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DIRAC-CONCENTRATIONS IN AN INTEGRO-PDE MODEL FROM EVOLUTIONARY GAME THEORY

King-Yeung Lam

Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA

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Abstract. Nonlocal Lotka-Volterra models have the property that solutions concentrate as Dirac masses in the limit of small diffusion. Motivated by the existence of moving Dirac-concentrations in the time-dependent problem, we study the qualitative properties of steady states in the limit of small diffusion. Under different conditions on the growth rate and interaction kernel as motivated by the framework of adaptive dynamics, we will show that as the diffusion rate tends to zero the steady state concentrates (i) at a single location; (ii) at two locations simultaneously; or (iii) at one of two alternative locations. The third result in particular shows that solutions need not be unique. This marks an important difference of the non-local equation with its local counterpart.

1. **Introduction.** This paper is concerned with the following reaction-diffusion model from evolutionary game theory:

$$\begin{cases} \varepsilon u_t = \varepsilon^2 \partial_x^2 u + u\left(r(x) - \int_\Omega K(x,y)u(y,t)\,dy\right) & \text{for } x \in \Omega, t > 0,\\ \partial_n u = 0 & \text{for } x \in \partial\Omega, t > 0,\\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

where $u(x,t)$ represents the population with trait $x \in \Omega$ for some bounded domain $\Omega \in \mathbb{R}^N$ with smooth boundary $\partial\Omega$ at time t. The intrinsic growth rate for individuals with trait x is given by $r(x) \in C^\infty(\bar\Omega)$, and the integral term models an additional contribution to the death rate due to competition with other phenotypes with different traits, with competition kernel $K(x,y) \in C^\infty(\bar\Omega \times \bar\Omega)$. Throughout this paper, we assume

(H): $\min\limits_{\bar\Omega} r > 0$, $\min_{\bar\Omega\times\bar\Omega} K > 0$.

In this model, individuals with trait x in a population $u(\cdot,t)$ has fitness $r(x) - \int_\Omega K(x,y)u(y,t)\,dy$, and reproduction is asexual and is subject to mutation with rate ε^2.

Equation (1) can be viewed as a competition model of infinitely many species. This can be seen by formally setting the mutation rate ε to be zero, while considering solutions of the form $\sum_{i=1}^N U_i(t)\delta_0(x - x_i)$, where $\{x_i\}$ is a set of N distinct

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strategies. Then

$$\varepsilon\partial_t U_i(t) = U_i(t)\left[r(x_i) - \sum_{j=1}^{N} K(x_i,x_j)U_j(t)\right] \quad \text{for } 1 \le i \le N, \text{ and } t > 0, \tag{2}$$

which is a version of the the Lotka-Voletrra model of N competing species.

The time-dependent problem (1) was considered in [20] in case $\Omega = \mathbb{R}^n$. Under convexity assumptions on the initial condition and on coefficients of the equation, it was shown that solutions of (1) concentrates as a single moving Dirac mass, as $\varepsilon \to 0$. Moreover, they showed that the movement of the Dirac mass can be well described by a form of canonical equation, which is connected to the framework of adaptive dynamics [6] underlying the selection process.

Motivated by the work on the time-dependent problem, we will show in this paper that (1) possesses Dirac-concentrated steady states. Furthermore, under three different set of conditions, we will show that the steady state concentrates (i) at a single location; (ii) at two locations simultaneously; or (iii) at two alternative locations. The third result in particular shows that solutions need not be unique. This marks an important difference of the non-local equation (1) with its local counterpart. The steady states in scenarios (i) and (iii) can be considered as evolutionary endpoints corresponding to the single moving Dirac mass found in [20]. The dimorphic steady Dirac mass in scenario (ii) motivates the study of moving Dirac masses supported at two points, which is currently open. We also refer the interested readers to [26] where the existence and structure of positive steady states of a related model is discussed using a bifurcation approach.

Reaction-diffusion equations modeling the evolution of a quantitative trait has a long history (see, e.g. [3, 12, 14, 21] for the case when $K \equiv 1$ is constant). The version studied in this paper, which involves a non-local interaction kernel, was introduced by [24] in the context of competition with neighbors, with

$$K(x,y) = \frac{\alpha_0}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-y)^2}{2\sigma^2}\right).$$

See also [1, 5, 11] for works on the pure selection case.

Furthermore, (1) can be rigorously derived from an individual-based, stochastic model in which a finite number of individuals may randomly die or produce an offspring with a rate depending on the competition among conspecifics. Taking the limit of an infinite number of individuals with the correct time scale, (1) can be obtained. We refer the interested reader to [4].

In the model of this paper, the growth rate $r(x)$ and interaction kernel $K(x,y)$ are prescribed rather than derived from density- and frequency-dependent interactions among phenotypes. In general, the relative advantage of a trait x against a different trait y depends on the context of their interaction. For instance, in [8, 10, 16, 27] the invasion fitness between phenotypes with different dispersal strategies is obtained in the context of reaction-diffusion equations modeling the two competiting species in a bounded spatial domain. Those results has implications in the mutation-selection framework [9, 17, 18, 23], which concerns populations structured by space and trait.

The remainder of this paper is organized as follows: The mathematical statement of the main results are presented in Section 2. Apriori estimates and the WKB transform are presented in Section 3. In Section 4, Theorems 1 and 2 are proved by the constrained Hamilton-Jacobi equation method pioneered by [7]. In Section 5, the existence of positive steady states and Theorem 3 are proved using a dynamical approach based on persistence theory and the construction of two forward-invariant regions of (1). Finally, the assumptions of our main results and their relation to the framework of adaptive dynamics are discussed in Section 6.

2. **Main Results.** In this paper, we focus on the existence, and multiplicity of steady states of (1) when the trait space is one-dimensional, i.e. $\Omega = (-1,1)$. When there is no ambiguity, we suppress the upper and lower limits in the integral and write, for $\rho(y) \in L^1((-1,1))$, $\int \rho(y)\,dy = \int_{-1}^{1} \rho(y)\,dy$. In such case, the steady state $\tilde{u}_\varepsilon(x)$ satisfies

$$\begin{cases} \varepsilon^2 \partial_{xx}\tilde{u}_\varepsilon + \tilde{u}_\varepsilon \left(r(x) - \int K(x,y)\tilde{u}_\varepsilon(y)\,dy \right) = 0 & \text{for } x \in (-1,1), \\ \partial_x \tilde{u}_\varepsilon = 0 & \text{for } x = \pm 1. \end{cases} \tag{3}$$

In the following we state our three main results.

Theorem 1. *Assume*

(A): $\frac{\partial^2}{\partial x^2}\left[\frac{K(x,y)}{r(x)}\right] > 0$ *for all* $x, y \in [-1,1]$, *and there exists* $\hat{x} \in (-1,1)$ *such that*

$$\frac{\partial}{\partial x}\left[\frac{K(x,y)}{r(x)}\right]_{x=x_0, y=x_0} = \begin{cases} < 0 & \text{for } x_0 \in [-1,\hat{x}), \\ 0 & \text{for } x_0 = \hat{x}, \\ > 0 & \text{for } x_0 \in (\hat{x}, 1]. \end{cases} \tag{4}$$

Then, as $\varepsilon \to 0$, *every positive solution* $\tilde{u}_\varepsilon(x)$ *of* (3) *satisfies*

$$\tilde{u}_\varepsilon(x) \to \frac{r(\hat{x})}{K(\hat{x},\hat{x})}\delta_0(x-\hat{x}) \quad \text{in distribution.}$$

Theorem 2. *Assume*

(B): $\frac{\partial^2}{\partial x^2}\left[\frac{K(x,y)}{r(x)}\right] < 0$ *for all* $x, y \in [-1,1]$, *and there exists* $\hat{x} \in (-1,1)$ *such that*

$$\frac{\partial}{\partial x}\left[\frac{K(x,y)}{r(x)}\right]_{x=x_0, y=x_0} = \begin{cases} < 0 & \text{for } x_0 \in [-1,\hat{x}), \\ 0 & \text{for } x_0 = \hat{x}, \\ > 0 & \text{for } x_0 \in (\hat{x}, 1]. \end{cases} \tag{5}$$

Then, as $\varepsilon \to 0$, *every positive solution* $\tilde{u}_\varepsilon(x)$ *of* (3) *satisfies*

$$\tilde{u}_\varepsilon(x) \to A\delta_0(x+1) + B\delta_0(x-1) \quad \text{in distribution,}$$

where the positive constants A *and* B *are unqiuely determined by*

$$\begin{pmatrix} K(1,-1) & K(1,1) \\ K(-1,-1) & K(-1,1) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} r(1) \\ r(-1) \end{pmatrix}.$$

Theorem 3. *Assume*

(C): $\sup_{-1<x<1} \frac{\partial}{\partial x}\left[\frac{K(x,1)}{r(x)}\right] < 0$ *and that* $\inf_{-1<x<1} \frac{\partial}{\partial x}\left[\frac{K(x,-1)}{r(x)}\right] > 0.$

Then, for all ε *sufficiently small,* (3) *has at least two positive solutions* $\tilde{u}_{\varepsilon,+}(x)$ *and* $\tilde{u}_{\varepsilon,-}(x)$. *Moreover, as* $\varepsilon \to 0$, *we have*

$$\tilde{u}_{\varepsilon,+}(x) \to \frac{r(1)}{K(1,1)}\delta_0(x-1) \quad \text{and} \quad \tilde{u}_{\varepsilon,-}(x) \to \frac{r(-1)}{K(-1,-1)}\delta_0(x+1) \quad \text{in distribution.}$$

For the ease of exposition, we will postpone the proof for the existence of steady state to Corollary 5.3 in Section 5.

