Concentration Phenomena of a Semilinear Elliptic Equation with Large Advection in an Ecological Model

King-Yeung Lam

School of Mathematics, University of Minnesota

127 Vincent Hall, 206 Church St. S.E., Minneapolis, MN 55455 USA

Email: adrian@math.umn.edu Phone Number: 612-625-0356

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Abstract

We consider a reaction-diffusion-advection equation arising from a biological model of migrating species. The qualitative properties of the globally attracting solution are studied and in some cases the limiting profile is determined. In particular, a conjecture of Cantrell, Cosner and Lou on concentration phenomena is resolved under mild conditions. Applications to a related parabolic competition system are also discussed.

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1 Introduction

In mathematical ecology, reaction-diffusion equations are often used to determine the factors behind the survival and extinction of animal populations. (See for examples [1, 2, 3, 4]). One well-known example is the following logistic reaction-diffusion model for population dynamics (See [5]):

$$
\begin{cases} u_t = d\Delta u + u[m(x) - u] & in\ \ \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = 0 & on\ \ \partial\Omega \times (0,\infty), \end{cases} \tag{1}
$$

where $u(x,t)$ represents the population density, $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^N$, $d > 0$ is the dispersal rate, $m(x)$ accounts for the local growth rate, Ω is the habitat of the population and is assumed to be a bounded region of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and ν is the outward unit normal vector on $\partial\Omega$. The Neumann boundary condition, which coincides with the no-flux boundary condition, is imposed on $\partial\Omega$.

If the environment is spatially heterogeneous, i.e. $m(x)$ is non-constant, then it seems reasonable to assume that the population has a tendency to move up the gradient of $m(x)$ in addition to random dispersal. In this direction, Belgacem and Cosner [6] proposed the following reaction-diffusion-advection equation:

$$
\begin{cases} u_t = \nabla \cdot (d\nabla u - \alpha u \nabla m) + u(m - u) & in\ \ \Omega \times (0,\infty), \\ d\frac{\partial u}{\partial \nu} - \alpha u \frac{\partial m}{\partial \nu} = 0 & on\ \ \partial\Omega \times (0,\infty), \end{cases} \tag{2}
$$

where the parameter $\alpha \geq 0$ measures the rate at which the population moves up the gradient of $m(x)$. Again, the corresponding no-flux boundary condition, is imposed. For discussions on the modeling aspects, we refer to [6, 7] and the references therein.

The dynamics of (2) seems simple. In fact, it was established in [6, 8] that *if we assume that*

(H1) $m(x) \in C^3(\overline{\Omega})$, *and is positive somewhere,*

then for any $d > 0$, (2) *has a unique positive steady-state* u *for all large* α. *Moreover,* u *is globally asymptotically stable among all nonnegative, nonzero solutions.* In other words, the steady-state u of (2) determines the long-time behavior of all solutions of (2). We shall always assume **(H1)** throughout this paper.

From both mathematical and biological points of view, it seems important to understand the qualitative properties of u. In particular, it would be interesting to describe the shape of u. There has been considerable effort in this direction. Recently, it was proved in [9] that *if the set of critical points of* $m(x)$ *has Lebesgue measure zero, then*

$$
\lim_{\alpha \to \infty} \int_\Omega u(x)dx = 0.
$$

That is, the total population size tends to 0 despite the fact that the species is tracking the resources more accurately. To understand the mechanism behind such phenomenon, again a better description of the shape of u is desired. To this end, the following results were proved.

Theorem 1.1 (Cantrell-Cosner-Lou). *Suppose $m(x) > 0$ in $\overline{\Omega}$. Let u be the unique positive steady-state of* (2)*.*

(i) If $\alpha > d/\min_{\overline{\Omega}} m$, then $u(x) > \max_{\overline{\Omega}} m \cdot e^{\alpha(m(x)-\max_{\overline{\Omega}} m)}/d$ for every $x \in \overline{\Omega}$. In particular, $\max_{\overline{\Omega}} u > \max_{\overline{\Omega}} m$.

(ii) Suppose $\Omega = (-1, 1)$, and $m(x)$ has finitely many critical points $\{x_i\}_{i=1}^n$, then $u \to 0$ uniformly in compact subsets of $\Omega \setminus \{x_i\}_{i=1}^n$ as $\alpha \to \infty$.

Based on these results, the following conjecture was proposed in [9] and Section 3.2 in [10].

Conjecture 1.2. *u concentrates precisely on the set of (positive) local maximum points of $m(x)$ as $\alpha \to \infty$.*

Remark 1.3. *We have modified the concentration set to be the set of positive local maximum points instead of local maximum points stated in* [9]*, since we are considering a more general situation where $m(x)$ can change sign on the set of its local maximum points.*

In this paper we shall establish Conjecture 1.2 under mild conditions on $m(x)$.

Let $\mathfrak{M}$ be the set of all positive strict local maximum points of $m(x)$ (i.e. those lying in $\{x \in \Omega : m(x) > 0\}$).

