\frac{1}{1+\alpha^* z^{(1-d_I/d_S)}}\right] \mathrm{d}z = \frac{\gamma(L)}{\beta(L)}.$$

(ii) *For each fixed* $q > 0$*, suppose* $d_S, d_I \to 0$ *such that* $d_I/d_S \to \eta \in [0,\infty)$*, then*

$$(S_e(\cdot), I_e(\cdot)) \to \left(\frac{N}{1+\eta\alpha^*}\delta(\cdot - L), \frac{\eta\alpha^* N}{1+\eta\alpha^*}\delta(\cdot - L)\right) \quad \textit{in distribution sense,}$$

*where* $\alpha^* = \alpha^*(\eta)$ *is uniquely determined by* (1.8).

**Remark 2.** (1) If $\beta(L) < \gamma(L)$, there is no EE when the advection $q$ is large, i.e., when the downstream end is low risk, then there is no endemic equilibrium for large $q$. Part (i) implies that when the downstream site is high risk, then the susceptible and infected individuals concentrate at the downstream end for large advection.

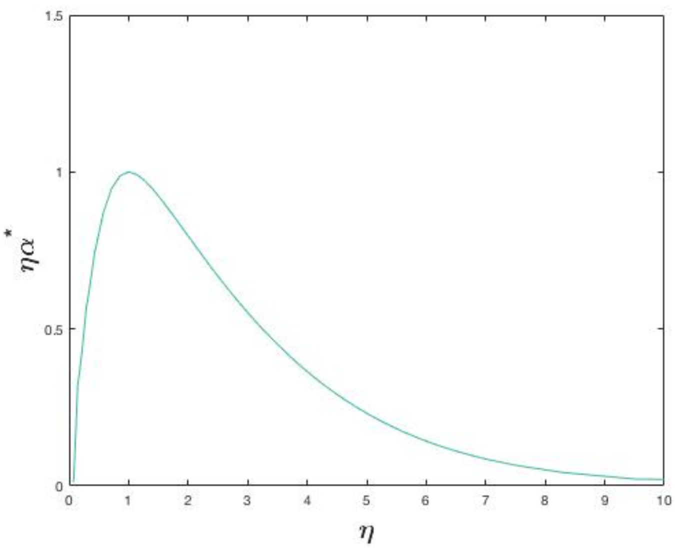


Figure 1: The graph for $\eta \mapsto \eta\alpha^*(\eta)$ when $\beta(L)/\gamma(L) = 2$.

(2) Part (ii) implies that when diffusion rates of susceptible and infected populations are both small and $d_I/d_S \le O(1)$, advection will transport individuals to the downstream end so that the individuals concentrate around the downstream end.

Corollary 1.3 has some interesting implications for the disease persistence. It shows that, under various limits, the finial size of the infected population is approximately given by $\eta\alpha^*(\eta)N/(1+\eta\alpha^*(\eta))$, where $\eta$ is approximately the ratio of $d_I$ and $d_S$. Figure 1 shows that the final size of the infected population becomes negligible when one of the diffusion rates is much smaller than the other; when two diffusion rates are comparable, their final sizes are also comparable. In particular, part (i) implies that when $d_I/d_S$ or $d_S/d_I$ is small, increasing the advection may keep the final size of the infected population small. Part (ii) implies that given any advection rate, if both diffusion rates are small but one of which is much smaller, the final size of the infected population also becomes negligible.

Next we consider the asymptotic profiles of the EE when the diffusion rate of the susceptible population approaches zero. We caution the readers that $\mathcal{R}_0$ does not depend on $d_S$, so we need to assume $\mathcal{R}_0 > 1$ (this holds, e.g. if $\int_0^L \beta(x)\,dx > \int_0^L \gamma(x)\,dx$ and $q/d_I$ is sufficiently small) to ensure the existence of EE.

**Theorem 1.4.** (i) *Assume $q = 0$ and $\mathcal{R}_0(d_I, 0) > 1$. Then there exist positive constants $\overline{C}_3, \underline{C}_3$ such that for $0 < d_S < 1$,*

$$\underline{C}_3 \le \frac{I_e(x)}{d_S} \le \overline{C}_3 \quad \textit{for } 0 \le x \le L$$

*holds for any EE of* (1.1).

(ii) *Suppose that* $\beta(L) < \gamma(L)$ *and* $\mathcal{R}_0(d_I, q) > 1$. *Then there exist constants* $C_4, C_5 > 0$ *such that for* $0 < d_S < 1$,

$$I_e(x) \le C_4 e^{-C_5 \frac{q}{d_S}} \quad \text{for } 0 \le x \le L \tag{1.13}$$

*holds for any EE of* (1.1). *Moreover,*

$$\lim_{d_S \to 0} \left\| S_e(x) - \frac{qN}{d_S} e^{-\frac{q}{d_S}(L-x)} \right\|_{L^\infty((0,L))} = 0.$$

**Remark 3.** (1) *The basic reproduction number* $\mathcal{R}_0 = \mathcal{R}_0(d_I, q)$ *is independent of* $d_S > 0$. *i.e. If* $\mathcal{R}_0(d_I, q) > 1$ *for some* $q \ge 0$, *then the DFE is unstable for all* $d_S > 0$. *In particular, the existence of EE is guaranteed for all* $d_S > 0$.

(2) *When the dimension of the underlying spatial domain is greater than one, the analogue of part (i) of Theorem 1.4 remains valid.*

(3) *Suppose there exists* $q_0 > 0$ *such that* $\mathcal{R}_0(d_I, q) > 1$ *for all* $0 < q \le q_0$. *Then for all* $q/d_S \gg 1$, (1.13) *remains valid.*

Theorem 1.4 shows that the infected individuals of the endemic equilibrium vanish as the diffusion rate of the susceptible individuals approaches zero when the downstream end which is a *low-risk* site. Biologically, in the model with advection, since the diffusion rate of the susceptible individuals is very small, then the advection transports the susceptible individuals to the downstream end which is a “healthy” site, thus the disease is eliminated much more quickly, in comparison with the case of no advection. We notice that the susceptible individuals concentrate at the downstream end. This is also different from the case without advection in [2], where the endemic equilibrium converges to some disease free equilibrium which remains positive at all *low-risk* sites.

The rest of the paper is organized as follows: Sect. 2 is devoted to the proof of Theorem 1.1. Theorem 1.2 is proved in Sect. 3. Theorems 1.4 is established in Sect. 4.

# 2 Proof of Persistence Results (Theorem 1.1)

For notational convenience, in what follows we denote various constants by $C, C_i (i = 1, 2, \ldots)$. The constants $C, C_i$ may be different for different purposes.

**Lemma 2.1.** *Let* $\int_0^L [S_0(x) + I_0(x)]\,\mathrm{d}x = N$. *Then*

$$\liminf_{t \to \infty} \int_0^L \overline{S}(x, t)\,\mathrm{d}x \ge \frac{(\inf_{0<x<L} \gamma) N}{\|\beta\| + \inf_{0<x<L} \gamma}.$$

*In particular, for any solution* $(S(x), I(x))$ *of* (1.4), *we have*

$$\int_0^L S(x)\,\mathrm{d}x \geq \frac{(\inf_{0<x<L}\gamma)N}{\|\beta\| + \inf_{0<x<L}\gamma}. \tag{2.1}$$

*Proof.* It suffices to observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \overline{S}(x,t)\,\mathrm{d}x \geq (\min\gamma)N - [(\max\beta) + (\min\gamma)]\int_0^L \overline{S}(x,t)\,\mathrm{d}x$$

for all $t \geq 0$. □

## 2.1 Equation (1.1) generates a semiflow

Recall $X = \left\{(S_0, I_0) \in C([0,L];\mathbb{R}_+) : \int_0^L [S_0(x) + I_0(x)]\,\mathrm{d}x = N\right\}$.

**Definition 2.1.** (i) *We define* $\Phi$ *to be the semiflow generated by* (1.1). *i.e. for initial data* $P_0 = (S_0, I_0) \in X$ *and each* $t \geq 0$ *for which the solution remains in* $X$*, define* $\Phi_t(P_0) = (\bar{S}(x,t), \bar{I}(x,t))$*, where* $(\bar{S}, \bar{I})$ *is the corresponding solution of* (1.1).

(ii) *We say that* $\Phi$ *is point-dissipative if there exists* $C > 0$ *independent of initial condition such that*

$$\limsup_{t\to\infty}\left(\|\bar{S}(\cdot,t)\| + \|\bar{I}(\cdot,t)\|\right) \leq C.$$

(iii) *We say that* $\Phi$ *is eventually bounded on* $X$ *if* $\cup_{t\geq t_0}\Phi_t(X)$ *is bounded for some* $t_0 \geq 0$.

(iv) *For each* $t > 0$*, we say that* $\Phi_t : X \to X$ *is compact if* $\Phi_t(B)$ *is precompact for every bounded subset* $B$ *of* $X$.

**Proposition 2.2.** *Equation* (1.1) *generates a semiflow* $\Phi$ *in* $X$*. Moreover,* $\Phi$ *is (i) point-dissipative, (ii) eventually bounded on* $X$*, and (iii)* $\Phi_t : X \to X$ *is compact for each* $t > 1$.

*Proof.* By the local existence results for semilinear parabolic equations, for each initial data $P_0(S_0, I_0) \in X$, there exists $T = T(\|P_0\|_X) > 0$ such that $(\bar{S}, \bar{I})$ exists and remains bounded in $X$ for $t \in [0, T)$ (note also that the integral constraint (1.3) is always enforced).

