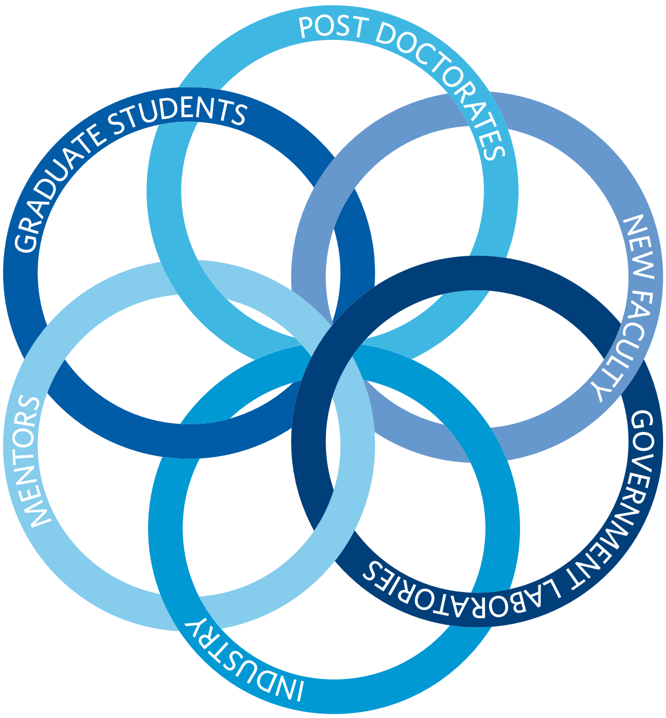
EARLY CAREER

The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Krystal Taylor and Ben Jaye serve as the editors of this section. Next month's theme will be Research.



Math Instruction

Teaching Mathematics Content Courses for Future Elementary Teachers: An Invitation to Innovation

Gabriela Dumitrascu

Since the development of the academic field of mathematics education, a fundamental tension between subject matter and teaching method has existed and continues to be ongoing. There are conflicting answers to the question of whether teaching should rely more on subject matter knowledge or pedagogical methods, which has led to divergent policies in teacher education.

The Complexity: Teaching Future Elementary School Teachers

Some argue that subject matter knowledge for teaching is defined by the content that elementary school-grade students are expected to learn. Most likely, this perspective is shared by preservice teachers[1] who are enrolled in the mathematics content courses required by their teacher preparation program. Others argue that teachers should have a broader perspective and deeper content knowledge so that they can understand where their students are heading in their learning journey. This perspective is more likely to be shared by instructors who teach content courses for future teachers.

Add to this tension the facts, that 1) the preservice teacher population is extremely homogeneous (mostly white females) often oriented towards nonmathematical areas [6], and 2) instructors of mathematics content courses for future teachers tend to have doctorates in mathematics with no experience teaching children in grades K–6 [5]. It becomes easy to see why teaching mathematics content courses to future elementary school teachers is one of the most challenging jobs in higher education. The overall homogeneity of elementary math teachers tips the scales in favor of method and pedagogy, but developing the right set of practical subject matter skills is equally important and should be central to any math teaching program.

Two Frameworks for Curriculum Development: MKT and LT

MKT: Mathematics Knowledge for Teaching. With the technological tools that we now have at hand we can find ideas about how to develop a curriculum specifically for content courses for future teachers. For example, on ChatGPT, you might find the following guidelines: establish a strong foundation (deep and accurate understanding of elementary-level mathematical concepts, number sense, arithmetic operations, geometry, measurement, data analysis, algebraic thinking), model effective teaching strategies, provide opportunities for active learning, encourage

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[1]*Preservice teachers, teacher candidates, or future teachers are college students who are currently enrolled in teacher education programs or courses and are in the process of preparing to become licensed or certified teachers.*

reflection and self-assessment, incorporate real-life connections, differentiate instruction (adapt instruction for individual learning styles), foster problem-solving skills, stay up to date, supervise practice, collaborate, and mentor. Did I miss anything? Surely, I did! But how do we put those steps into practice in the reality of a classroom setting?

From a theoretical perspective, the "mathematical knowledge for teaching" (MKT) framework has been widely accepted in the past two decades of American mathematics education. The framework "weaves" four main threads of knowledge [1]:

- Common mathematical knowledge (expected to be known by any well-educated adult)
- Specialized mathematical knowledge (strictly mathematical knowledge that is particular to the work of teaching, yet not required, or known, in other mathematically intensive professions: e.g., how to represent the steps and the reasoning behind the division algorithm using base-ten blocks, or illustrate the division of fractions using an aria model)
- Knowledge of mathematics and students (how children learn mathematics; what are common mistakes, misconceptions, or naïve interpretations)
- Knowledge of mathematics and teaching (strategies and teaching practices that are successful in teaching mathematics effectively).

The work done to develop the MKT framework is extensive and impressive, and so far, it has been used primarily to analyze teachers' work or to develop instruments for measuring teachers' knowledge. In my teaching, however, I use MKT as a blueprint for creating a more effective curriculum for instructing future mathematics teachers. This means that I start every lesson planning with a list of learning goals that correspond to each category of knowledge.

LT: Learning Trajectories. Clements and Sarama [4] propose learning trajectories as a framework to guide teachers in helping children develop their math skills effectively. It emphasizes the importance of understanding child development and bridging the gap between research and practice to provide equitable math education for all students. Effective teaching involves meeting students where they are in terms of their mathematical knowledge and helping them build on what they already know. Learning trajectories are proposed as a solution to address these challenges. They consist of three parts:

- Specific mathematical goal (each trajectory has a specific mathematical goal that students are working toward)
- Development path (there is a path along which children develop their mathematical understanding to reach the goal)
- Instructional activities (along this path, there are instructional activities that are fine-tuned for each step to help children progress to the next level).

Learning trajectories are useful recourses for exploring the impact of the mathematical content progression on the development of mathematical understanding. In the section Developing Mathematical Knowledge of this article, I describe how I used LTs to teach arithmetic operations in base-r systems.

Note that I used the verb "weave" to describe the MKT framework. The reason is that it requires imagination and creativity to combine all four strains—common, specialized, student and teaching knowledge—and the LTs into a single lesson. In that vein, this essay is an invitation for both new and old generations of instructors to use these frameworks in addition to their own passion and creative powers when they train future mathematics teachers. What follows is an example of how I have incorporated the MKT and LT methods into my teaching, and how doing so has helped to balance theory with practice, which opens creative opportunities that have made my teaching more effective. Others may use or adapt this model to help them bridge the divides of subject matter and pedagogy, to achieve their teaching goals, and maximize their students' math teaching potential The example addresses three main concerns: developing in-depth mathematical knowledge, reducing math anxiety, and increasing sensitivity in future elementary math teachers.

The Challenge: Developing Awareness, Trust, and Mathematical Knowledge in Future Teachers

Developing awareness. In my experience, the hardest part of instructing future elementary teachers is figuring out how to help them discover and recognize *the need* to engage with mathematical content on a deeper level, as doing so will help them to more effectively teach mathematics to young children. As a mathematics educator who also teaches methods courses, I can see (or I've noticed) that recognition for this need develops in future teachers only once they've had direct interaction with children. However, content courses are usually taken before the methods courses, and direct interaction with children is not part of the requirements for a content course. For this reason, I start my content courses with two readings that underscore how mastering impacts both children's mathematical development and teachers' professional development: "Sean's Numbers" is an extract from the article "Mathematics, mathematicians, and mathematics education" [2] and "Invitation to Learn and Grow" is a section from the textbook *Elementary and Middle School Mathematics: Teaching Developmentally* [7].

In our first meeting, students are divided into groups and asked to read "Sean's Numbers," which reveals the types of mathematical knowledge that enable a teacher to navigate complex mathematical interactions skillfully and adaptively in a diverse classroom. The reading describes a classroom episode where students are exploring the concepts of even and odd numbers. One student, Sean, questions whether six can be both an even and an odd number (six can be divided fairly into two groups of three and, also, into three groups of two). The teacher guides the discussion by encouraging the students to articulate their ideas and arguments. Mei, another student, engages in a thoughtful and well-expressed critique of Sean's argument. Mei's astute argument leads to an ironic exchange with Sean which mathematically is a debate about mod 2 arithmetic and 2 mod 4 numbers.

Each group is then asked to answer four questions, listed below.

