Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

DYNAMICS OF A REACTION-DIFFUSION-ADVECTION MODEL FOR TWO COMPETING SPECIES

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(Communicated by the associate editor name)

Abstract. We study the dynamics of a reaction-diffusion-advection model for two competing species in a spatially heterogeneous environment. The two species are assumed to have the same population dynamics but different dispersal strategies: both species disperse by random diffusion and advection along the environmental gradient, but with different random dispersal and/or advection rates. Given any advection rates, we show that three scenarios can occur: (i) If one random dispersal rate is small and the other is large, two competing species coexist; (ii) If both random dispersal rates are large, the species with much larger random dispersal rate is driven to extinction; (iii) If both random dispersal rates are small, the species with much smaller random dispersal rate goes to extinction. Our results suggest that if both advection rates are positive and equal, an intermediate random dispersal rate may evolve. This is in contrast to the case when both advection rates are zero, where the species with larger random dispersal rate is always driven to extinction.

1. Introduction.

In this paper we consider

$$\begin{cases} u_t = \nabla\cdot(\mu\nabla u - \alpha u\nabla m) + u(m(x)-u-v) & \text{in } \Omega\times(0,\infty),\\ v_t = \nabla\cdot(\nu\nabla v - \beta v\nabla m) + v(m(x)-u-v) & \text{in } \Omega\times(0,\infty),\\ \mu\frac{\partial u}{\partial n} - \alpha u\frac{\partial m}{\partial n} = \nu\frac{\partial v}{\partial n} - \beta v\frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x)\geq 0, \quad v(x,0)=v_0(x)\geq 0, \end{cases} \tag{1}$$

where ∇ is the gradient operator, $\nabla\cdot$ is the divergence operator; u and v, representing the population densities of two competing species with random dispersal rates

2000 *Mathematics Subject Classification.* Primary: 35J57, 35B40; Secondary: 92D40.

Key words and phrases. Directed movement; competing species; reaction-diffusion-advection; exclusion; evolution of dispersal.

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μ, ν respectively, are therefore non-negative functions of x and t; $m(x)$ represents the local intrinsic growth rate and is assumed to be the same for both species; $\alpha, \beta > 0$ measure the tendency of the biased movement of species along the environmental gradient; no-flux boundary condition is imposed on the boundary $\partial\Omega$ of a bounded smooth domain Ω in $\mathbf{R}^N$ representing the habitat; $\partial/\partial n := n \cdot \nabla$ where n is the outward unit normal vector on $\partial\Omega$.

Throughout this paper, we assume $m \in C^{2,\gamma}(\bar{\Omega})$ is nonconstant to reflect the spatial heterogeneity, $m > 0$ in $\bar{\Omega}$, and u_0 and v_0 are not-identically zero. By the maximum principle [28] and parabolic regularity theory, (1) has a solution $u(x,t)$ and $v(x,t)$ in $C^{2,1}(\bar{\Omega} \times (0,\infty))$ and are strictly positive for $x \in \bar{\Omega}$ and $t > 0$. We are interested in the dynamics of (1). To this end, we first note that (1) has two semi-trivial steady states, denoted by $(\theta_{\mu,\alpha}, 0)$ and $(0, \theta_{\nu,\beta})$, where $\theta_{\mu,\alpha}$ is the unique coexistence steady state of

$$\begin{cases} \theta_t = \nabla \cdot (\mu \nabla \theta - \alpha \theta \nabla m) + \theta(m - \theta) & \text{in } \Omega \times (0,\infty), \\ \mu \frac{\partial \theta}{\partial n} - \alpha \theta \frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{2}$$

It is shown in [3] that for all $\mu > 0$ and $\alpha \geq 0$, a unique coexistence state $\theta_{\mu,\alpha}$ for (2) exists. Moreover, $\theta_{\mu,\alpha}$ is globally asymptotically stable among non-negative, non-trivial initial data.

When $\alpha = \beta = 0$, (1) is reduced to the Lotka-Volterra model with diffusion,

$$\begin{cases} u_t = \mu \Delta u + u(m(x) - u - v) & \text{in } \Omega \times (0,\infty), \\ v_t = \nu \Delta v + v(m(x) - u - v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, \end{cases} \tag{3}$$

where $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ is the Laplace operator. It is shown in Dockery et al. [16] that if $\mu < \nu$, the semi-trivial steady state $(\theta_{\mu,0}, 0)$ is globally asymptotically stable; i.e., the slower diffuser always drives the faster diffuser to extinction. In particular, in a spatially inhomogeneous but temporally constant environment an exotic species with random movement can invade when rare if and only if it is the slower diffuser [19].

However, the movement of organisms can often be biased, e.g., the movement upward along the gradient of resources, as resources are usually not distributed uniformly in space. Belgacem and Cosner introduced (2) in [3] to describe the biased movement of a single species. Subsequently, the two species model (1) was introduced and studied in Cantrell et al. [7, 8] and Chen et al. [12]. For more recent progress on (1) and (2) and related models, see [2, 4, 9, 10, 11, 13, 18, 23, 24, 25]. This current study is mainly motivated by the following result (Theorem 3.1, [18]).

Theorem 1.1. *Suppose that* $\alpha = \beta$, $\Omega = (0,1)$ *and* $m_x > 0$ *in* $[0,1]$.

1. *If* $0 \leq \alpha < \mu / \max_{\bar{\Omega}} m$, *there exists some* $\delta_1 > 0$ *such that for* $\nu \in (\mu, \mu + \delta_1)$, $(\theta_{\mu,\alpha}, 0)$ *is globally asymptotically stable.*
2. *If* $\alpha > \max\{\mu / \min_{\bar{\Omega}} m, \max_{\bar{\Omega}} m / \min_{\bar{\Omega}} m_x\}$, *there exists some* $\delta_2 > 0$ *such that for* $\nu \in (\mu, \mu + \delta_2)$, $(0, \theta_{\nu,\beta})$ *is globally asymptotically stable.*

Part 1 of Theorem 1.1 says that when the advection is weak relative to random diffusion, the species with the smaller random diffusion rate drives the other species to extinction. This is similar to the case $\alpha = \beta = 0$. It is interesting to observe that by part 2 of the theorem, when the advection is strong relative to random diffusion, the fate of two species is interchanged: the species with large random diffusion rate is always the winner. A critical assumption in Theorem 1.1 is that both random diffusion rates are close to each other. A natural question is: What is the dynamics of (1) if $\alpha = \beta$ but μ and ν are very different? More generally, given any two pairs of parameters (α, μ) and (β, ν), how do we determine the dynamics of (1)? For general values of α, β and ν, we shall consider two cases: (i) sufficiently large μ and (ii) sufficiently small μ.

For sufficiently large μ, we first establish the following result.

Theorem 1.2. *Given any $\alpha > 0$ and $\beta > 0$, there exists ν_1 such that if $\nu > \nu_1$, then for sufficiently large μ, $(0, \theta_{\nu,\beta})$ is globally asymptotically stable.*

Theorem 1.2 says that for any fixed advection rates, if both random diffusion rates are sufficiently large, then the species with the smaller random diffusion rate will drive the other species to extinction. In particular, when both species have the same advection rate, large random diffusion rate is selected against.

What happens if ν is small but μ is sufficiently large? To describe our result, we need to further restrict our choices of $m(x)$. Denote the set of local maximum points of m by $\mathfrak{M}$ and by $\Sigma_0 = \{x \in \Omega : \nabla m = 0 \text{ and } x \notin \mathfrak{M}\}$. Most of the times we also make additional hypotheses on m:

(M1) Every critical points of m are non-degenerate, and $\Delta m > 0$ on Σ_0. Moreover, $\frac{\partial m}{\partial n} < 0$ on $\partial\Omega$.

(M2) m has a unique critical point $x_0 \in \Omega$ which is a non-degenerate global maximum point. Moreover, $\frac{\partial m}{\partial n} < 0$ on $\partial\Omega$.

In contrast to Theorem 1.2, we have the following result.

Theorem 1.3. *Assume* (M2). *Given any $\alpha > 0$, $\beta > 0$, there exists ν_2 such that if $0 < \nu < \nu_2$, then for sufficiently large μ, both $(\theta_{\mu,\alpha}, 0)$ and $(0, \theta_{\nu,\beta})$ are unstable, and* (1) *has at least one asymptotically stable coexistence steady state.*

Remark 1. In Theorem 1.3, if we assume (M1) instead of (M2), and replace the assumption $\beta > 0$ by β suitably large, then the conclusion still holds true.

Theorem 1.3 says that for any given advection rates, if one of the random diffusion rates is small and the other is sufficiently large, two species can coexist! In particular, when both species have the same advection rate, two species with one small and one large random diffusion rate can coexist.

We now turn to the case when μ is sufficiently small.

Theorem 1.4. *Assume* (M2). *For all $\alpha > 0$ and $\beta > 0$, if $\nu > \beta \sup m$, then for all μ sufficiently small, both $(\theta_{\mu,\alpha}, 0)$ and $(0, \theta_{\nu,\beta})$ are unstable, and* (1) *has at least one asymptotically stable coexistence state.*

Remark 2. In Theorem 1.4, if we assume (M1) instead of (M2), and replace the assumption $\alpha > 0$ by α suitably large, then the conclusion still holds true.

Theorem 1.4 implies that if one of the random diffusion rates is suitably large and the other is sufficiently small, then both species can coexist. This result is in the same spirit as Theorem 1.3, though the proofs are rather different. In contrast to this result and Theorem 1.2, our next result shows that if both random diffusion rates are small, then the species with larger random diffusion rate drives the other species to extinction!