**Claim 2.3.** *For each* $t_1 > 0$*, there exists* $C_1 = C_1(N, t_1)$ *independent of initial data* $(S_0, I_0) \in X$ *such that* $\|\bar{S}(\cdot,t)\| + \|\bar{I}(\cdot,t)\| \leq C_1$ *for all* $t \geq t_1$*. In fact,* $C_1$ *depends on* $N$ *and* $\min\{t_1, 1\}$ *only.*

Let $t_1 > 0$ be given. Applying the local maximum principal for parabolic equations [14, Theorem 7.36] to (1.1), there exists a constant $C_2 = C_2(p, \min\{t_1, 1\})$ such that

$$\sup_{\substack{0<x<L \\ \max\{2t_1/3,\ t_1-1\}<t<t_1+1}} |\bar{I}(x,t)| \le C_2 \int_{\max\{t_1-2,\ t_1/3\}}^{t_1+1} \int_0^L |\bar{I}(x,t)|\,\mathrm{d}x\,\mathrm{d}t \le 3C_2 N, \quad (2.2)$$

and for any $p > 0$,

$$\sup_{\substack{0<x<L \\ t_1<t<t_1+1}} \bar{S} \le C_2 \left( \|\bar{S}\|_{L^p([0,L]\times[\max\{t_1/2,\ t_1-1\},t_1+1])} + \|\bar{I}\|_{L^\infty([0,L]\times[\max\{2t_1/3,\ t_1-1\},\infty))} \right). \quad (2.3)$$

Since $C_2$ is independent of $t_1 \ge 1$, (2.2) says that $\|\bar{I}\|_{L^\infty([0,L]\times[\max\{2t_1/3,\ t_1-1\},\infty))}$ is bounded. Thus $\|\bar{S}\|_{L^\infty([0,L]\times[t_1,\infty))}$ is also bounded, by (2.3). The proof of Claim 2.3 is completed.

By the above arguments, (1.1) generates a semiflow $\Phi$ in $X$. Furthermore, Claim 2.3 says that $\Phi$ is point-dissipative and eventually bounded in $X$.

Finally, it follows from Sobolev inequalities and parabolic $L^p$ estimates that for each $t_0 > 1$ and $\alpha \in (0,1)$, there exists $C_2'$ such that

$$\|(\bar{S},\bar{I})\|_{C^{\alpha,\alpha/2}([0,L]\times[t_0-1/2,t_0+1])} \le C_2' \|(\bar{S},\bar{I})\|_{L^\infty([0,L]\times[t_0-1,t_0+1])}. \quad (2.4)$$

Since $C_2'$ is independent of $t_0 > 1$ and the initial data $(S_0, I_0) \in X$, we have

$$\|\Phi_t((S_0,I_0))\|_{C^\alpha([0,L])} \le C_2' N.$$

In particular, $\Phi_t : X \to X$ is compact for each $t > 1$. □

**Remark 4.** *By* (2.4)*, for each* $P_0 = (S_0, I_0) \in X$*, the family of limiting total trajectories, defined below, is non-empty:*

$$\left\{ (\hat{S},\hat{I}) \in C(\mathbb{R};X) :\ \exists t_k \to \infty,\ \Phi_{t+t_k}(P_0) \to (\hat{S}(\cdot,t),\hat{I}(\cdot,t)) \ \ \text{in } C_{loc}(\mathbb{R};X) \right\}.$$

**Corollary 2.4** (Existence of compact global attractor)**.** *The semiflow* $\Phi$ *has a compact attractor* $A$ *of* $X$*, i.e.* $\mathrm{dist}_X(\Phi_t(X), A) \to 0$ *as* $t \to \infty$*.*

*Proof.* By [24, P.41, Theorem 2.30 and P.39, Remark 2.26(b)], it remains to verify that the semiflow $\Phi$ is (i) point-dissipative; (ii) eventually bounded in $X$, and that (iii) $\Phi_t : X \to X$ is compact for some $t > 0$. All of which are proved in Proposition 2.2. □

## 2.2 $\mathcal{R}_0 \le 1$: The Global Asymptotic Stability of the DFE $(\tilde{S}, 0)$

We decompose the state space $X$ according to the persistence and extinction of the infected population:

$$X_0 = \{(S_0,I_0) \in X : I_0 \not\equiv 0\} \quad \text{and} \quad \partial X_0 = X \setminus X_0 = \{(S_0,I_0) \in X : I_0 \equiv 0\}.$$

It is easy to see that $\partial X_0$ is a forward invariant set with respect to $\Phi$. The following lemma shows that $(\tilde{S}, 0)$ is globally asymptotically stable among solutions in $\partial X_0$.

**Lemma 2.5.** *If $I_0 \equiv 0$, then the solution $(\bar{S}, \bar{I})$ of* (1.1) *converges to the DFE $(\tilde{S}, 0)$ as $t \to \infty$, where $\tilde{S}(x) = \frac{Nq}{d_S(1-e^{-qL/d_S})} e^{-q(L-x)/d_S}$.*

*Proof.* It is easy to see that $\bar{I}(x,t) = 0$ for all $0 \le x \le L$ and $t \ge 0$. By spectral decomposition, and the fact that 0 is the principal eigenvalue of

$$\begin{cases} d_S \varphi_{xx} - q\varphi_x + \mu\varphi = 0, & 0 < x < L, \\ d_S \varphi_x - q\varphi = 0, & x = 0,\ L, \end{cases}$$

with corresponding eigenfunction $e^{qx/d_S}$, there exists $C \in \mathbb{R}$ such that $\bar{S}(x,t) \to Ce^{qx/d_S}$ as $t \to \infty$. By the integral constraint $\int_0^L (S + I)\, \mathrm{d}x = N$, we must have $C = \frac{Nqe^{-qL/d_S}}{d_S(1-e^{-qL/d_S})}$. This finishes the proof. $\square$

Before we prove Theorem 1.1(a), we give the following definitions.

**Definition 2.2.** (i) *Let $(\lambda_1, \phi_1)$ be the principal eigenvalue and positive eigenfunction of*

$$\begin{cases} d_I \phi_{xx} - q\phi_x + (\beta - \gamma)\phi + \lambda\phi = 0, & 0 < x < L, \\ d_I \phi_x - q\phi = 0, & x = 0,\ L. \end{cases} \tag{2.5}$$

*By [5, Lemma 2.2], $\mathcal{R}_0 \le 1$ if and only if $\lambda_1 \ge 0$.*

(ii) *For each $t \ge 0$ and each solution $(\bar{S}, \bar{I})$ of* (1.1)*, define*

$$c(t; \bar{S}, \bar{I}) := \inf \left\{ \tilde{c} \in \mathbb{R}_+ : \bar{I}(x,t) \le \tilde{c}\phi_1(x) \ \ \textit{for all } 0 \le x \le L \right\}. \tag{2.6}$$

(iii) *A forward invariant set $A \subset X$ is said to be chain transitive if for any $P, P' \in A$, any $0 < \epsilon < 1$ and $T > 1$, there is a finite sequence of points $P = P_1, ..., P_{m+1} = P'$ and times $t_1, ..., t_m$ such that*

$$t_j \ge T, \qquad \textit{and} \qquad \|\Phi_{t_j}(P_j) - P_{j+1}\| \le \epsilon, \qquad \textit{for } j = 1, ..., m.$$

*Proof of Theorem 1.1(a).* Let $(\bar{S}, \bar{I})$ be the solution of (1.1) with initial data $(S_0, I_0) \in X$. We will show that $(\bar{S}, \bar{I}) \to (\tilde{S}, 0)$ as $t \to \infty$ in four steps. By Lemma 2.5, we may assume $I_0 \not\equiv 0$.

**Step 1:** For each solution $(\bar{S}, \bar{I})$ for which $I_0 \not\equiv 0$, $\bar{S} > 0$ for all $0 \le x \le L$ and $t > 0$.

By the strong maximum principle, $\bar{I}(x,t) > 0$ for all $0 \le x \le L$ and $t > 0$. Thus the trivial solution is a strict lower solution of the first equation of (1.1), and Step 1 follows.

**Step 2:** If $\mathcal{R}_0 \leq 1$, then for each solution $(\bar{S}, \bar{I})$ for which $I_0 \not\equiv 0$, $c(t; \bar{S}, \bar{I})$ is strictly decreasing in $t$. In particular $c_* := \inf_{t>0} c(t; \bar{S}, \bar{I}) = \lim_{t\to\infty} c(t; \bar{S}, \bar{I}) \geq 0$ exists.

Fix $t_0 \geq 0$. Since $\mathcal{R}_0 \leq 1$, it follows from [5, Lemma 2.2] that the principal eigenvalue $\lambda_1 \geq 0$. One can then verify that $I^*(x,t) := c(t_0; \bar{S}, \bar{I})\phi_1(x)$ satisfies

$$\begin{cases} I^*_t - d_I I^*_{xx} + q I^*_x - \left[\beta(x)\bar{S}/(\bar{S}+I^*) - \gamma(x)\right] I^* > 0, & 0 \leq x \leq L,\ t \geq t_0, \\ d_I I^*_x - q I^* = 0, & x = 0,\ L,\ t \geq t_0, \\ I^*(x,t_0) \geq \bar{I}(x,t_0), & 0 \leq x \leq L. \end{cases}$$

By the strong maximum principle, it follows that $I^* > \bar{I}$ for all $0 \leq x \leq L$ and $t > t_0$. This shows that $c(t; \bar{S}, \bar{I}) < c(t_0; \bar{S}, \bar{I})$ for all $t > t_0$. Step 2 is completed.

**Step 3:** For any solution $(\bar{S}, \bar{I})$ of (1.1) for which $I_0 \not\equiv 0$, we have $c_* = 0$.

Suppose to the contrary that for some $P_0 = (S_0, I_0)$, $c_* > 0$. Choose, by Remark 4, a limiting total trajectory $(\hat{S}, \hat{I}) \in C(\mathbb{R}; X)$, i.e. there exists $t_k \to \infty$ such that $\Phi_{t+t_k}(P_0) \to (\hat{S}(\cdot,t), \hat{I}(\cdot,t))$ in $C_{loc}(\mathbb{R}; X)$. Then it follows that $c(t; \hat{S}, \hat{I}) \equiv c_* > 0$ for all $t \in \mathbb{R}$, i.e. $\hat{I}(x,t) > 0$ for all $x, t$. By Lemma 2.1, one also have $\hat{S} > 0$ for all $x, t$. One can then repeat Step 1 to show that $c(t; \hat{S}, \hat{I})$ is also strictly decreasing for all $t \geq 0$. This contradiction completes the proof of Step 3.

**Step 4:** $\Phi_t(P_0) = (\bar{S}(\cdot,t), \bar{I}(\cdot,t)) \to (\tilde{S}(\cdot), 0)$ in $X$ as $t \to \infty$ for each $P_0 = (S_0, I_0) \in X$ with $I_0 \not\equiv 0$.