- What are some characteristics of even numbers? Why, in a sequence of consecutive numbers, do the even and odd numbers alternate? Why do $even + even = even$; $odd + odd = even$; $even + odd = odd$? Do Sean's numbers (odd multiples of two) have similar properties? (Common mathematical knowledge)
- In the article we find three definitions for even numbers: fair share (a number is even if it can be split into two equal groups); pair (a number is even if it is composed of groups of two); and alternating (the even and odd numbers alternate on the number line, with zero being even) (pg. 427). Using visual representations (graphical illustrations), how do you show that the three definitions of even numbers are equivalent? (Specialized mathematical knowledge)
- Follow Sean's mathematical idea about numbers that are odd multiples of two (Sean's numbers) and check if Sean's number + Sean's number = Sean's number (Knowledge of mathematics and students)
- What is the position of Sean's numbers on the number line? How would you use this argument to conclude the discussion? (Knowledge of mathematics and teaching)

The ensuing discussion is focused on the mathematical knowledge that a teacher would need to use Sean's question as an opportunity to help their students grow in their mathematical thinking. In my closing statement of that session, I talk about modular arithmetic as a special type of arithmetic that should be part of the knowledge they draw on to support, and not to impinge on, children's mathematical explorations, regardless of whether they end up teaching it to K-6 students.

Developing trust. In our second meeting, students read "Invitation to Learn and Grow," which describes what skills are required to teach mathematics in the 21st century. The article lists seven skills, in the form of characteristics, habits, and abilities, in the following order: knowledge of mathematics, persistence, positive disposition, readiness for change, willingness to be a team player, devotion to lifelong learning, focused time to reflect and become self-aware. After we discuss the basic meaning of each skill, I offer my own perspective: that the list represents a learning trajectory for becoming an effective math teacher, but the skills should be developed in reverse order. That is, examining ourselves for areas of improvement and reflecting on our successes are both the markings of a lifelong learner, as well as behaviors that foster intellectual growth and continued development. Additionally, working together as a team and supporting each other will both motivate us and provide us with the support we need to pursue a difficult endeavor.

To learn and to grow is to continually change. But a readiness to change relies on trust—trust in ourselves and our peers—which is why collaboration and support are so important to the learning process. They help build that trust and encourage all of us to take a chance on new ideas, even if those ideas disrupt our equilibrium. Once we become comfortable and willing to learn something new, we will be able to cultivate a positive attitude towards the subject of mathematics. A positive attitude facilitates a person's ability to persist, reflect on, and engage with problem solving and other mathematical thinking. This will lead us to develop a more flexible and adaptive mathematical thinking, which will help expanding our mathematical knowledge. As with any other human skill, these skills can become automatic only through repeated practice and reinforcement. Once they become ingrained in the brain, they can be executed quickly and efficiently without conscious thought. For this reason, focused time to reflect and become self-aware, devotion to lifelong learning, willingness to be a team player, readiness for change, positive disposition, persistence, and knowledge of mathematics define the tone and describe the norms for my classrooms. I have designed a poster listing these norms, which I insert into each course syllabus and website.

Developing mathematical knowledge. The next challenge, for me when instructing future elementary teachers is implementing the MKT framework, which is intended to help future teachers learn to

- Create meaningful learning experiences by navigating complex mathematical interactions skillfully and adaptively.
- Present concepts in multiple ways.
- Make connections between content.
- Think beyond their own perspective and instead focus on:
    - Knowledge of mathematics and students
    - Knowledge of mathematics and teaching

Since in the K–6 mathematics curriculum the largest section is focused on developing numbers and operations sense, a part of the mathematical content for future teachers is also about understanding number systems and operations. Presenting numbers and arithmetic operations in different number systems provides unique insights that future teachers need to be efficient in developing numbers sense and operations sense in young children. Since using numbers and performing operations in base 10 are already automatic reflexes for an adult, one way to challenge these reflexes is by examining how numbers and operations work in bases other than 10. I have tried several approaches to engage my students, which I failed because some students avoided answering the exam questions related with non-base 10 number systems. Others, who were successfully solving problems with numbers represented in a base other than 10, were doing so by referring to their base 10 understanding instead of using grouping and place-value.

The bitter taste of failing and the firm conviction that the MKL framework should be the baseline for efficiently preparing mathematics teachers compelled me to look for a better way to introduce numbers and number systems. Consequently, I found an approach that gave me more satisfaction regarding my students' level of engagement.

I start the section on numbers and number systems by presenting the research done on children's mathematics learning trajectories, and I introduce my students to the Learning and Teaching with Learning Trajectories [LT]^2 website [3] where they watch a couple of videos of children who are at different levels on their learning trajectory for counting. Then, I invite my students to experience for themselves the process of learning to count as the young children do: I tell them they will learn a new way to count, using new names for numbers and new symbols. In the "new" numerical system, which is base 5, the numbers are alpha for card({x}), beta for card({x,x}), gamma for card({x,x,x}), delta for card({x,x,x,x}), and epsilon for card({}). I use the Greek alphabet, but you have the freedom to invent your own symbols and words to express the cardinality of sets. As they struggle counting (especially backward and skip counting) they also share their feelings and reflect on what they are experiencing. The goal of the discussion is to understand how children might feel when they are first learning about numbers and operations (MKT framework).

To address knowledge of mathematics and teaching (MKT framework), I divide the class into groups. Each group is given a bag of Q-tips that they are asked to count, then report back to me the number of Q-tips in their bag. Afterward, each group receives an addition and a subtraction problem, each of which they need to solve and represent using Q-tips. Once they are done solving the problems, I reorganize the groups using the jigsaw strategy (each new group has one member from each of the previous groups). Within their new groups, each student must explain and model how they used the Q-tips to solve the problems.

After these activities, the engagement of future teachers is noticeably higher in exploring how the arithmetic operations work in different numerical systems and they start to recognize the importance of the mathematical concepts of grouping and place-value in the development of number sense.

In conclusion, my process for mathematics teacher preparation blends the following three perspectives:

- Practice to develop skills and habits of mind for effective mathematics teaching.
- Design instruction using the MKT and LT theoretical frameworks.
- Commit to collaboration between research mathematicians and mathematics educators (which should be the topic for another article).

These three perspectives bring the disparate worlds of theory and knowledge and teaching together and open a whole universe where our creativity and imagination can become limitless.

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Credits

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Interdisciplinary Teaching of Mathematics with Primary Historical Sources

Richard A. Edwards

A Great Divide

In 1959, the chemist and novelist C. P. Snow (1905–1980) identified what he saw as an increasing and unproductive isolation between scholars of different disciplines. "We have two polar groups: at one pole we have the literary intellectuals, at the other scientists. Between the two there is a gulf of mutual incomprehension." [Sn, p. 4] Snow recalled moments in his career when literary elites would scoff at scientists who were unfamiliar with the sonnets of Shakespeare, while they were themselves ignorant of comparable scientific ideas such as the laws of thermodynamics.

Whether or not Snow accurately described intellectual life in the mid-20th century, and there is an argument that the way he concretized the gulf merely increased academic tribalism, I am fortunate to work at an institution that actively encourages interdisciplinary research and teaching. I have derived great benefit from rubbing shoulders with colleagues in the humanities and social sciences. Their perspectives on education, history, philosophy, and ethics have shaped my views on what effective teaching looks like, and what it means to learn.

What of my students? Do they appreciate interdisciplinary teaching, or wonder about how scientists and novelists can productively collaborate? Many of my students take a consumerist view toward their courses. By this I mean that their primary goal is to pass my class and get on with their degree. The actual content of the course is less important than the fact that it moves them one step closer to their career aspirations. If they think about it at all, many tend to think of mathematics primarily as a tool for solving problems in science. They might enjoy my class, but very few of them will think much more about it once the semester has come to a close. Is there a place in my classroom for the humanities? When I began my career this wasn't something I thought about. The only teaching I did that could be considered interdisciplinary amounted to little more than occasionally teaching a history of mathematics course, and infusing my calculus lectures with anecdotes (of questionable veracity) about famous mathematicians. Then a quote from a British educator named Charlotte Mason (1842–1923) captured my imagination:

> There is a region of apparent sterility in our intellectual life. Science says of literature, *I'll have none of it*, and science is the preoccupation of our age. When we present theorems divested to the bone of all superfluous trappings, we lose the vitality along with what we've stripped away. History expires in the process, poetry cannot come to birth, religion faints; we sit down to the dry bones of science and say, *here is knowledge, all the knowledge there is to know.* [Ma, p. 317]

Snow challenged his associates in the humanities, while Mason chided the scientists. Can I, as a mathematician, teach in ways that draw from the best of both traditions? I'd like to teach in ways that retain the vitality and effervescence of mathematics. I would like to restore humanity to theorems. I want my students, as Polya described, to experience the tension and enjoy the triumph of mathematical discovery [Po].

I believe I have found a path forward by teaching mathematics via primary historical sources. Moving toward teaching in this way has been one of the most rewarding, yet challenging, efforts of my career. It has also changed how I personally think about learning mathematics.