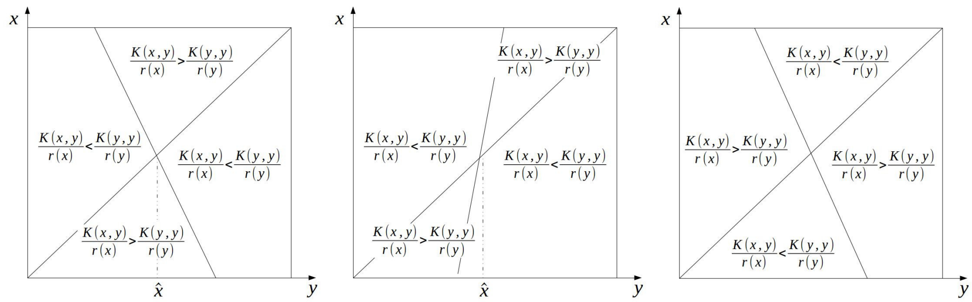

FIGURE 1. The left, center and right panels illustrate the sign of $\frac{K(x,y)}{r(x)} - \frac{K(y,y)}{r(y)}$ as a function of x and y, under the assumptions of Theorems 1, 2 and 3 respectively. Here x and y are the strategy of the invader and resident species respectively. $\frac{K(x,y)}{r(x)} - \frac{K(y,y)}{r(y)} < 0$ (resp. > 0) means invasion of resident with strategy "y" by invader with strategy "x" is a success (resp. failure).

3. **WKB-Ansatz and Apriori Estimates.**

Consider the WKB transform

$$\tilde{v}_\varepsilon(x) = \varepsilon \log \tilde{u}_\varepsilon(x), \quad \text{i.e.} \quad \tilde{u}_\varepsilon(x) = \exp\left(\frac{\tilde{v}_\varepsilon(x)}{\varepsilon}\right),$$

then $\tilde{v}_\varepsilon(x)$ satisfies the equation

$$\begin{cases} -\varepsilon \partial_x^2 \tilde{v}_\varepsilon - |\partial_x \tilde{v}_\varepsilon|^2 = \tilde{H}_\varepsilon(x) := r(x) - \int K(x,y)\tilde{u}_\varepsilon(y)\,dy & \text{for } -1 < x < 1, \\ \partial_x \tilde{v}_\varepsilon = 0 & \text{for } x = \pm 1. \end{cases} \tag{6}$$

We first develop some apriori estimates of $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$.

Lemma 3.1. *Let $\tilde{u}_\varepsilon$ be a positive solution of (3), then*

$$\frac{r_*}{K^*} \leq \int \tilde{u}_\varepsilon(y)\,dy \leq \frac{r^*}{K_*},$$

where the positive constants r^*, r_*, K^*, K_* are given respectively by

$$r^* = \sup_{(-1,1)} r(x), \quad r_* = \inf_{(-1,1)} r(x), \quad K^* = \sup_{(-1,1)^2} K(x,y), \quad K_* = \inf_{(-1,1)^2} K(x,y). \tag{7}$$

Proof. Integrating (3) over $x \in (-1,1)$, we obtain

$$\int r(x)\tilde{u}_\varepsilon(y)\,dy = \iint K(x,y)\tilde{u}_\varepsilon(x)\tilde{u}_\varepsilon(y)\,dxdy\,.$$

Then we have

$$r_* \int \tilde{u}_\varepsilon(y)\,dy \leq K^* \left(\int \tilde{u}_\varepsilon(y)\,dy\right)^2$$

from which the lower bound follows. The upper bound of $\int \tilde{u}_\varepsilon\,dy$ can be derived analogously. □

Lemma 3.2. *There exists C independent of $\varepsilon > 0$, such that $\|\tilde{H}_\varepsilon(x)\|_{C^3([-1,1])} \leq C$.*

Proof. Fix $k = 0, 1, 2$, or 3. By definition of $\tilde{H}_\varepsilon$, we have

$$\left|\partial_x^k \tilde{H}_\varepsilon(x)\right| = \left|\partial_x^k r(x)\right| + \left|\int \partial_x^k K(x,y)\tilde{u}_\varepsilon(y)\,dy\right|$$
$$\leq \|r\|_{C^k([-1,1])} + C\|K\|_{C^k([-1,1]^2)}\int \tilde{u}_\varepsilon(y)\,dy.$$

In view of the L^1 bound of $\tilde{u}_\varepsilon$ (Lemma 3.1), the right hand side is bounded independent of $\varepsilon > 0$ and $x \in (-1,1)$. □

Lemma 3.3. *There exists C independent of $\varepsilon > 0$ such that $\sup_{(-1,1)} |\partial_x \tilde{v}_\varepsilon(x)| \leq C$. In particular, the family $\{\tilde{v}_\varepsilon\}$ is equicontinuous in the variable $x \in [-1,1]$.*

Proof. If $\partial_x \tilde{v}_\varepsilon(x) \equiv 0$, there is nothing to prove. Otherwise, there exists $x_\varepsilon \in (-1,1)$ such that $\sup |\partial_x \tilde{v}_\varepsilon| = |\partial_x \tilde{v}_\varepsilon(x_\varepsilon)| > 0$, then $\partial_x^2 \tilde{v}_\varepsilon(x_\varepsilon) = 0$, and by equation (3), we have

$$|\partial_x \tilde{v}_\varepsilon(x_\varepsilon)|^2 = |\tilde{H}_\varepsilon(x_\varepsilon)| \leq \|\tilde{H}_\varepsilon\|_{C([-1,1])}.$$

In view of Lemma 3.2, the right hand side of the above equation is bounded independent of ε. This proves the lemma. □

Lemma 3.4. $\lim_{\varepsilon\to 0}\left[\sup_{-1<x<1} \tilde{v}_\varepsilon\right] = 0.$

Proof. Suppose to the contrary that there exists $\varepsilon_k \to 0$ such that

$$\lim_{\varepsilon_k\to 0}\left[\sup_{-1<x<1} \tilde{v}_{\varepsilon_k}\right] > 0 \quad \text{or} \quad \lim_{\varepsilon_k\to 0}\left[\sup_{-1<x<1} \tilde{v}_{\varepsilon_k}\right] < 0.$$

Since $\tilde{u}_{\varepsilon_k} = \exp(\tilde{v}_{\varepsilon_k}/\varepsilon_k)$ and since $\{\tilde{v}_\varepsilon\}$ is equicontinuous (Lemma 3.4), we have

$$\int \tilde{u}_{\varepsilon_k}\,dy \to +\infty \quad \text{or} \quad \int \tilde{u}_{\varepsilon_k}\,dy \to 0.$$

But both cases are impossible, in view of Lemma 3.1. □

Corollary 3.5. *The families $\{\tilde{v}_\varepsilon\}$ and $\{\tilde{H}_\varepsilon\}$ are precompact in $C([-1,1])$ and $C^2([-1,1])$ respectively.*

Proof. By Lemmas 3.3 and 3.4, the family $\{\tilde{v}_\varepsilon\} \subset C([-1,1])$ is equibounded and equicontinuous. By Arzelà-Ascoli Theorem, it is precompact in $C([-1,1])$. Similarly, the precompactness of $\{\tilde{H}_\varepsilon\}$ in $C^2([-1,1])$ follows from its boundedness in $C^3([-1,1])$ (Lemma 3.2). □

Proposition 1. *By passing to a subsequence $\varepsilon_k \to 0$, there exists $\tilde{v}(x) \in C([-1,1])$ and $\tilde{H}(x) \in C^2([-1,1])$ such that*

$$\tilde{v}_{\varepsilon_k}(x) \to \tilde{v}(x) \quad \text{in } C([-1,1]), \quad \text{and} \quad \tilde{H}_{\varepsilon_k}(x) \to \tilde{H}(x) \quad \text{in } C^2([-1,1]).$$

Moreover,

(i) $\sup_{-1<x<1} \tilde{v}(x) = 0$;

(ii) $\tilde{v}(x)$ *is a viscosity solution of*

$$-|\partial_x \tilde{v}|^2 = \tilde{H}(x) \quad \text{for } -1 < x < 1; \tag{8}$$

(iii) $\max_{-1\leq x\leq 1} \tilde{H}(x) = 0$;

(iv) *If* $\sup_{-1<x<1} \tilde{v}_{\varepsilon_k}(x) = \tilde{v}_{\varepsilon_k}(x_k)$ *for each k, then* $\operatorname{dist}\left(x_k, \left\{x : \tilde{H}(x) = 0\right\}\right) \to 0$.