Theorem 1.5. *Assume* (M2). *For all $\alpha > 0$, $\beta > 0$, if $0 < \nu < \beta \inf m$, then for all μ sufficiently small, $(0, \theta_{\nu,\beta})$ is globally asymptotically stable.*

Remark 3. In Theorem 1.5, if we assume (M1) instead of (M2), and replace the assumption $\alpha > 0$ by α suitably large and $\nu < \beta \inf_\Omega m$ by $\nu < K$ for some positive constant K depending on β, then the conclusion still holds true.

This paper is organized as follows. In Section 2, we give some important a priori estimates on the upper and lower bounds of solutions of (2). Section 3 is devoted to the case when μ is sufficiently large. In Section 4 we consider the case when μ is sufficiently small.

2. **Some key a priori estimates.** In this section we relax the assumption that $m > 0$ in $\bar{\Omega}$ to $\int_\Omega m \geq 0$. By [3], (2) has a unique positive steady state $\theta_{\mu,\alpha}$ for every $\mu > 0$ and $\alpha \geq 0$. We shall establish some upper and lower bounds of $\theta_{\mu,\alpha}$, which are independent of μ and/or α. We start by stating the following lemma which is contained in [12].

Lemma 2.1. *Suppose $m \in C^2(\bar{\Omega})$ is a nonconstant function.*
- (a) *If $\beta/\nu \leq 1/\sup m$, then $\theta_{\nu,\beta}(x) < (\sup m) e^{\beta[m(x)-\sup m]/\nu}$ for all $x \in \Omega$.*
- (b) *If $\beta/\nu \geq 1/\inf m$, then $\theta_{\nu,\beta}(x) > (\sup m) e^{\beta[m(x)-\sup m]/\nu}$ for all $x \in \Omega$.*

2.1. **Upper bound.** Denote the set of all positive local maximum points of m by $\mathfrak{M}_+$ (i.e. $\mathfrak{M}_+ = \{x \in \mathfrak{M} : m(x) > 0\}$). The main result of this subsection is the following upper estimate of $\theta_{\mu,\alpha}$. (See also [23].)

Theorem 2.2. *Assume* (M1). *There exists some positive constants α_1, C, r, γ and $\delta^* < 1$ such that for all $\mu > 0$ and $\alpha \geq \alpha_1$,*

$$\theta_{\mu,\alpha}(x) \leq \begin{cases} Ce^{\alpha\delta^*[m(x)-m(x_0)]/\mu} & \text{in } B_r(x_0), \text{ for any } x_0 \in \mathfrak{M}_+, \\ e^{-\gamma\alpha/\mu} & \text{in } \Omega \setminus \cup_{x_0 \in \mathfrak{M}_+} B_r(x_0). \end{cases}$$

Corollary 1. *Assume* (M2). *For any $\alpha_2 > 0$ and any $0 < \delta^* < 1$, there exist some constants $C, r, \gamma > 0$ depending on α_2 (but independent of μ and α) such that for all $\mu > 0$ and $\alpha \geq \alpha_2$,*

$$\theta_{\mu,\alpha}(x) \leq \begin{cases} Ce^{\alpha\delta^*[m(x)-m(x_0)]/\mu} & \text{in } B_r(x_0), \\ e^{-\gamma\alpha/\mu} & \text{in } \Omega \setminus B_r(x_0), \end{cases}$$

where x_0 is the unique critical point of m.

Remark 4. Corollary 1 is applicable to the case when $\alpha > 0$ is arbitrary but fixed and $\mu \to 0$.

Proof of Theorem 2.2. Transform the equation by $w(x) = e^{-\alpha m(x)/\mu}\theta_{\mu,\alpha}$ (See, e.g. [3, 26]) which satisfies

$$\begin{cases} \mu\nabla \cdot (e^{\alpha m/\mu}\nabla w) + e^{\alpha m/\mu} w \left[m(x) - e^{\alpha m/\mu} w\right] = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

If α/μ is bounded, by applying the maximum principle, we have

$$\|\theta_{\mu,\alpha}\|_{L^\infty(\Omega)} \leq C\|e^{\alpha m/\mu}\|_{L^\infty(\Omega)}\|w\|_{L^\infty(\Omega)} \leq C\|e^{\alpha m/\mu}\|_{L^\infty(\Omega)}\|me^{-\alpha m/\mu}\|_{L^\infty(\Omega)}. \tag{4}$$

Next we consider $\alpha/\mu \to \infty$. As a consequence of (M1), $\mathfrak{M}$ consists of finitely many points. Denote

$$\{m(x) : x \in \mathfrak{M}\} = \{m_1, m_2, ..., m_n\}, \quad m_1 < m_2 < ... < m_n;$$
$$\mathfrak{M}_i = \{x \in \mathfrak{M} : m(x) = m_i\}.$$

By the non-degeneracy of critical points of m, there exist $r > 0$, $K > 0$ such that for any $z \in \mathfrak{M}$,

$$\begin{cases} \frac{1}{K}|z-x|^2 \le m(z) - m(x) \le K|z-x|^2 \\ \frac{1}{K}|z-x| \le |\nabla m(x)| \le K|z-x| \end{cases} \tag{5}$$

for all $x \in B_r(z)$. Set $m_0 = \min_{\overline{\Omega}} m$ and choose $0 < \eta < \min_{1\le i\le n}\{m_i - m_{i-1}, r^2/K\}$ such that $\{m_i - \eta\}_i$ are regular values of m. Fix $0 < \delta_1 < 1$. Define recursively

$$\delta_{i+1} = \frac{\delta_i \eta}{m_{i+1} - m_i + \eta}\ , \qquad i = 1, 2, ..., n-1.$$

Then we have

$$1 > \delta_1 > \delta_2 > \cdots > \delta_n \equiv \delta^* = \delta_1 \prod_{i=1}^{n-1} \frac{\eta}{m_{i+1} - m_i + \eta} > 0.$$

Furthermore, by (5) and (M1) there exists a large constant K_1 independent of μ, α such that

$$\frac{\delta^* \alpha}{\mu}|\nabla m|^2 + \Delta m > 0 \quad \text{ in } \overline{\Omega \setminus D},\ D = \cup_{z\in\mathfrak{M}} \overline{B_{\sqrt{\frac{\mu}{\alpha}}K_1}(z)}. \tag{6}$$

Define

$$\Omega_1 = \Omega, \quad \Omega_{i+1} = \{x \in \Omega : m(x) > m_i - \eta\} \setminus \cup_{z\in\mathfrak{M}_i} \overline{B_r(z)}.$$

Note that $\Omega_{i+1} \subset \Omega_i$, since $\{x \in \Omega : m(x) > m_{i+1} - \eta\} \subset \Omega_i$ from the definition of η.

Define

$$M = \|\theta_{\mu,\alpha}\|_{L^\infty(\Omega)}, \quad d = KK_1^2, \quad \phi_i = Me^d e^{\alpha\delta_i(m(x)-m_i)/\mu}$$

and

$$N[\phi] := -\nabla \cdot (\mu \nabla \phi - \alpha \phi \nabla m) - \phi(m - \theta_{\mu,\alpha}).$$

Then we have

$$N[\phi_i] \ge \phi_i \left[\alpha(1-\delta_i)\left(\frac{\delta_i \alpha}{\mu}|\nabla m|^2 + \Delta m\right) - m\right] \ge 0 \tag{7}$$

in $\Omega_1 \setminus D = \Omega \setminus D$ for $i = 1, ..., n$ by (6) and by choosing $\alpha \ge \alpha_1$ large. Moreover, by (M1) we see that

$$\mu \frac{\partial \phi_i}{\partial n} - \alpha \phi_i \frac{\partial m}{\partial n} = \alpha(\delta_1 - 1)\phi_i \frac{\partial m}{\partial n} > 0 \quad \text{ on } \partial\Omega. \tag{8}$$

Note that in $D \cap \Omega_i$, $m(x) - m_i \ge -K(K_1\sqrt{\mu/\alpha})^2$. Hence for all i,

$$\phi_i(x) = Me^d e^{\delta_i\alpha(m(x)-m_i)/\mu} \ge Me^d e^{\delta_i\alpha(-KK_1^2\mu/\alpha)/\mu} \ge M \ge \theta_{\mu,\alpha} \quad \text{ in } D \cap \Omega_i. \tag{9}$$

We shall show by induction that $\theta_{\mu,\alpha} \le \phi_i$ in Ω_i, for $i = 1, ..., n$. Consider ϕ_1 on $\Omega_1 = \Omega$. By (9), it remains to prove that $\phi_1 \ge \theta_{\mu,\alpha}$ in $\Omega_1 \setminus D$. We shall do so by using a sharp characterization of the strong maximum principle. As $N[\theta_{\mu,\alpha}] = 0$, the principal eigenvalue of the operator N on Ω, under the boundary operator $\mathcal{B}u|_{\partial\Omega} := \mu\frac{\partial u}{\partial n} - \alpha u \frac{\partial m}{\partial n}$, denoted by $\sigma[N; \mathcal{B}, \Omega]$, must be zero. So $\sigma[N; \mathcal{B}, \Omega] = 0$. As $\Omega_1 \setminus D$ is a nice subdomain of Ω, the boundary operator

$$\mathcal{B}_1 u := \begin{cases} \mathcal{B}u & \text{ on } \partial(\Omega_1 \setminus D) \cap \partial\Omega, \\ u & \text{ on } \partial(\Omega_1 \setminus D) \cap \Omega, \end{cases}$$

on $\partial(\Omega_1 \setminus D)$ is well-defined, and by Proposition 3.2 of [5], it is apparent that

$$\sigma[N; \mathcal{B}_1, \Omega_1 \setminus D] > \sigma[N; \mathcal{B}, \Omega_1] = 0.$$

As we have $N[\phi_1 - \theta_{\mu,\alpha}] \geq 0$ in $\Omega_1 \setminus D$ by (7) and $\mathcal{B}_1[\phi_1 - \theta_{\mu,\alpha}] \geq 0$ in $\partial(\Omega_1 \setminus D)$ by (8) and (9), according to Theorem 2.4 of [1], one can infer by the strong maximum principle that $\phi_1 - \theta_{\mu,\alpha} \geq 0$ in $\Omega_1 \setminus D$. Combining with (9), we have proved $\phi_1 \geq \theta_{\mu,\alpha}$ in $\Omega_1 = \Omega$.