Theorem 1.4. *Assume that u is the unique positive steady-state of* (2)*. If $x_0 \in \mathfrak{M}$, then for any ball B centered at x_0,*

$$\liminf_{\alpha\to\infty} \sup_B u \geq m(x_0). \tag{3}$$

In other words, u concentrates at each point of $\mathfrak{M}$. The proof of Theorem 1.4 is based on the observation that u solves a corresponding eigenvalue problem and is given in Section 2.

To prove that u concentrates precisely on $\mathfrak{M}$, we impose the following assumptions on $m(x)$.

(H2) $\frac{\partial m}{\partial \nu} \leq 0$ on $\partial\Omega$.

(H3) $m(x)$ has finitely many local maximum points in $\overline{\Omega}$, all being strict local maxima located in the interior of Ω.

(H4) $\Delta m(x_0) > 0$ if $x_0 \in \overline{\Omega}$ is a local minimum or a saddle point of $m(x)$.

Theorem 1.5. *Assume $m(x)$ satisfies* **(H2)**, **(H3)** *and* **(H4)**, *then for any compact subset K of $\overline{\Omega} \setminus \mathfrak{M}$, there exists $\gamma = \gamma(K) > 0$, such that*

$$0 < u(x) \leq e^{-\gamma\alpha}, \quad \textit{for all } x \in K.$$

In particular, $u \to 0$ uniformly and exponentially in K, as $\alpha \to \infty$.

Theorems 1.4 and 1.5 together guarantee that u concentrates precisely on $\mathfrak{M}$, the set of positive local maximum points of $m(x)$, thereby Conjecture 1.2 is established. Theorem 1.5 is proved in Section 2 by the construction of an upper solution closely related to the shape of $m(x)$.

The question of determining the profile of u is, however, far more challenging. We only have the following result by a very interesting method introduced in [11] for the special case when $m(x)$ is constant on the set of local maximum points of $m(x)$.

Theorem 1.6. *If $m(x)$ satisfies* **(H2)**, **(H3)** *and* **(H4)** *and moreover,*

$$\det D^2 m(x_0) \neq 0 \textit{ for all } x_0 \in \mathfrak{M},$$

with $m(x_0) \equiv m_1 > 0$ for all local maximum points $x_0 \in \Omega$, then

$$\lim_{\alpha\to\infty} \| \, u(x) - 2^{N/2} m_1 e^{\alpha[m(x)-m_1]/d} \, \|_{L^\infty(\Omega)} = 0. \tag{4}$$

Remark 1.7. *The factor $2^{N/2}m_1$, though mysterious at first glance, is actually the consequence of the profile of u at each of its "weights", which is like a Gaussian distribution $e^{\alpha[(x-x_0)^T D^2 m(x_0)(x-x_0)]/2d}$, as well as the integral constraint $\int_{B(x_0)} u^2 - um \, dx = O(e^{-\gamma\alpha})$ for each $x_0 \in \mathfrak{M}$.*

As in [9, 12], our resolution of Conjecture 1.2 has implications for the following competition system.

$$\begin{cases} U_t = \nabla \cdot (d_1 \nabla U - \alpha U \nabla m) + U(m - U - V) & \textit{in } \Omega \times (0, \infty), \\ V_t = d_2 \Delta V + V(m - U - V) & \textit{in } \Omega \times (0, \infty), \\ d_1 \frac{\partial U}{\partial \nu} - \alpha U \frac{\partial m}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \textit{on } \partial\Omega \times (0, \infty). \end{cases} \tag{5}$$

This system was introduced to model the competition of two species whose population densities are denoted by $U(x,t)$ and $V(x,t)$ respectively. The two species have identical local growth rate $m(x)$ and competition abilities, but different dispersal strategies: the species with density V disperses randomly, whereas the other species U disperses, in addition to random diffusion, by a directed movement towards more favorable locations, i.e. where $m(x)$ is large. The goal of this model is to understand how different dispersal strategies affect the outcome of the competition in a heterogeneous environment.

When $\alpha = 0$, it is well-known [13] that *if* $d_1 > d_2$, *then* (5) *has no coexistence steady-states, and solution* (U_α, V_α) *of* (5) *always converges to* $(0, \theta_{d_2})$ *as* $t \to \infty$*, where* θ_{d_2} *is the unique positive solution to*

$$\begin{cases} d_2\Delta\theta + \theta(m-\theta) & = 0 \qquad \text{in } \Omega, \\ \frac{\partial\theta}{\partial\nu} & = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{6}$$

However, for any $d_1, d_2 > 0$, the existence of the positive steady-states $U_\alpha, V_\alpha > 0$ of (5) was established in [9, 11] for all large values of α. Moreover, they proved that *at least one of the co-existence steady-sates is stable!* Some qualitative properties of these co-existence steady-states were also obtained under extra hypotheses on $m(x)$.