Proof. By Corollary 3.5, we may pass to a subsequence so that the solution $(\tilde{v}_{\varepsilon_k}, \tilde{H}_{\varepsilon_k})$ of (6) converges to some $(\tilde{v}, \tilde{H})$ in $C([-1,1]) \times C^2([-1,1])$. Assertion (i) follows from Lemma 3.4. Since $\tilde{v}_\varepsilon$ is a classical solution of (6), we may apply the stability theorem (see, e.g. [2, Theorem 4.1]) to conclude that the limit function $\tilde{v}(x)$ is a viscosity solution of the Hamilton-Jacobi equation (8). This proves assertion (ii).

We next prove (iv). By assumption, x_k is a local maximum point of $\tilde{v}_{\varepsilon_k}$, so that

$$\partial_x^2 \tilde{v}_{\varepsilon_k}(x_k) \leq 0 = \partial_x \tilde{v}_{\varepsilon_k}(x_k) \quad \Longrightarrow \quad \tilde{H}_{\varepsilon_k}(x_k) \geq 0.$$

Since, by the equation (8), we also have $\tilde{H}(x) \leq 0$ for all x, we see that

$$0 \leq \liminf_{k\to\infty} \tilde{H}_{\varepsilon_k}(x_k) \leq \limsup_{k\to\infty} \tilde{H}_{\varepsilon_k}(x_k) \leq 0.$$

It follows from the uniform convergence of $\tilde{H}_{\varepsilon_k} \to \tilde{H}$ in $[-1,1]$ that any limit point x_0 of $\{x_k\}$ satisfies $\tilde{H}(x_0) = 0$. This proves (iv). Since $\tilde{H}(x) \leq 0$ and the nodal set of H is nonempty, (iii) is also proved. ☐

Next, we prove a result in the special case when $\tilde{H}(x)$ has a unique maximum point.

Proposition 2. *Suppose, in addition to the hypotheses of Proposition 1, that for some $x' \in [-1,1]$,*

$$\tilde{H}(x) \leq 0 \quad \text{in } [-1,1], \text{ and equality holds if and only if } x = x'.$$

Then

$$\tilde{u}_{\varepsilon_k}(x) \to \frac{r(x')}{K(x',x')}\delta_0(x-x') \quad \text{in distribution sense,} \tag{9}$$

and $\tilde{H}(x) = r(x) - \frac{K(x,x')r(x')}{K(x',x')}$ *for* $-1 \leq x \leq 1$.

Proof. We first show a property of the limit function $\tilde{v}(x)$.

Claim 1. *$\tilde{v}(x') = 0$, and $\tilde{v} < 0$ for $x \in [-1,x') \cup (x',1]$.*

Let the maximum of $\tilde{v}_{\varepsilon_k}$ be attained at $x_k \in [-1,1]$. Then, by Proposition 1(iv), $x_k \to x'$, so that

$$\tilde{v}(x') = \lim_{k\to\infty} \tilde{v}_{\varepsilon_k}(x_k) = \lim_{k\to\infty}\left[\max_{-1\leq x\leq 1} \tilde{v}_{\varepsilon_k}\right] = 0,$$

where we used Lemma 3.4 for the last equality. Next, suppose to the contrary that $\tilde{v}(x'') = 0$ for some $x'' \in [-1,1]\setminus\{x'\}$. The fact that $\tilde{v} \leq 0$ implies that x'' is a local maximum of $\tilde{v}$. We discuss the two cases separately: (i) $x'' \in (-1,1)\setminus\{x'\}$; (ii) $x'' \in \{-1,1\}\setminus\{x'\}$ and that $\tilde{v}(x) < 0$ for $x \in (-1,1)\setminus\{x'\}$. In case (i) x'' is an interior local maximum point of $\tilde{v}$. Since $\tilde{v}$ is viscosity solution of (8), we have $H(x'') \geq 0$. But this can only happen if $x'' = x'$, which is a contradiction. In case (ii), $\tilde{v}$ attains a strict local maximum at $x'' = \pm 1$ and there is a sequence $x''_k \to x''$ such that $\tilde{v}_{\varepsilon_k}$ attains a local max at x''_k. This implies that $\tilde{H}_{\varepsilon_k}(x''_k) \geq 0$. Letting $k \to \infty$, we have $\tilde{H}(x'') \geq 0$ for some $x'' \in \{1,-1\}\setminus\{x'\}$. This again is a contradiction to the assumption on $\tilde{H}$. Claim 1 is proved.

By Claim 1 and Lemma 3.1, we may pass to a subsequence and assume that $\tilde{u}_{\varepsilon_k}(x) = \exp(\tilde{v}_{\varepsilon_k}(x)/\varepsilon_k) \to C'\delta_0(x-x')$ in distribution sense for some $C' > 0$. By integrating (3), and letting $\varepsilon_k \to 0$, we have

$$r(x')C' = \lim_{k\to\infty}\int r(x)\tilde{u}_{\varepsilon_k}(x)\,dx = \lim_{k\to\infty}\iint K(x,y)\tilde{u}_{\varepsilon_k}(x)\tilde{u}_{\varepsilon_k}(y)\,dxdy = K(x',x')(C')^2.$$

Since $C' > 0$, we deduce that $C' = r(x')/K(x',x')$. Since the limit is independent of subsequences of $\{\varepsilon_k\}$, the convergence $\tilde{u}_\varepsilon(x) \to \frac{r(x')}{K(x',x')}\delta_0(x-x')$ holds for the full sequence $\varepsilon_k \to 0$. Finally,

$$\tilde{H}(x) = \lim_{k\to\infty}\left[r(x) - \int K(x,y)u_{\varepsilon_k}(y)\,dy\right] = r(x) - \frac{K(x,x')r(x')}{K(x',x')}.$$

This concludes the proof of Proposition 2. $\square$

4. **Proof of Theorems 1 and 2.**

Proof of Theorem 1. By Proposition 1, we pass to a sequence $\varepsilon_k \to 0$ so that $\tilde{v}_{\varepsilon_k} \to \tilde{v}$ in $C([-1,1])$ and $\tilde{H}_{\varepsilon_k} \to \tilde{H}$ in $C^2([-1,1])$. First, we claim that $\tilde{H}(x)/r(x)$ is strictly concave, since

$$\partial_x^2\left(\frac{\tilde{H}_{\varepsilon_k}(x)}{r(x)}\right) = \partial_x^2\left(1 - \int \frac{K(x,y)}{r(x)}\tilde{u}_{\varepsilon_k}(y)\,dy\right) \leq -\inf_{\substack{-1<x<1\\0<y<1}} \partial_x^2\left[\frac{K(x,y)}{r(x)}\right]\int \tilde{u}_{\varepsilon_k}(y)\,dy.$$

By assumption **(A)** and Lemma 3.1, we may let $\varepsilon_k \to 0$ to conclude the strict concavity of $\tilde{H}(x)/r(x)$.