Next we prove $\phi_2 \geq \theta_{\mu,\alpha}$ in Ω_2. By (9), it remains to show that $\phi_2 \geq \theta_{\mu,\alpha}$ in $\Omega_2 \setminus D$. Define the boundary operator $\mathcal{B}_2$ by

$$\mathcal{B}_2 u = \begin{cases} \mathcal{B}u & \text{on } \partial(\Omega_2 \setminus D) \cap \partial\Omega, \\ u & \text{on } \partial(\Omega_2 \setminus D) \cap \Omega, \end{cases}$$

then similar as before, $\sigma[N; \mathcal{B}_2, \Omega_2 \setminus D] > \sigma[N; \mathcal{B}, \Omega] = 0$. By (7), we have $N[\phi_2 - \theta_{\mu,\alpha}] \geq 0$ in $\Omega_2 \setminus D$ for $\alpha \geq \alpha_1$. It remains to check the boundary condition. It is immediate from (8) that

$$\mathcal{B}_2[\phi_2 - \theta_{\mu,\alpha}] = \mathcal{B}[\phi_2 - \theta_{\mu,\alpha}] \geq 0 \quad \text{on } \partial(\Omega_2 \setminus D) \cap \partial\Omega.$$

Now observe that

$$\partial(\Omega_2 \setminus D) \cap \Omega \subset (\Omega_2 \cap \partial D) \cup [(\partial\Omega_2) \cap \Omega].$$

In $\Omega_2 \cap \partial D$, $\mathcal{B}_2[\phi_2 - \theta_{\mu,\alpha}] = \phi_2 - \theta_{\mu,\alpha} \geq 0$ by (9). Whereas in $(\partial\Omega_2) \cap \Omega$, we have $m(x) \geq m_1 - \eta$. We either have (i) $x \in \cup_{z \in \mathfrak{M}_1} \partial B_r(z)$; or (ii) $x \notin \cup_{z \in \mathfrak{M}_1} \partial B_r(z)$ and $m(x) = m_1 - \eta$. But (i) is impossible, since on $\cup_{z \in \mathfrak{M}_1} \partial B_r(z)$,

$$m(x) \leq m_1 - \frac{1}{K}|x - z|^2 = m_1 - \frac{r^2}{K} < m_1 - \eta.$$

So we must have (ii), i.e. $m(x) = m_1 - \eta$. Consequently on $\partial\Omega_2 \cap \Omega$,

$$\frac{\phi_2}{\phi_1} = \exp\{\delta_2 \alpha (m(x) - m_2)/\mu - \delta_1 \alpha (m(x) - m_1)/\mu\} = 1.$$

Hence $\phi_2 = \phi_1 \geq \theta_{\mu,\alpha}$ on $\partial\Omega_2 \cap \Omega \subset \Omega_1$. Therefore we have proved that $\mathcal{B}_2[\phi_2 - \theta_{\mu,\alpha}] \geq 0$ on $\partial(\Omega_2 \setminus D)$. By the strong maximum principle, we infer that $\phi_2 \geq \theta_{\mu,\alpha}$ in $\Omega_2 \setminus D$. Together with (9), we have shown $\phi_2 \geq \theta_{\mu,\alpha}$ in Ω_2.
Next, suppose for induction that

$$\phi_i \geq \theta_{\mu,\alpha} \quad \text{in } \Omega_i. \tag{10}$$

By (9), it remains to show that $\phi_{i+1} \geq \theta_{\mu,\alpha}$ in $\Omega_{i+1} \setminus D$. By (7), we have $N[\phi_{i+1} - \theta_{\mu,\alpha}] \geq 0$ in $\Omega_{i+1} \setminus D$. Define the boundary operator $\mathcal{B}_{i+1}$ by

$$\mathcal{B}_{i+1} u = \begin{cases} \mathcal{B}u & \text{on } \partial(\Omega_{i+1} \setminus D) \cap \partial\Omega, \\ u & \text{on } \partial(\Omega_{i+1} \setminus D) \setminus \partial\Omega. \end{cases}$$

We verify that $\mathcal{B}_{i+1}[\phi_{i+1} - \theta_{\mu,\alpha}] \geq 0$ in $\partial(\Omega_{i+1} \setminus D)$. Firstly, $\mathcal{B}_{i+1}[\phi_{i+1} - \theta_{\mu,\alpha}] \geq 0$ on $\partial(\Omega_{i+1} \setminus D) \cap \partial\Omega$ by (8). Secondly, observe that $\partial(\Omega_{i+1} \setminus D) \cap \Omega \subset (\Omega_{i+1} \cap \partial D) \cup (\partial\Omega_{i+1} \cap \Omega)$. And $\mathcal{B}_{i+1}[\phi_{i+1} - \theta_{\mu,\alpha}] = \phi_{i+1} - \theta_{\mu,\alpha} \geq 0$ in $\partial D \cap \Omega_{i+1}$ by (9).
Similar as before one can deduce that $m(x) = m_{i+1} - \eta$ in $\partial\Omega_{i+1} \cap \Omega$. Hence $\phi_{i+1} = \phi_i$ in $\partial\Omega_{i+1} \cap \Omega$ by construction. So by (10),

$$\mathcal{B}_{i+1}[\phi_{i+1} - \theta_{\mu,\alpha}] = \phi_{i+1} - \theta_{\mu,\alpha} = \phi_i - \theta_{\mu,\alpha} \geq 0 \qquad \text{on } \partial\Omega_{i+1} \cap \Omega \subset \Omega_i.$$

Therefore, as $\sigma[N; \mathcal{B}_{i+1}, \Omega_{i+1} \setminus D] > \sigma[N, \mathcal{B}, \Omega] = 0$, by the strong maximum principle, $\phi_{i+1} \geq \theta_{\mu,\alpha}$ in $\Omega_{i+1} \setminus D$ and hence in Ω_{i+1}, by (9).

In conclusion, $\phi_i \geq \theta_{\mu,\alpha}$ on Ω_i, $i = 1, ..., n$. Hence there exists $r_1 \in (0, r]$ such that

$$\text{for all } i, \quad \theta_{\mu,\alpha}(x) \leq M e^d e^{\delta^* \alpha (m(x) - m_i)/\mu} \quad \text{in } \cup_{z \in \mathfrak{M}_i} B_{r_1}(z), \tag{11}$$

$$\theta_{\mu,\alpha}(x) \le Me^d e^{-\delta^* \alpha r_1^2/(\mu K)} \quad \text{in } \Omega \setminus \cup_{z\in\mathfrak{M}} B_{r_1}(z). \tag{12}$$

Next we claim that M is bounded independent of $\mu > 0$ and $\alpha \ge \alpha_1$. Firstly, there exists $R_0 > 0$ such that for each $z \in \mathfrak{M}$,

$$d - \frac{\delta^*\alpha(m(x)-m_i)}{\mu} < d - \frac{\delta^*\alpha|x-z|}{\mu K} < -\log 2 \quad \text{in } B_{r_1}(z) \setminus B_{\sqrt{\frac{\mu}{\alpha}}R_0}(z).$$

Secondly, since $\alpha/\mu \to \infty$, we may assume $d - \frac{\delta^*\alpha r_1^2}{\mu K} < -\log 2$. Hence, by (11) and (12),

$$\theta_{\mu,\alpha}(x) \le \frac{M}{2} \qquad \text{in } \Omega \setminus \left(\cup_{z\in\mathfrak{M}} B_{\sqrt{\frac{\mu}{\alpha}}R_0}(z)\right)$$

and $M = \|\theta_{\mu,\alpha}\|_{L^\infty(\Omega)}$ must be assumed in $B_{\sqrt{\frac{\mu}{\alpha}}R_0}(z_{\mu,\alpha})$ for some $z_{\mu,\alpha} \in \mathfrak{M}$. Set $x = z_{\mu,\alpha} + \sqrt{\frac{\mu}{\alpha}}y$, then

$$\mu\left(\frac{\alpha}{\mu}\Delta_y\theta_{\mu,\alpha}\right) - \alpha\sqrt{\frac{\alpha}{\mu}}\nabla_x m \cdot \nabla_y\theta_{\mu,\alpha} + \theta_{\mu,\alpha}(m - \theta_{\mu,\alpha} - \alpha\Delta_x m) = 0.$$

Divide the above equation by α,

$$\Delta_y\theta_{\mu,\alpha} - \sqrt{\frac{\alpha}{\mu}}\nabla_x m \cdot \nabla_y\theta_{\mu,\alpha} + \left(\frac{m - \theta_{\mu,\alpha} - \alpha\Delta m}{\alpha}\right)\theta_{\mu,\alpha} = 0. \tag{13}$$