Primary Source Projects

The benefits and challenges of teaching with primary sources have long been a source of discussion among those interested in the history and pedagogy of mathematics [Ja]. Reading primary source texts allows students to see how individuals first conceptualized an idea, and how mathematical ideas have evolved over time. Many textbooks, almost by their very nature, present mathematical ideas as refined and finished products. In contrast, original sources help to foreground the motivations, cultural contexts, and intellectual atmosphere of their source authors. Primary sources display the patterns of communication that have

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characterized the mathematical community, can reveal how those standards have changed over time, and why.

In response to these benefits, over 100 primary source projects (PSPs) have been developed under the NSF-funded TRIUMPHS project (https://digitalcommons.ursinus.edu/triumphs/) and its predecessor grants. PSPs are classroom projects designed to replace standard textbook-driven presentations of important mathematical topics. Each PSP features selections from one or more historical sources, supplementary text from the project author that provides both historical and mathematical background, and a series of tasks which help students interact productively with the historical source and learn its mathematical content. Each PSP has been designed to help students reach a level of fluency with a mathematical topic that is at least as strong as if they had learned it via a traditional textbook approach, in roughly the same amount of time. They are replacements, not additions, to my syllabus.

The series of tasks in a PSP offers students opportunities to engage in doing mathematics in a variety of powerful ways. These include activities which model how mathematicians actually work; for example, conjecturing, testing, refining, proving, and generalizing relationships between objects. PSPs also include tasks that allow students to interpret results as they were originally presented, and then reformulate these results in modern terms. Such activities encourage robust understanding of mathematics by immersing students in an ongoing conversation which can sometimes span centuries.

For example, I implement "Fermat's Method of Finding Maxima and Minima" [Mo] in order to help students better understand the extreme value theorem, learn methods for finding extrema of functions, and practice their derivative rules. In addition to these object-level themes, the project can help break students out of recipe-thinking with regards to optimization, show how a technique has evolved over time, and generate discussion around the question of what counts as a general method. The source material comes from the writings of Pierre de Fermat (1607–1665), along with some commentary on Fermat's work that Rene Descartes (1596–1650) sent to Marin Mersenne (1588–1648). Students get a glimpse into the personalities of these mathematicians as they struggle to understand and explain a topic (optimization) that first-year college students struggle to understand and explain today. After presenting his optimization process, Fermat boasted:

> We can hardly be provided with a more general method.

However, at this early stage in the project, most of my students appreciate the critique raised by Descartes (in a private letter to Mersenne):

> If he [Fermat] speaks of wanting to send you still more papers, I beg of you to ask him to think them out more carefully than those preceding!

I like to use these primary source excerpts to motivate student discussion: Was Fermat's method robust, or did it only work for the specific examples he chose? Was Descartes right to question the generalizability of the method? How is Fermat's method similar to, or different from, the method in our modern textbook? By the end of their correspondence on this subject, Descartes seemed happy with Fermat's method, and wrote,

> Seeing the last method that you [Fermat] use for finding tangents to curved lines, I can reply to it in no other way than to say that it is very good and that, if you had explained it in this manner at the outset, I would not have contradicted at all.

Certainly a ringing endorsement for students and faculty alike to communicate our ideas clearly…and show our work.

Primary source projects take students to pivotal moments in the history of mathematics. For example, the PSP "Rigorous Debates Over Debatable Rigor: Monster Functions in Introductory Analysis" [Ba] transports students to the late nineteenth century when mathematicians were just beginning to think about *properties of functions* as something worthy of study in their own right. As with every PSP in the TRIUMPHS collection, the goal is to teach mathematics, not its history. This PSP features core object-level themes such as continuity, differentiability, the Intermediate Value Property, Darboux's theorem, and uniform differentiability.[1] Because the results are presented in their human and historical contexts, instructors can use it to talk about important meta-level themes such as: Why might someone take a critical view of the basic ideas of calculus? Why did mathematicians need to develop new vocabulary, techniques, and theorems in calculus? Other scholars, notably [BCC], have analyzed this particular project in detail with respect to its ability to promote student discussion of metadiscursive rules in Introductory Analysis. Here I restrict myself to sharing some excerpts which illustrate how the project presents the human drama of mathematical correspondence.

The discussion between Darboux and Hoüel began cordially enough…but then descended into a flurry of colorful phrases as the mathematicians become increasingly frustrated with each other.

Darboux began:

> Go on then and explain to me a little, I beg you, why it is that when one uses the rule for

[1] *In many modern texts the notion of uniform differentiability does not appear explicitly. A function is continuously differentiable if and only if its derivative is uniformly continuous.*

composition of functions, the derivative of $y = x^2 \sin\frac{1}{x}$ is found to be $-\cos\frac{1}{x} + 2x\sin\frac{1}{x}$, which is indeterminate for $x = 0$ even though the true value is $\lim \frac{y}{x} = 0$?

Darboux to Hoüel, January, 1875

Darboux then critiqued certain proofs which Hoüel had previously provided, using delightful phrases such as "Here is what I reproach in your reasoning…" and "…your proof has nothing but the appearance of rigor." Hoüel's responses hinted at a growing frustration over what he perceived as Darboux throwing up pointless counterexamples, and it seems as if the two correspondents were beginning to talk past each other.

> Yes, I admit as a fact of experience (without looking to prove it in general, which might be difficult) that in the functions that I treat, one can always find h satisfying the inequality $\frac{f(x+h)-f(x)}{h} - f'(x) < \epsilon$, no matter what the value of x, and I avow to you that I am ignorant of what the word derivative would mean if it is not this. I believe this hypothesis is identical with that of the existence of a derivative.

Hoüel to Darboux, January 1875

Hoüel's response did not satisfy Darboux, who seemed more concerned with attending to the dependencies between variables in a proof. He tried to get Hoüel to reflect on how variables are introduced, especially those variables that carry universal quantifiers. This is something many students today also struggle with at this point in their mathematical studies, which is one reason why having them read this source material can be powerful.

> You have not addressed the nature of my objection…For your methods to be sound, you will need to explain very clearly what part of your reasoning is deficient in this particular case. Without that, your proofs are not proof. As for the question of the derivative, this time you change the question. It is clear that for a value x_0 of x, that saying
>
> $$\lim \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$
>
> is the same as saying: One can find h such that
>
> $$\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0) < \epsilon$$
>
> for this value of h and for all values that are smaller. But there is an abyss between this proposition and the following: Being given a function $f(x)$ for which the derivative exists for all values of x between a and b, to every quantity ϵ, one can find a corresponding quantity h such that
>
> $$\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0) < \epsilon$$
>
> for all values of x between a and b.

Darboux to Hoüel, January, 1875

In addition to exploring issues such as the proper placement of quantifiers in assertions involving multiple variables (e.g., those that define today's properties of pointwise and uniform differentiability), [BCC] note that this PSP gives students opportunities to discuss important questions related to mathematics: What is the purpose of examples? What intuitions are refined by studying them? The Darboux-Hoüel correspondence represents an important turning point in the history of analysis. I love using PSPs such as this one, which take students to the forefront of mathematical developments. I'm convinced that living on this ragged edge is both more exciting for students, and more mathematically satisfying, than some textbook-driven lectures.

Although many PSPs feature work from names my students recognize, such as Euler, Gauss, Cauchy, etc., many of the primary source texts give students exposure to geographically and culturally diverse authors. When I implement *A Genetic Context for Understanding the Trigonometric Functions* [Ot], my precalculus students get to work through selections from Greek mathematicians Hipparchus and Ptolemy, from Hindu mathematicians such as Varāhamahira, and read selections from *The Exhaustive Treatise on Shadows*, written in the court of a Turkish sultan in the year 1021. My second-semester calculus students solidify their understanding of series convergence through the PSP *Bhāskara's Approximation to and Mādhava's Series for Sine* [Mo2]. I'll never forget the day when six of my students simultaneously burst out in delightful surprise at seeing source material written in Sanskrit—a language they had learned to read growing up in India (of course the PSP also provides an English translation). In my multivariable calculus course, we often end the semester with *Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem* [Ed] in which students read the work of the enigmatic George Green (1793–1841), a working-class miller who didn't begin his college career until middle age, but whose ideas about electromagentism led to the theorem which now bears his name. This year, I look forward to implementing at least one project based on the work of Maria Agnesi [Mo3].