This, and Proposition 1(iii), implies the existence of some $x' \in [-1,1]$, such that $\tilde{H}(x) \leq 0$ and equality holds iff $x = x'$. (Note that x' may depend on the subsequence.) By Proposition 2, we deduce that $\tilde{u}_{\varepsilon_k}(x) \to \frac{r(x')}{K(x',x')}\delta_0(x-x')$ in distribution sense. Moreover,

$$\tilde{H}(x) = \lim_{k\to\infty}\left[r(x) - \int K(x,y)u_{\varepsilon_k}(y)\,dy\right] = r(x) - \frac{r(x')K(x,x')}{K(x',x')} \quad \text{for } -1 \leq x \leq 1.$$

The fact that $\tilde{H}(x)$ is non-positive (Proposition 1(iii)) implies that

$$\frac{K(x,x')}{r(x)} \geq \frac{K(x',x')}{r(x')} \quad \text{for all } x \in [-1,1].$$

By **(A)**, we must have $x' = \hat{x}$. Since the limit point $x' = \hat{x}$ is independent of subsequence $\varepsilon_k \to 0$, we deduce that in the full limit $\varepsilon \to 0$, $\tilde{u}_\varepsilon(x) \to \frac{r(\hat{x})}{K(\hat{x},\hat{x})}\delta_0(x-\hat{x})$ in distribution sense. $\square$

Proof of Theorem 2. By Proposition 1, we pass to a sequence $\varepsilon_k \to 0$ so that $\tilde{v}_{\varepsilon_k} \to \tilde{v}$ in $C([-1,1])$ and $\tilde{H}_{\varepsilon_k} \to \tilde{H}$ in $C^2([-1,1])$. We claim that $\tilde{H}(x)/r(x)$ is strictly convex. To this end, we compute

$$\partial_x^2\left(\frac{\tilde{H}_{\varepsilon_k}(x)}{r(x)}\right) = \partial_x^2\left(1 - \int \frac{K(x,y)}{r(x)}\tilde{u}_{\varepsilon_k}(y)\,dy\right) \geq -\sup_{\substack{-1<x<1\\0<y<1}} \partial_x^2\left[\frac{K(x,y)}{r(x)}\right]\int \tilde{u}_{\varepsilon_k}(y)\,dy,$$

and observe that the strict convexity of $\tilde{H}(x)/r(x)$ follows from hypothesis **(B)** and Lemma 3.1. Combining with the facts that $H(x) \leq 0$ and $r(x) > 0$ in $[-1,1]$, $\tilde{H}(x)/r(x)$, and hence $\tilde{H}(x)$, are strictly negative in $(-1,1)$.

Claim 2. $\tilde{v}(x) < 0$ *for* $-1 < x < 1$.

Suppose to the contrary that $\tilde{v}(x') = 0$ for some $x' \in (-1,1)$, then x' is an interior local maximum point of $\tilde{v}$. By the fact that $\tilde{v}$ is viscosity solution of (8), we deduce that $\tilde{H}(x') \geq 0$ for the interior point $x' \in (-1,1)$. This is a contradiction, as $\tilde{H}(x) < 0$ in $(-1,1)$. Thus $\tilde{v}(x) < 0$ for $-1 < x < 1$ and, by Lemma 3.1,

$$\tilde{u}_{\varepsilon_k}(x) \to A\delta_0(x+1) + B\delta_0(x-1) \quad \text{in distribution sense.} \tag{10}$$

Claim 3. *$A > 0$ and $B > 0$.*

Otherwise suppose $B = 0$, then $\tilde{u}_{\varepsilon_k} \to A\delta_0(x+1)$. By the arguments in the proof of Proposition 2, we deduce that $\tilde{H}(x) = r(x) - \frac{K(x,-1)r(-1)}{K(-1,-1)}$. By Proposition 1(iii), $\tilde{H}(x) \leq 0$ for all $x \in [-1,1]$, and hence

$$\frac{K(-1,-1)}{r(-1)} \leq \frac{K(x,-1)}{r(x)} \quad \text{for } -1 \leq x \leq 1.$$

But this is a contradiction to $\partial_x \left[\frac{K(x,y)}{r(x)}\right]_{x=y=-1} < 0$ (by **(B)**). Hence $B > 0$. Similarly, one can show that $A > 0$ as well.

To determine the value of the positive constants A and B, we first prove the following estimate.

Claim 4. . $\lim\limits_{k\to\infty} |\partial_x \tilde{u}_{\varepsilon_k}(0)| = 0$.

To see the claim, let $\delta = -\frac{1}{2}\inf_{|x|<1/2} \tilde{v}$, then by Claim 2 we have $\delta > 0$. For all k large,

$$\sup_{|x|<\frac{1}{2}} \tilde{v}_{\varepsilon_k} < -\delta, \quad \text{and} \quad \sup_{|x|<1/2} \tilde{u}_{\varepsilon_k} < \exp\left(-\frac{\delta}{\varepsilon_k}\right).$$

Now, let $U_k(z) := \tilde{u}_{\varepsilon_k}(\varepsilon_k z)$, then $\sup_{|z|<2} |U_k(z)| < \exp\left(-\frac{\delta}{\varepsilon_k}\right)$, and

$$-\partial_z^2 U_k(z) = \tilde{H}_{\varepsilon_k}(\varepsilon_k z) U_k(z) \quad \text{for } |z| < 1/\varepsilon_k.$$

Since $\|\tilde{H}_{\varepsilon_k}\|_{C([-1,1])} \leq C$ (Lemma 3.2), we deduce that

$$\sup_{|z|<2} |\partial_z^2 U_k(z)| \leq C \exp\left(-\frac{\delta}{\varepsilon_k}\right).$$

By interpolation, $\varepsilon_k |\partial_x \tilde{u}_{\varepsilon_k}(0)| = |\partial_z U_k(0)| \leq C\exp\left(-\frac{\delta}{\varepsilon_k}\right)$. This yields Claim 4.

We conclude the proof by determining A and B. To this end we integrate (3) over $-1 < x < 0$, then

$$-\varepsilon_k^2 \partial_x \tilde{u}_{\varepsilon_k}(0) = \int_{-1}^{0} \tilde{u}_{\varepsilon_k}(x) \left(r(x) - \int_{-1}^{1} K(x,y)\tilde{u}_{\varepsilon_k}(y)\,dy \right) dx.$$

Using (10) and using Claim 4, we may let $k \to \infty$ to obtain

$$0 = A\left[r(-1) - K(-1,-1)A - K(-1,1)B\right]. \tag{11}$$

Similarly, we may repeat integrate (3) over $0 < x < 1$ and repeat the above arguments to obtain

$$0 = B\left[r(1) - K(1,-1)A - K(1,1)B\right]. \tag{12}$$

Solving (11) and (12), we have

$$A = \frac{\frac{K(1,1)}{r(1)} - \frac{K(-1,1)}{r(-1)}}{\frac{K(-1,-1)K(1,1)}{r(-1)r(1)} - \frac{K(-1,1)K(1,-1)}{r(-1)r(1)}} = \frac{K(1,1)r(-1) - K(-1,1)r(1)}{K(-1,-1)K(1,1) - K(-1,1)K(1,-1)},$$

and

$$B = \frac{\frac{K(-1,-1)}{r(-1)} - \frac{K(1,-11)}{r(1)}}{\frac{K(-1,-1)K(1,1)}{r(-1)r(1)} - \frac{K(-1,1)K(1,-1)}{r(-1)r(1)}} = \frac{K(-1,-1)r(1) - K(1,-1)r(-1)}{K(-1,-1)K(1,1) - K(-1,1)K(1,-1)}.$$

Since A and B are uniquely determined and is independent of subsequences, we deduce in the full limit $\varepsilon \to 0$, $\tilde{u}_\varepsilon(x) \to A\delta_0(x+1) + B\delta_0(x-1)$ holds in distribution sense. This proves Theorem 2. □

5. **Proof of Theorem 3.** Consider now the time-dependent problem (1) in case $\Omega = (-1, 1)$.

$$\begin{cases} \varepsilon\partial_t u_\varepsilon = \varepsilon^2\partial_x^2 u_\varepsilon + u_\varepsilon\left(r(x) - \int K(x,y)u_\varepsilon(y,t)\,dy\right) & \text{for } -1 < x < 1, t > 0, \\ \partial_x u_\varepsilon = 0 & \text{for } x = \pm 1, t > 0, \\ u_\varepsilon(x,0) = u_0(x) & \text{for } -1 < x < 1. \end{cases} \tag{13}$$

In this section, let $u_\varepsilon(x,t)$ be a solution of (13).