By applying the maximum principle to $\theta_{\mu,\alpha}$ and using $\frac{\partial m}{\partial n} \le 0$, we have $M = \|\theta_{\mu,\alpha}\|_{L^\infty(\Omega)} \le \|m\|_{L^\infty(\Omega)} + \alpha\|\Delta m\|_{L^\infty(\Omega)}$. The middle term $\sqrt{\alpha/\mu}\nabla_x m(z_{\mu,\alpha} + \sqrt{\frac{\mu}{\alpha}}y)$ in the above equation is bounded by $2\|D^2 m\|_{L^\infty(\Omega)}\|y\|$. Hence the coefficients of (13) are bounded in $L^\infty(B_{4R_0}(0))$. By the Harnack Inequality (Theorem 8.20, [17]), there exists a constant $c = c(N, R_0) > 0$ (N being the dimension) such that

$$\theta_{\mu,\alpha}(x) \ge cM \quad \text{in } B_{\sqrt{\frac{\mu}{\alpha}}R_0}(z_{\mu,\alpha}).$$

Hence

$$cM^2\left(\frac{\mu}{\alpha}\right)^{N/2} R_0^N \le \int_{B_{\sqrt{\frac{\mu}{\alpha}}R_0}(z_{\mu,\alpha})} \theta_{\mu,\alpha}^2 \le \int_\Omega \theta_{\mu,\alpha}^2. \tag{14}$$

Moreover, by (11) and (12),

$$\int_\Omega \theta_{\mu,\alpha} m \le \|m\|_{L^\infty(\Omega)} \int_\Omega \theta_{\mu,\alpha} \le CM\left(\frac{\mu}{\alpha}\right)^{N/2}. \tag{15}$$

Now integrating the equation of $\theta_{\mu,\alpha}$ to obtain

$$\int_\Omega \theta_{\mu,\alpha}^2 = \int_\Omega \theta_{\mu,\alpha} m. \tag{16}$$

Combining (14), (15) and (16) we infer

$$cM^2\left(\frac{\mu}{\alpha}\right)^{N/2} R_0^N \le CM\left(\frac{\mu}{\alpha}\right)^{N/2},$$

which gives the boundedness of M as $\alpha/\mu \to \infty$. This proves the theorem in the case $\mathfrak{M} = \mathfrak{M}_+$, i.e. $m(x) > 0$ for all $x \in \mathfrak{M}$. If it is not the case, assume

$$m_1 < m_2 < ... < m_{k-1} \le 0 < m_k < ... < m_n, \qquad \text{for some } k \ge 2.$$

Then define $\phi_0 = Me^d e^{\alpha(m(x)-\hat\eta)}/\mu$ where $-\hat\eta$ is a regular value of m chosen such that $\mathfrak{M} \cap [-\hat\eta, 0) = \emptyset$ and

$$0 < \hat\eta < \min\left\{\eta, \frac{\delta_k m_k}{2-\delta_k}\right\}. \tag{17}$$

Now consider $\Omega_0 = \{x \in \Omega : m < -\hat{\eta}\} \cup (\cup_{z\in\mathfrak{M}_0} B_r(z))$ where $\mathfrak{M}_0 := \{x \in \mathfrak{M} : m(x) = 0\}$ (possibly empty). Note that by similar considerations as before $\partial\Omega_0 \setminus \partial\Omega \subset \{x \in \Omega : m(x) = -\hat{\eta}\}$ and it is regular as $-\hat{\eta}$ is a regular value of m. Since $m \leq 0$ in Ω_0, it is easy to see that $N[\phi_0 - \theta_{\mu,\alpha}] \geq 0$ in Ω_0. Define

$$\mathcal{B}_0 u = \begin{cases} \mathcal{B}u & \text{on } \partial\Omega_0 \cap \partial\Omega, \\ u & \text{on } \partial\Omega_0 \setminus \partial\Omega. \end{cases}$$

Then $\mathcal{B}_0[\phi_0 - \theta_{\mu,\alpha}] = \mathcal{B}[\phi_0 - \theta_{\mu,\alpha}] = 0$ on $\partial\Omega_0 \cap \partial\Omega$ by simple calculation, and on $\partial\Omega_0 \cap \Omega \subset \{x \in \Omega : m(x) = -\hat{\eta}\} \cap \Omega_k$,

$$\begin{aligned} \phi_0 &= Me^d e^{\alpha(-\hat{\eta}-\hat{\eta})/\mu} \\ &> Me^d e^{\delta_k \alpha(-\hat{\eta}-m_k)/\mu} \qquad \text{by (17)} \\ &= \phi_k \geq \theta_{\mu,\alpha}. \end{aligned}$$

Therefore, by the strong maximum principle, $\phi_0 - \theta_{\alpha,\mu} \geq 0$ in Ω_0. Hence the proposition is proved.

□

Remark 5. Corollary 1 follows by observing that in the proof of Theorem 2.2, if (M1) is replaced by (M2), then it is sufficient to have $\alpha > 0$ bounded away from zero in (7) and that $\delta^* = \delta_1 < 1$ can be chosen arbitrary close to 1.

Consider the following variation of (1) with the local intrinsic growth rate $p(x)$.

$$\begin{cases} \nabla \cdot (\mu\nabla u - \alpha u \nabla m) + u(p - u) = 0 & \text{in } \Omega, \\ \mu\frac{\partial u}{\partial n} - \alpha u \frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

The argument in the proof of Theorem 2.2 can actually yield

Theorem 2.3. *There exists positive constants α, C, r, γ and $\delta^* < 1$ such that if a positive solution $\tilde{\theta}_{\mu,\alpha}$ exists for all $\mu > 0$ and $\alpha \geq \alpha_1$, then*

$$\tilde{\theta}_{\mu,\alpha} \leq \begin{cases} Ce^{\alpha\delta^*[m(x)-m(x_0)]/\mu} & \text{in } B_r(x_0), \text{ for any } x_0 \in \mathfrak{M}_+, \\ e^{-\gamma\alpha/\mu} & \text{in } \Omega \setminus \cup_{x_0\in\mathfrak{M}_+} B_r(x_0). \end{cases}$$

In the next subsection, we see that if $p = \chi(m)$ for some increasing positive χ in (18), then $\tilde{\theta}_{\mu,\alpha}$ exists. Moreover, $\tilde{\theta}_{\mu,\alpha} > p$ in $\mathfrak{M}$ for large values of α/μ.

2.2. **Lower bound.** Consider the equation

$$\begin{cases} \nabla \cdot (\mu\nabla u - \alpha u \nabla m) + u(\chi(m) - u) = 0 & \text{in } \Omega, \\ \mu\frac{\partial u}{\partial n} - \alpha u \frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where χ satisfies $\chi' > 0$ and $\int_\Omega \chi(m) > 0$. Then there exists a unique coexistence state $\tilde{\theta}_{\mu,\alpha}$ of (19) which is globally asymptotically stable. (See, e.g. Theorem 10, [4].) The following is a generalization of a result in [4], where it was proved for non-degenerate local maximum points and $\mu = 1$. We are going to show that the same result holds for any kind of local maximum, whenever the ratio α/μ is large.

Theorem 2.4. *Suppose $m \in C^2(\bar{\Omega})$ assumes a positive local maximum value M in a (closed) set $\Omega_M \subset\subset \Omega$. More precisely, let $\Omega_M^\epsilon := \{x \in \Omega : \mathrm{dist}(x, \Omega_M) < \epsilon\}$, we have*

$$m(x) \begin{cases} = M & \text{in } \Omega_M, \\ \in (M/2, M) & \text{in } \Omega_M^\epsilon \setminus \Omega_M. \end{cases}$$

Then for all $K_1 > 0$, *there exists* $K_2 > 0$ *such that whenever* $0 < \mu \le K_1$ *and* $\alpha/\mu \ge K_2$, *then*

$$\tilde{\theta}_{\mu,\alpha}(x) > \chi(M)e^{\alpha[m(x)-M]/\mu} \qquad \text{for all } x \in \Omega_M^{\epsilon/2},$$

where $\tilde{\theta}_{\mu,\alpha}$ *is the unique coexistence state of* (19).

Corollary 2. [4] *If* x_0 *is a non-degenerate local maximum point of* m *and* $\chi(m) = m$, *then given any* $K_1 > 0$, *there exists* $K_2, r > 0$ *such that whenever* $0 < \mu \le K_1$ *and* $\alpha/\mu \ge K_2$, *we have*

$$\theta_{\mu,\alpha}(x) > m(x_0)e^{\alpha[m(x)-m(x_0)]/\mu} \qquad \text{for all } x \in B_r(x_0),$$

where $\theta_{\mu,\alpha}$ *is the unique positive steady state of* (2).

In particular, it shows that $\theta_{\mu,\alpha}(x) > m(x)$ at every local maximum point of m.

Proof of Theorem 2.4. We define $\bar{u}$ to be an upper solution of (19) if

$$\begin{cases} \nabla \cdot (\mu \nabla \bar{u} - \alpha \bar{u} \nabla m) + \bar{u}(\chi(m) - \bar{u}) \le 0 & \text{in } \Omega, \\ \mu \frac{\partial \bar{u}}{\partial n} - \alpha \bar{u} \frac{\partial m}{\partial n} \ge 0 & \text{on } \partial\Omega. \end{cases}$$

We define similarly $\underline{u}$ to be a lower solution by reversing the above inequalities. We have the following lemma concerning the sublinear character of (19).

Lemma 2.5. *If* $\bar{u} \ge 0$ *such that* $\bar{u} \not\equiv 0$ *(resp.* $\underline{u} \ge 0$*) is an upper solution (resp. lower solution) of* (19), *then* $\bar{u} \ge \underline{u}$ *in* Ω.