Challenges, and What it Means to "Learn"

Despite their great potential, teaching with PSPs brings its own set of challenges. Many of my students struggle with reading. Asking them to read (translated) excerpts from long ago is a heavy lift for some of them. It helps to have them work through the projects in groups. It takes me longer to prepare for class when I'm going to teach with a PSP, although each project features detailed notes for instructors as well as an implementation plan. I don't use a PSP for every topic in a course (although I know instructors who teach courses using only PSPs). In a typical semester, I find time for three or four projects, depending on the course. One method I've recently found success with is putting students into groups and giving each group a different PSP (but all related to a similar topic, such as series convergence) to complete. At the end of the week, we have a "PSP Showcase" in which each group gets to do a mini-presentation of their work to the rest of the class.

Not all of the PSPs are well-aligned with my online homework problem system. However, since each PSP is intended to be a replacement for my standard lesson, I can also use the student's written work to replace the online homework for that lesson. One challenge that I am very aware of is trying to avoid interpreting history through the prejudices of today. This kind of presentism—judging the past by today's standards—can inadvertently give students a sense that all of history was an inevitable sequence of events, of which the 21st century student is its pinnacle.

The greatest—but most rewarding—challenge that I have had to wrestle with in teaching with PSPs has been re-orienting my thoughts about what it looks like to *learn* mathematics. Of course I want my students to become fluent in the discourse of modern mathematics. Yet being familiar with the modern conception of an idea can sometimes be tantamount to knowing only the last page of a long and richly complex story. Instead, I find it helpful to conceptualize learning as increased participation in the mathematical community [La]. This includes knowing both the current standards of our community, but also how our mathematical ideas have changed over time. PSPs are one means to facilitate that kind of learning.

PSPs give students opportunities to witness mathematicians at work, to "imitate [their] moves while trying to figure out the reasons for the strange things [they are] doing" [Sf, p. 202]. This may be an important step in helping students tell new stories about the world of mathematics, and their own place in that world. I close with a quote attributed to Descartes that frequently comes to mind while I'm teaching:

> Scientific truths are battles won.
>
> —Rene Descartes, quoted in [We, p. 162]

His words remind me that much of what we teach has a rich and important history. Perhaps by immersing students in that history, we can help give them a more robust understanding of our subject. PSPs may not bridge Snow's "gulf of mutual incomprehension," but I can testify to their ability to generate enthusiasm and excitement for learning mathematics.

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Richard A. Edwards

Credits

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Getting Your Hands Dirty: Teaching Math Biology with Active Learning Strategies

Adrian Lam

Nowadays, mathematical and computational methods are ubiquitous in many areas of biological research, such as genomics, ecology, evolutionary biology, neuroscience, and systems biology, to name a few. It is therefore important to introduce students to the interdisciplinary field of mathematical biology at an early stage, typically during their freshman or sophomore years. It is no surprise that an increasing number of universities are recognizing the importance of mathematical biology and integrating it into their undergraduate curriculum. The integration of mathematics and biology opens a world of possibilities for students to explore various biological phenomena using quantitative techniques. By grounding the mathematical concepts in real-life biological scenarios, students also gain a deeper appreciation for the role of mathematics in shaping their understanding of the living world.

I am an associate professor at the department of mathematics at OSU. My research interest lies in the analysis of partial differential equations. I have worked on systems of reaction-diffusion equations and free-boundary problems which are inspired by applications in biology. I have had the pleasure of teaching and advising students in the math bio track since I joined the faculty at the Ohio State University in 2014. In this article, I aim to share some of my personal experiences with teaching the course Introduction to Mathematical Biology with my colleague Avner Friedman (founding director of Mathematical Biosciences Institute) since 2018. While coteaching mathematical biology can be quite different from teaching other more traditional mathematics courses, in terms of the syllabus, audience, and teaching goals, it also offers ample opportunities to apply active learning techniques. Here, I would like to share some of our recent experiences and personal take-aways in interacting with our students.

One of the major differences in teaching mathematical biology compared to traditional mathematics courses lies in the scope and emphasis of the syllabus. In a math biology course, it is crucial to provide students with a thorough understanding of the biological context behind mathematical models. Traditional applied math courses may mention motivation briefly before diving into theorems and proofs. However, in a math biology class, the goal is to establish a strong connection between mathematical methods and their applicability to biological problems. Rigorous proofs are still valuable but take a back seat to explaining the biological rationale behind the models.

Another significant difference is the audience in a math biology course. Many students who enroll in this course do not major in mathematics. While they can be bright individuals (many of whom are premed students), they often find mathematical concepts challenging to grasp or even intimidating. As instructors, building rapport with such students and ensuring effective communication can be a challenging but rewarding task.

The traditional way of teaching mathematics involves presenting the subject logically, defining precise mathematical objects, deriving results, and providing examples of alternative solution methods. After that, students can work on problem sets independently to improve their familiarity with the techniques. For a math biology class, however, there are opportunities to apply active learning methods and dedicate more time to inquiry- and problem-based labs.

To foster a more interactive and engaging learning environment, we structured our course into weekly modules. Each week, we introduce a mathematical method alongside one or more biological motivations for its use. For instance, for the module focused on epidemiology in week 6, we introduce the SIR (Susceptible-Infected-Recovered) model, which is a set of ordinary differential equations depicting the transition of the overall infection status among members of a population. Given that most students were affected by the COVID-19 pandemic, lively discussions ensued when exploring how to incorporate real-world details into the model, such as how to incorporate an asymptomatic period in the SIR model before an infected individual becomes symptomatic and can be detected.

In addition to theoretical discussions, we emphasize numerical computation during our teaching. This approach allows students to witness how models work and how to

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interpret results in a biological context. For example, while covering birth-death processes, we immediately used laptops to compute the survival probability of right whales in the North Atlantic from given data based on the work of Caswell et al. [Caswell, Fujiwara, and Brault, PNAS, 1999]. This exercise illustrated how small changes in birth rates can significantly impact the outcome.

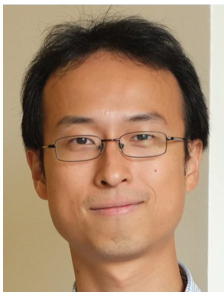
Numerical computation not only provides students with a practical understanding of mathematical models but also equips them with valuable computational skills that are increasingly essential in modern biology. In today's data-driven scientific landscape, the ability to analyze data and simulate models computationally is critical for making informed decisions and conducting cutting-edge research. To further enhance students' programming skills, our course includes a hands-on computing component primarily using MATLAB. On Fridays, the lecture shifts to a laboratory section with weekly programming assignments. These assignments are individual small projects based on classroom demonstrations. Students are not expected to prove theorems but rather to utilize MATLAB to compute the model with given parameters. Our goal is for them to engage in practical problem-solving using mathematical tools. During the lab, we demonstrate how to translate a biological problem into a mathematical model and guide students on numerically computing and presenting results accurately. The students then work on their assignments on their own computers, and this setting provides ample opportunities for personal interaction. We debug code together, discuss code rationale and purpose, and explore effective presentation and interpretation of results. The numerical assignments take the form of short reports, where students are expected to present and explain the numerical results in terms of how they address the biological question. This setup empowers students to experience real-world problem-solving and leaves them feeling accomplished and knowledgeable.

The course culminates in a final project, spanning four to six weeks. In small groups, students are assigned concrete biological problems along with small data sets. Their first task is to research the biological background to identify where mathematics could be helpful in providing answers. The second step is to select an appropriate mathematical model that represents the relevant biological process. The final step is to use mathematical theories and computational methods in order to fit the model and derive predictions from the model. The choice of model includes those discussed in the course, but students are not limited to them. This part of the course actively involves students in the scientific process of applying mathematical theory to solve biological problems. Regular meetings with the instructor in and outside of class offer opportunities for progress updates and feedback. During the final two weeks of the semester, the groups take turns presenting their findings to their peers. Here, the various groups take the podium and teach their peers about the biological problem, explain their rationale for mathematical model selection, and showcase their analytical and numerical results. With adequate feedback, most groups successfully complete their projects, providing students with a taste of scientific research in the context of mathematical biology.

As an instructor, teaching this course is a refreshing experience due to the flexibility it offers in content selection. This allows the instructor to include current research topics or address emerging environmental issues like the transmission of COVID-19 or global climate change. Moreover, modeling biological phenomena provides an open-ended inquiry experience that does not carry a single correct answer and demands a different set of skills than those required for solving well-defined mathematical problems. I am delighted by the quality of questions I receive from students, and the course fosters extensive discussions concerning the relation of mathematical models to specific biological situations. The final group project provides an opportunity for me as the instructor to meet with small groups of students outside of regular lectures. While working on their projects, students share with me their reasons for taking the course and their long-term goals. While many took this course to fulfill the math biology track requirements, I am surprised that a large number of them took this course purely out of curiosity. Since a substantial number of them are planning to further their education in medicine and biology, I hope that this experience can train them to think critically as well as quantitatively. Also, this personalized interaction allows me to get to know my students on a deeper level. In fact, I am working with a group of particularly successful students to write up their project findings for submission to a journal dedicated to undergraduate research.