5.1. **Persistence theory and the existence of equilibrium.**

Lemma 5.1. *The function $\rho_\varepsilon(t) := \int u_\varepsilon(y,t)\,dy$ satisfies*

$$\min\left\{\rho_\varepsilon(0), \frac{r_*}{K^*}\right\} \le \rho_\varepsilon(t) \le \max\left\{\rho_\varepsilon(0), \frac{r^*}{K_*}\right\}, \tag{14}$$

where r^*, r_*, K^*, K_* are given in (7), and, letting $t \to \infty$,

$$\frac{r_*}{K^*} \le \liminf_{t\to\infty} \rho_\varepsilon(t) \le \limsup_{t\to\infty} \rho_\varepsilon(t) \le \frac{r^*}{K_*}. \tag{15}$$

Proof. By integrating (13) over x, we see that ρ_ε satisfies

$$\varepsilon\partial_t\rho_\varepsilon = \int u_\varepsilon r\,dy + \iint K(x,y)u_\varepsilon(x)u_\varepsilon(y)\,dxdy \tag{16}$$

and hence also the differential inequalities

$$\varepsilon\partial_t\rho_\varepsilon \le \rho_\varepsilon(r^* - K_*\rho_\varepsilon), \qquad \text{and} \qquad \varepsilon\partial_t\rho_\varepsilon \ge \rho_\varepsilon(r_* - K^*\rho_\varepsilon),$$

from which the lemma follows by ODE comparison. □

Lemma 5.2. *There exists $C > 0$, such that for any $t_0 > 1$,*

$$\|u_\varepsilon(\cdot, t_0)\|_{L^\infty((-1,1))} \le C\int_{t_0-1}^{t_0}\int_{-1}^{1} u_\varepsilon(x,t)\,dxdt.$$

Proof. For each y and t, extend $u_\varepsilon(x,t)$, $r(x)$ and $K(x,y)$ on the boundary $x = \pm 1$ by reflection, we may assume that u_ε satisfies the same equation in $(-3,3)\times[0,\infty)$. Hence, we have

$$\|u_\varepsilon(\cdot,t_0)\|_{L^\infty((-1,1))} \le C\int_{t_0-1}^{t_0}\int_{-2}^{2} u_\varepsilon(x,t)\,dxdt = 2C\int_{t_0-1}^{t_0}\int_{-1}^{1} u_\varepsilon(x,t)\,dxdt$$

by application of the local maximum principle [19, Theorem 7.36]. □

The following proposition from persistence theory, which is a special case of [25, Theorem 6.2], is the key to proving Theorem 3.

Proposition 3. *Fix $\varepsilon > 0$. Suppose*

(i) *X is a closed convex subset of $C([0,1];[0,\infty))$.*
(ii) *X is forward-invariant with respect to the semiflow generated by (13) in $C([0,1];[0,\infty))$.*
(iii) *X is not the singleton set of the trivial function.*

Then (3) has a positive solution $\tilde{u}_\varepsilon(x)$ lying in X.

Proof. In the context of persistence theory, for each $u_0(x) \in X$, we define the persistence function $\rho : X \to [0,\infty)$ by $\rho(u_0) = \inf_{-1<x<1} u_0$. Then ρ is continuous and concave. First, we prove the following claim, which asserts that the semiflow $\Phi_t : X \to X$, generated by (13) in $C([0,1];[0,\infty))$, is uniformly strongly ρ- persistent (see [25, Definition 3.1]). Here for each $u_0 \in C([-1,1])$, $\Phi_t(u_0) = u_\varepsilon(\cdot,t)$, where u_ε is the solution of (13) with initial data u_0.

Claim 5. *There exists $\delta > 0$ independent of (non-trivial) initial condition $u_0 \geq 0$ such that*
$$\liminf_{t\to\infty} \rho(u_\varepsilon(\cdot,t)) \geq \delta.$$

To see the claim, we apply the Harnack inequality (for parabolic equations on bounded domain with Neuman boundary condtiions), due to J. Huska [13, Theorem 2.5], to obtain
$$\inf_{-1<x<1} u_\varepsilon(x,t) \geq C \sup_{-1<x<1} u_\varepsilon(x,t) \geq \frac{C}{2}\int u_\varepsilon(y,t)\,dy \qquad \text{for } t > 1.$$
Claim 5 thus follows upon taking $t \to \infty$, and using Lemma 5.1.

Claim 6. *The semiflow Φ_t, restricted to the forward-invariant set X, has a compact attractor $\mathcal{A}$ of neighborhood of compact sets. i.e. every compact subsets $K_0 \subset\subset X$ has a neighborhood N such that*
$$\lim_{t\to\infty} \operatorname{dist}(\Phi_t(u_0),\mathcal{A}) = 0 \qquad \textit{uniformly for } u_0 \in N,$$
where $\operatorname{dist}(\Phi_t(u_0),\mathcal{A}) = \inf_{v_0\in\mathcal{A}} \|\Phi_t(u_0) - v_0\|_{C([-1,1])}$.

We use [25, Theorem 2.30] to show the claim. It suffices to show that the semiflow Φ_t is (i) point-dissipative; (ii) asymptotically smooth; and (iii) eventually bounded on every compact subset K_0 of X. Here we refer the readers to [25, Definition 2.25] for the definitions of (i) - (iii). Point-dissipativity is a direct consequence of Lemmas 5.1 and 5.2.

Next, we prove asymptotic smoothness. First, we combine the parabolic Krylov-Safanov estimate [15] (see also [19, Corollary 7.36]) and the local maximum principle (Lemma 5.2) to obtain, for each $\varepsilon > 0$, $0 < \gamma < 1$ and $0 < \delta < T$, the existence of a constant $C > 0$ such that for any $t_0 \geq 0$,
$$\begin{aligned} \|u_\varepsilon\|_{C^\gamma([-1,1]\times[t_0+\delta,t_0+T))} &\leq C\|u_\varepsilon(\cdot,\cdot)\|_{L^\infty((-1,1)\times(t_0+\delta/2,t_0+T))} \\ &\leq C\|u_\varepsilon(\cdot,\cdot)\|_{L^1((-1,1)\times(t_0,t_0+T))}. \end{aligned} \tag{17}$$
Now, let X_1 be a forward-invariant, bounded, closed subset of X, let $t_i \to \infty$ and $p_i \in X_1$, then by (17) and Lemma 5.1,
$$\|\Phi_{t_i}(p_i)\|_{C^\gamma([-1,1])} \leq C\int_{t_i-1}^{t_i}\int_{-1}^{1} u_\varepsilon(x,t)\,dxdt \leq C,$$
i.e. the family $\{\Phi_{t_i}(p_i)\}_i$ is uniformly bounded in $C^\gamma([-1,1])$ and hence has a convergent subsequence in $C([0,1])$. This demonstrates that Φ_t is asymptotically smooth.

Finally, let K_0 be a compact susbet of X, then there exists $M > 0$ such that $\sup_{u_0\in K_0}\int u_0(y)\,dy \leq M$, and Lemma 5.1 implies that
$$\sup_{t\geq 0}\|u_\varepsilon(\cdot,t)\|_{L^1((-1,1))} \leq \max\left\{M, \frac{\max_{[-1,1]} r}{\min_{[-1,1]\times[-1,1]} K}\right\}.$$

Hence, Lemma 5.2 says that if $u_0 \in K_0$, then

$$\sup_{t>1} \|u_\varepsilon(\cdot,t)\|_{C([-1,1])} \leq C \sup_{t\geq 0} \|u_\varepsilon(\cdot,t)\|_{L^1((-1,1))} \leq C \max\left\{M, \frac{\max_{[-1,1]} r}{\min_{[-1,1]\times[-1,1]} K}\right\},$$

i.e. the semiflow Φ_t is eventually bounded on every compact subset K_0 of X. This proves Claim 6.

Claim 7. *For each $t \in (0,1]$, $\Phi_t : X \to X$ is compact.*

Fix $t \in (0,1]$ and a bounded subset B of X, then by (17), there exists $C = C(t)$ such that

$$\|\Phi_t(u_0)\|_{C^\gamma([-1,1])} = \|u_\varepsilon(\cdot,t)\|_{C^\gamma([-1,1])} \leq C\|u_\varepsilon\|_{L^1((-1,1)\times(0,1))}$$

where $u_\varepsilon(x,t)$ is the solution of (13) with initial condition u_0. By Lemma 5.1, the last term can be estimated by $C\max\{\|u_0\|_{L^1((-1,1))}, r^*/K_*\}$. Hence we may take supremum over $u_0 \in B$, so that $\Phi_t(B)$ is a bounded subset of $C^\gamma([-1,1])$ and is precompact in X. This proves Claim 7.

Claim 8. *If $u_0 \in C([-1,1])$ satisfies $\inf_{-1<x<1} u_0 > 0$, then $\inf_{-1<x<1} u_\varepsilon(x,t) > 0$ for all $t > 0$.*

This is a direct consequence of the strong maximum principle [19, Theorem 2.7].