Proof. Let $w := e^{-\alpha m/\mu}(\bar{u} - \tilde{\theta}_{\mu,\alpha})$, then w satisfies

$$\mathcal{N}_1[w] := \mu\nabla\cdot(e^{\alpha m/\mu}\nabla w) + e^{\alpha m/\mu}(\chi(m) - \bar{u} - \tilde{\theta}_{\mu,\alpha})w \le 0 \text{ in } \Omega \text{ and } \mathcal{B}_n w = 0 \text{ on } \partial\Omega,$$

where $\mathcal{B}_n = \frac{\partial w}{\partial n}$ is the Neumann boundary operator on $\partial\Omega$. Now $w' = e^{\alpha m/\mu}\theta_{\mu,\alpha}$ satisfies

$$\mathcal{N}_2[w'] := \mu\nabla\cdot(e^{\alpha m/\mu}\nabla w') + e^{\alpha m/\mu}(\chi(m) - \tilde{\theta}_{\mu,\alpha})w' = 0 \text{ in } \Omega \text{ and } \mathcal{B}_n w' = 0 \text{ on } \partial\Omega.$$

By the notations in the proof of Theorem 2.2, we have by comparison, $\sigma[\mathcal{N}_1, \mathcal{B}_n, \Omega] > \sigma[\mathcal{N}_2, \mathcal{B}_n, \Omega] = 0$. It follows by the strong maximum principle that $w \ge 0$, i.e. $\bar{u} \ge \tilde{\theta}_{\mu,\alpha}$. Similarly, $\underline{u} \le \tilde{\theta}_{\mu,\alpha}$ and the lemma is proved. □

The following lemma is a direct consequence of Lemma 2.5.

Corollary 3. *Let* (u, v) *be a coexistence steady state of* (1), *then* $\theta_{\mu,\alpha}$ *and* $\theta_{\nu,\beta}$ *exist and* $u \le \theta_{\mu,\alpha}$ *and* $v \le \theta_{\nu,\beta}$ *in* Ω.

Since $\tilde{\theta}_{\mu,\alpha}$ is the unique positive solution of (19), it suffices to construct a lower solution $\underline{\theta}$. Let $\bar{\delta} > 0$ be chosen small such that $M - \bar{\delta}$ is a regular value of m and satisfies

$$\sup_{\partial\Omega_M^\epsilon} m < M - \bar{\delta}, \qquad \chi(M - \bar{\delta}) > 0.$$

Since $M - \bar{\delta}$ is a regular value of m, consider

$$O_1 = \{x \in \Omega_M^\epsilon : m(x) > M - \bar{\delta}\}.$$

Then ∂O_1 is a smooth $(N-1)-$dimensional manifold and the unit outer normal $n = -\nabla m/\|\nabla m\|$ is well-defined on ∂O_1, so

$$\frac{\partial m}{\partial n} = \nabla m \cdot n = -|\nabla m| < 0 \qquad \text{on } \partial O_1.$$

Therefore, there exists $\underline{\delta} \in (0, \bar{\delta})$ and $O_2 := \{x \in O_1 : M - \bar{\delta} < m(x) < M - \underline{\delta}\}$ such that

$$\nabla m \neq 0 \quad \text{in } \bar{O}_2.$$

Define a smooth cut-off $\rho : \mathbf{R} \to \mathbf{R}$ by

$$\rho(t) = \begin{cases} 1 & t \geq M - \underline{\delta} \\ 0 & t \leq M - \bar{\delta} \end{cases}$$

and satisfies in addition

$$\rho, \rho' > 0 \text{ in } (M - \bar{\delta}, M - \underline{\delta}) \qquad \text{and} \qquad \rho'' > 0 \text{ in } \left(M - \bar{\delta}, M - \frac{\bar{\delta} + \underline{\delta}}{2}\right).$$

Define

$$G(x) := \begin{cases} 1 & \text{in } O_1 \setminus O_2, \\ \rho(m(x)) & \text{in } O_2, \\ 0 & \text{in } \Omega \setminus O_1 \end{cases} \tag{20}$$

and

$$\underline{\theta}(x) := Me^{\alpha[m(x)-M]/\mu} G(x).$$

Then $\underline{\theta} \in C^2(\bar{\Omega})$. Now we calculate

$$\begin{aligned} &\mu \nabla \underline{\theta} - \alpha \underline{\theta} \nabla m \\ =& Me^{\alpha[m(x)-M]/\mu} \alpha (\nabla m) G + \mu M e^{\alpha[m(x)-M]/d} \nabla G - \alpha M e^{\alpha[m(x)-M]/\mu} (\nabla m) G \\ =& \mu M e^{\alpha[m(x)-M]/\mu} \nabla G. \end{aligned}$$

Then,

$$\begin{aligned} &\nabla \cdot (\mu \nabla \underline{\theta} - \alpha \underline{\theta} \nabla m) + \underline{\theta}(\chi(m) - \underline{\theta}) \\ =& \nabla \cdot (\mu M e^{\alpha[m(x)-M]/\mu} \nabla G) + Me^{\alpha[m(x)-M]/\mu} (\chi(m) - \underline{\theta}) G \\ =& \alpha M e^{\alpha[m(x)-M]/\mu} \nabla G \cdot \nabla m + \mu M e^{\alpha[m(x)-M]/\mu} \Delta G + Me^{\alpha[m(x)-M]/\mu} (\chi(m) - \underline{\theta}) G \\ =& Me^{\alpha[m(x)-M]/\mu} \{\alpha \nabla G \cdot \nabla m + \mu \Delta G + G(\chi(m) - \underline{\theta})\}. \end{aligned}$$

In a neighborhood of the boundary $\partial\Omega$, $\underline{\theta}$ is identically zero and the boundary condition for lower solution is satisfied automatically. It suffices to show that

$$\alpha \nabla G \cdot \nabla m + \mu \Delta G + (\chi(m) - \underline{\theta}) G \geq 0. \tag{21}$$

Denote

$$g(t) = \frac{e^{\alpha t/\mu}}{\chi(t)} \qquad \text{with} \qquad g'(t) = e^{\alpha t/\mu} \left(\frac{\alpha}{\mu \chi(t)} - \frac{\chi'(t)}{\chi(t)^2} \right),$$

then $g(t)$ is increasing in $\chi^{-1}\{[M - \bar{\delta}, M]\}$ if $\alpha/\mu > \sup_{[M-\bar{\delta},M]} \chi'/\chi$. Hence if $x \in O_1$, then

$$\frac{e^{\alpha m(x)/\mu}}{\chi(m(x))} \leq \frac{e^{\alpha M/\mu}}{\chi(M)},$$

which implies

$$\chi(m(x)) \geq \chi(M) e^{\alpha[m(x)-M]/\mu} \geq \chi(M) e^{\alpha[m(x)-M]/\mu} G(x) = \underline{\theta}.$$

Therefore (21) is satisfied automatically when $G \equiv 0, 1$. By (20) it suffices to show (21) in O_2. Now,

$$\alpha \nabla G \cdot \nabla m = \alpha \rho' |\nabla m|^2 \geq 0, \qquad \Delta G = \rho'' |\nabla m|^2 + \rho' \Delta m.$$

Choose $\delta' \in (\underline{\delta}, \bar{\delta})$ such that $\inf_{M-\delta' \le m \le M-\underline{\delta}} \rho > 1/2$ and

$$\sup_{M-\delta' \le m \le M-\underline{\delta}} \rho' |\Delta m| < \frac{\chi(M-\bar{\delta})}{6\mu}, \qquad \sup_{M-\delta' \le m \le M-\underline{\delta}} |\rho''||\nabla m| < \frac{\chi(M-\bar{\delta})}{6\mu}.$$

Note that δ' depends on K_1, but is independent of $0 < \mu \le K_1$. Then, on the one hand, in $\{x \in O_2 : M - \delta' \le m(x) \le M - \underline{\delta}\}$, since $0 \le \frac{1}{2}\underline{\theta} \le \frac{1}{2} M e^{-\alpha \underline{\delta}/\mu} \to 0$, $\chi(m) - \underline{\theta} > 0$,

$$\begin{aligned}
&\alpha \nabla G \cdot \nabla m + \mu \Delta G + (\chi(m) - \underline{\theta}) G \\
&\ge 0 + \mu \rho'' |\nabla m|^2 + \mu \rho' \Delta m + \frac{1}{2}(\chi(M-\bar{\delta}) - \underline{\theta}) \\
&\ge \left[\frac{\chi(M-\bar{\delta})}{6} - \mu |\rho''||\nabla m|^2\right] + \left[\frac{\chi(M-\bar{\delta})}{6} - \mu \rho' |\Delta m|\right] + \left[\frac{\chi(M-\bar{\delta})}{6} - \frac{1}{2}\underline{\theta}\right] \\
&\ge 0
\end{aligned}$$

whenever $0 < \mu \le K_1$ and $\alpha/\mu \ge K_2$ for some K_2. On the other hand, for $\{x \in O_2 : M - \bar{\delta} \le m \le M - \delta'\}$, we have

$$\frac{\alpha \rho'}{2} + \mu \rho'' \ge 0 \qquad \text{and} \qquad \frac{\alpha |\nabla m|^2}{2} + \mu \Delta m \ge 0$$

for α/μ large. Then

$$\begin{aligned}
&\alpha \nabla G \cdot \nabla m + \mu \Delta G + (m - \underline{\theta}) G \\
&\ge \alpha \rho' |\nabla m|^2 + \mu \rho'' |\nabla m|^2 + d \rho' \Delta m + 0 \\
&= |\nabla m|^2 \left(\frac{\alpha \rho'}{2} + \mu \rho''\right) + \rho' \left(\frac{\alpha}{2} |\nabla m|^2 + \mu \Delta m\right) \\
&\ge 0
\end{aligned}$$

for α/μ large. □

3. Local and global stability of $(\theta_{\mu,\alpha}, 0)$ and $(0, \theta_{\nu,\beta})$ as $\mu \to \infty$.

In this section we discuss the stability of semi-trivial steady states $(\theta_{\mu,\alpha}, 0)$ and $(0, \theta_{\nu.\beta})$ of (1) and also the existence and non-existence of coexistence states for sufficiently large μ. We first prove the following result which is a generalized version of Lemma 2.1 of [22]:

Lemma 3.1. *For any $\alpha \ge 0$,*

$$\lim_{\mu \to \infty} \theta_{\mu,\alpha} = \overline{m} := \frac{1}{|\Omega|} \int_\Omega m \qquad \text{in } C^{2,\gamma}(\bar{\Omega}).$$

Proof. For each α, $\theta_{\mu,\alpha}$ is bounded in $L^\infty(\Omega)$ as $\mu \to \infty$ by (4). Since

$$\begin{cases} \Delta \theta_{\mu,\alpha} - \frac{1}{\mu}\left[\alpha \nabla \theta_{\mu,\alpha} \cdot \nabla m + \theta_{\mu,\alpha}(m(x) - \theta_{\mu,\alpha} - \alpha \Delta m)\right] = 0 & \text{in } \Omega, \\ \frac{\partial \theta_{\mu,\alpha}}{\partial n} - \frac{\alpha}{\mu} \theta_{\mu,\alpha} \frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

it follows by standard elliptic estimates [17] and (16) that $\theta_{\mu,\alpha} \to \overline{m}$ in $C^{2,\gamma}(\bar{\Omega})$. □

Lemma 3.2. *Given any $\alpha \ge 0$, $\beta > 0$ and $\nu > 0$, $(\theta_{\mu,\alpha}, 0)$ is unstable for all μ sufficiently large.*

Proof. It suffices to consider the following eigenvalue problem

$$\begin{cases} \nabla\cdot(\nu\nabla\psi_1 - \beta\psi_1\nabla m) + (m-\theta_{\mu,\alpha})\psi_1 = -\lambda_1\psi_1 & \text{in } \Omega, \\ \nu\frac{\partial\psi_1}{\partial n} - \beta\psi_1\frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

where λ_1 is the principal eigenvalue with positive eigenfunction ψ_1. Let $\varphi = e^{-\beta m/\nu}\psi_1$, then (22) becomes

$$\begin{cases} \nu\nabla\cdot(e^{\beta m/\nu}\nabla\varphi) + e^{\beta m/\nu}(m-\theta_{\mu,\alpha})\varphi = -\lambda_1 e^{\beta m/\nu}\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

Divide (23) by φ and integrate over Ω, we have, after integration by parts,

$$\nu\int_\Omega \frac{e^{\beta m/\nu}|\nabla\varphi|^2}{\varphi^2} + \int_\Omega e^{\beta m/\nu}(m-\theta_{\mu,\alpha}) = -\lambda_1\int_\Omega e^{\beta m/\nu}.$$

Since $\int_\Omega \frac{e^{\beta m/\nu}|\nabla\varphi|^2}{\varphi^2} > 0$ and $\theta_{\mu,\alpha} \to \overline{m} = \frac{1}{|\Omega|}\int_\Omega m$ by Lemma 3.1, it remains to show

$$\lim_{\mu\to\infty}\int_\Omega e^{\beta m/\nu}(m-\theta_{\mu,\alpha}) = \int_\Omega e^{\beta m/\nu}(m-\overline{m}) > 0. \tag{24}$$

To establish our assertion, set $f(t) = \int_\Omega e^{t(m-\overline{m})}(m-\overline{m})$. Then, $f(0)=0$, and

$$f'(t) = \int_\Omega e^{t(m-\overline{m})}(m-\overline{m})^2 > 0$$

for all t. Therefore, $\int_\Omega e^{\beta m/\nu}(m-\overline{m}) = e^{\beta\overline{m}/\nu}f(\beta/\nu) > 0$.
In conclusion, for μ sufficiently large, we have $\lambda_1 < 0$; i.e. $(\theta_{\mu,\alpha}, 0)$ is unstable. □

The local stability of $(0, \theta_{\nu,\beta})$ is determined by the following eigenvalue problem

$$\begin{cases} \nabla\cdot(\mu\nabla\phi - \alpha\phi\nabla m) + (m-\theta_{\nu,\beta})\phi = -\lambda\phi & \text{in } \Omega, \\ \mu\frac{\partial\phi}{\partial n} - \alpha\phi\frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{25}$$

We denote $\tilde{\lambda}_1$ to be the principal eigenvalue to (25) and ϕ_1 to be the positive eigenfunction.

Lemma 3.3. *For any $\nu > 0$, $\alpha, \beta \geq 0$,*

$$\lim_{\mu\to\infty}\tilde{\lambda}_1 = \frac{1}{|\Omega|}\int_\Omega(\theta_{\nu,\beta} - m).$$

Proof. Integrating (25), we have

$$-\tilde{\lambda}_1\int_\Omega\phi_1 = \int_\Omega(m-\theta_{\nu,\beta})\phi_1. \tag{26}$$

This implies that $|\tilde{\lambda}_1| \leq \|m-\theta_{\nu,\beta}\|_{L^\infty(\Omega)}$. If we normalize $\|\phi_1\|_{L^\infty(\Omega)} = 1$, similar to Lemma 3.1, we see that $\phi_1 \to 1$ in $C^2(\overline{\Omega})$. Hence the lemma follows from (26). □

Proof of Theorem 1.3. By Corollary 1 and Remark 4, there exists $\nu_1 > 0$ small such that if $\nu \in (0, \nu_1)$, $\int_\Omega \theta_{\nu,\beta} < \int_\Omega m$, which implies the instability of $(0, \theta_{\nu,\beta})$ by Lemma 3.3. By Lemma 3.2, $(\theta_{\mu,\alpha}, 0)$ is also unstable. By the theory of monotone dynamical system [15, 20, 21, 27, 29], this guarantees the existence of at least one stable coexistence state of (1). Remark 1 follows by invoking Theorem 2.2 in place of Corollary 1.

□

Definition 3.4. For each $\kappa \in [0, \bar{m}]$, define $\tilde{v}_\kappa$ to be the unique positive solution to

$$\begin{cases} \nabla \cdot (\nu \nabla \tilde{v} - \beta \tilde{v} \nabla m) + \tilde{v}(m - \kappa - \tilde{v}) = 0 & \text{in } \Omega, \\ \nu \frac{\partial \tilde{v}}{\partial n} - \beta \tilde{v} \frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

We first establish the following result:

Lemma 3.5. *For any $\beta \geq 0$, there exists $\nu_2 > 0$ such that for any $\kappa \in [0, \overline{m}]$,*

$$\int_\Omega \tilde{v}_\kappa > \left(\int_\Omega m \right) - |\Omega|\kappa \quad \text{for all } \nu \geq \nu_2,$$

where $\bar{m} = \frac{1}{|\Omega|}\int_\Omega m$ and $\tilde{v}_\kappa$ is defined in Definition 3.4. Moreover,

$$\int_\Omega (m - \kappa - \tilde{v}_\kappa) = 0.$$

In particular, taking $\kappa = 0$, we have $\int_\Omega \theta_{\nu,\beta} > \int_\Omega m$, which, in view of Lemmas 3.3 and 3.5, gives the following stability result of $(0, \theta_{\nu,\beta})$.

Corollary 4. *For any $\alpha, \beta \geq 0$, there exists $\nu_2 > 0$ such that if $\nu \geq \nu_2$, then $(0, \theta_{\nu,\beta})$ is stable for all μ sufficiently large.*

Proof of Lemma 3.5. Divide (27) by $e^{-\beta m/\nu}\tilde{v}_\kappa$ and integrate by parts, we have, for any $\nu > 0$,

$$\int_\Omega e^{\beta m/\nu}(m - \kappa - \tilde{v}_\kappa) = -\frac{1}{\nu}\int_\Omega e^{\beta m/\nu}|\nu \nabla \tilde{v}_\kappa - \beta \tilde{v}_\kappa \nabla m|^2 < 0. \tag{28}$$

If $\beta = 0$, then we are done. Now assume $\beta > 0$ and let

$$\tilde{v}_\kappa = \bar{m} - \kappa + \frac{1}{\nu} v_1(\kappa) + O(\frac{1}{\nu^2}).$$

Then by (28),

$$\begin{aligned} 0 &> \int_\Omega \left(1 + \frac{\beta}{\nu} m + O(\frac{1}{\nu^2})\right)\left(m - \kappa - \overline{m} + \kappa - \frac{1}{\nu} v_1(\kappa) + O(\frac{1}{\nu^2})\right) \\ &= \frac{1}{\nu}\int_\Omega [-v_1(\kappa) + \beta m(m - \overline{m})] + O(\frac{1}{\nu^2}) \\ &= \frac{1}{\nu}\int_\Omega \left[-v_1(\kappa) + \beta(m - \overline{m})^2\right] + O(\frac{1}{\nu^2}) \end{aligned}$$

where we used $\int_\Omega (m - \overline{m}) = 0$ in the last equality. This implies $\int_\Omega v_1(\kappa) \geq \beta \int_\Omega (m - \overline{m})^2 > 0$. Hence, there exists $\nu_2 > 0$ such that $\int_\Omega \tilde{v}_\kappa > \int_\Omega m - |\Omega|\kappa$ for all $\kappa \in [0, \overline{m}]$ and for all $\nu \geq \nu_2$. □

Our next lemma concerns the limiting behavior of every coexistence steady state of (1) as $\mu \to \infty$.