In conclusion, teaching mathematical biology with hands-on, computational component has proven to be effective in enhancing student engagement and understanding. Problem-based labs and projects offer students the skills and experiences needed to use mathematics effectively, which can be challenging to teach using lecture-style approaches. Additionally, these methods promote communication between math faculty and students. By grounding mathematical concepts in real-life biological scenarios, students gain a deeper appreciation for the significance of mathematics in modern biology. As the field continues to evolve, these courses play a pivotal role in shaping the next generation of biologists equipped with quantitative expertise and problem-solving skills.

ACKNOWLEDGMENTS. This article is dedicated to my late colleague Professor Ching-Shan Chou. Ching-Shan Chou and Avner Friedman first designed and pioneered this undergraduate mathematical biology course at the Ohio State University.



Adrian Lam

Credits

Photo of Adrian Lam is courtesy of Adrian Lam.

Extreme Cases: Math Education Within the US Prison System

Katherine J. Pearce

Pulling into a visitor parking space for the first time, I watch a stray dog hesitantly making her way toward the barbed wire fences. The surroundings are austere and unforgiving. Lockhart Correctional Facility is about an hour's drive from the city of Austin and miles from the nearest town center, situated, intentionally, in the middle of nowhere. I wonder briefly how the dog ended up here and feel a pang of sadness that I have nothing to offer her. It will not be the last time I struggle with these feelings in this parking lot; they only intensify when I think about my students also being left here to fend for themselves.

I pocket my ID and car keys, grab the box of printouts I made for class, and head toward the barbed wire gate. With sunlight still streaming through the lobby's glass doors behind me, the first checkpoint feels similar to airport security. I empty my pockets, place everything on a table to be searched, remove my belt and shoes, and walk slowly through the metal detector. A female correctional officer (CO) outlines my body with her wand before patting me down, paying special attention to the bottoms of my feet, while another CO searches my class materials. He warns me that highlighters and papers with "too much ink" are not permitted, but thankfully all my materials are allowed through today. I gather them up and head toward the next security checkpoint, where any glimpse of daylight or sense of familiarity is gone.

The main guard station remotely buzzes me through a door after surveilling me on camera, and I hand over my ID in exchange for a badge. I am buzzed through one last door and suddenly step out into prison, narrowly avoiding the lines of people herded in and out of the cafeteria. I hear my pulse in my ears until I notice that they are all smiling and welcoming me. Someone exclaims, "God, I miss wearing jeans," and we laugh. One of the incarcerated women approaches me to show me to my classroom, and we walk down a hallway under fluorescent lighting and a large hemispherical mirror that shows everything around the corner. "We are so grateful and excited you are here!" she tells me. I feel the same.

When we arrive at the classroom, she explains apologetically that I will have to wait while the COs release my students to come to class. Although class technically begins at 6 p.m., we usually don't start until 6:30, but I enjoy the wait. That half hour has become the most important part of teaching for me; without a book, phone, or electronics to kill time, I end up studying all the art and handouts displayed on the walls. Our classroom is also used by vocational and reentry programs, and the exercises that the students complete are somber reminders of the real stakes here. "My goal is to get sober for my children." "I want to learn how to forgive myself and earn forgiveness from others." "When I get out, I will start a new career with my certifications to support my family." These messages reiterate to me that I am here primarily to build their confidence in themselves and their problem-solving abilities. As a formerly incarcerated friend will later share with me, that confidence gives them a sense of freedom even in incarceration.

The twelve students in my class trickle in, and we introduce ourselves. Even though I learned about average state prison populations during the Texas Prison Education Initiative's (TPEI) orientation, I am still surprised at the large variance in age, from about early 20s to mid 60s. I feel a flutter of anxiety about the pace I've decided on for my lectures. The students need to pass my course as a prerequisite for any credit-bearing TPEI math course offered through UT Austin's extension program, so I want to focus on building their abstract problem-solving abilities. In particular, I have decided to spend a lot of time developing their intuition in subject areas they had likely seen before, like integer addition; I am worried the students will be bored with my decision.

"So, just out of curiosity, what do y'all think about math in general?" I ask, quickly adding, "No judgment here, I

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do it for a living and still feel love-hate about it." We laugh, but it's hard to keep smiling as I hear about their previous experiences in math classes. The students are diverse in age, race, ethnicity, and sexual orientation, but they unfortunately had one major thing in common: all of them proclaimed to be "bad at math," and many had dropped or failed out of school specifically because of math.[1]

I tell them my plan for the course: we are going to start over with math and look at it from a new perspective. Anything that they need to know for the course, I will teach them, and it would actually be preferable if they could wipe away any memory of subjects where they had unsuccessful learning outcomes. Even things they have seen before, like integer addition, I want us to consider with fresh eyes. "Our course objective, and the real advantage of math," I tell them, "is using specific instances of a problem to understand how it works in full generality." And that is exactly what we do.

As mathematicians, we often look to "extreme cases" of a problem to gain intuition for a general solution. Mathematics education within the prison system is perhaps the most extreme case, and I believe it sheds light on how we can teach more effectively in a university classroom.

1. Satisfying Constraints

In addition to the usual concerns of teaching math, teaching math in prison comes with a unique set of challenges. With respect to the aforementioned issue of bringing class materials inside, there are certain canonical classroom resources that are prohibited or impossible to obtain. There is no computer, internet access, or even a desk for lecture materials, only a small whiteboard at the front of the room which, just like in university classrooms, is unreliably equipped with dry erase markers. There are no office hours, recitations, or means of communicating with students outside of class time. If students miss class for a variety of legitimate reasons, like having to work late at their prison jobs or being confined to their cells during a lockdown, they have no way to access material or get in touch with me until the next class.

During the summer, for example, I taught an elective course called "The Art of Mathematics," in which students investigated several math topics, like algorithms and infinite set cardinalities, that served as inspiration for their own art work. TPEI provided sketchbooks, folders, and colored pencils, but I was not permitted to leave the colored pencils with the students when class was over. The other class materials required certain stickers to denote that TPEI had given them to these students, in hopes they would not accidentally be confiscated.

I also brought in art books and textbooks that I had at home to pass around the room, which I was told to collect at the end of each class. Though I did not, I was very tempted to ignore this rule on two occasions. Once when an older student was poring through *The Math Book* by Clifford A. Pickover and asked me excitedly if she could check it out like a library book. Then again when a younger student asked me if she could borrow Baby Rudin[2] because she was curious about analysis proofs; she'd also asked me earlier that evening which prerequisite textbooks or classes she'd need in order to teach herself analysis some day. Later in the course, I learned the latter student was trying to study astronomy in college before her arrest; unfortunately, she'd been ousted from the STEM major due to her grades in the required math classes.

The culmination of the math art elective course was to be a gallery night, a classroom exhibition of their work on the last day. Together, we would reflect on the various topics we'd covered, discussing the mathematical ideas and artistic choices that spoke to us in each other's work. I'd invited the TPEI program coordinators, Max and Chloe, who were also excited to see what the students had created.