Theorem 1.8 (Chen-Lou)**.** *Suppose that* $\int_\Omega m(x)dx > 0$ *and all critical points of* m *are non-degenerate* $(\det D^2 m(x_0) \neq 0)$*. Then for any positive steady-state* (U_α, V_α) *of* (5)*,*

$$\liminf_{\alpha\to\infty} \max_{\bar\Omega} U_\alpha \geq \max_{\mathfrak{M}}[m - \theta_{d_2}] > 0,$$

where θ_{d_2} *is the unique positive solution to* (6)*.*

Assume further that $m(x)$ *satisfies* **(H2)** *and that* $m(x)$ *has exactly one critical point* x_0 *which is a non-degenerate local maximum in the interior of* Ω*, then for any positive steady-state* (U_α, V_α) *of* (5)*,*

$$\forall\beta \in (0,1): \quad \lim_{\alpha\to\infty} \| V_\alpha - \theta_{d_2} \|_{C^{1+\beta}(\bar\Omega)} = 0, \ \text{and}$$

$$\lim_{\alpha\to\infty} \| U_\alpha(x) e^{\alpha[\max_{\bar\Omega} m - m(x)]/d_1} - 2^{N/2}[m(x_0) - \theta_{d_2}(x_0)] \|_{L^\infty(\Omega)} = 0.$$

Note that the condition $\int_\Omega m(x)dx > 0$ is there to ensure the existence of θ_{d_2}. (See [9].) It is interesting that our methods for (2) can be applied to study the coexistence steady-states.

Theorem 1.9. *Assume* $\int_\Omega m(x)dx > 0$*.*

(i) Assume that **(H3)** *holds. Given any positive steady-state* (U_α, V_α) *of* (5)*, if* $x_0 \in \mathfrak{M}$*, then for any ball* B *centered at* x_0*,*

$$\liminf_{\alpha\to\infty} \sup_B U_\alpha \geq m(x_0) - \theta_{d_2}(x_0). \tag{7}$$

If in addition, **(H2)** *and* **(H4)** *hold, then, for each compact subset* K *of* $\overline{\Omega} \setminus \mathfrak{M}$*, there exists a constant* $\gamma = \gamma(K) > 0$ *such that whenever* (U_α, V_α) *is a positive steady-state of* (5)*,*

$$U_\alpha(x) \leq e^{-\gamma\alpha} \quad \textit{for every } x \in K.$$

(ii) If **(H2)**, **(H3)** *and* **(H4)** *hold,* $\det D^2 m(x_0) \neq 0$ *for all* $x_0 \in \mathfrak{M}$*, and* $m(x_0) \equiv m_1 > 0$ *for all local maximum points* $x_0 \in \Omega$*, then*

$$\lim_{\alpha\to\infty} \| V_\alpha - \theta_{d_2} \|_{C^{1+\beta}(\bar{\Omega})} = 0 \qquad \forall \beta \in (0,1), \tag{8}$$

$$\lim_{\alpha\to\infty} \| U_\alpha(x) - 2^{N/2}(m_1 - \theta_{d_2}(x_0)) e^{\alpha[m(x)-m_1]/d_1} \|_{L^\infty(O_i)} = 0, \tag{9}$$

where O_i *is any open neighborhood of* x_0 *such that* $\tilde{x}_0 \notin \overline{O}_i$ *for any other* $\tilde{x_0} \in \mathfrak{M}$*.*

Remark 1.10. *(i)* (7) *is useful only when* $m(x_0) > \theta_{d_2}(x_0)$*. And this is true on* $\mathfrak{M}$ *if* $d_2 > 0$ *is sufficiently small and* $\Delta m(x_0) > 0$*. (The proof of this fact is included in Appendix A.)*

(ii) The choice of γ *in Part (i) of Theorem* 1.9 *is independent of choice of positive steady-state* (U_α, V_α)*.*

(iii) By maximum principle, $m_1 - \theta_{d_2}(x_0) > 0$ *in* (9) *for any* $d > 0$*.*

The rest of the paper is organized as follows. In Section 2 we provide the proofs for Theorems 1.4, 1.5, and 1.6. Section 3 will be devoted to proving Theorem 1.9. Finally, some concluding remarks will be included in Section 4.

2 Proofs of Theorems 1.4, 1.5, and 1.6

To simplify the presentation, we set $d = 1$ in the proofs. This assumption can be removed with minor corrections. We first obtain the following equation for u:

$$\begin{cases} \nabla \cdot (\nabla u - \alpha u \nabla m) + u(m-u) = 0 & \textit{in } \Omega, \\ \frac{\partial u}{\partial \nu} - \alpha u \frac{\partial m}{\partial \nu} = 0 & \textit{on } \partial\Omega. \end{cases} \tag{10}$$

Proof of Theorem 1.4. Let u be the unique solution to (10), and x_0 be a strict local maximum of $m(x)$. Then u is the principal eigenfunction of the following eigenvalue problem with principal eigenvalue 0:

$$\begin{cases} \nabla\cdot(\nabla\phi - \alpha\phi\nabla m) + (m-u)\phi + \lambda\phi = 0 & \textit{in } \Omega, \\ \frac{\partial\phi}{\partial\nu} - \alpha\phi\frac{\partial m}{\partial\nu} = 0 & \textit{on } \partial\Omega. \end{cases} \tag{11}$$