By Step 3, we have $c_* = 0$ which implies that $\bar{I}(x,t) \to 0$ as $t \to \infty$. Therefore

$$\omega(P_0) \in \left\{ (\hat{S}, \hat{I}) \in C(\mathbb{R}; X) : \hat{I} \equiv 0 \right\}.$$

Thus $\omega(P_0)$ is a compact, and chain transitive ([24, P.81, Proposition 8.6]; see also [29, P.8, Lemma 1.2.1']) subset of $\partial X_0 = \{(S_0, I_0) \in X : I_0 \equiv 0\}$. Since the DFE $(\tilde{S}, 0)$ is globally asymptotically stable among solutions in $\partial X_0$ (Lemma 2.5), it follows that $\omega(P_0) = \{(\tilde{S}, 0)\}$. This completes the proof of the theorem. □

### 2.3 $\mathcal{R}_0 > 1$: Persistence and Existence of EE

**Lemma 2.6.** *There exists $C_3$ independent of initial data $(S_0, I_0) \in X$ such that for any $t_1 \geq 4$,*

$$N \leq C_3 \left( \inf_{\substack{0<x<L \\ t>t_1+2}} \overline{S} + \|\overline{I}\|_{L^\infty([0,L]\times[t_1-2,\infty))} \right). \tag{2.7}$$

*Proof.* By the weak Harnack inequality [14, Theorem 7.37], there exist positive constants

$C_3'$, $p_0$ independent of $t_1 \geq 4$, and initial data $S_0, I_0$ such that

$$\|\overline{S}\|_{L^{p_0}([0,L]\times[t_1-1,t_1+1])} \leq C_3' \left( \inf_{\substack{0\,<\,x\,<\,L \\ t_1\,+\,2\,<\,t\,<\,t_1\,+\,3}} \overline{S} + \|\overline{I}\|_{L^\infty([0,L]\times[t_1-2,\infty))} \right). \tag{2.8}$$

Combining it with

$$N - \int_0^L \overline{I}(x,t)\,\mathrm{d}x = \int_0^L \overline{S}(x,t)\,\mathrm{d}x \leq L \sup_{0<x<L} \overline{S}(x,t),$$

(which follows from (1.3)) and also choosing $p = p_0$ in (2.3) we have, for some $C_3 > 0$,

$$N \leq C_3 \left( \inf_{\substack{0\,<\,x\,<\,L \\ t_1\,+\,2\,<\,t\,<\,t_1\,+\,3}} \overline{S} + \|\overline{I}\|_{L^\infty([0,L]\times[t_1-2,\infty))} \right). \tag{2.9}$$

Since $t_1 \geq 4$ is arbitrary, (2.7) is proved. $\square$

**Definition 2.3.** (i) *Define function $\rho : X \to \mathbb{R}_+$ by $\rho((S,I)) = \inf_{0<x<L} I(x)$.*

(ii) *We say that $\Phi$ is uniformly weakly $\rho$-persistent if there exists $\epsilon > 0$ independent of initial condition $(S_0, I_0) \in X$ such that any solution to (1.1) satisfies*

$$\limsup_{t\to\infty} \rho(\bar{S}(\cdot,t), \bar{I}(\cdot,t)) \geq \epsilon.$$

**Lemma 2.7.** *$\Phi$ is uniformly weakly $\rho$-persistent if $\mathcal{R}_0 > 1$.*

*Proof.* Suppose $\mathcal{R}_0 > 1$. By [5, Lemma 2.2], the DFE $(\tilde{S}, 0)$ is linearly unstable, i.e. the principal eigenvalue of the problem

$$\begin{cases} d_I\phi_{xx} - q\phi_x + (\beta-\gamma)\phi + \lambda\phi = 0 & \text{for } 0 < x < L, \\ d_I\phi_x - q\phi = 0 & \text{for } x = 0, L \end{cases}$$

is negative. Therefore, there exists $0 < \delta_1 < 1$ such that the principal eigenvalue $\hat{\lambda}_1$ of

$$\begin{cases} d_I\phi_{xx} - q\phi_x + [(1-\delta_1)\beta - \gamma]\phi + \hat{\lambda}\phi = 0 & \text{for } 0 < x < L, \\ d_I\phi_x - q\phi = 0 & \text{for } x = 0, L \end{cases} \tag{2.10}$$

is negative. We denote by $\hat{\phi}_1$ a positive eigenfunction corresponding to the principal eigenvalue $\hat{\lambda}_1$ of (2.10).

**Claim 2.8.** *Let $\delta_2 = \frac{N\delta_1}{2C_3}$, where $C_3$ is given by Lemma 2.6, then*

$$\limsup_{t\to\infty} \inf_{0<x<L} I(x,t) \geq \delta_2.$$

Suppose to the contrary that for some $t_1 \geq 4$, $\inf_{0<x<L} \bar{I}(x,t) < \delta_2$ for all $t \geq t_1 - 2$, then by Lemma 2.6, we deduce that

$$\bar{S}(x,t) \geq \frac{N}{C_3} - \delta_2 \quad \text{for } 0 < x < L,\, t \geq t_1 + 2.$$

Hence

$$\frac{\bar{S}(x,t)}{\bar{S}(x,t) + \bar{I}(x,t)} \geq 1 - \delta_1 \quad \text{for } 0 < x < L,\, t \geq t_1 + 2.$$

Therefore, we deduce that $I(x,t)$ is a supersolution of

$$\begin{cases} w_t = d_I w_{xx} - q w_x + [(1-\delta_1)\beta - \gamma] I, & 0 < x < L,\, t > t_2 + 2, \\ d_I w_x - q w = 0, & x = 0, L,\, t > t_2 + 2. \end{cases}$$

Since for each $\epsilon > 0$, $I_*(x,t) = \epsilon e^{-\hat{\lambda}_1 (t - t_1 - 2)} \hat{\phi}_1(x)$ is a (sub)solution of the above problem with $\hat{\lambda}_1 < 0$, it is impossible that $\bar{I}(x,t) \leq \delta_2$ for all $0 < x < L$ and $t \geq t_1 + 2$. This contradiction establishes Claim 2.8, i.e. $\Phi$ is uniformly weakly $\rho$-persistent. □

**Remark 5.** *Part of the arguments of the above proof can be simplified further by invoking [25, Theorem 3].*

Finally, we use topology to show Theorem 1.1(b)(ii) and then focus on proving Theorem 1.1 (b)(iii) in Subsection 2.4.

**Proposition 2.9.** *Suppose $\mathcal{R}_0 > 1$, then there exists $(S_e, I_e) \in X$ such that $\rho((S_e, I_e)) = \inf_{0<x<L} I_e(x) > 0$ and $\Phi_t((S_e, I_e)) = (S_e, I_e)$ for all $t > 0$.*

*Proof.* Assume $\mathcal{R}_0 > 1$. We have shown that (i) the semiflow $\Phi$ is uniformly weakly $\rho$-persistent (Lemma 2.7), (ii) $\Phi_t : X \to X$ is compact for each $t > 1$ (Proposition 2.2), and (iii) $\Phi$ has a compact attractor of $X$ (Corollary 2.4). Observe in additional the following facts:

- $X$ is a closed convex subset of the Banach space $C([0, L]; \mathbb{R}^2)$.
- $\rho : X \to \mathbb{R}_+$ is continuous and concave. Here concave means $\rho(\lambda(S_1, I_1) + (1 - \lambda)(S_2, I_2)) \geq \lambda \rho((S_1, I_1)) + (1 - \lambda)\rho((S_2, I_2))$, which is true for infimums.

The existence of an EE $(S_e, I_e)$ then follows from [24, P. 158, Theorem 6.2]. □

## 2.4 $\mathcal{R}_0 > 1$: Global convergence to $EE$ when $d_S = d_I = d$.

In this subsection, we will prove Theorem 1.1(b), when $d_S = d_I = d > 0$.

**Lemma 2.10.** *Let $d_S = d_I = d > 0$ and $\mathcal{R}_0 > 1$, then there exists a unique EE $(S_e, I_e)$ of* (1.1)*, and that*

$$N_e(x) := S_e(x) + I_e(x) = \frac{qNe^{-q(L-x)/d}}{d(1-e^{-qL/d})}. \tag{2.11}$$

*Proof.* First, let $(S_e, I_e)$ be an EE given by Theorem 1.4. First, $N_e$ satisfies the equation

$$\begin{cases} d(N_e)_{xx} - q(N_e)_x = 0, & 0 < x < L, \\ d(N_e)_x - qN_e = 0, & x = 0,\ L, \\ \int_0^L N_e \, \mathrm{d}x = N, \end{cases}$$

which implies (2.11). Hence, $I_e$ is a positive steady state of

$$\begin{cases} \bar{I}_t = d\bar{I}_{xx} - q\bar{I}_x + \frac{\beta(N_e(x)-\bar{I})\bar{I}}{N_e(x)} - \gamma\bar{I} \quad , & 0 < x < L,\ t > 0, \\ d\bar{I}_x - q\bar{I} = 0, & x = 0,\ L,\ t > 0, \\ \bar{I}(x,0) = \bar{I}_0(x), & 0 < x < L. \end{cases} \tag{2.12}$$

Since (2.12) is of the logistic type, it possesses at most one positive steady state (see, e.g. [4, P.148, Proposition 3.3]). Thus $I_e$ and $S_e = N_e - I_e$ are uniquely determined. $\square$

*Proof of Theorem 1.1(b).* By Lemma 2.7, $\Phi$ is uniformly weakly $\rho$-persistent. Since also $\rho \circ \Phi$ is continuous, we may apply [24, P.126, Theorem 5.2] to conclude (i). The assertion (ii) follows from Proposition 2.9. To prove assertion (iii), assume $d_S = d_I = d > 0$, $q \geq 0$, and $\mathcal{R}_0 > 1$, and consider the following system with respect to $(\bar{I}, W)$, which is equivalent to (1.1) via the relation $W := \bar{S} + \bar{I}$:

$$\begin{cases} \bar{I}_t = d\bar{I}_{xx} - q\bar{I}_x + \frac{\beta(W-\bar{I})\bar{I}}{W} - \gamma\bar{I}, & 0 < x < L,\ t > 0, \\ W_t = dW_{xx} - qW_x, & 0 < x < L,\ t > 0, \\ d\bar{I}_x - q\bar{I} = dW_x - qW = 0, & x = 0,\ L,\ t > 0, \\ \bar{I}(x,0) = \bar{I}_0(x), \quad W(x,0) = W_0(x) := \bar{S}_0(x) + \bar{I}_0(x), & 0 < x < L. \end{cases} \tag{2.13}$$

Our goal is to show that for each initial condition $(I_0, W_0)$ with $I_0 \not\equiv 0$, the $\omega$-limit set $\omega((I_0, W_0)) = \{(I_e, I_e + S_e)\}$, where $(S_e, I_e)$ is the unique EE given by Lemma 2.10. By Lemma 2.5, $W(x,t) \to N_e(x)$ as $t \to \infty$, uniformly in $0 \leq x \leq L$, where $N_e$ is given by (2.11). Therefore, the equation of $I$ in (2.13) is asymptotic to (2.12).