Lemma 3.6. *For any coexistence steady state (u, v) of (1) with $\alpha, \beta \geq 0$, by passing to a subsequence, when $\mu \to \infty$,*

$$\|u - \kappa\|_{C^2(\overline{\Omega})} \to 0 \quad \text{and} \quad \|v - \tilde{v}_\kappa\|_{C^2(\overline{\Omega})} \to 0$$

for some $\kappa \in [0, \overline{m})$ and $\tilde{v}_\kappa$ is defined in Definition 3.4.

Proof. By Corollary 3 and (4), we have

$$\|u\|_{L^\infty(\Omega)} \le e^{\alpha\|m\|_{L^\infty(\Omega)}/\mu}\|m\|_{L^\infty(\Omega)} \ \text{ and } \ \|v\|_{L^\infty(\Omega)} \le e^{\beta\|m\|_{L^\infty(\Omega)}/\nu}\|m\|_{L^\infty(\Omega)}.$$

As in Lemma 3.1, by passing to a subsequence, we may assume that u converge to a constant $\kappa \in [0, \|m\|_{L^\infty(\Omega)}]$ in $C^2(\bar{\Omega})$ as $\mu \to \infty$. Since v is the positive solution to the equation

$$\begin{cases} \nabla\cdot(\nu\nabla v - \beta v\nabla m) + v(m-u-v) = 0 & \text{in } \Omega,\\ \nu\frac{\partial v}{\partial n} - \beta v\frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

we see that $v \to \tilde{v}_\kappa$ in $C^2(\bar{\Omega})$ as $\mu \to \infty$, where $\tilde{v}_\kappa$ is defined in Definition 3.4. Denote $u_1 = u/\|u\|_{L^\infty(\Omega)}$, then $u_1 \to 1$ in $C^2(\bar{\Omega})$. If we divide the equation for u by $\|u\|_{L^\infty(\Omega)}$ and integrate, we see that $\int_\Omega u_1(m-u-v) = 0$. By passing to the limit, we infer that $\int_\Omega(m-\kappa-\tilde{v}_\kappa) = 0$. This implies $\kappa \le \overline{m}$. Now $\kappa \ne \bar{m}$, since $\int_\Omega(m-\overline{m}-v_{\overline{m}}) = -\int_\Omega v_{\overline{m}} < 0$. Therefore $\kappa \in [0, \overline{m})$. □

Proof of Theorem 1.2. By Lemma 3.2, $(\theta_{\mu,\alpha}, 0)$ is unstable when μ is sufficiently large. Given $\alpha > 0$ and $\beta > 0$, if $\nu \ge \nu_2$, then by Corollary 4, $(0, \theta_{\nu,\beta})$ is locally stable for all μ large. It suffices to show that for any $\nu \ge \nu_2$, there is no coexistence steady state of (1) for all μ large. This would imply that $(0, \theta_{\nu,\beta})$ is globally asymptotically stable. Assume to the contrary that for a sequence $\mu = \mu_k \to \infty$, there exists coexistence steady state (u, v) of (1) with $\mu = \mu_k$. Then by Lemma 3.6 we may assume that there exists $\kappa \in [0, \overline{m})$ such that $u \to \kappa$ and $v \to \tilde{v}_\kappa$ as $\mu_k \to \infty$. Moreover, we have

$$\int_\Omega (m - \kappa - \tilde{v}_\kappa) = 0,$$

which is a contradiction to the choice of ν_2 and Lemma 3.5. Theorem 1.2 is proved. □

4. Local and global stability of $(\theta_{\mu,\alpha}, 0)$ and $(0, \theta_{\nu,\beta})$ as $\mu \to 0$.

In this section we discuss the stability of steady states $(\theta_{\mu,\alpha}, 0)$ and $(0, \theta_{\nu.\beta})$ of (1) and also the existence and non-existence of coexistence states for sufficiently small μ.

Lemma 4.1. *[Local stability of $(\theta_{\mu,\alpha}, 0)$ when $\mu \to 0$.]*

(a) *Assume* (M1). *There exists a positive constant α_1 such that for any $\nu > 0$, $\alpha \ge \alpha_1$ and $\beta > 0$, $(\theta_{\mu,\alpha}, 0)$ is unstable for sufficiently small μ.*
(b) *Assume* (M2). *For any $\nu > 0$, $\alpha > 0$, $\beta > 0$, $(\theta_{\mu,\alpha}, 0)$ is unstable for small μ.*

Proof. The local stability of $(\theta_{\mu,\alpha}, 0)$ depends on (22). Let λ_1 be the principal eigenvalue of (22). Divide (22) by $e^{-\beta m/\nu}\psi_1$ and integrate, we have

$$\int_\Omega e^{\beta m/\nu}(m - \theta_{\mu,\alpha}) + \lambda_1 \int_\Omega e^{\beta m/\nu} = -\int_\Omega \frac{e^{\beta m/\nu}|\nu\nabla\psi_1 - \beta\psi_1\nabla m|^2}{\nu\psi_1^2} < 0.$$

As we assume either (M1) with $\alpha \ge \alpha_1$ or (M2) with $\alpha > 0$, $\|\theta_{\mu,\alpha}\|_{L^1(\Omega)} \to 0$ as $\mu \to 0$. Therefore, $\lambda_1 < 0$ as $\mu \to 0$. □

Lemma 4.2. *[Local stability of $(0, \theta_{\nu,\beta})$ when $\mu \to 0$.] Given $\alpha > 0$ and $\beta > 0$,*

(a) *if $\nu \ge (\sup m)\beta$, $(0, \theta_{\nu,\beta})$ is unstable for sufficiently small μ;*
(b) *assuming* (M2), *if $0 < \nu \le (\inf m)\beta$, $(0, \theta_{\nu,\beta})$ is stable for sufficiently small μ;*
(c) *assuming* (M1), *there exists constants $K, \alpha_2 > 0$ (depending on β) such that for all $\alpha \ge \alpha_2$ and $0 < \nu \le K$, $(0, \theta_{\nu,\beta})$ is stable for sufficiently small μ.*

Proof. Let $\tilde{\lambda}_1$ be the principal eigenvalue of (25). Recall the following result from [14]:

Theorem 4.3. *Suppose that $\nabla m \neq 0$ on $\partial\Omega$. Let $\alpha > 0$ be a fixed positive constant. Consider the following eigenvalue problem*

$$\begin{cases} \nabla \cdot (\mu \nabla \phi - \alpha \phi \nabla m) - V\phi + \lambda \phi = 0 & \text{in } \Omega, \\ \mu \frac{\partial \phi}{\partial n} - \alpha \phi \frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

Then the principal eigenvalue of (29), denoted by $\tilde{\lambda}_1$, satisfies

$$\lim_{\mu \to 0} \tilde{\lambda}_1 = \inf_{\mathfrak{M} \cup \Sigma_0} \left\{ V(x) + \frac{\alpha}{2} \sum_{i=1}^{N} (|\kappa_i(x)| + \kappa_i(x)) \right\}$$

where $\mathfrak{M}$ is the set of local maximum points of m,

$$\Sigma_0 = \{x \in \Omega : |\nabla m| = 0 \text{ and } x \notin \mathfrak{M}\}$$

and $\{\kappa_i(x)\}_{i=1}^N$ are eigenvalues of $D^2 m(x)$.

Let x_0 be chosen such that $m(x_0) = \sup_\Omega m$, then by Theorem 4.3,

$$\begin{aligned} \lim_{\mu \to 0} \tilde{\lambda}_1 &= \inf_{\mathfrak{M} \cup \Sigma_0} \left\{ \theta_{\nu,\beta}(x) - m(x) + \frac{\alpha}{2} \sum_{i=1}^{N} (|\kappa_i(x)| + \kappa_i(x)) \right\} \\ &\leq \theta_{\nu,\beta}(x_0) - m(x_0) < 0. \end{aligned}$$

The last inequality follows from Lemma 2.1 (a) and that $\nu \geq (\sup m)\beta$. This shows the instability of $(0, \theta_{\nu,\beta})$ and proves (a).

Next, assume (M2). By Lemma 2.1 (b), whenever $\nu, \beta > 0$ satisfies $\nu \leq (\inf m)\beta$, then let x_0 be the unique critical point,

$$\theta_{\nu,\beta}(x) > m(x_0) e^{\beta[m(x) - m(x_0)]/\nu}$$

in $B_r(x_0)$. This in particular implies that $\theta_{\nu,\beta}(x_0) > m(x_0)$ and hence

$$\lim_{\mu \to 0} \tilde{\lambda}_1 = \inf_{\mathfrak{M} \cup \Sigma_0} \left\{ \tilde{\theta}_{\nu,\beta}(x) - m(x) + \frac{\alpha}{2} \sum_{i=1}^{N} (|\kappa_i(x)| + \kappa_i(x)) \right\} = \theta_{\nu,\beta}(x_0) - m(x_0) > 0.$$

This shows (b).

Finally, assume (M1). By Corollary 2, fix $K_1 = \beta$, there exists $K, r > 0$ such that whenever $\nu, \beta > 0$ satisfies $\nu \leq \min\{\beta/K, \beta\}$, then for any $x_0 \in \mathfrak{M}$,

$$\theta_{\nu,\beta}(x) > m(x_0) e^{\beta[m(x) - m(x_0)]/\nu} \qquad \text{in } B_r(x_0).$$

This implies that $\theta_{\nu,\beta}(x_0) > m(x_0)$ for all $x_0 \in \mathfrak{M}$. Now choosing

$$\alpha_2 > \sup_{\Sigma_0} \frac{2 \max\{m(x) - \theta_{\nu,\beta}(x), 0\}}{\displaystyle\sum_{i=1}^{N} (|\kappa_i(x)| + \kappa_i(x))}, \tag{30}$$

then for all $\alpha \geq \alpha_2$,

$$\lim_{\mu \to 0} \tilde{\lambda}_1 = \inf_{\mathfrak{M} \cup \Sigma_0} \left\{ \tilde{\theta}_{\nu,\beta}(x) - m(x) + \frac{\alpha}{2} \sum_{i=1}^{N} (|\kappa_i(x)| + \kappa_i(x)) \right\} > 0.$$

This proves (c). ∎

Proof of Theorem 1.4. The instability of $(\theta_{\mu,\alpha},0)$ and $(0,\theta_{\nu,\beta})$ are proved in Lemmas 4.1 and 4.2, respectively. The existence of at least one stable coexistence state (u,v) of (1) follows from the theory of monotone dynamical system. $\square$

We have the following limiting profile of coexistence state of (1).