Finally, by the above setup, and Claims 5, 6, 7 and 8, we may apply [25, Theorem 6.2] to conclude the existence of at least one positive solution $\tilde{u}_\varepsilon(x)$ of (3) in X. □

Corollary 5.3. *Under our hypotheses* **(H)** *on $K(x,y)$ and $r(x)$. The equation (3) has at least one positive solution.*

Proof. Take $X = C([-1,1];[0,\infty))$ to be the set of nonnegative continuous functions in Proposition 3. □

5.2. Proof of Theorem 3.

Lemma 5.4. *Define*

$$h_-(x) := \partial_x\left[K(x,-1)\frac{r(-1)}{K(-1,-1)} - r(x)\right], \quad h_+(x) := \partial_x\left[K(x,1)\frac{r(1)}{K(1,1)} - r(x)\right]. \tag{18}$$

Under the assumption **(C)**, *there exists $g_-(x), g_+(x) \in C^2([-1,1])$ (both are independent of ε), such that*

(i) $g_-(x) < h_-(x)$ *for all* $x \in [-1,1]$;

(ii) $\int_{-1}^{x} g_-(y)\,dy > 0$ *for* $-1 < x < 1$;

(iii) $\int_{-1}^{1} g_-(y)\,dy = 0$;

(iv) $g_-(x) = x+1$ *in some neighborhood of* -1,

and that

(i') $g_+(x) > h_+(x)$ *for all* $x \in [-1,1]$;

(ii') $\int_{-1}^{x} g_+(y)\,dy > 0$ *for* $-1 < x < 1$;

(iii') $\int_{-1}^{1} g_+(y)\,dy = 0$;

(iv') $g_+(x) = x - 1$ *in some neighborhood of* 1.

Proof. We will first construct $g_-(x)$. By assumption **(C)**, we have

$$h_-(-1) > 0 \quad \text{and} \quad \int_{-1}^{x} h_-(y)\,dy > 0 \ \text{ for } \ -1 < x \le 1.$$

Hence, By subtracting a small positive constant from $h_-(x)$ and modifying in a small neighborhood of -1, one may obtain a smooth function g_0 such that $g_0(x) = x + 1$ in a small neighborhood of -1 and $g_0(x) < h_-(x)$ in $[-1, 1]$ and $\int_0^x g_0(y)\,dy > 0$ for $x \in (-1, 1)$. Finally, further subtract from g_0 a positive function supported in a neighborhood of 1, we obtain $g_-(x)$ with all desired properties (i) to (iv).

To construct $g_+(x)$, we first observe by assumption **(C)** that

$$h_+(1) < 0 \quad \text{and} \quad \int_{x}^{1} h_+(y)\,dy < 0 \quad \text{for } \ -1 \le x < 1.$$

By repeating the steps in constructing $g_-(x)$, we obtain $g_+(x)$ satisfying properties (i'), (iii'), (iv'), and that

$$\int_x^1 g_+(y)\,dy < 0 \quad \text{in } (-1, 1).$$

In view of (iii'), the last property is equivalent to (ii"). $\square$

For each sufficiently small ε, we construct the sets $X^{\delta}_{\varepsilon,\pm}$, which will then shown to be forward-invariant. For this purpose, fix

$$\delta_0 := \frac{c_0}{3\|K\|_{C^2([-1,1]^2)}}, \quad \text{where } \ c_0 = \frac{1}{2}\min\left\{\inf_{(-1,1)} |g_- - h_-|, \inf_{(-1,1)} |g_+ - h_+|\right\} > 0 \tag{19}$$

and choose

$$0 < \delta < \min\{\frac{(\min r)^3}{2(\max K)^2\|r'\|}, \delta_0\} \quad \text{and} \quad 0 < \eta_0 < \min\left\{\frac{2r^*}{K_*}, \frac{K_*\delta_0}{r_*}\right\}, \tag{20}$$

with η_0 small enough so that

$$\max\left\{\sup_{[-1,-1+\eta_0]^3}\left|\frac{r(z)}{K(x,y)} - \frac{r(-1)}{K(-1,-1)}\right|, \sup_{[1-\eta_0,1]^3}\left|\frac{r(z)}{K(x,y)} - \frac{r(1)}{K(1,1)}\right|\right\} < \frac{\delta}{2}. \tag{21}$$

Next, define

$$w_-(x) = -\sqrt{\int_{-1}^{x} g_-(y)\,dy} < 0 \quad \text{and} \quad w_+(x) = \sqrt{\int_{-1}^{x} g_+(y)\,dy} > 0. \tag{22}$$

where $g_\pm$ is from Lemma 5.4, and define the spaces

$$X_{\varepsilon,+} := \left\{ u_0 \in C^1([-1,1];[0,\infty)) \ : \ \begin{array}{ll} \frac{r_*}{K^*} \le \int u_0\,dy \le \frac{r^*}{K_*} & \text{and} \\ \varepsilon \partial_x u_0 \ge w_+ u_0 & \text{for } -1 < x < 1. \end{array} \right\},$$

$$X_{\varepsilon,-} := \left\{ u_0 \in C^1([-1,1];[0,\infty)) \ : \ \begin{array}{ll} \frac{r_*}{K^*} \le \int u_0\,dy \le \frac{r^*}{K_*} & \text{and} \\ \varepsilon \partial_x u_0 \le w_- u_0 & \text{for } -1 < x < 1. \end{array} \right\},$$

$$X^{\delta}_{\varepsilon,+} := \left\{ u_0 \in X_{\varepsilon,+} : \left|\int u_0\,dy - \frac{r(1)}{K(1,1)}\right| \le \delta \right\},$$

and

$$X^{\delta}_{\varepsilon,-} := \left\{ u_0 \in X_{\varepsilon,-} : \left|\int u_0\,dy - \frac{r(-1)}{K(-1,-1)}\right| \le \delta \right\}.$$

Lemma 5.5. *Let η_0 satisfy (20) and (21), and let $\eta_3 = \frac{1}{2}\int_{-1+\eta_0/2}^{-1+\eta_0} -w_-(y)\,dy > 0$, where w_- is defined in (22). Then for all $\varepsilon < \varepsilon_0 := \eta_3/\log\left(\frac{2r^*}{\eta_0 K_*}\right)$ and any $u_0 \in X_{\varepsilon,-}$, we have*

$$\varepsilon \log u_0(x_0) \le -\eta_3 \quad \text{for all } x_0 \in [-1+\eta_0, 1]. \tag{23}$$

Proof. Now, let $x \in [-1, -1+\eta_0/2]$ and $x_0 \in [-1+\eta_0, 1]$. By definition of $X_{\varepsilon,-}$,

$$\varepsilon \log u_0(x) = \varepsilon \log u_0(x_0) - \int_x^{x_0} \frac{\varepsilon \partial_x u_0(y)}{u_0(y)}\,dy \ge \varepsilon \log u_0(x_0) - \int_{-1+\eta_0/2}^{-1+\eta_0} w_-(y)\,dy.$$

Hence, letting $\eta_3 = \frac{1}{2}\int_{-1+\eta_0/2}^{-1+\eta_0} -w_-(y)\,dy > 0$,

$$u_0(x) \ge \exp\left(\frac{\varepsilon \log u_0(x_0) + 2\eta_3}{\varepsilon}\right).$$

Integrating over $x \in [-1, -1+\eta_0/2]$, and using the integral constraint in $X_{\varepsilon,-}$, we have

$$\frac{r^*}{K_*} \ge \frac{\eta_0}{2} \exp\left(\frac{\varepsilon \log u_0(x_0) + 2\eta_3}{\varepsilon}\right).$$

By our choices of η_0 and ε, we have

$$\eta_3 > \varepsilon \log\left(\frac{2r^*}{\eta_0 K_*}\right) \ge \varepsilon \log u_0(x_0) + 2\eta_3.$$

This proves (23). $\square$

Lemma 5.6. *Assume $u_\varepsilon(\cdot,t) \in X^\delta_{\varepsilon,-}$ for $0 \le t \le T$, then when $\varepsilon \in (0, \varepsilon_0']$, we have*

$$\sup_{-1<x<1} [\partial_x H_\varepsilon(x,t) + g_-(x)] < -c_0 \quad \text{for } \; 0 \le t \le T, \tag{24}$$

where c_0 is given in (19), and

$$\varepsilon_0' := \frac{\eta_3}{\log 8 - \log \delta_0}, \qquad H_\varepsilon(x,t) := r(x) - \int K(x,y) u_\varepsilon(y,t)\,dy. \tag{25}$$