**Claim 2.11.** *Let $A_2$ be a compact, invariant, internal chain-transitive subset of $C([0, L]; \mathbb{R}_+)$ with respect to the semiflow generated by* (2.12)*, then $A_2 = \{0\}$ or $\{I_e(x)\}$.*

By the proof of Lemma 2.10, $I_e$ is the unique positive steady state of (2.12). In fact, by the remarks [4, P.150], $I_e$ attracts all solutions of (2.12) with non-negative, non-trivial initial data. This proves the claim.

Now, fix an initial data $(I_0, W_0) \in X'$ such that $I_0 \not\equiv 0$, where

$$X' := \left\{ (\tilde{I}, \tilde{W}) \in C([0, L]; \mathbb{R}^2_+) : \tilde{I} \leq \tilde{W} \quad \text{and} \quad \int_0^L \tilde{W} \, \mathrm{d}x = N \right\}.$$

(Note that $(S_0, I_0) \in X$ iff $(I_0, S_0 + I_0) \in X'$.) And let $B = \omega((I_0, W_0))$ be the $\omega$-limit set of the point $(I_0, W_0)$ with respect to the semiflow generated by (2.13). Since the solution $(\bar{I}, W)$ satisfies $W(x, t) \to N_e(x)$ as $t \to \infty$, we have $B = \tilde{B} \times \{N_e(x)\}$. By nature of being an $\omega$-limit set, $B$ is compact, invariant and chain transitive with respect to the semiflow generated by (2.13). This implies that $\tilde{B}$ is compact, invariant and chain transitive with respect to the semiflow generated by (2.12). Thus we may conclude from Claim 2.11 that $\tilde{B} = \{0\}$ or $\{I_e\}$ and that $B = \{(0, N_e)\}$ or $\{(I_e, N_e)\}$. By Lemma 2.7 (specifically Claim 2.8), $\bar{I} \not\to 0$, and hence $B = \{(I_e, N_e)\}$. i.e. $(\bar{I}(x, t), W(x, t)) \to (I_e(x), N_e(x))$ as $t \to \infty$ uniformly in $0 \leq x \leq L$. This completes the proof of Theorem 1.1(b). $\square$

# 3 Concentration phenomenon

This section is devoted to the proof of Theorem 1.2. From now on, for any given continuous function $f(x)$ on $[0, L]$, denote $\|f\| = \|f\|_{L_\infty((0,L))}$. For the rest of the paper, for any $q > 0$, $d_S > 0$ and $d_I > 0$, we drop the subscript "e" and denote the endemic equilibrium of (1.4) by $(S(x), I(x))$.

First we start with an elementary lemma.

**Lemma 3.1.** *Let $q^2/d \geq 4 \sup_{0<x<L} c(x)$ and*

$$\begin{cases} du_{xx} - qu_x + c(x)u \leq 0, & 0 < x < L, \\ -du_x(0) + qu(0) \geq 0, \;\; u(L) \geq 0, & \end{cases}$$

*then either $u \equiv 0$ or $u > 0$ in $[0, L)$.*

*Proof.* Let $v(x) = e^{-qx/2d} u(x)$, then $v(x)$ satisfies

$$\begin{cases} dv_{xx} + \tilde{c}(x)v \leq 0, & 0 < x < L, \\ -dv_x(0) + \frac{q}{2}v(0) \geq 0, \;\; v(L) \geq 0, & \end{cases}$$

where $\tilde{c}(x) = c(x) - q^2/4d \leq 0$ by the assumption. By the strong maximum principle, either $v(x) \equiv 0$ or $v(x) > 0$ for $x \in [0, L)$. $\square$

**Lemma 3.2.** *Set $C_* = \|\beta\| + \|\gamma\| + 2$ and consider any EE $(S(x), I(x))$ of (1.4).*

(i) *Assume $q^2/d_I \geq C_*^2$. Then $I^-(x) \leq I(x) \leq I^+(x)$ for $0 \leq x \leq L$, where*

$$I^\pm(x) := I(L) e^{-\left(\frac{q}{d_I} \mp \frac{C_*}{q}\right)(L-x)}; \tag{3.1}$$

(ii) *Assume $q^2/d_I \geq C_*^2$. For each $\eta_0 > 0$, if $d_S \geq d_I/\eta_0$, then*

$$\max\{S^-(x), 0\} \leq S(x) \leq S^+(x) \quad \textit{for } 0 \leq x \leq L,$$

*where*

$$S^\pm(x) := S(L)e^{-\frac{q}{d_S}(L-x)} \pm C_4 \frac{d_S I(L)}{q^2} e^{-\frac{\delta q}{d_S}(L-x)}, \tag{3.2}$$

*where $\delta = \frac{1}{2}\min\{1, 1/\eta_0\}$ and $C_4 = \max\{\|\beta\|, \|\gamma\|\}/[\delta(1-\delta)]$.*

*Proof.* We only prove the upper bound of (i) in detail, and briefly comment on proof for the lower bound, which follows from analogous arguments. For $q^2/d_I \geq C_*^2$,

$$\begin{aligned}
L_I[I^+] &:= d_I I^+_{xx} - qI^+_x + \Big(\beta\frac{S}{S+I} - \gamma\Big)I^+ \\
&= \Big[d_I\Big(\frac{q}{d_I} - \frac{C_*}{q}\Big)^2 - q\Big(\frac{q}{d_I} - \frac{C_*}{q}\Big) + \Big(\beta\frac{S}{S+I} - \gamma\Big)\Big]I^+ \\
&= \Big[-C_* + C_*^2\frac{d_I}{q^2} + \Big(\beta\frac{S}{S+I} - \gamma\Big)\Big]I^+ \\
&\leq \Big[-C_* + C_*^2\frac{d_I}{q^2} + \|\beta\|\Big]I^+ \\
&\leq \Big[-1 + C_*^2\frac{d_I}{q^2}\Big]I^+ \leq 0,
\end{aligned}$$

(the second inequality used the choice of $C_*$, the last inequality used assumption $q^2/d_I \geq C_*^2$)

$$\begin{cases} -d_I I^+_x(0) + qI^+(0) = \frac{d_I C_*}{q} I^+(0) \geq 0, \\ I^+(L) = I(L). \end{cases}$$

This allows the application of Lemma 3.1 to $I^+(x) - I(x)$, proving the upper bound. One can similarly check that $I^-(x)$ is a lower solution with respect to the equation of $I$ in (1.1), and show the lower bound by applying Lemma 3.1 to $I(x) - I^-(x)$. This concludes the proof of (i).

Next, we prove the assertion (ii). Observe by the choices of $\delta$ and $C_*$ that

$$\frac{\delta q}{d_S} \leq \frac{d_S}{2d_I}\cdot\frac{q}{d_S} = \frac{q}{2d_I} = \left(\frac{q}{d_I} - \frac{C_*}{q}\right) - \left(\frac{q}{2d_I} - \frac{C_*}{q}\right) \leq \frac{q}{d_I} - \frac{C_*}{q}. \tag{3.3}$$

By combining assertion (i) of the lemma and (3.3), we have

$$I(x) \leq I(L)e^{-\left(\frac{q}{d_I} - \frac{C_*}{q}\right)(L-x)} \leq I(L)e^{-\frac{\delta q}{d_S}(L-x)}. \tag{3.4}$$

Recall the definition of $S^-(x) = S(L)e^{-\frac{q}{d_S}(L-x)} - C_4\frac{d_S}{q^2}I(L)e^{-\frac{\delta q}{d_S}(L-x)}$, then

$$
\begin{aligned}
d_S S^-_{xx} - qS^-_x + \Big(-\beta\frac{S}{S+I} + \gamma\Big)I &\geq d_S S^-_{xx} - qS^-_x - \|\beta\| I \\
&= C_4\delta(1-\delta)I(L)e^{-\frac{\delta q}{d_S}(L-x)} - \|\beta\| I \\
&\geq [C_4\delta(1-\delta) - \|\beta\|]I(L)e^{-\frac{\delta q}{d_S}(L-x)} \geq 0,
\end{aligned}
$$

where we used (3.4) for the second inequality, and the choice of $C_6$ for the last inequality. For the boundary conditions, by the facts that $C_4 > 0$ and $0 < \delta < 1$ we obtain

$$
d_S S^-_x(0) - qS^-(0) = C_4\frac{(1-\delta)d_S}{q}e^{-\frac{\delta q}{d_S}(L-x)} > 0,
$$

and $S^-(L) < S(L)$. Hence, the function $W_1(x) = S(x) - S^-(x)$ satisfies

$$
\begin{cases}
d_S(W_1)_{xx} - q(W_1)_x \leq 0 & \text{for } 0 < x < L, \\
d_S(W_1)_x(0) - qW_1(0) \leq 0 & \text{and} \quad W_1(L) \geq 0.
\end{cases}
$$

We may then apply Lemma 3.1 to $W_1(x) = S(x) - S^-(x)$ to conclude that $S(x) \geq S^-(x)$ for $x \in [0, L]$. Since $S(x) \geq 0$ is always satisfied, the lower bound is proved.

Finally, we can similarly apply Lemma 3.1 to $S^+(x) - S(x)$, where $S^+(x)$ is given in (3.2), to show the upper bound assertion in (ii). We omit the details. □

**Lemma 3.3.** *For each $\eta_0 \geq 1$, there exist constants $C_5, C_5' > 0$ such that if $q/d_I \geq 1/\eta_0, q^2/d_S \geq C_5$ and $d_I/d_S \leq \eta_0$, then $I(L) \leq C_5' S(L)$.*

*Proof.* Suppose to the contrary that there exist some $\eta_0 > 0$, a sequence of parameters $(d_{S,j}, d_{I,j}, q_j)$ satisfying $q_j/d_{I,j} \geq 1/\eta_0, q_j^2/d_{S,j} \to \infty, d_{I,j}/d_{S,j} \leq \eta_0$, and $(S_j, I_j)$, which is a sequence of EE of (1.4) with $(d_S, d_I, q) = (d_{S,j}, d_{I,j}, q_j)$, satisfying $I_j(L)/S_j(L) \to \infty$.