Now by the transformation $\phi = e^{\alpha m}\psi$, (11) is equivalent to

$$\begin{cases} \nabla\cdot(e^{\alpha m}\nabla\psi) + (m-u)\psi e^{\alpha m} + \lambda e^{\alpha m}\psi = 0 & \textit{in } \Omega, \\ \frac{\partial\psi}{\partial\nu} = 0 & \textit{on } \partial\Omega. \end{cases} \tag{12}$$

with principal eigenvalue equal to 0. The variational characterization of the principal eigenvalue of (12) implies

$$0 = \lambda = \inf_{\psi\in H^1}\left\{\frac{\int e^{\alpha m}(|\nabla\psi|^2 + (u-m)\psi^2)}{\int e^{\alpha m}\psi^2}\right\}$$

Given any small ball $B = B_{r_0}(x_0)$ centered at x_0, since $m(x)$ attains a strict maximum at x_0, $\max_{\partial B_{r_0}(x_0)} m < m(x_0)$. For any ϵ such that $0 < \epsilon < m(x_0) - \max_{\partial B_{r_0}(x_0)} m$, define

$$\begin{aligned} M_1 :=& m(x_0) - \frac{\epsilon}{3} > m(x_0) - \frac{2\epsilon}{3} := M_2, \\ U_1 :=& \{x \in B_{r_0}(x_0) : m(x) > m(x_0) - \frac{\epsilon}{3}\} \\ U_2 :=& \{x \in B_{r_0}(x_0) : m(x) > m(x_0) - \frac{2\epsilon}{3}\} \\ U_3 :=& \{x \in B_{r_0}(x_0) : m(x) > m(x_0) - \epsilon\}. \end{aligned}$$

Note that we have $U_1 \subset\subset U_2 \subset\subset U_3 \subset\subset B_{r_0}(x_0)$. Now take a smooth test function ψ such that,

$$\psi(x) = \begin{cases} 1 & \text{if } x \in U_2 \\ 0 & \text{if } x \in \Omega\setminus U_3 \end{cases} \qquad 0 \le \psi(x) \le 1 \qquad |\nabla\psi| \le C(\epsilon)$$

Then,

$$\begin{aligned} 0 &\le \frac{\int e^{\alpha m}|\nabla\psi|^2 + \int e^{\alpha m}(u-m)\psi^2}{\int e^{\alpha m}\psi^2} \\ &\le \frac{\int_{U_3} e^{\alpha M_2}C(\epsilon)^2}{\int_{U_1} e^{\alpha M_1}} + \frac{\int_{U_3} e^{\alpha m}(u-m)\psi^2}{\int_{U_3} e^{\alpha m}\psi^2} \\ &\le C'(\epsilon)e^{\alpha(M_2-M_1)} + \max_{\overline{U_3}}(u-m) \\ &\le C'(\epsilon)e^{-\frac{\epsilon\alpha}{3}} + \max_{\overline{U_3}} u - m(x_0) + \epsilon. \end{aligned}$$

For α sufficiently large, the first term in the last line will become less than ϵ, hence (3) follows.

$\square$

Next, we turn to the proof of Theorem 1.5. We first give the following definition of an upper solution. Denote from now on

$$L\phi \equiv \nabla \cdot (\nabla \phi - \alpha \phi \nabla m) + (m - \phi)\phi.$$

Definition 2.1. *$\overline{u}$ is said to be an upper solution of* (10) *if (i) $\sim$ (iii) below hold:*

(i) There exists an open cover $\{U_i\}$ of $\overline{\Omega}$, i.e. $\overline{\Omega} = \bigcup U_i$ where U_i's are relatively open in $\overline{\Omega}$, and, $\phi_i \in C^2(U_i)$, $L\phi_i \leq 0$, such that

$$\overline{u} = \min_i \{\phi_i\} \text{ is continuous in } \overline{\Omega}.$$

(ii) Denote $\Omega_i = \{x \in \Omega : \overline{u} = \phi_i\}$. $\partial \Omega_i$ is piecewise C^1, and

$$\Omega_i \subset\subset U_i \text{ for all } i. \tag{13}$$

(iii) $\frac{\partial \overline{u}}{\partial \nu} - \alpha \overline{u} \frac{\partial m}{\partial \nu} \geq 0$ for any $x \in \partial\Omega$, whenever the normal derivative $\frac{\partial \overline{u}}{\partial \nu}$ is defined.

The definition of lower solution can be obtained by reversing all the inequalities above and replacing min *by* max.

The following is the key to obtaining an upper bound of u.

Lemma 2.2. *Fix α sufficiently large so that the unique positive solution u of* (10) *exists. If $\overline{u} > 0$ is an upper solution of* (10) *in the sense of Definition* 2.1*, then $\overline{u} \geq u$.*

To prove Lemma 2.2, we first relate the above definition of upper solution to that of a weak upper solution from [14].