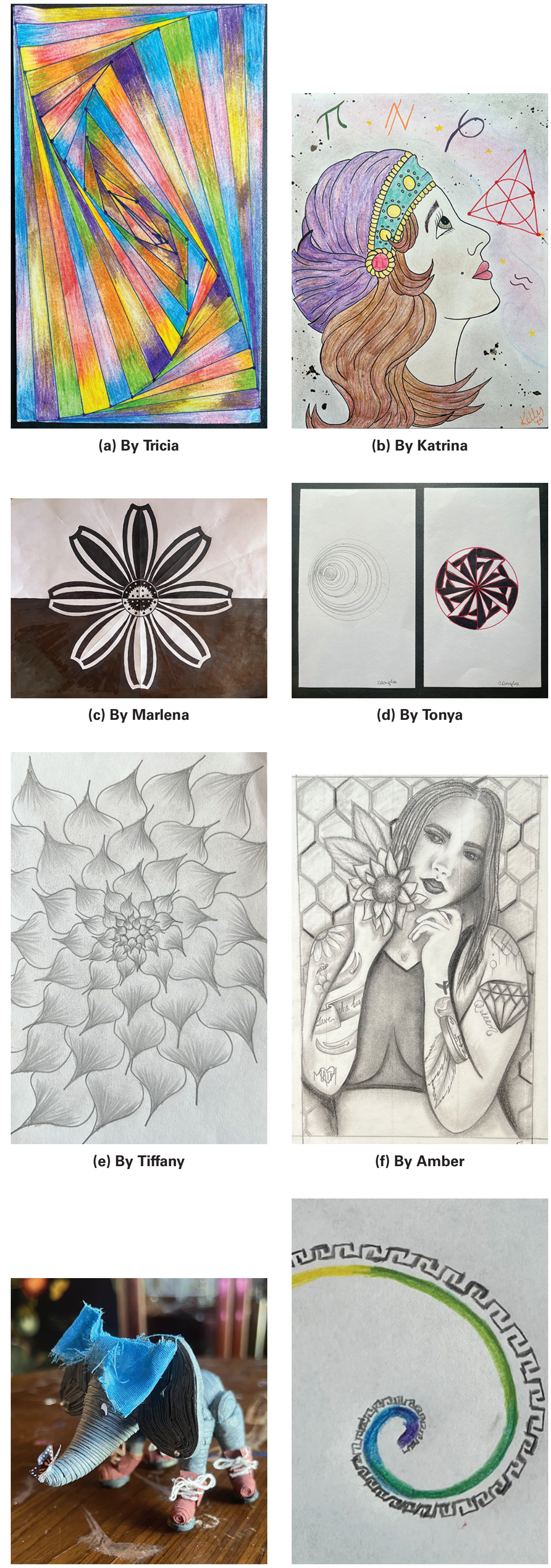
When we arrived at the classroom, we waited for an unusually long time before a couple of students came in and mentioned they'd had a difficult time getting to our room. Around 6:45, when Max went back to the central guard station to ask if all TPEI students had been called for class, we were told that a fight had broken out in one of the dorms, and it was now on lockdown. As a result, despite no involvement in the fight, about half of my students were unable to come and present the art pieces they'd worked on all summer. By the time the ones who could make it finally arrived, we only had about 45 minutes left. Everyone involved was disappointed, but those of us in attendance tried to make the best out of the remainder of the evening. The exhibit was breathtaking, and the influences and creativity in every piece were truly awe-inspiring. Max and Chloe were blown away and agreed that the students' exhibit looked and felt professionally curated.

Since we couldn't bring cameras inside to take pictures, I proposed to the students that, with their permission, I'd borrow their sketchbooks overnight, take photos of their work, then return their sketchbooks to them the next evening at another TPEI instructor's class. That way, students wouldn't have to tear out pages from their sketchbooks and would only be without them for less than a day. However, I was honest about my concern that there could be some unforeseen issue getting the sketchbooks back to them, which was always a risk. I told them I understood completely if they wanted to hold on to their work just in case. I was moved close to tears when every one of them gifted me their art work, tearing pages out of their

[1]*According to the Bureau of Justice Statistics, fewer than 4 out of 10 incarcerated people have completed high school, versus 9 out of 10 in the general population. The average incarcerated person in state prison is 39 years old with a 10th grade education. [WSHW22]*

[2]Principles of Mathematical Analysis *by Walter Rudin (1953)*

sketchbooks for me to take home and keep. Those pieces are now framed and displayed in my office, and photos of some of the pieces are shown in Figure 1. In (a), the sketch was created with a straight-edge, colored pencils, and deodorant for shading, inspired by M.C. Escher's work with H.S.M. Coxeter on the "limit of infinite smallness" [Wie10]. Figure 1(b) shows a self-portrait of the aspiring astronomy student contemplating math topics from the course. In (c), the concept of yin and yang is illustrated, playing off of mathematical symmetries. In Figure 1(d), the sketch on the left is inspired by the Poincaré disk, created with circular objects of different radii that the student collected from around the prison. On the right in Figure 1(d), the same student interprets Escher's limit of infinite smallness. In (e), we see an artistic rendering of a Fibonacci spiral in nature, and in (f), a self-portrait of the student in front of a tiled background, with a new (imaginary) tattoo showcasing her love of math on her right elbow. The sculpture in (g) is a 3-D quilling of an elephant representing me, named "Lil Kate," made out of small paper strips dyed with food coloring and adhered with a mixture of coffee creamer and water. In (h), the student outlines a proportionally accurate Fibonacci spiral with a meander [CL16].



(a) By Tricia (b) By Katrina (c) By Marlena (d) By Tonya (e) By Tiffany (f) By Amber (g) By Heather (h) By Erica

Figure 1. Students' math-inspired art work.

2. Developing Intuition

One of the biggest challenges of teaching in prison is being diligent in my language choices. I actively try to avoid using idioms or explanations that could trigger mental blocks for students, especially about math subjects in which they'd experienced difficulty or unsuccessful learning outcomes. Since I am hoping to build up their intuition on a new mathematical foundation, it's important not to repeat the same explanations that caused them confusion the first time. Moreover, because of the already difficult environment we are in, I don't want to say something that would cause students any more distress.

A small but pervasive example of being more conscious of my language in the classroom was swapping out the term "homework" for "assignment" in my college algebra course. But I also wanted to make sure my writing achieved the same objectives as my classroom language and tone. So every week, I wrote lecture notes for the students to keep that read exactly like I would speak during class, while still including the usual textbook definitions, examples, formulas, etc.—and while trying to avoid anything that would elicit a mental block toward the subject. This style of lecture notes was especially important in situations where students missed class. Since there are no dedicated times like office hours, appointments, or recitations for them to get help, I tried to make my notes reminiscent of our classroom: thorough, but not dry, while still containing the explanations they need to understand the material.

Out loud and in text, before introducing any new topic in the course, I wanted to provide motivation for why we were talking about it, which really addresses why they should care about it. As a stereotypical example, I did not have to ask how my students felt about fractions; they all groaned loudly when I said, "Today, we'll be talking about fractions." I had anticipated this, mostly because of certain family members and friends who have had an identical response to dealing with fractions. I jokingly threw up my hands and acquiesced over the groans: "Okay, fine, we'll talk about one of my favorite things in math first instead ... prime numbers!" We then detoured through prime numbers, prime factorizations, division trees, greatest common divisors, and least common multiples before we finally circled back to fractions. Suddenly, the two biggest obstacles to them mastering fractions previously (adding and simplifying) were reduced to problems they had just tackled in a very different setting. Once we'd gotten around the initial hurdle to approaching a subject they'd already convinced themselves they'd "never understand," my students no longer experienced the same mental block toward it. Instead, I witnessed them appreciate the power of abstraction, to the point that one of them exclaimed, "Holy crap, I can actually help my kids with their homework now!" after adding three fractions with different denominators using their least common multiple.

By the time we got to solving linear equations and word problems, the students had a completely different outlook on the material. After hearing it so many times on the first day, I had banned the phrase "bad at math" from our classroom, but even if I hadn't, that was not how the students felt anymore. We had spent multiple class periods developing intuition for the properties of addition, subtraction, multiplication, and division of real numbers, from an abstract perspective but also with the concrete example of debt to understand computations with negative numbers. What I'd worried would seem boring to the students was actually what initially piqued their curiosity: we took a specific instance of a problem, then abstracted it away with algebraic tools to study the problem in full generality. Once they'd seen a given topic from these two different angles, they also had plenty of "low-stakes" opportunities to practice. I frequently reminded them that it's normal to make mistakes when trying something new. So by the time I said to them, "Today, we are going to solve some word problems," the students were no longer groaning, or feeling anxious about using variables to represent unknowns; their confidence in their abstract problem-solving abilities had been strengthened by the time we spent in the low-stakes material, building a solid foundation from which to work. Upon solving one of the linear systems arising from a word problem, one of my students proudly announced, "I feel like a mathematician!" "You are," I replied.

When I taught the art elective course, I ascribed to a similar ideology. Each mathematical topic (algorithms, proportion, infinity, and abstraction) was paired with complementary art work from different time periods and regions of the world to illustrate the idea. It was surprising how much of the material, which I'd created with the intention of introducing them to advanced undergraduate math they wouldn't have seen before, illuminated other mathematical concepts for them that I hadn't even anticipated. When we talked about proportions, for instance, we discussed how the ancient Greeks thought the ideal relationship between the width w and height h of a building is given by the "golden proportion"

$$\varphi = \frac{w}{h} = \frac{w+h}{w}.$$

After we talked about its relationship to the Fibonacci sequence, I mentioned offhandedly that we can solve explicitly for the golden ratio φ using the quadratic formula; a student immediately raised her hand and asked to see how that would work. After I showed them the trick of setting $h = 1$, they were amazed to see an "elementary" formula they'd rotely memorized in school being used to answer such a seemingly unrelated question. Later in the same class, when we were discussing the Archimedean method of approximating π, I wrote the more familiar equation $C = \pi d$ as $\pi = C/d$ to emphasize the ratio. To my surprise and dismay, the vast majority of them had never seen or considered π as a ratio, and it was emotional to see them appreciating the amazing property of circles that every mathematician discovers at some point in their career. I began to think deeply about how we motivate these concepts when students first encounter them.