Integrating the first equation of (1.4) in $(0, L)$, applying the boundary condition in (1.4) and dividing the result by $I_j(L)$, we have

$$
\int_0^L \gamma(x)\frac{I_j(x)}{I_j(L)}\,\mathrm{d}x = \frac{1}{I_j(L)}\int_0^L \beta(x)\frac{S_j(x)I_j(x)}{S_j(x) + I_j(x)}\,\mathrm{d}x.
$$

By the fact that the function $g(S, I) = \frac{SI}{S+I}$ is increasing in both $S \geq 0$ and $I \geq 0$, we may use the upper and lower bounds $S^{\pm}$ and $I^{\pm}$ of $S$ and $I$ obtained in Lemma 3.2 to get

$$
\int_0^L \gamma(x)\frac{I^-(x)}{I_j(L)}\,\mathrm{d}x \leq \frac{1}{I_j(L)}\int_0^L \beta(x)\frac{S^+(x)I^+(x)}{S^+(x) + I^+(x)}\,\mathrm{d}x, \tag{3.5}
$$

where $q, d_I, d_S, S(L), I(L)$ in (3.1) and (3.2) are being replaced by $q_j, d_{I,j}, d_{S,j}, S_j(L), I_j(L)$, respectively.

Let $y = q_j(L-x)/d_{I,j}$, $\eta_j = d_{I,j}/d_{S,j}$ so that $\eta_j y = q_j(L-x)/d_{S,j}$, and (3.5) becomes

$$\begin{aligned}
&\big(\inf_{0<x<L}\gamma\big)\int_0^{q_jL/d_{I,j}} e^{-(1+C_* d_{I,j}/q_j^2)y}\,\mathrm{d}y \\
&\le \|\beta\|\int_0^{q_jL/d_{I,j}} \frac{S_j(L)e^{-\eta_j y}+C_4 I_j(L)\frac{d_{S,j}}{q_j^2}e^{-\delta\eta_j y}}{S_j(L)e^{-\eta_j y}+C_4 I_j(L)\frac{d_{S_j}}{q_j^2}e^{-\delta\eta_j y}+I_j(L)e^{-(1-C_* d_{I,j}/q_j^2)y}}e^{-(1-C_* d_{I,j}/q_j^2)y}\,\mathrm{d}y.
\end{aligned} \tag{3.6}$$

Rewrite the right hand of (3.6) and estimate it by again making use of the monotonicity of $\frac{SI}{S+I}$ in $S$ and $I$, we have (denoting $\alpha_j = I_j(L)/S_j(L)$ and $C_{5,j} = d_{S,j}/q_j^2$)

$$\begin{aligned}
&\|\beta\|\int_0^{q_jL/d_{I,j}} \frac{e^{-\eta_j y}/\alpha_j + C_4\frac{d_{S,j}}{q_j^2}e^{-\delta\eta_j y}}{e^{-\eta_j y}/\alpha_j + C_4\frac{d_{S,j}}{q_j^2}e^{-\delta\eta_j y}+e^{-(1-C_* d_{I,j}/q_j^2)y}}e^{-(1-C_* d_{I,j}/q_j^2)y}\,\mathrm{d}y \\
&\le \|\beta\|\int_0^{q_jL/d_{I,j}} \frac{e^{-\eta_j y}/\alpha_j + C_4C_{5,j}e^{-\delta\eta_j y}}{e^{-\eta_j y}/\alpha_j + C_4C_{5,j}e^{-\delta\eta_j y}+e^{-\frac{y}{2}}}e^{-\frac{y}{2}}\,\mathrm{d}y \\
&\le \|\beta\|\int_0^{q_jL/d_{I,j}} \frac{1/\alpha_j + C_4C_{5,j}}{1/\alpha_j + C_4C_{5,j}+e^{-\frac{y}{2}}}e^{-\frac{y}{2}}\,\mathrm{d}y,
\end{aligned} \tag{3.7}$$

where the first inequality follows from, for sufficiently large $j$,

$$\frac{d_{I,j}}{q_j^2} = \frac{d_{I,j}}{d_{S,j}}\cdot\frac{d_{S,j}}{q_j^2} \le \eta_0\cdot\frac{d_{S,j}}{q_j^2} \le \frac{1}{2C_*}.$$

Again from $q_j^2/d_{I,j} \ge 2C_*$, the left hand of (3.6) satisfies

$$\big(\inf_{0<x<L}\gamma\big)\int_0^{q_jL/d_{I,j}} e^{-(1+C_* d_{I,j}/q_j^2)y}\,\mathrm{d}y \ge \big(\inf_{0<x<L}\gamma\big)\int_0^{q_jL/d_{I,j}} e^{-\frac{3y}{2}}\,\mathrm{d}y. \tag{3.8}$$

From (3.7) and (3.8) and letting $j\to\infty$, while using Lebesgue's Dominated Convergence and the fact that

$$1/\eta_0 \le \liminf_{j\to\infty}\frac{q_j}{d_{I,j}} \le \limsup_{j\to\infty}\frac{q_j}{d_{I,j}} \le +\infty,$$

we have

$$\begin{aligned}
\frac{2}{3}\big(\inf_{0<x<L}\gamma\big)\Big(1-e^{-\frac{3L}{2\eta_0}}\Big) &= \big(\inf_{0<x<L}\gamma\big)\int_0^{\frac{L}{\eta_0}} e^{-\frac{3y}{2}}\,\mathrm{d}y \le \lim_{j\to\infty}\big(\inf_{0<x<L}\gamma\big)\int_0^{q_jL/d_{I,j}} e^{-\frac{3y}{2}}\,\mathrm{d}y \\
&\le \lim_{j\to\infty}\|\beta\|\int_0^{q_jL/d_{I,j}} \frac{1/\alpha_j + C_4C_{5,j}}{1/\alpha_j + C_4C_{5,j}+e^{-\frac{y}{2}}}e^{-\frac{y}{2}}\,\mathrm{d}y \\
&= 0,
\end{aligned}$$

as $\alpha_j\to\infty, C_{5,j}\to 0$. This contradiction establishes the boundedness of $\alpha_j$. i.e. $I(L)/S(L) = O(1)$. $\square$

**Lemma 3.4.** *For each $\eta_0 > 0$, there exist $C_6, C_6' > 0$ such that if $q/d_S, q^2/d_S \geq C_6$ and $d_I/d_S \leq \eta_0$, then $S(L) \leq C_6' q/d_S$.*

*Proof.* Integrating the estimates of $S(x)$ in Lemma 3.2 from 0 to $L$, we get

$$\left(1 - e^{-\frac{q}{d_S}L}\right) - C_4 C_5' \frac{d_S}{q^2}\left(1 - e^{-\frac{\delta q}{d_S}L}\right) \leq \frac{q\int_0^L S(x)\,\mathrm{d}y}{d_S S(L)}, \tag{3.9}$$

where $C_4$ and $C_5'$ are given in Lemmas 3.2 and 3.3 respectively. From (1.5), we know

$$S(L) \leq \frac{q}{d_S} \frac{N}{\left(1 - e^{-\frac{q}{d_S}L}\right) - C_4 C_5' \frac{d_S}{q^2}\left(1 - e^{-\frac{\delta q}{d_S}L}\right)} \leq \frac{q}{d_S} \cdot 2N,$$

provided $q/d_S$ and $q^2/d_S$ are sufficiently large. □

**Lemma 3.5.** *There exists $C_7, C_7' > 0$ such that for any EE of (1.1), if $q/d_I, q^2/d_I \geq C_7$, then*

$$\left| I(x) - I(L)e^{-\frac{q}{d_I}(L-x)} \right| \leq C_7' I(L) \frac{d_I}{q^2} e^{-\frac{q}{2d_I}(L-x)} \tag{3.10}$$

*for all* $0 \leq x \leq L$.

*Proof.* By Lemma 3.2(i),

$$I(L)^{-\frac{q}{d_I}(L-x)}\left[e^{-\frac{C_*}{q}(L-x)} - 1\right] \leq I(x) - I(L)e^{-\frac{q}{d_I}(L-x)} \leq I(L)^{-\frac{q}{d_I}(L-x)}\left[e^{\frac{C_*}{q}(L-x)} - 1\right] \tag{3.11}$$

for $0 \leq x \leq L$, where $C_* = \|\beta\| + \|\gamma\| + 2$. Next, we choose $C_7' = 3C_*$, and define

$$g_\pm(y) = e^{\pm\frac{C_*}{q}y} - 1 \mp \frac{C_7' d_I}{q^2} e^{\frac{q}{2d_I}y}.$$

**Claim 3.6.** *$g_+(y) \leq 0$ and $g_-(y) \geq 0$ for $0 \leq y \leq L$.*

We only show the $g_+(y) \leq 0$. The proof for $g_-(y) \geq 0$ is analogous. Now,

$$g_+'(y) = \frac{C_*}{q} e^{\frac{C_*}{q}y} - \frac{C_7'}{2q} e^{\frac{q}{2d_I}y} \quad \text{for } 0 \leq x \leq L.$$

Since $g_+'(0) = \frac{C_*}{q} - \frac{C_7'}{2q} = \frac{C_*}{q}\left(1 - \frac{3}{2}\right) < 0$, and $g_+'$ changes sign at most once, it suffices to check that $g_+(L) < 0$. We consider two cases: (A) $0 < q \leq 1$; (B) $q > 1$.

In Case (A): $0 < q \leq 1$,

$$\begin{aligned} g_+(L) &\leq \frac{e^{\frac{C_* L}{q}}}{q}\left(q - C_7' \frac{d_I}{q} e^{\frac{qL}{2d_I} - \frac{C_* L}{q}}\right) \\ &\leq \frac{e^{\frac{C_* L}{q}}}{q}\left(1 - C_7' \frac{d_I}{q} e^{\frac{qL}{3d_I}}\right) < 0, \end{aligned}$$

where the first inequality follows from $q \le 1$ and $q/d_I \gg 1/q$, and the last from $q/d_I \gg 1$.