Definition 2.3. *$\overline{u} \in W^{1,2}(\Omega)$ is said to be a weak upper solution of* (10) *if it satisfies*

$$\begin{cases} \int_\Omega [-(\nabla \overline{u} - \alpha u \nabla m) \cdot \nabla \psi + \overline{u}(m - \overline{u})\psi] \leq 0, \text{ for any } \psi \in W^{1,2}(\Omega),\ \psi \geq 0 \\ \frac{\partial \overline{u}}{\partial \nu} - \alpha \overline{u} \frac{\partial m}{\partial \nu} \geq 0 \text{ on } \partial\Omega, \end{cases}$$

The definition of weak lower solution can be obtained by reversing the inequalities appropriately. Note that by **(H2)**, *$-\alpha \frac{\partial m}{\partial \nu} \geq 0$ on $\partial\Omega$.*

The following lemma can be proved via integration by parts.

Lemma 2.4. *Suppose $\overline{u}$ is an upper solution of* (10) *in the sense of definition* 2.1*, then it is a weak upper solution of* (10)*.*

Remark 2.5. *Lemma* 2.4 *is true even if we drop the C^1 regularity of $\partial\Omega_i$ in Definition* 2.1*, provided we use the arguments in Lemma 4.10 of [15]. This observation will not be used in this paper.*

We recall the following well-known theorem on upper and lower solutions.

Theorem 2.6 (Sattinger)**.** *If $\overline{u}$ and $\underline{u}$ are weak upper and lower solutions of* (10) *respectively, and $\overline{u} \geq \underline{u}$, then there exists a classical solution u of* (10) *such that $\underline{u} \leq u \leq \overline{u}$. Moreover, u is stable from above.*

We can now prove Lemma 2.2 by making use of the dynamics of (2).

Proof of Lemma 2.2. Since $\overline{u}$ and 0 are weak upper and lower solutions of (10) respectively. By Theorem 2.6, there exists a solution u' which is stable from above such that $0 \leq u' \leq \overline{u}$. Since 0 is unstable in (10) (by the global stability of u), $u' \not\equiv 0$. Hence, $u' \equiv u$ (by the uniqueness of u). Therefore, we have $u \leq \overline{u}$. □

To prove Theorem 1.5, it remains to construct an appropriate upper solution of (10) according to Definition 2.1. To avoid complicated notations and to illustrate the ideas more clearly, we shall only prove in detail the cases:

(a) When $m(x) \equiv m_1 > 0$ on $\mathfrak{M}$ and $m > 0$ at each of its critical points,

(b) When $m(x) \equiv m_1 > 0$ on $\mathfrak{M}$ and $m \leq 0$ at some of its critical points,

(c) When $m(x)$ has two distinct values $0 < m_1 < m_2$ on $\mathfrak{M}$ and $m \leq 0$ at some of its critical points.

We remark that the same technique can be applied to prove the general case when $m(x)$ has any (finite) number of distinct values on $\mathfrak{M}$. The precise statement of the lemma that leads to Theorem 1.5 and some comments on its proof are included in Appendix B.

Proof of Theorem 1.5. Case (a): When $m(x) \equiv m_1 > 0$ on $\mathfrak{M}$ and $m > 0$ at each of its critical points.

Lemma 2.7. *Suppose that $m(x)$ satisfies* **(H2)***,* **(H3)** *and* **(H4)***. Assume $m(x) \equiv m_1$ on $\mathfrak{M}$ and $m > 0$ at each of its critical points. Then for any $c < 1$, sufficiently close to 1, and for any $0 < \epsilon < 1$, there exists $\alpha_0(\epsilon, c) > 0$ such that*

$$\overline{u}_1 = e^{\epsilon\alpha(m(x)-cm_1)}$$

is an upper solution of (10) *in the sense of definition* 2.1 *for all $\alpha \geq \alpha_0$.*

Proof.

$$\begin{aligned}
L\overline{u}_1 &= \Delta\overline{u}_1 - \alpha\nabla m \cdot \nabla\overline{u}_1 + (m - \overline{u}_1 - \alpha\Delta m)\overline{u}_1 \\
&= \overline{u}_1\big\{(\epsilon^2 - \epsilon)\alpha^2|\nabla m|^2 + (\epsilon - 1)\alpha\Delta m + m - e^{\epsilon\alpha(m-cm_1)}\big\} \\
&= \overline{u}_1\big\{(\epsilon - 1)\alpha[\epsilon\alpha|\nabla m|^2 + \Delta m] + m - e^{\epsilon\alpha(m-cm_1)}\big\}.
\end{aligned}$$

It suffices now to prove that the sum in the large parenthesis is negative.

In $\{x \in \Omega : m(x) \leq c^{\frac{1}{2}}m_1\}$, by **(H4)**, there exists $k_1 > 0$ such that

$$\epsilon\alpha|\nabla m|^2 + \Delta m > k_1 \qquad \text{for all } \alpha \text{ large.}$$

While $m - e^{\epsilon\alpha(m-cm_1)}$ is bounded from above by $|m|_\infty$, therefore $L\overline{u}_1 \leq 0$ for all α sufficiently large.