Proof. Recall the definition of $h_-(x)$ in (18), we compute

$$\begin{aligned}
&\partial_x H_\varepsilon(x,t) + h_-(x) \\
&= \partial_x K(x,-1)\frac{r(-1)}{K(-1,-1)} - \int_{-1}^1 \partial_x K(x,y) u_\varepsilon(y,t)\,dy \\
&= \partial_x K(x,-1)\left[\frac{r(-1)}{K(-1,-1)} - \int_{-1}^1 u_\varepsilon(y,t)\,dy\right] + \int_{-1}^{-1+\eta_0} [\partial_x K(x,-1) - \partial_x K(x,y)]\, u_\varepsilon(y,t)\,dy \\
&\qquad + \int_{-1+\eta_0}^1 [\partial_x K(x,-1) - \partial_x K(x,y)]\, u_\varepsilon(y,t)\,dy \\
&\le \delta \|K\|_{C^1} + \eta_0 \|K\|_{C^2} \int_{-1}^{-1+\eta_0} u_\varepsilon\,dy + 2\|K\|_{C^1} \int_{-1+\eta_0}^1 u_\varepsilon\,dy \\
&\le \|K\|_{C^2}\left(\delta + \eta_0 \frac{r^*}{K_*} + 4e^{-\eta_3/\varepsilon}\right),
\end{aligned}$$

where we used the integral constraint in the definition of $X_{\varepsilon,-}$, and Lemma 5.5, in the last inequality. By the definition of δ and η_0 in (20) and that $\varepsilon \in (0, \varepsilon_0']$, we have

$$\partial_x H_\varepsilon(x,t) + h_-(x) < 3\delta_0 \|K\|_{C^2} = c_0,$$

and thus

$$\partial_x H_\varepsilon(x,t) + g_-(x) + < c_0 - h_-(x) + g_-(x) \leq -c_0 \quad \text{for } (x,t) \in [-1,1]\times[0,T],$$

where the last inequality follows from the definition of c_0 in (19) and the fact that $g_-(x) - h_-(x) < -2c_0$ in $[-1,1]$ (Lemma 5.4). □

Lemma 5.7. *Let c_0 be given by (19) and $\varepsilon \leq \varepsilon_0'' := c_0/(\|w_-\|_{C^2} + \|w_+\|_{C^2})$. Assume $u_\varepsilon(\cdot,0) \in X^\delta_{\varepsilon,-}$ and*

$$\sup_{-1<x<1} [\partial_x H_\varepsilon(x,t) + g_-(x)] < -c_0 \quad \textit{for } \; 0 \leq t \leq T,$$

Then $u_\varepsilon(\cdot,t) \in X_{\varepsilon,-}$ for $0 \leq t \leq T$.

Proof. Define $v_\varepsilon(x,t) = \varepsilon \log u_\varepsilon(x,t)$, then v_ε satisfies

$$\begin{cases} \partial_t v_\varepsilon = \varepsilon \partial_x^2 v_\varepsilon + |\partial_x v_\varepsilon|^2 + H_\varepsilon(x,t) & \text{for } x \in (-1,1), t > 0, \\ \partial_x v_\varepsilon = 0 & \text{for } x = \pm 1, t > 0, \\ v_\varepsilon(x,0) = \varepsilon \log u_\varepsilon(x,0) & \text{for } x \in (-1,1). \end{cases} \tag{26}$$

By Lemma 5.1 and the definition of $X_{\varepsilon,-}$, it is enough to show

$$\partial_x v_\varepsilon(x,t) \leq w_-(x) < 0 \text{ for } (x,t) \in (-1,1)\times[0,T]. \tag{27}$$

Now, differentiate (26) with respect to x, and use Lemma 5.6, we have

$$\begin{cases} \partial_t w - \varepsilon \partial_{xx} w - 2w\partial_x w - \partial_x H_\varepsilon(x,t) = 0 & \text{in } (-1,1)\times[0,T], \\ w = 0 & \text{on } \{-1,1\}\times[0,T], \\ w(x,0) \leq w_-(x) & \text{in } (-1,1). \end{cases} \tag{28}$$

where $w(x,t) = \partial_x v_\varepsilon(x,t) = \varepsilon \partial_x u_\varepsilon(x,t)/u_\varepsilon(x,t)$. Moreover, we verify that w_- (as given in (22)) satisfies

$$\begin{cases} -\varepsilon(\partial_{xx} w_-) - 2(w_-)\partial_x(w_-) - \partial_x H_\varepsilon(x,t) = -\varepsilon(\partial_{xx} w_-) - g_-(x) - \partial_x H_\varepsilon(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (-1,1), \\ w_- = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x = \pm 1. \end{cases} \tag{29}$$

Now, by the hypotheses of the lemma,

$$-\varepsilon(\partial_{xx} w_-) - 2(w_-)\partial_x(w_-) - \partial_x H_\varepsilon(x,t) = -\varepsilon(\partial_x^2 w_-) - g(x) - \partial_x H_\varepsilon(x,t) > -c_0 + c_0 = 0.$$

Hence w_- is an upper solution of (28), from which it follows that $\partial_x v_\varepsilon = w \leq w_-$ for $(x,t) \in (-1,1)\times[0,T]$. □

Lemma 5.8. *Let $\varepsilon \leq \varepsilon_0''' := \frac{\eta_3}{\log 2C_1 - \log \delta}$, where*

$$C_1 = \max\left\{4 + \frac{2K^*}{K_*}\left(2 + \frac{r^*}{r_*}\right), 2\frac{r^* K^*}{r_* K_*} + 4\frac{(K^*)^2}{K_* r_*}\right\}.$$

Suppose

$$u_\varepsilon(\cdot,0) \in X^\delta_{\varepsilon,-}, \quad \textit{and} \quad u_\varepsilon(\cdot,t) \in X_{\varepsilon,-} \;\; \textit{for } 0 \leq t \leq T,$$

then $u_\varepsilon(\cdot,t) \in X^\delta_{\varepsilon,-}$ for $0 \leq t \leq T$.

Proof. Recall that $\rho_\varepsilon(t) := \int_{-1}^1 u_\varepsilon(y,t)\,dy$, then $\frac{r_*}{K^*} \le \rho_\varepsilon(t) \le \frac{r^*}{K_*}$. By integrating (13) over $-1 < x < 1$, we have

$$\begin{aligned}
\varepsilon\partial_t\rho_\varepsilon \le& \left(\sup_{(-1,-1+\eta_0)} r\right)\int_{-1}^{-1+\eta_0} u_\varepsilon\,dy + r^*\int_{-1+\eta_0}^1 u_\varepsilon\,dy\\
&- \left(\inf_{(-1,-1+\eta_0)^2} K\right)\left[\int_{-1}^{-1+\eta_0} u_\varepsilon\,dy\right]^2 + 2K^*\left[\int_{-1}^1 u_\varepsilon\,dy\right]\left[\int_{-1+\eta_0}^1 u_\varepsilon\,dy\right]\\
=& \left(\inf_{(-1,-1+\eta_0)^2} K\right)\left[\frac{\sup_{(-1,-1+\eta_0)} r}{\inf_{(-1,-1+\eta_0)^2} K} - \int_{-1}^{-1+\eta_0} u_\varepsilon\,dy\right]\int_{-1}^{-1+\eta_0} u_\varepsilon\,dy\\
&+ [r^* + 2K^*\rho_\varepsilon(t)]\int_{-1+\eta_0}^1 u_\varepsilon\,dy.
\end{aligned}$$

Using Lemma 5.1, we have

$$\int_{-1}^{-1+\eta_0} u_\varepsilon\,dy = \rho_\varepsilon - \int_{-1+\eta_0}^1 u_\varepsilon\,dy, \quad \text{and} \quad \frac{K^*}{r_*}\rho_\varepsilon \ge 1,$$

so that

$$\begin{aligned}
\varepsilon\partial_t\rho_\varepsilon \le& \left(\inf_{(-1,-1+\eta_0^2)} K\right)\left[\frac{\sup_{(-1,-1+\eta_0)} r}{\inf_{(-1,-1+\eta_0)^2} K}\right]\int_{-1}^{-1+\eta_0} u_\varepsilon\,dy - \left(\inf_{(-1,-1+\eta_0^2)} K\right)\rho_\varepsilon^2\\
&+ \left(\inf_{(-1,-1+\eta_0^2)} K\right)\left[\rho_\varepsilon^2 - \left(\int_{-1}^{-1+\eta_0} u_\varepsilon\,dy\right)^2\right] + [r^* + 2K^*]\,\rho_\varepsilon\int_{-1+\eta_0}^1 u_\varepsilon\,dy\\
\le& \left(\inf_{(-1,-1+\eta_0^2)} K\right)\left[\frac{\sup_{(-1,-1+\eta_0)} r}{\inf_{(-1,-1+\eta_0)^2} K}\right]\rho_\varepsilon - \left(\inf_{(-1,-1+\eta_0^2)} K\right)\rho_\varepsilon^2\\
&+ \left(\inf_{(-1,-1+\eta_0^2)} K\right)\left[2\rho_\varepsilon\int_{-1+\eta_0}^1 u_\varepsilon\,dy\right] + \left[\frac{K^*r^*}{r_*} + 2K^*\right]\rho_\varepsilon\int_{-1+\eta_0}^1 u_\varepsilon\,dy\\
\le& \left(\inf_{(-1,-1+\eta_0^2)} K\right)\left[\frac{\sup_{(-1,-1+\eta_0)} r}{\inf_{(-1,-1+\eta_0)^2} K} - \rho_\varepsilon\right]\rho_\varepsilon\\
&+ \left[2\left(\inf_{(-1,-1+\eta_0^2)} K\right) + \frac{K^*r^*}{r_*} + 2K^*\right]\rho_\varepsilon\int_{-1+\eta_0}^1 u_\varepsilon\,dy\\
\le& \left(\inf_{(-1,-1+\eta_0^2)} K\right)\left[\frac{\sup_{(-1,-1+\eta_0)} r}{\inf_{(-1,-1+\eta_0)^2} K} + \frac{C_1}{2}\int_{-1+\eta_0}^1 u_\varepsilon\,dy - \rho_\varepsilon\right]\rho_\varepsilon,
\end{aligned}$$