In Case (B): $q > 1$,

$$g_+(L) \le O\left(\frac{1}{q}\right) - C_7' \frac{d_I}{q^2} e^{\frac{qL}{2d_I}} = \frac{1}{q}\left(O(1) - C_7' \frac{d_I}{q} e^{\frac{qL}{2d_I}}\right) < 0,$$

where the last inequality follows from $q/d_I \gg 1$. Hence $g_+(y) \le 0$. The proof for $g_-(y) \ge 0$ is analogous and we skip the details. This proves Claim 3.6. Apply Claim 3.6 to (3.11), we obtain

$$\left| I(x) - I(L) e^{-\frac{q}{d_I}(L-x)} \right| \le I(L) \left[ \frac{C_7' d_I}{q^2} e^{-\frac{q}{2d_I}(L-x)} \right].$$

This proves (3.10). □

**Lemma 3.7.** *Assume $\beta(L) > \gamma(L)$. For each $\eta \ge 0$, suppose that*

$$q/d_I \to \infty, \quad q^2/d_S \to \infty, \quad \text{and} \quad d_I/d_S \to \eta, \tag{3.12}$$

*then $I(L)/S(L) \to \alpha^*(\eta)$, where $\alpha^*(\eta)$ is given in* (1.8).

*Proof.* Similar to the proof of Lemma 3.3, it suffices to prove the lemma for each sequence $(d_{S,j}, d_{I,j}, q_j)$ such that (3.12) holds. Fix such a sequence of parameters and let $(S_j, I_j)$ be a sequence of EE of (1.4) with parameters $(d_S, d_I, q) = (d_{S,j}, d_{I,j}, q_j)$. Denote also

$$\eta_j = \frac{d_{I,j}}{d_{S,j}} \quad \text{and} \quad \alpha_j = \frac{I_j(L)}{S_j(L)}.$$

Recall $y = q_j(L-x)/d_{I,j}$, $\eta_j y = q_j(L-x)/d_{S,j}$ and $\alpha_j = \frac{I_j(L)}{S_j(L)}$, then (3.5) becomes

$$\begin{aligned}
&\int_0^{q_j L/d_{I,j}} \gamma\Big(L - \frac{d_{I,j}}{q_j} y\Big) e^{-(1+o(1))y} \,\mathrm{d}y \\
&\le \int_0^{q_j L/d_{I,j}} \frac{e^{-\eta_j y} + C_4 \alpha_j o(1) e^{-\delta \eta_j y}}{e^{-\eta_j y} + C_4 \alpha_j o(1) e^{-\delta \eta_j y} + \alpha_j e^{-(1-o(1))y}} \beta\Big(L - \frac{d_{I,j}}{q_j} y\Big) e^{-(1-o(1))y} \,\mathrm{d}y.
\end{aligned} \tag{3.13}$$

Let

$$\underline{\alpha} = \liminf_{j\to\infty} \alpha_j \quad \text{and} \quad \overline{\alpha} = \limsup_{j\to\infty} \alpha_j.$$

By Lemma 3.3, $0 \le \underline{\alpha} \le \overline{\alpha} < +\infty$ are both finite. Now, pass to the limit superior of $\alpha_j$ as $j \to \infty$ in (3.13), we obtain

$$\gamma(L) \int_0^\infty e^{-y} \,\mathrm{d}y \le \beta(L) \int_0^\infty \frac{e^{-(1+\eta)y}}{e^{-\eta y} + \overline{\alpha} e^{-y}} \,\mathrm{d}y. \tag{3.14}$$

Similar to the inequality (3.5), we can obtain

$$\int_0^L \gamma(x)\frac{I^+(x)}{I(L)}\,\mathrm{d}x \geq \frac{1}{I(L)}\int_0^L \beta(x)\frac{\max\{S^-(x),0\}I^-(x)}{\max\{S^-(x),0\}+I^-(x)}\,\mathrm{d}x. \tag{3.15}$$

Again, by $y = q_j(L-x)/d_{I,j}$, $\eta_j y = q_j(L-x)/d_{S,j}$ and $\alpha_j = \frac{I_j(L)}{S_j(L)}$, we can rewrite (3.15) as

$$\begin{aligned}&\int_0^{q_jL/d_{I,j}} \gamma\Big(L-\frac{d_{I,j}}{q_j}y\Big)e^{-(1-o(1))y}\,\mathrm{d}y\\ &\geq \int_0^{q_jL/d_{I,j}} \frac{\max\{e^{-\eta_j y}-C_4\alpha_j o(1)e^{-\delta\eta_j y},0\}}{\max\{e^{-\eta_j y}-C_4\alpha_j o(1)e^{-\delta\eta_j y},0\}+\alpha_j e^{-(1+o(1))y}}\beta\Big(L-\frac{d_{I,j}}{q_j}y\Big)e^{-(1+o(1))y}\,\mathrm{d}y.\end{aligned} \tag{3.16}$$

By passing to the limit inferior as $j\to\infty$ in (3.16), we obtain

$$\gamma(L)\int_0^\infty e^{-y}\,\mathrm{d}y \geq \beta(L)\int_0^\infty \frac{e^{-(1+\eta)y}}{e^{-\eta y}+\underline{\alpha}e^{-y}}\,\mathrm{d}y. \tag{3.17}$$

It follows from (3.14) and (3.17) that $\overline{\alpha}\leq\underline{\alpha}$, i.e. $\limsup_{j\to\infty}\alpha_j \leq \liminf_{j\to\infty}\alpha_j$. Hence $\alpha^* := \lim_{j\to\infty}\alpha_j$ (for the full sequence) exists, and is uniquely determined by

$$\int_0^\infty \Big[\beta(L)\frac{e^{-\eta y}}{e^{-\eta y}+\alpha^* e^{-y}}-\gamma(L)\Big]e^{-y}\,\mathrm{d}y = 0,$$

which is equivalent to (1.8) (by the transformation $z = e^{-y}$). This proves the lemma. □

**Lemma 3.8.** *For each $\eta_0 > 0$, the following limits hold:*

$$\lim_{q/d_I\to\infty,\, q^2/d_I\to\infty} \frac{q}{d_I I(L)}\int_0^L I(x)\,\mathrm{d}x = 1, \tag{3.18}$$

$$\lim_{\substack{q/d_I\to\infty,\, q^2/d_S\to\infty\\ d_I/d_S\leq\eta_0}} \frac{q}{d_S S(L)(1-e^{-qL/d_S})}\int_0^L S(x)\,\mathrm{d}x = 1. \tag{3.19}$$

*Proof.* To prove (3.18), we apply part (i) of Lemma 3.2, so that

$$e^{-\frac{q}{d_I}\left(1+\frac{d_I}{q^2}C_*\right)(L-x)} \leq \frac{I(x)}{I(L)} \leq e^{-\frac{q}{d_I}\left(1-\frac{d_I}{q^2}C_*\right)(L-x)}.$$

Letting $y = q(L-x)/d_I$ and using the fact that $d_I/q^2 = o(1)$, we have

$$e^{-(1-o(1))y} \leq \frac{I(L-\frac{d_I}{q}y)}{I(L)} \leq e^{-(1+o(1))y} \quad \text{for} \quad 0\leq y\leq qL/d_I.$$

(3.18) thus follows by simply integrating the above.

For the second assertion, apply part (ii) of Lemma 3.2 to get

$$e^{-\frac{q}{d_S}(L-x)} - C_4\frac{d_S}{q^2}\frac{I(L)}{S(L)}e^{-\frac{\delta q}{d_S}(L-x)} \le \frac{S(x)}{S(L)} \le e^{-\frac{q}{d_S}(L-x)} + C_4\frac{d_S}{q^2}\frac{I(L)}{S(L)}e^{-\frac{\delta q}{d_S}(L-x)}.$$

Multiply by $q/d_S$ and integrate, while using the fact that $d_S/q^2 \to 0$, we have

$$\frac{q}{d_S S(L)}\int_0^L S(x)\,\mathrm{d}x = \left(1 - e^{-\frac{qL}{d_S}}\right) + o(1)\delta^{-1}\left(1 - e^{-\frac{\delta q L}{d_S}}\right).$$

This proves (3.19). $\square$

**Lemma 3.9.** *Suppose for some* $\eta \in [0,\infty)$,

$$\frac{q}{d_S} \to \infty, \quad \frac{q^2}{d_S} \to \infty, \quad \frac{d_I}{d_S} \to \eta,$$

*then*

$$\lim\left(\int_0^L S(x)\,\mathrm{d}x, \int_0^L I(x)\,\mathrm{d}x\right) = \left(\frac{N}{1+\eta\alpha^*(\eta)}, \frac{\eta\alpha^*(\eta)N}{1+\eta\alpha^*(\eta)}\right), \tag{3.20}$$

*and*

$$\lim\left[\frac{d_S}{q}\left(S(L), I(L)\right)\right] = \left(\frac{N}{1+\eta\alpha^*(\eta)}, \frac{\alpha^*(\eta)N}{1+\eta\alpha^*(\eta)}\right). \tag{3.21}$$

*Proof.* By Lemmas 3.7 and 3.8,

$$\frac{\int_0^L I(x)\,\mathrm{d}x}{\int_0^L S(x)\,\mathrm{d}x} = (1+o(1))\left[\frac{d_I I(L)}{q}\right]\left[\frac{q}{d_S S(L)(1-e^{-qL/d_S})}\right] = (1+o(1))\frac{d_I}{d_S}\frac{I(L)}{S(L)} \to \eta\alpha^*(\eta). \tag{3.22}$$

By (1.5), we see that the limits

$$A_I := \lim \int_0^L I(x)\,\mathrm{d}x \quad \text{and} \quad A_S := \lim \int_0^L S(x)\,\mathrm{d}x$$

exist, and satisfy $A_I/A_S = \eta\alpha^*(\eta)$ and $A_I + A_S = N$. This implies (3.20). Next, we combine (3.19) and (3.20) to get

$$\lim \frac{d_S}{q}S(L) = \lim \frac{d_S}{q}S(L)(1-e^{-qL/d_S}) = A_S = \frac{N}{1+\eta\alpha^*(\eta)}.$$

Combining this and Lemma 3.7, we obtain (3.21). $\square$

Next, we prove Theorem 1.2.