In $\{x \in \Omega : m(x) > \sqrt{c}m_1\}$, $e^{\epsilon\alpha(m-cm_1)} \geq e^{\epsilon\alpha(\sqrt{c}m_1-cm_1)} = e^{k_2\alpha}$ for some $k_2 > 0$. Whereas $(\epsilon - 1)\alpha[\epsilon\alpha|\nabla m|^2 + \Delta m] + m$ grows at most in the order α^2, therefore, $L\overline{u}_1 \leq 0$ if α is sufficiently large. Combining, $L\overline{u}_1 \leq 0$ in Ω if α is sufficiently large.

It remains to check the boundary condition,

$$\frac{\partial\overline{u}_1}{\partial\nu} = \frac{\partial}{\partial\nu}e^{\epsilon\alpha(m(x)-cm_1)} = \overline{u}_1\epsilon\alpha\frac{\partial m}{\partial\nu} \geq \overline{u}_1\alpha\frac{\partial m}{\partial\nu}$$

making use of **(H2)** and $0 < \epsilon < 1$. The proof is completed. $\square$

Notice that $\overline{u}_1$ tends to zero uniformly in any compact subset of $\{x \in \Omega : m(x) < cm_1\}$. On the other hand, fix any compact subset K of $\Omega \setminus \mathfrak{M}$,

$$K \subseteq \{x \in \Omega : m(x) \leq c^2 m_1\},$$

if we take $c < 1$ sufficiently close to 1, since all local maximum points of $m(x)$ are strict. Therefore, in this case, Theorem 1.5 is a consequence of Lemma 2.2 and Lemma 2.7.

Case (b): When $m(x) \equiv m_1 > 0$ on $\mathfrak{M}$ and $m \leq 0$ at some of its critical points.

Lemma 2.8. *Assume $m(x)$ satisfies* **(H2)**, **(H3)** *and* **(H4)**, *and that $m(x) \equiv m_1 > 0$ on $\mathfrak{M}$. For each $c < 1$ close to 1, there exists, for all α large, an upper solution $\overline{u}_2 > 0$ in the sense of Definition 2.1 such that*

$$\overline{u}_2(x) \leq \begin{cases} e^{\epsilon\alpha(m(x)-cm_1)} & \text{when } m(x) > 0, \\ e^{\alpha(m(x)-k)} & \text{when } m(x) \leq 0, \end{cases}$$

where $0 < \epsilon < 1, k > 0$ are appropriately chosen constants independent of α.

Notice that in $\{x \in \Omega : m(x) < cm_1\}$, $\overline{u}_2 \to 0$ as $\alpha \to \infty$. We see that in this case, Theorem 1.5 follows as before from Lemma 2.8 and Lemma 2.2.

Proof of Lemma 2.8. Given $c < 1$, let

$$\phi_1 := e^{\epsilon\alpha(m(x)-cm_1)} \quad \text{and} \quad \phi_0 := e^{\alpha(m(x)-k)},$$

$\mathfrak{M}_0 =$ {strict local maximum points x_0 of $m(x)$ s.t. $m(x_0) = 0$}
$\Lambda_1 =$ The union of all connected components of $\{x \in \Omega : m(x) > -\delta_0\}$ not intersecting $\mathfrak{M}_0$

where $0 < \delta_0 < -\frac{1}{2}\max\{m(x_0) : x \in \Omega \text{ s.t. } \nabla m(x_0) = 0 \text{ and } m(x_0) < 0\}$ is chosen small enough so that each connected component of $\{x \in \Omega : m(x) > -\delta_0\}$ intersecting $\mathfrak{M}_0$ lies in $\{x \in \Omega : m(x) \leq 0\}$. This is possible since all local maxima are strict. And $0 < \epsilon < 1$ is chosen to satisfy

$$\epsilon < \frac{\delta_0}{cm_1 + \delta_0}, \tag{14}$$

k is chosen such that

$$0 < k < \epsilon c m_1. \tag{15}$$

Set

$$\overline{u}_2 = \begin{cases} \phi_1 & \text{in } \{x \in \Omega : m(x) > 0\} \\ \phi_0 & \text{in } \Omega \setminus \Lambda_1 \\ \min\{\phi_0, \phi_1\} & \text{in } \Lambda_1 \setminus \{x \in \Omega : m(x) > 0\}. \end{cases}$$

As before, $L\phi_1 \leq 0$ in Λ_1 for all α large. On the other hand, by a direct computation,

$$L\phi_0 = \phi_0(m - \phi_0) \leq 0 \quad \text{on } \{x \in \Omega : m(x) \leq 0\}.$$

Hence, $L\overline{u}_2 \leq 0$ for all α large, whenever it is C^2. Also, the boundary condition $\frac{\partial \overline{u}_2}{\partial \nu} - \alpha \overline{u}_2 \frac{\partial m}{\partial \nu} \geq 0$ is satisfied on $\partial\Omega$ whenever it is well-defined.