where C_1 is given in the statement of the lemma. Using also Lemmas 5.5 and (21),

$$\varepsilon\partial_t\rho_\varepsilon \le \left(\inf_{(-1,-1+\eta_0^2)} K\right)\left[\frac{r(-1)}{K(-1,-1)} + \frac{\delta}{2} + C_1 e^{-\eta_3/\varepsilon} - \rho_\varepsilon\right]\rho_\varepsilon \quad \text{for } 0 \le t \le T.$$

By the smallness of ε specified in the lemma, we deduce that

$$\varepsilon\partial_t\rho_\varepsilon \le \left(\inf_{(-1,-1+\eta_0^2)} K\right)\left(\frac{r(-1)}{K(-1,-1)} + \delta - \rho_\varepsilon\right)\rho_\varepsilon \quad \text{for } 0 \le t \le T.$$

Since also $\rho_\varepsilon(0) \le \frac{r(-1)}{K(-1,-1)} + \delta$, we have $\rho_\varepsilon(t) \le \frac{r(-1)}{K(-1,-1)} + \delta$ for $t \in [0,T]$. Similarly, we can show that $\rho_\varepsilon(t) \ge \frac{r(-1)}{K(-1,-1)} - \delta$ for $t \in [0,T]$. Hence we have $u_\varepsilon(\cdot,t) \in X^\delta_{\varepsilon,-}$ for $t \in [0,T]$. ☐

Proposition 4. *For all ε sufficiently small, the semiflow Φ_t generated by (13) is forward-invariant in $X^\delta_{\varepsilon,+}$ and also in $X^\delta_{\varepsilon,-}$.*

Proof. Fix $\varepsilon \leq \min\{\varepsilon_0, \varepsilon_0', \varepsilon_0'', \varepsilon_0'''\}$, where the latter four constants are as specified in Lemmas 5.5 to 5.8. Assume $u_\varepsilon(\cdot, 0) \in X^\delta_{\varepsilon,-}$, and define I to be the maximal interval for which $u_\varepsilon(\cdot, t) \in X^\delta_{\varepsilon,-}$, i.e.

$$I := \{t_0 \geq 0 : u_\varepsilon(\cdot, t) \in X^\delta_{\varepsilon,-} \quad \text{for } 0 \leq t \leq t_0\},$$

and I' to be the maximal interval for which $\sup_{-1<x<1}[\partial_x H_\varepsilon(x, t) + g_-(x)] < -c_0$,

$$I' := \left\{ t_0 \geq 0 : \sup_{-1<x<1} [\partial_x H_\varepsilon(x, t) + g_-(x)] < -c_0 \quad \text{for } 0 \leq t \leq t_0 \right\}.$$

By the definition of $X^\delta_{\varepsilon,-}$ it is clear that I is closed, $I \ni 0$ is nonempty, and that I' is open.

We claim that $I = I'$, so that I is non-empty, open and closed. Now Lemma 5.6 implies $I \subset I'$; and Lemmas 5.7 and 5.8 together implies $I \supset I'$. We thus have $I = I'$. Being a non-empty, open and closed subset of $[0, \infty)$, it must be the case that $I = [0, \infty)$. i.e. $X^\delta_{\varepsilon,-}$ is forward-invariant.

The proof for the forward-invariance of $X^\delta_{\varepsilon,+}$ is similar and is omitted. □

Proof of Theorem 3. By the forward-invariance of $X^\delta_{\varepsilon,-}$ (Proposition 4) and the fact that $X^\delta_{\varepsilon,-}$ is closed and convex, we may apply Proposition 3 to yield, for every sufficiently small ε, a positive solution $\tilde{u}_{\varepsilon,-} \subset X_{\varepsilon,-}$. Let $\tilde{v}_{\varepsilon,-} = \varepsilon \log \tilde{u}_{\varepsilon,-}$, then by the proof of Proposition 1, we may pass to a sequence $\varepsilon_k \to 0$ and assume $\tilde{v}_{\varepsilon,-} \to \tilde{v}_-$ such that $\sup_{-1<x<1} \tilde{v}_- = 0$. Furthermore, the fact that $\tilde{u}_{\varepsilon,-} \in X_{\varepsilon,-}$ implies that

$$\partial_x \tilde{v}_{\varepsilon_k,-}(x) \leq w_- \leq 0 \quad \text{ and } \quad \tilde{v}_{\varepsilon_k,-}(x) \leq \tilde{v}_{\varepsilon_k,-}(-1) + \int_{-1}^x w_-(y)\, dy \quad \text{ for } -1 < x < 1.$$

Passing $\varepsilon_k \to 0$, we deduce that $\tilde{v}_-(x) \leq \int_{-1}^x w_-(y)\, dy < 0$ for $-1 < x \leq 1$. Hence, we must have

$$\tilde{u}_{\varepsilon,-}(x) \to A\delta_0(x + 1) \quad \text{ in distribution sense.}$$

Since $\int \tilde{u}_{\varepsilon_k,-}\, dy \geq \frac{\min r}{\max K} > 0$, we may deduce as in proof of Proposition 2 that $A = \frac{r(-1)}{K(-1,-1)}$. Since A is independent of subsequences $\varepsilon_k \to 0$, we deduce that $\tilde{u}_{\varepsilon,-}(x) \to \frac{r(-1)}{K(-1,-1)}\delta_0(x + 1)$ in distribution, as $\varepsilon \to 0$. Similarly, the forward invariance of $X_{\varepsilon,+}$ implies the existence of another positive solution of (3) $\tilde{u}_{\varepsilon,+}$ such that $\tilde{u}_{\varepsilon,+}(x) \to \frac{r(1)}{K(1,1)}\delta_0(x - 1)$ in distribution, as $\varepsilon \to 0$. □

6. **Discussion.** In this paper, the existence, multiplicity, and qualitative behavior of a nonlocal competition model are studied. Sufficient conditions are obtained in Theorems 1, 2 and 3, which guarantee the concentration of steady states (i) at a single location; (ii) at two locations simultaneously ; (iii) at two alternative locations. In the following, we briefly discuss the meaning of the assumptions in terms of the adaptive dynamics framework.

The adaptive dynamics framework focuses on the competition between two different phenotypes of the same species. For this purpose, let $x, y \in [-1, 1]$ be two different phenotypes. The *invasion fitness* $\lambda(x, y)$ is defined as the exponential growth rate of the rare invader phenotype x in the environment where the resident phenotype y is at equilibrium. By considering the linear stability of $\left(\frac{r(x)}{K(x,x)}, 0\right)$ in the two-species competition ODE system

$$\begin{cases} U'(t) = U(t)(r(x) - K(x, x)U(t) - K(x, y)V(t)), \\ V'(t) = V(t)(r(y) - K(y, x)U(t) - K(y, y)V(t)), \end{cases}$$

we easily deduce

$$\lambda(x,y) = r(x) - K(x,y)\frac{r(y)}{K(y,y)} = \frac{r(x)r(y)}{K(y,y)}\left(\frac{K(y,y)}{r(y)} - \frac{K(x,y)}{r(x)}\right).$$

Hence the sign of $\lambda(x,y)$, which determines the invasion success or failure of the phenotype x against phenotype y at equilibrium, is equivalent to the sign of $\frac{K(y,y)}{r(y)} - \frac{K(x,y)}{r(x)}$. This partially justifies the use of derivatives of $\frac{K(x,y)}{r(x)}$, with respect to x, in the assumptions **(A)**, **(B)** and **(C)**. To a certain extent, it is sufficient to impose assumptions on the nodal sets of $\frac{K(y,y)}{r(y)} - \frac{K(x,y)}{r(x)}$, which is the same as the nodal set of $\lambda(x,y)$.

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E-mail address: lam.184@math.ohio-state.edu