*Proof of Theorem 1.2.* Fix $\eta_0 \geq 1$ and let $C_1 = \max\{1, C_*^2, C_5, C_6, C_7\}$, where $C_*, C_5, C_6, C_7$ are given in Lemmas 3.2 - 3.5. First, we assume $\frac{q}{d_S}, \frac{q^2}{d_S} \geq C_1$, $\frac{q}{d_I}, \frac{q^2}{d_I} \geq C_7$ and $d_I/d_S \leq \eta_0$. By Lemmas 3.3 and 3.5, we have

$$\left| I(x) - I(L)e^{-\frac{q}{d_I}(L-x)} \right| \leq C_7' I(L) \frac{d_I}{q^2} e^{-\frac{q}{2d_I}(L-x)} \leq O\left( S(L) \frac{d_I}{q^2} \right) e^{-\frac{q}{2d_I}(L-x)} \tag{3.23}$$

for $0 \leq x \leq L$. By Lemma 3.4, $S(L) = O(q/d_S)$, so $O\left(S(L)\frac{d_I}{q^2}\right) = O\left(\frac{d_I}{d_S q}\right) = O\left(\frac{1}{q}\right)$. This proves

$$\left| I(x) - I(L)e^{-\frac{q}{d_I}(L-x)} \right| \leq O\left(\frac{1}{q}\right) e^{-\frac{q}{2d_I}(L-x)} \quad \text{for } 0 \leq x \leq L. \tag{3.24}$$

Next, by Lemma 3.2, there exists $\delta \in (0, 1/2]$ such that

$$\left| S(x) - S(L)e^{-\frac{q}{d_S}(L-x)} \right| \leq O\left(\frac{I(L)d_S}{q^2}\right) e^{-\frac{\delta q}{d_S}(L-x)} \quad \text{for } 0 \leq x \leq L.$$

By Lemmas 3.3 and 3.4, $O\left(\frac{I(L)d_S}{q^2}\right) = O\left(\frac{S(L)d_S}{q^2}\right) = O\left(\frac{1}{q}\right)$, hence

$$\left| S(x) - S(L)e^{-\frac{q}{d_S}(L-x)} \right| \leq O\left(\frac{1}{q}\right) e^{-\frac{\delta q}{d_S}(L-x)}. \tag{3.25}$$

Since $d_I/d_S \leq \eta_0$, we may replace $\delta$ to be the smaller of $\delta$ and $1/(2\eta_0)$, so that (3.24) and (3.25) can be combined to get

$$\left| I(x) - I(L)e^{-\frac{q}{d_I}(L-x)} \right| + \left| S(x) - S(L)e^{-\frac{q}{d_S}(L-x)} \right| \leq O\left(\frac{1}{q}\right) e^{-\frac{\delta q}{d_S}(L-x)}.$$

and (1.9) follows from the fact that $1/q \leq C_1/q \leq q/d_S$. Next, we assume in addition that (1.10) holds, hence,

$$\frac{q}{d_I} \to \infty, \quad \frac{q}{d_S} \to \infty, \quad \frac{q^2}{d_I} \to \infty, \quad \frac{q^2}{d_S} \to \infty, \quad \frac{d_I}{d_S} \to \eta \in [0, \infty).$$

By (1.9) and (3.20), one can deduce (1.11). Also, (1.12) follows from (3.21). This concludes the proof of Theorem 1.2. □

### 3.1 Limiting profile of EE when $q/d_S \to \xi \in [0, \infty)$

In this subsection, we discuss the counterpart of Theorem 1.2 in the case when $\limsup q/d_S$ is finite.

**Theorem 3.10.** *Assume that $\beta(L) > \gamma(L)$. Then there exists some positive constant $C$, independent of $d_S, d_I$ and $q$, such that* (1.4) *has at least one EE whenever $q/d_I \geq C$. Assume that*

$$\frac{q}{d_I} \to \infty, \quad \frac{q^2}{d_S} \to \infty, \quad \text{and} \quad \frac{q}{d_S} \to \xi \in [0, \infty), \tag{3.26}$$

*then any EE $(S(x), I(x))$ of* (1.4) *satisfies*

(i) $\left(\int_0^L S(x)\,\mathrm{d}x, \int_0^L I(x)\,\mathrm{d}x\right) \to (N, 0)$;

(ii) *The susceptible population component of the EE satisfies*

$$S(x) = \begin{cases} \frac{N}{L} + o(1) & \text{if } \xi = 0, \\ \frac{N\xi}{1-e^{-\xi L}} e^{-\xi(L-x)} + o(1) & \text{if } \xi \in (0,\infty); \end{cases} \tag{3.27}$$

(iii) *The infected population component of the EE satisfies*

$$I(x) = \begin{cases} \frac{\alpha^*(0)N}{L} e^{-\frac{q}{d_I}(L-x)} + o(1) & \text{if } \xi = 0, \\ \frac{\alpha^*(0)N\xi}{1-e^{-\xi L}} e^{-\frac{q}{d_I}(L-x)} + o(1) & \text{if } \xi \in (0,\infty). \end{cases} \tag{3.28}$$

*Proof.* By the hypothesis (3.26),

$$q \to \infty, \quad \frac{q}{d_I} \to \infty, \quad \frac{q^2}{d_S} \to \infty, \quad \frac{q}{d_S} \to \xi \in [0,\infty), \quad \frac{d_I}{d_S} \to 0, \quad \frac{q^2}{d_I} \to \infty.$$

By Lemmas 3.7 and 3.8,

$$\frac{\int_0^L I(x)\,\mathrm{d}x}{\int_0^L S(x)\,\mathrm{d}x} = (1+o(1))\frac{d_I I(L)}{q}\frac{q}{d_S S(L)(1-e^{-qL/d_S})} = (1+o(1))\frac{d_I}{q}\frac{I(L)}{S(L)}\frac{q/d_S}{(1-e^{-qL/d_S})} \to 0 \tag{3.29}$$

since $d_I/q \to 0$, Lemma 3.7 says that $I(L)/S(L) \to \alpha^*(0) > 0$. Combining with (1.5), we obtain part (i).

**Claim 3.11.**

$$\lim\,(S(L), I(L)) = \begin{cases} \left(\frac{N}{L}, \frac{\alpha^*(0)N}{L}\right), & \text{for } \xi = 0, \\ \left(\frac{\xi N}{1-e^{-\xi L}}, \frac{\alpha^*(0)\xi N}{1-e^{-\xi L}}\right), & \text{for } \xi \in (0,\infty). \end{cases} \tag{3.30}$$

Combine (3.19) and part (i) to get

$$\lim \frac{d_S}{q} S(L)(1-e^{-qL/d_S}) = N.$$

This determines the (finite) limit of $S(L)$. By Lemma 3.7, $\lim I(L) = \alpha^*(0) \lim S(L)$. The claim is proved. Finally, (3.27) and (3.28) can be derived from Lemma 3.2, since $S(L), I(L)$ have finite limits. □

## 4 Asymptotic Profiles of the EE when $\beta(L) < \gamma(L)$

In this section, we study the case $\beta(L) < \gamma(L)$ and establish Theorem 1.4 in a series of lemmas.

**Lemma 4.1.** *If $q = 0$, there exist two positive constants $C_8, C_8'$ such that*

$$C_8 \le \frac{I(x)}{d_S} \le C_8' \quad \text{for } 0 \le x \le L$$

*for sufficiently small $d_S$.*

*Proof.* When $q = 0$, from Lemma 3.1 in [2], the pair $(S, I)$ is a solution if and only if $(S, I)$ satisfies

$$\begin{cases} d_S S + d_I I = \kappa, & 0 < x < L, \\ d_I I_{xx} + \Big(\beta - \gamma - \beta \dfrac{I}{S+I}\Big) I = 0, & 0 < x < L, \\ I_x(0) = I_x(L) = 0, \ S_x(0) = S(L) = 0, \\ \int_0^L [S(x) + I(x)]\,\mathrm{d}x = N, \end{cases} \tag{4.1}$$

for some positive constant $\kappa$. Moreover, [2, Lemmas 4.2 and 4.4] assert that as $d_S \to 0$,

$$\lim_{d_S \to 0} \kappa \to 0, \quad \lim_{d_S \to 0} I(\cdot) \to 0 \quad \text{and} \quad \lim_{d_S \to 0} \frac{d_I I(\cdot)}{\kappa} \to I^*(\cdot),$$

with the last two limits being in $C([0, L])$, and $I^* \in C([0, L])$ satisfies $0 < I^* \le 1$ on $x \in [0, L]$ and $\{x \in (0, L) : \beta(x) < \gamma(x)\} \subseteq \{x \in [0, L] : 0 < I^* < 1\}$ has positive measure. From the first equation in (4.1),

$$S + I = \frac{\kappa - d_I I}{d_S} + I = \frac{\kappa}{d_S} - \frac{(d_I - d_S) I}{d_S}. \tag{4.2}$$

Integrating and using the integral constraint (1.5), we obtain

$$N = \frac{\kappa}{d_S} \int_0^L \Big(1 - \frac{d_I I}{\kappa}\Big)\,\mathrm{d}x + \int_0^L I\,\mathrm{d}x. \tag{4.3}$$

Hence $\kappa / d_S \le N / \int_0^L (1 - d_I I/\kappa)\,\mathrm{d}x$ and

$$\lim_{d_S \to 0} \frac{I}{d_S} \le \lim_{d_S \to 0} \frac{\kappa}{d_I d_S} < \frac{N}{d_I \int_0^L (1 - d_I I/\kappa)\,\mathrm{d}x} = \frac{N}{d_I \int_0^L (1 - I^*)\,\mathrm{d}x}.$$

Since the right hand side is a constant, then there exists a constant $C_8$ such that $\|I\|/d_S \le C_8$ for sufficiently small $d_S$. Next we prove $(\inf I)/d_S \not\to 0$ as $d_S \to 0$ by contradiction. Suppose

$\inf I = o(d_S)$, then by Harnack inequality $\|I\| = o(d_S)$. Since $\|I\| = o(d_S)$, we deduce from (4.3) that $\kappa/d_S \to N/\int_0^L (1-I^*)\mathrm{d}x$ as $d_S \to 0$. Using (4.3) once again, we have

$$N = \lim_{d_S\to 0} \frac{L\kappa}{d_S} - \lim_{d_S \to 0} \frac{(d_I - d_S)\int_0^L I\,\mathrm{d}x}{d_S} = \frac{LN}{\int_0^L (1-I^*)\,\mathrm{d}x}.$$

This implies that $I^* \equiv 0$, which is a contradiction as $\beta(L) < \gamma(L)$ and the set $\{x \in [0, L] : 0 < I^* < 1\} \supset \{x \in (0, L) : \beta(x) < \gamma(x)\}$ has positive measure. $\square$

**Lemma 4.2.** *For each $q_0 > 0$, there exist $C_9 > 0$, $\delta > 0$ all independent of $d_S$ such that $\|I\| \le C_9 e^{-\frac{q\delta}{d_S}}$ for $0 < q \le q_0$ and $q/d_S$ is sufficiently large.*

*Proof.* Since $\gamma(L) > \beta(L)$, there exist $\epsilon > 0, \delta > 0$ such that $\gamma(x) - \beta(x) > \epsilon$ for $x \in (L-2\delta, L)$. For any $x \in (L-2\delta, L)$, integrating the first equation of (1.4) from $x$ to $L$, we have

$$\begin{aligned} d_S S_x(x) - qS(x) &= \int_x^L \Big[\gamma(\tau) - \beta(\tau)\frac{S(\tau)}{S(\tau)+I(\tau)}\Big] I(\tau)\,\mathrm{d}\tau \\ &\ge \int_x^L [\gamma(\tau) - \beta(\tau)] I(\tau)\,\mathrm{d}\tau \\ &\ge \epsilon \int_x^L I(\tau)\,\mathrm{d}\tau \\ &\ge \epsilon C_9' \|I\| (L-x), \end{aligned}$$

where the last inequality is obtained by the Harnack inequality and $C_9' = C_9'(d_I, q_0)$ is a constant independent of $d_S$ and $q \le q_0$.