To see that $\overline{u}_2$ is an upper solution in the sense of Definition 2.1, it remains to show the continuity of $\overline{u}_2$ and (13). To this end, it suffices to check the following:

(i) $\phi_1 > \phi_0$ in $\{x \in \Omega : m(x) = -\delta_0\} \bigcap \partial(\Lambda_1 \setminus \{x \in \Omega : m(x) > 0\})$;

(ii) $\phi_1 < \phi_0$ in $\{x \in \Omega : m(x) = 0\} \bigcap \partial(\Lambda_1 \setminus \{x \in \Omega : m(x) > 0\})$.

More precisely,

(i): When $m(x) = -\delta_0$, by (14),

$$e^{\epsilon\alpha(m(x)-cm_1)} = e^{\epsilon\alpha(-\delta_0-cm_1)} > e^{\alpha(-\delta_0-k)} = e^{\alpha(m(x)-k)}.$$

Hence, $\overline{u}_2 = \phi_0$ in a neighborhood of $\{x \in \Omega : m(x) = -\delta_0\} \bigcap \partial(\Lambda_1 \setminus \{x \in \Omega : m(x) > 0\})$.

(ii): When $m(x) = 0$, by (15),

$$e^{\epsilon\alpha(m(x)-cm_1)} = e^{-\epsilon\alpha cm_1} < e^{-\alpha k} = e^{\alpha(m(x)-k)}.$$

Hence, $\overline{u}_2 = \phi_1$ in a neighborhood of $\{x \in \Omega : m(x) = 0\} \bigcap \partial(\Lambda_1 \setminus \{x \in \Omega : m(x) > 0\})$.

(Notice that ϕ_i are strictly increasing functions of $m(x)$. Hence (possibly making δ_0 smaller) the non-differentiable regions of $\overline{u}_2$ are regular level surfaces of $m(x)$ by the implicit function theorem.) $\square$

Case (c): When $m(x)$ has two distinct values $0 < m_1 < m_2$ on $\mathfrak{M}$ and $m \leq 0$ at some of its critical points.

We first decompose Ω according to the value of $m(x)$. Write $\mathfrak{M} = \mathfrak{M}_1 \bigcup \mathfrak{M}_2$, where $\mathfrak{M}_i = \{x_0 \in \mathfrak{M} : m(x_0) = m_i\}$, $i = 1, 2$. And define

$$\mathfrak{M}_0 = \{\text{strict local maximum points } x_0 \text{ of } m(x) \text{ s.t. } m(x_0) = 0\},$$

which is possibly empty. Given any $c < 1$ close to 1, define

$\Gamma_1 =$	$\{x \in \Omega : m(x) > 0\}$
$\Lambda_1 =$	The union of all connected components of $\{x \in \Omega : m(x) > -\delta_0\}$ not intersecting $\mathfrak{M}_0$
$\Gamma_2 =$	The union of all connected components of $\{x \in \Omega : m(x) > cm_1\}$ not intersecting $\mathfrak{M}_1$
$\Lambda_2 =$	The union of all connected components of $\{x \in \Omega : m(x) > c^2 m_1\}$ not intersecting $\mathfrak{M}_1$

where δ_0 is chosen as in the proof of Lemma 2.8. We have a partition:

$$\Omega = (\Omega \setminus \Lambda_1) \cup (\Lambda_1 \setminus \Gamma_1) \cup (\Gamma_1 \setminus \Lambda_2) \cup (\Lambda_2 \setminus \Gamma_2) \cup \Gamma_2.$$

Lemma 2.9. *Given $m(x)$ satisfying* **(H2)**, **(H3)** *and* **(H4)**, *and that $m(x)$ attains exactly two distinct values $0 < m_1 < m_2$ on $\mathfrak{M}$. For each $c < 1$ close to 1, for all α large, there exists an upper solution $\overline{u}_3 > 0$ in the sense of Definition* 2.1 *such that*

$$\overline{u}_3(x) \leq \begin{cases} e^{\alpha(m(x)-k)} & \text{in } \Omega \setminus \Lambda_1 \\ e^{\epsilon_1 \alpha(m(x)-cm_1)} & \text{in } \Lambda_1 \setminus \Lambda_2, \\ e^{\epsilon_2 \alpha(m(x)-cm_2)} & \text{in } \Lambda_2, \end{cases}$$

where $0 < \epsilon_i < 1, k > 0$ are appropriately chosen constants independent of α.

Notice that in $\{x \in \Lambda_2 : m(x) < cm_2\} \bigcup \{x \in \Omega \setminus \Lambda_2 : m(x) < cm_1\}$,

$$\overline{u}_3 \to 0 \quad \text{ as } \alpha \to \infty.$$

We see that in the case $m(x)$ having two distinct values $m_1 < m_2$ on $\mathfrak{M}$, Theorem 1.5 follows as before from Lemma 2.9 and Lemma 2.2.