My students' comments in class cast many mathematical concepts in a new light for me as well. For example, in talking about algorithmic constructions with *girih* tiles [LS07], we discussed how it is possible with rudimentary tools to fabricate these tiles with precise angular measurements and arrange them into astonishingly intricate designs. We talked about how to decompose the polygonal tiles into triangles, and how an understanding of triangles unlocks a lot of possibilities for the tiles' fabrication and design. One of the students mused casually, "So that's why they teach an entire class about triangles. They're like the atoms of shapes." I've since begun borrowing that phrase to motivate the subject of trigonometry.

During another class period, we talked about the cardinality of the natural numbers. I chuckled to myself as I listened to a familiar debate among the students. *"Why would 0 be included? You don't count anything with zero fingers."* Fortunately, they were satisfied that it didn't much matter with respect to cardinality after we wrote out the bijection between the natural numbers and the integers. However, that awareness came back to bite me in the form of a deeply profound and unexpected question that I received while explaining Cantor's diagonalization argument. I had just demonstrated the proof by contradiction: if we try to enumerate all of the elements $s_1, s_2, s_3, \dots$ of the set T of infinite sequences of 0's and 1's, it is always possible to construct an element $s \in T$ that differs from s_k in the kth position, so that element s of T actually wasn't enumerated in our list. "But why can't you just call that element s_0? Then wouldn't you be able to count them all with counting numbers since it doesn't matter if we include 0?" The question was so subtle and clever that it caught me off guard. "You're thinking like a mathematician," I replied, before we spent the next few minutes verifying that Cantor had indeed gotten it right, though probably not on his first try.

3. Solving Problems

Once I was talking about my experience teaching math in prison with a friend of mine who was formerly incarcerated. He'd served a ten-year sentence over the entirety of his 20s, during which time he had the opportunity to take several math courses for high school degree equivalency. He's now earning his bachelors degree while working full-time as a water treatment facility operator. While math courses equipped him with necessary skills for his new career, he credits those classes with something even more important. As a creative writer, my friend had always felt more passionate about writing classes in school. He admitted to me that he'd never liked that there was "only one right answer" in his math classes. But once he was incarcerated, solving math problems became a mentally stimulating and comforting activity. With so much time to think and reflect, he realized how many problems in life lack a clear-cut solution, and he began to appreciate the existence and uniqueness of the solutions to the problems in his math assignments.

I told my students about this conversation with my friend, and I asked them if they felt similarly about learning math while incarcerated. Several of them participate in entrepreneurial and vocational training programs, and those students echoed the importance of math for their new career paths. In fact, one student made parole during our spring algebra class, and she still wanted to finish the course remotely postincarceration to help in earning her EMT certifications. (This situation is one of many in which the TPEI program managers' administrative efforts are essential to our success.)

Several other students said that my friend's point about taking comfort in having a right answer really resonated with them, even though they'd had terrible previous experiences with math. One of them was the student who'd had to drop her astronomy major because of her math classes, who'd asked about teaching herself Baby Rudin. Another student had been concurrently earning her high school diploma during our class ("only 50 years late!" she'd often say), and she explained to me how learning math again had helped her find balance in her life. "Everything in math has an opposite," she said, "I am a 'yin and yang' type of person, and I guess that's why I love math. You can always add back anything you subtract, or multiply anything you divide." Her friend chimed in, "Plus, even just knowing there are infinitely many possible choices to start from gives you options for all of the gray areas in life." Yet another student, who was nervous about her upcoming parole hearing the following week, reiterated that being able to solve a problem and find a correct answer out of infinitely many possible choices was going to be vital to her success postincarceration. Like so many of her classmates, she'd also been interested in STEM growing up; she wanted to be an astronaut when she entered high school, before dropping out because of math. "I'm nervous about finding a job and a permanent place to live, but I have people to help me for now," she said. "I probably can't be an astronaut anymore because of my felony charge, but I still want to enroll in college and earn my BS once I'm back on my feet." In spite of all the problems and uncertainties that she faced postincarceration, she'd already begun identifying solutions.

On the last day of the art elective class, after the exhibition, I announced that I would be teaching a brand new course in the fall that had never been offered by TPEI. It is a credit-bearing math course at UT Austin that is the prerequisite for the calculus sequence. My student—the aspiring astronomer enthralled by Baby Rudin—came up to talk to me after the other students had said their thank-you's and goodbye's. "I really want to take the course, but I've failed precalculus before and I'm worried I will again," she said nervously. "I haven't thought about math in a long time." Remembering all of the insights she'd shared during the course, I told her that I knew she could do it and reassured her that I would be there to help her succeed this time. "Besides," I added, "you've been thinking like a mathematician this whole time."

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Katherine J. Pearce

Credits

Figure 1 is courtesy of students of TPEI.
Photo of Katherine J. Pearce is courtesy of Katherine J. Pearce.

Tips for Saving Time with Grading

Jennifer Schaefer and Rebecca Swanson

Grading is a way for faculty to both assess student work and provide feedback to students on their level of understanding of a course's material. Whether your grading system is traditional or nontraditional, the feedback-giving process is often one of the most time consuming aspects of teaching. As early career faculty transition from lighter graduate school teaching loads to more sizable teaching loads as new faculty, finding time to grade an increased amount of student work can be challenging. Having a student grader can reduce your grading burden, but not all institutions offer this support. Our goal is to provide you with some tips and tricks to make the grading process more smooth and efficient regardless of institution.

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Think About Your Policies

A lot of energy can be saved by carefully crafting grading policies when you develop your syllabus. For instance, you could decide to grade student work on proficiency, something both authors do in their classes. What this means is that homework is graded using a scale of proficiency, e.g., mastery, near mastery, tentative, unsure, or didn't attempt. This form of grading saves time because the instructor can focus on feedback instead of agonizing over the number of points to deduct for mistakes. The following are a few other policies you can implement in any course. Regardless of what you decide, we advise that you create grading policies that work for you and your course and stick to them.

Determine whether shorter, more frequent assessments or longer, less frequent assignments work better for your course and your grading style. One of the authors assigns and collects homework assignments each class period. She does this because she finds these shorter assignments less overwhelming, and she can grade the assignments more quickly. However, one of her colleagues prefers to collect a longer homework assignment once per week.

Allow for a small number of low-stakes assignments to be dropped. This removes some of the anxiety when students receive lower grades than they would like and can eliminate the request for make-up assignments.

Think about your late work policy. Trying to balance flexibility with fairness is tough and determining what to do on a case-by-case basis can waste time and emotional energy. If you do decide to allow for make-up assignments or extensions, only give a limited number and keep the extension window short. This reduces the number of submissions you need to keep track of. One of the authors uses a policy that she has found to be equitable and easily enforced. Homework submissions are due on Wednesdays, but solutions are posted on Saturdays. As long as a student asks for an extension before the due date, it is granted, until the solution set is posted, at no penalty.

Incorporate group assignments into your course. This allows you to assess student understanding while grading fewer submissions. At the same time, students develop the skill of working with others, which benefits them outside of class.

Once your policies are in place, how can you make grading and giving feedback easier? We've compiled lists of both tech-free and tech-required tools that can help.

Tech-Free Grading Tips

Grade a smaller selection of the assigned problems, or grade some assignments and not others. It is OK if you don't grade everything you assign. Let us say that again. It is **OK** if you don't grade everything you assign. After all, the goal is for the bulk of student learning to happen when students complete the assignment to begin with. You could do this by grading a subset of the problems you assigned. Or you could consider having some assignments that receive feedback without a grade and other assignments that receive grades without feedback. Either way, you can provide solution sets for students to see where they made mistakes and how they can correct them.

Grade one problem at a time and group problem solutions based on mistakes made. When grading problem sets, your mental load will be higher if you attempt to grade more than one problem at a time. This is because you have to change what you are looking for as you move between problems, and this shift takes time and mental energy. As you move through students' submissions for a given problem, it is helpful to group the assignments by the mistakes the students made. This allows you to quickly provide the same feedback to all students who made similar mistakes and helps ensure fairness with your grading.

Use a rubric. Designing a rubric will take a bit of time up front, but utilizing a rubric while grading can save you a great deal of time overall. This is because a rubric delineates exactly what you are looking for into a limited number of categories and requirements. The fewer buckets you have, the easier and more consistent your grading will be. Moreover, if you pair a rubric with student presentations—group work, homework, or projects—you can fill out the rubric while the presentations are happening so that once a student is done, you are done grading as well. This can allow you to assess work in real time without needing an additional block of time just for grading. By giving your rubric to your students ahead of time, you also communicate what your expectations are, which helps them correlate your feedback with the grade they receive.