Lemma 4.4. *Assume* (M2) *and* $\alpha,\beta,\nu>0$. *Let* (u,v) *be any coexistence state of* (1). *For all* $\delta\in(0,1)$ *and* $\epsilon>0$, *there exists* $C>0$ *such that for all* $\mu>0$,

$$u(x)\le Ce^{\delta\alpha[m(x)-m(x_0)]/\mu}$$

in Ω. *Furthermore, for any* $\gamma\in(0,1)$, $v\to\theta_{\nu,\beta}$ *in* $C^{1,\gamma}(\bar\Omega)$ *as* $\mu\to0$.

Lemma 4.5. *Assume* (M1), $\alpha\ge\alpha_1$ *and* $\beta,\nu>0$. *Let* (u,v) *be any coexistence state of* (1). *Then for any* $r>0$ *small, there exists positive constants* C *and* $\delta^*<1$ *such that for any* $\mu>0$,

$$u(x)\le\begin{cases} Ce^{\delta^*\alpha[m(x)-m(x_0)]/\mu} & \text{in } B_r(x_0),\ \text{for any } x_0\in\mathfrak{M},\\ e^{-\delta^*\alpha/\mu} & \text{in } \Omega\setminus\cup_{z\in\mathfrak{M}}B_r(z).\end{cases}$$

Furthermore, for any $\gamma\in(0,1)$, $v\to\theta_{\nu,\beta}$ *in* $C^{1,\gamma}(\bar\Omega)$ *as* $\mu\to0$. *Here* α_1 *is defined in Theorem* 2.2.

Proof of Lemmas 4.4 and 4.5. The upper estimate of Lemma 4.4 follows from Corollary 1 and Corollary 3. Now since $u\to0$ in $L^p(\Omega)$ for any $p>1$, and is uniformly bounded as $\mu\to0$, by standard elliptic estimates, $v\to\theta_{\nu,\beta}$ in $C^{1,\gamma}(\bar\Omega)$ for all $\gamma\in(0,1)$.

Lemma 4.5 can be proved similarly, with Theorem 2.2 in place of Corollary 1. $\square$

Proof of Theorem 1.5. Since $(\theta_{\mu,\alpha},0)$ is unstable (Lemma 4.1) and $(0,\theta_{\nu,\beta})$ is locally stable (Lemma 4.2), it suffices to show that there are no coexistence steady state to (1). Suppose to the contrary that for $\mu=\mu_k\to0$, (1) has a coexistence steady state (u,v).

Firstly, Assume (M2) and $0<\nu\le\beta\inf m$. By Lemma 4.4, as $\mu\to0$, $u\to0$ in $L^p(\Omega)$ for any $p>1$ and $v\to\theta_{\nu,\beta}$ in $C^{1,\gamma}(\bar\Omega)$ for all $\gamma\in(0,1)$. Given any $\epsilon>0$ small, let λ_ϵ be the smallest eigenvalue of the following linear eigenvalue problem

$$\begin{cases}\nabla\cdot(\mu\nabla\phi_\epsilon-\alpha\phi_\epsilon\nabla m)+\phi_\epsilon(m-\theta_{\nu,\beta}+\epsilon)+\lambda_\epsilon\phi_\epsilon=0 & \text{in } \Omega,\\ \mu\frac{\partial\phi_\epsilon}{\partial n}-\alpha\phi_\epsilon\frac{\partial m}{\partial n}=0 & \text{on } \partial\Omega.\end{cases}\tag{31}$$

We claim that for any given $\epsilon>0$, $\lambda_\epsilon<0$ for sufficiently small μ.

Rewrite the equation of ϕ_ϵ as

$$\mu\nabla\cdot[e^{\alpha m/\mu}\nabla(\phi_\epsilon e^{-\alpha m/\mu})]+\phi_\epsilon(m-\theta_{\nu,\beta}+\epsilon)=-\lambda_\epsilon\phi_\epsilon.\tag{32}$$

Rewrite the equation of u as

$$\mu\nabla\cdot[e^{\alpha m/\mu}\nabla(ue^{-\alpha m/\mu})]+u(m-u-v)=0.\tag{33}$$

Multiplying (32) by $ue^{-\alpha m/\mu}$ and (33) by $\phi e^{-\alpha m/\mu}$, then integrate and subtract the result, we have

$$\int_\Omega u\phi_\epsilon e^{-\alpha m/\mu}(u+v-\theta_{\nu,\beta}+\epsilon)=-\lambda_\epsilon\int_\Omega u\phi_\epsilon e^{-\alpha m/\mu}.$$

For any given $\epsilon>0$, since $v\to\theta_{\nu,\beta}$ as $\mu\to0$, we see that if μ is small, then $v-\theta_{\nu,\beta}+\epsilon>0$. This shows that $\lambda_\epsilon<0$ when μ is small.

Set $\varphi := \phi_\epsilon e^{-\alpha m/\mu}$. Then φ satisfies

$$\begin{cases} -\mu\Delta\varphi - \alpha\nabla m\cdot\nabla\varphi + \varphi(\theta_{\nu,\beta} - \epsilon - m) = \lambda_\epsilon\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{34}$$

By (M2), $|\nabla m| \geq |\frac{\partial m}{\partial n}| > 0$ on $\partial\Omega$. Denote the unique local maximum point of m by $x_0 \in \Omega$. Theorem 4.3 shows that

$$\lim_{\mu\to 0}\lambda_\epsilon = \theta_{\nu,\beta}(x_0) - \epsilon - m(x_0).$$

Since $\lambda_\epsilon < 0$, we have $\theta_{\nu,\beta}(x_0) - \epsilon - m(x_0) \leq 0$. Since ϵ is arbitrary, we have $\theta_{\nu,\beta}(x_0) - m(x_0) \leq 0$, which is a contradiction since if $0 < \nu < \beta \inf m$, $\theta_{\nu,\beta}(x_0) - m(x_0) > 0$ by Lemma 2.1 (b). This proves the first part of the theorem.

Alternatively, assume (M1) and let (u, v) be any coexistence state of (1). Then for $\alpha \geq \alpha_1$, as $\mu \to 0$, $u \to 0$ in $L^p(\Omega)$ and $v \to \theta_{\nu,\beta}$ uniformly by Lemma 4.5. Given any $\epsilon > 0$ small, consider (31) again with the current m. Given any ϵ, similar to the above procedure, $\lambda_\epsilon < 0$ for sufficiently small μ.

Set $\varphi := \phi_\epsilon e^{-\alpha m/\mu}$. Then φ satisfies

$$\begin{cases} -\mu\Delta\varphi - \alpha\nabla m\cdot\nabla\varphi + \varphi(\theta_{\nu,\beta} - \epsilon - m) = \lambda_\epsilon\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

By (M1), $|\nabla m| \geq |\frac{\partial m}{\partial n}| > 0$. Denote the unique local maximum point of m by $x_0 \in \Omega$. Theorem 4.3 shows that

$$\lim_{\mu\to 0}\lambda_\epsilon = \inf_{\mathfrak{M}\cup\Sigma_0}\left\{\theta_{\nu,\beta}(x_0) - \epsilon - m(x_0) + \frac{\alpha}{2}\sum_{i=1}^{N}(|\kappa_i(x)| + \kappa_i(x))\right\}.$$

By the choice of α_2 in (30), if $\alpha \geq \alpha_2$, then since $\lambda_\epsilon < 0$,

$$\lim_{\mu\to 0}\lambda_\epsilon = \inf_{x_0\in\mathfrak{M}}\left\{\theta_{\nu,\beta}(x_0) - \epsilon - m(x_0)\right\},$$

and we must have $\theta_{\nu,\beta}(x_0) - \epsilon - m(x_0) \leq 0$ for some $x_0 \in \mathfrak{M}$. Since ϵ is arbitrary, we have $\theta_{\nu,\beta}(x_0) - m(x_0) \leq 0$ for some $x_0 \in \mathfrak{M}$. On the other hand, by Corollary 2, there exists $K > 0$ such that if $0 < \nu \leq \min\{\beta/K, \beta\}$, $\theta_{\nu,\beta}(x_0) - m(x_0) > 0$ for all $x_0 \in \mathfrak{M}$. This contradiction proves the nonexistence of coexistence state of (1). This concludes the proof of the second part of the theorem. □

Acknowledgments. We are very grateful to an anonymous reviewer for the careful readings and helpful suggestions which help improve our manuscript. Parts of the work were done during the visit of XFC, KYL and YL to National Center for Theoretical Sciences at Tsinghua University, the visit of KYL to Mathematical Research Institute at Ohio State University, and the visit of YL to the Center for PDE at Eastern Normal China University. The authors thank these Institutions for the hospitality. This research was partially supported by the NSF grants DMS-1008905 (XFC) and DMS-1021179 (YL) and has been supported in part by the Mathematical Biosciences Institute and the National Science Foundation under grant DMS-0931642 (YL).

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Received xxxx 20xx; revised xxxx 20xx.

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