For any $x \in (L-2\delta, L)$, multiplying the above inequality by $e^{q(L-x)/d_S}$ and integrating over $(x, L)$, we have

$$d_S \Big[S(L) - e^{\frac{q}{d_S}(L-x)} S(x)\Big] \ge \epsilon C_9' \|I\| \int_x^L (L-\tau) e^{\frac{q}{d_S}(L-\tau)}\,\mathrm{d}\tau.$$

By direct calculation,

$$\int_x^L (L-\tau) e^{\frac{q}{d_S}(L-\tau)}\,\mathrm{d}\tau = \frac{d_S}{q} e^{\frac{q}{d_S}(L-x)}(L-x) + \frac{d_S^2}{q^2}\Big[1 - e^{\frac{q}{d_S}(L-x)}\Big]. \tag{4.4}$$

Then for $L - 2\delta < x < L$,

$$\begin{aligned} S(L) &\ge \frac{\epsilon C_9' \|I\|}{q} e^{\frac{q}{d_S}(L-x)}(L-x) + \frac{\epsilon C_9' \|I\| d_S}{q^2}\Big[1 - e^{\frac{q}{d_S}(L-x)}\Big] \\ &\ge \frac{\epsilon C_9' \|I\|}{q} e^{\frac{q}{d_S}(L-x)}(L-x) - \frac{\epsilon C_9' \|I\| d_S}{q^2} e^{\frac{q}{d_S}(L-x)} \\ &= \frac{\epsilon C_9' \|I\|}{q} e^{\frac{q}{d_S}(L-x)}\Big[(L-x) - \frac{d_S}{q}\Big]. \end{aligned}$$

Choosing $x = L - 3\delta/2$, for any $0 < q \le q_0$ and $q/d_S > 1/\delta$, we have

$$S(L) \ge \frac{C_9''}{q}\|I\|e^{\frac{3q\delta}{2d_S}}, \tag{4.5}$$

where $C_9'' = \delta\epsilon C_9'/2$ is a constant independent of $d_S$.

On the other hand, for any $x \in (0, L)$, integrating the equation of $S$ in (1.4) over $(x, L)$, we have

$$d_S S_x(x) - qS(x) \le \int_x^L \gamma(\tau)I(\tau)\,\mathrm{d}\tau \le \|\gamma\|\|I\|(L-x).$$

Multiplying the above inequality by $e^{q(L-x)/d_S}$ and integrating over $(x, L)$, we have

$$S(L) - e^{\frac{q}{d_S}(L-x)}S(x) \le \frac{\|\gamma\|\|I\|}{q}e^{\frac{q}{d_S}(L-x)}(L-x) + \frac{\|\gamma\|\|I\|d_S}{q^2}\Big[1 - e^{\frac{q}{d_S}(L-x)}\Big].$$

Then,

$$\begin{aligned}
S(x) &\ge S(L)e^{-\frac{q}{d_S}(L-x)} - \frac{\|\gamma\|\|I\|}{q}(L-x) + \frac{\|\gamma\|\|I\|d_S}{q^2}\Big[1 - e^{-\frac{q}{d_S}(L-x)}\Big] \\
&\ge S(L)e^{-\frac{q}{d_S}(L-x)} - \frac{\|\gamma\|\|I\|}{q}(L-x) \\
&\ge \left(\frac{C_9''}{q}\|I\|e^{\frac{3q\delta}{2d_S}}\right)e^{-\frac{q}{d_S}(L-x)} - \frac{\|\gamma\|\|I\|}{q}(L-x),
\end{aligned}$$

where the last inequality follows from (4.5). Integrating it over $x \in (L - \delta/2, L)$, we get

$$\int_0^L S(x)\,\mathrm{d}x \ge \frac{\|I\|}{q_0}\left[\frac{d_S}{q}C_9''e^{\frac{3q\delta}{d_S}}(1 - o(1)) - \frac{\|\gamma\|\delta^2}{8}\right].$$

Since $\int_0^L S(x)\,\mathrm{d}x \le N$, we deduce that for $q \le q_0$ and $q/d_S \gg 1$, we have $\|I\| = O(e^{-\frac{\delta q}{d_S}})$. $\square$

**Lemma 4.3.** *Given $d_I, q_0 > 0$. For any $0 < q \le q_0$ and $q/d_S \gg 1$,*

$$\left\|S(x) - \frac{qN}{d_S}e^{-\frac{q}{d_S}(L-x)}\right\| = o\left(\frac{1}{q}\right).$$

*Proof.* Multiplying the equation of $S$ in (1.4) by $e^{-qx/d_S}$ and integrating the result over $(x, L)$, we have

$$d_S\left[e^{-\frac{q}{d_S}L}S_x(L) - e^{-\frac{q}{d_S}x}S_x(x)\right] + \int_x^L e^{-\frac{q}{d_S}\tau}\Big[\gamma(\tau) - \beta(\tau)\frac{S(\tau)}{S(\tau)+I(\tau)}\Big]I(\tau)\,\mathrm{d}\tau = 0.$$

By the boundary condition, we may rewrite the equation as

$$S_x(x) = \frac{q}{d_S} S(L) e^{-\frac{q}{d_S}(L-x)} + \frac{e^{\frac{q}{d_S}x}}{d_S} \int_x^L e^{-\frac{q}{d_S}\tau} \Big[\gamma(\tau) - \beta(\tau)\frac{S(\tau)}{S(\tau)+I(\tau)}\Big] I(\tau)\, \mathrm{d}\tau.$$

By Lemma 4.2 and direct calculations, for any $0 < q \le q_0$ and sufficiently large $q/d_S$, we have

$$\begin{aligned} S_x(x) &= \frac{q}{d_S} S(L) e^{-\frac{q}{d_S}(L-x)} + \frac{e^{\frac{q}{d_S}x}}{d_S} \int_x^L e^{-\frac{q}{d_S}\tau} \Big[\gamma(\tau) - \beta(\tau)\frac{S(\tau)}{S(\tau)+I(\tau)}\Big] I(\tau)\, \mathrm{d}\tau \\ &= \frac{q}{d_S} S(L) e^{-\frac{q}{d_S}(L-x)} + O\Big(\frac{e^{-\frac{q}{d_S}\delta}}{d_S}\Big) e^{\frac{q}{d_S}x} \int_x^L e^{-\frac{q}{d_S}\tau}\, \mathrm{d}\tau \\ &= \frac{q}{d_S} S(L) e^{-\frac{q}{d_S}(L-x)} + O\left(\frac{1}{q} e^{-\frac{q}{d_S}\delta}\right) \left(1 - e^{-\frac{q}{d_S}(L-x)}\right). \end{aligned}$$

Integrating the last equation again over $(x, L)$, we obtain

$$S(x) = S(L) e^{-\frac{q}{d_S}(L-x)} + O\left(\frac{1}{q} e^{-\frac{q}{d_S}\delta}\right) \left[L - x - \frac{d_S}{q}\left(1 - e^{-\frac{q}{d_S}(L-x)}\right)\right],$$

for sufficiently large $q/d_S$. For large $q/d_S$, $L - x - \frac{d_S}{q}\left(1 - e^{-\frac{q}{d_S}(L-x)}\right)$ is bounded, which implies that

$$S(x) = S(L) e^{-\frac{q}{d_S}(L-x)} + O\left(\frac{1}{q} e^{-\frac{q}{d_S}\delta}\right). \tag{4.6}$$

Integrating the equation (4.6) and applying the integral constraint (1.5), then

$$S(L) = \frac{qN}{d_S\left(1 - e^{-\frac{q}{d_S}L}\right)} + O\Big(\frac{1}{d_S} e^{-\frac{q}{d_S}\delta}\Big).$$

Subtracting $qNe^{-\frac{q(L-x)}{d_S}}/d_S$ on the both sides of (4.6), we have

$$\begin{aligned} S(x) - \frac{qN}{d_S} e^{-\frac{q}{d_S}(L-x)} &= \Big[S(L) - \frac{qN}{d_S}\Big] e^{-\frac{q}{d_S}(L-x)} + O\big(\frac{1}{q} e^{-\frac{q}{d_S}\delta}\big) \\ &= \Big[\frac{qN}{d_S\left(1 - e^{-\frac{q}{d_S}L}\right)} - \frac{qN}{d_S} + O\Big(\frac{1}{d_S} e^{-\frac{q}{d_S}\delta}\Big)\Big] e^{-\frac{q}{d_S}(L-x)} + O\big(\frac{1}{q} e^{-\frac{q}{d_S}\delta}\big) \\ &= o(1) + O\big(\frac{1}{d_S} e^{-\frac{q}{d_S}\delta}\big) + O\big(\frac{1}{q} e^{-\frac{q}{d_S}\delta}\big) \\ &= o(1) + \frac{1}{q}\left[O\big(\frac{q}{d_S} e^{-\frac{q}{d_S}\delta}\big) + O\big(e^{-\frac{q}{d_S}\delta}\big)\right] = o(1) + o\left(\frac{1}{q}\right). \end{aligned}$$

This completes the proof. $\square$

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