Have your students self-grade their work. Allowing students to assess their own work can help students develop metacognition skills. It can also help reduce the amount of grading you are required to do. One of our colleagues pairs self-grading with student presentations of homework solutions. While a presentation is occurring in class, students are allowed to use a green pen to mark up their work based on their classmate's solution and any additional feedback the instructor offers. While this does take class time, grading goes more quickly because the students have already marked up their submissions. Another one of our colleagues in the mathematics community chooses to focus on student reflection when grading homework. She posts solutions once students submit work and has them self-grade and reflect on that work by asking questions about their understanding of what was done wrong, the mistakes they made, and what they still don't understand. This allows her students to think more intentionally about what they are learning. Students who thoughtfully complete this assignment get full credit.

Tech Tips

There are many ways that technology can help you assess student work too. Some are more conventional and others are relatively new. For instance, requiring students in major courses to complete their homework in LaTeX makes it easier for us to read their solutions. As an added bonus, it also provides them the opportunity to gain expertise in using our discipline's typesetting tool. As a newer option, you could use tools such as Audacity to record your comments over audio in place of or in addition to written feedback. The following are a few other options to consider.

Online homework. Most publishers have online homework systems associated with their textbooks. Such publisher-based systems come with large problem banks that offer students a great deal of practice and immediate feedback. The challenge is that there are often limits to the types of problems that can be posed, and that the focus of these problems is usually on the final answer, not on the work done to get that answer. Additionally, there is a cost to either the students or the institution. Some free alternatives do exist, though, that work with almost any textbook. Webwork is a free system supported by the Mathematical Association of America, and MyOpenMath is another free option that works with open educational resources in mathematics. Further, many campuses utilize a learning management system (LMS) in which you can create your own free online homework. Of course, then you are not able to make use of the large question banks available in the other systems. It should be noted that you do not have to go all or nothing with online homework. For instance, one of the authors uses online homework in her calculus courses but supplements it with written work as well.

Tools that help grade written work. Your LMS may also be capable of helping you grade written work. For instance, Canvas has a tool called Speed Grader, Blackboard uses BbAnnotate, and Moodle accepts file submissions through the Assignments feature. Each of these allows students to upload their own work and you to grade their submissions electronically, essentially the equivalent of paper grading, but online. This saves you from writing the same comment by hand repeatedly, entering grades into a gradebook, and spending time returning papers to students in class. An additional benefit is that students can submit work as a group, and your feedback is given to the entire group at once. Finally, you and your students don't have to pay extra to use your university's LMS!

There are additional resources that can help support faculty in giving feedback that are much more robust than the built-in LMS tools. Some examples include Gradescope, Assign2, Crowdmark, and Pearson's Freehand Grader. (As a disclaimer, both authors have experience working with Gradescope in their classes but have not had the opportunity to test the others.) What they have in common is that students complete work on paper, a tablet, or a computer, and then upload their work for you to grade electronically. For most, the instructor can also scan and upload student work. These tools all support multiple graders grading different problems from the same assignment at the same time, making them useful in coordinated settings. With any of these resources, you can edit your rubric as you go, and the changes apply to what you have already graded. You can also make use of saved comments as you give your feedback. All of them provide student data that allows the instructor to determine where students are struggling. Syncing with a standard LMS makes grade entry automatic, and as with the LMS tools, you aren't spending class time returning papers. Each of the resources below has one common challenge—they aren't free. Pricing plans vary and are often dependent upon a number of factors. However, you usually have an option to run a free pilot. If you are interested in continuing to use the product, you will need to further discuss those details with representatives at these companies. If you are using a non-traditional grading scheme that isn't points-based, these tools are not directly designed to help you, although each has its own workarounds. Below we highlight some of the differences among these tools:

- *Gradescope.* Gradescope is a well-developed grading platform by Turnitin. Gradescope provides a "grouping" feature that is currently not available in the other tools on this list. With this feature, the instructor can sort problems into groups based upon mistakes and then apply the same grade and feedback to all problems within a particular group. This can be especially useful when class sizes are large. Gradescope has two plans—a basic plan and an institutional plan. It should be noted that not all of the more desirable features are available through the free basic plan.
- *Assign2.* Assign2 seems to stand out in the robust statistics it can provide, including data on the amount of time the instructor spends grading a particular problem. For instructors planning to scan and upload student work, the system uses QR code technology to match submissions to students. Additionally, Assign2 is in the process of developing a number of new features, including tools to grade code and a grouping feature similar to that of Gradescope. They seem to have the easiest workaround for non-traditional grading schemes. Finally, they are the most budget-friendly option in this list.
- *Crowdmark.* Crowdmark, developed by a mathematician and his graduate student, aims to improve dialogue between students and instructors, as well as to scale human-to-human interactions. Crowdmark is very intentional about keeping the ownership of data in the hands of the students and faculty, making it readily available for distribution and assessment purposes. Crowdmark also utilizes QR-code technology that helps match students to their submissions, which is quite useful for instructors who scan and upload a large amount of student work. The pricing structure depends upon a number of factors. However, unlike with Gradescope, all features are available to all users.
- *Pearson's Freehand Grader.* For those of you already using Pearson's MyLab and Mastering online homework system, you have access to their Freehand Grader tool. While this tool is less developed than the others in this list, it has the advantage that it is free to those already using MyLab and Mastering. This is the one tool on the list that does not allow the instructor to easily scan and upload student work, as it is aimed more toward homework grading than exam grading, but it does have the other features that help grade student-submitted work more quickly, including the ability to build and alter the rubric as you go and to grade problem-by-problem, which is often better for consistency.

We hope you found something new that can save you some time while helping support your students as they learn. By carefully thinking about both your grading policies and which tools you will employ, you can make this aspect of your job easier, more supportive of student learning, and (maybe!) more fun. Happy Grading!

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Credits

Photo of Jennifer Schaefer is courtesy of Dickinson College.
Photo of Rebecca Swanson is courtesy of Rebecca Swanson.

Dear Early Career

How can I remain positive about my work after the rejection of a grant application?

—Down-in-the-dumps

Dear Down-in-the-dumps,

This is a tough one. After my first NSF rejection I worked very hard to write what I thought was an excellent resubmission which took into account the comments of the reviewers. My subsequent proposal was then promptly rejected with a collection of rather dismissive comments, and it was difficult to think about going through that process again. From a position where now I have had some more failure and a bit of success in the process, my coping strategy is to acknowledge the amount of randomness in the process, and just concentrate on the writing of the proposal itself as an important part of my research process.

To be more detailed, let's look at the process from the perspective of a reviewer. A reviewer on an NSF panel often has around 10–12 proposals to read and write reports on in advance of the panel (in addition to whatever their current workload is). The proposals are discussed in the panel session itself, and this discussion is typically summarized in a review. A panel might consist of around 10 people, and a proposal will typically be read by 3 panelists, but other panelists can make comments during the session. Additionally, external reviewers may be solicited for their opinions of the proposal. However, the views of the 3 panelists will be the predominant factor in the final evaluation of the proposal. I am not suggesting there is a better alternative to this system, but there is a lot of randomness associated to it: Some people are overly confident in their opinions on certain areas of research, and others will be biased by how "well-known" the submitter of the proposal is, whether this is conscious or not. It is much more difficult to read a proposal when you do not have direct research experience with problems closely related to it, and some reviewers misinterpret their own discomfort with a proposal as a weakness of the proposal. It could well be that if a proposal just fell into the inbox of three different panelists, then the outcome would be different. Additionally, it can happen that your proposal is great, but the competition is particularly tough that year, and some reviewers focus on creating reasons why your proposal was not ranked as highly as others, rather than giving some constructive feedback and encouraging resubmission.

Having said this, I do take feedback seriously and try to incorporate constructive comments in a resubmission. My rule is that if multiple reports all point to the same issue, then this should definitely be addressed, as even if I think their comments are completely off-base and the ravings of a lunatic, my proposal certainly did not express itself as I had intended. When I have received comments that were particularly difficult to process about research, then discussing these with senior colleagues that I trust has been useful.

Finally, remember that the process of writing a grant application is constructive. It can organize your thoughts on your research and force you to do the groundwork on the directions in which you would like to push your research. The work done putting your proposed research into a larger context will help you when you write research papers associated to those problems, and your efforts will help with other documents such as research statements. I look back at my proposals quite regularly when writing papers or thinking about projects to start new collaborations.

Best of luck with your resubmission!

—Early Career editors

Have a question that you think would fit into our Dear Early Career column? Submit it to `Taylor.2952@osu.edu` or `bjaye3@gatech.edu` with the subject Early Career.

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