Proof of Lemma 2.9. Let $\phi_0 := e^{\alpha(m(x)-k)}$ and $\phi_i := e^{\epsilon_i \alpha(m(x)-cm_i)}$ $(i = 1, 2)$, where $0 < \epsilon_1 < 1$ is chosen to satisfy

$$\epsilon_1 < \frac{\delta_0}{cm_1 + \delta_0}, \tag{16}$$

$k > 0$ and $0 < \epsilon_2 < 1$ are chosen such that

$$0 < k < \epsilon_1 cm_1, \tag{17}$$

$$0 < \epsilon_2 < \min\{\frac{\epsilon_1(c^2 m_1 - cm_1)}{c^2 m_1 - cm_2}, 1\}. \tag{18}$$

We can now define $\overline{u}_3$.

$$\overline{u}_3 := \begin{cases} \phi_0 & \text{in } \Omega \setminus \Lambda_1 \\ \phi_1 & \text{in } \Gamma_1 \setminus \Lambda_2 \\ \phi_2 & \text{in } \Gamma_2 \\ \min\{\phi_0, \phi_1\} & \text{in } \Lambda_1 \setminus \Gamma_1 \\ \min\{\phi_1, \phi_2\} & \text{in } \Lambda_2 \setminus \Gamma_2 \end{cases}$$

It can then be proved as before that

$$L\overline{u}_3 \leq 0 \text{ in } \Omega \quad \text{ and } \quad \frac{\partial \overline{u}_2}{\partial \nu} - \alpha \overline{u}_2 \frac{\partial m}{\partial \nu} \geq 0 \text{ on } \partial\Omega$$

whenever they are defined. It remains to show the continuity of $\overline{u}_3$, as well as (13). It suffices to show:

$$\begin{array}{lll}
\text{(i)} & \phi_0 < \phi_1 & \text{in } \{x \in \Omega : m(x) = -\delta_0\} \bigcap \partial(\Lambda_1 \setminus \Gamma_1); \\
\text{(ii)} & \phi_0 > \phi_1 & \text{in } \{x \in \Omega : m(x) = 0\} \bigcap \partial(\Lambda_1 \setminus \Gamma_1); \\
\text{(iii)} & \phi_1 < \phi_2 & \text{in } \{x \in \Omega : m(x) = c^2 m_1\} \bigcap \partial(\Lambda_2 \setminus \Gamma_2); \\
\text{(iv)} & \phi_1 > \phi_2 & \text{in } \{x \in \Omega : m(x) = c m_1\} \bigcap \partial(\Lambda_2 \setminus \Gamma_2).
\end{array}$$

(i), (ii) can be verified following similar lines as in the proof of Lemma 2.8, using (16) and (17).

(iii): When $m(x) = c^2 m_1$, by (18)

$$e^{\epsilon_1 \alpha(m(x)-cm_1)} = e^{\epsilon_1 \alpha(c^2 m_1 - cm_1)} < e^{\epsilon_2 \alpha(c^2 m_1 - cm_2)} = e^{\epsilon_2 \alpha(m(x)-cm_2)}, \text{ for } \alpha > 0.$$

(iv): When $m(x) = cm_1$

$$e^{\epsilon_1 \alpha(m(x)-cm_1)} = 1 > e^{\epsilon_2 \alpha c(m_1 - m_2)} = e^{\epsilon_2 \alpha(m(x)-cm_2)}, \text{ for } \alpha > 0.$$

□

Hence, Theorem 1.5 is proved for the cases when $m(x)$ attains 1 or 2 values on $\mathfrak{M}$. □

The proof of Theorem 1.6 is a modification of the proof in [11], overcoming the difficulty caused by the local minimum and saddle points of $m(x)$. We start with the following lemma.

Lemma 2.10. *With the assumption of Theorem* 1.6*, there exists* $C > 0$ *such that*

$$u(x) \le C e^{\alpha(m(x)-m_1)} \qquad \textit{for all } x \in \Omega \textit{ and all } \alpha \textit{ large.} \tag{19}$$

where m_1 *is the unique value of* $m(x)$ *on* $\mathfrak{M}$.

Proof. Consider $w = e^{(-\alpha+\epsilon)m(x)} u(x)$. Then in Ω, w satisfies

$$\Delta w + (\alpha - 2\epsilon)\nabla m \cdot \nabla w - \{\epsilon(\alpha - \epsilon)|\nabla m|^2 + \epsilon \Delta m + u - m\} w = 0 \tag{20}$$

Let $z^* = z^*(\alpha) \in \overline{\Omega}$ be such that $w(z^*) = \max_{\overline{\Omega}} w$. Then, for $x \in \Omega$,

$$u(x) \le u(z^*) e^{(-\alpha+\epsilon)(m(z^*)-m(x))}. \tag{21}$$

We notice that on $\partial\Omega$,

$$\begin{aligned}
\frac{\partial w}{\partial \nu} &= e^{(-\alpha+\epsilon)m(x)} (\frac{\partial u}{\partial \nu} + (-\alpha + \epsilon) u \frac{\partial m}{\partial \nu}) \\
&= e^{(-\alpha+\epsilon)m(x)} (\alpha u \frac{\partial m}{\partial \nu} + (-\alpha + \epsilon) u \frac{\partial m}{\partial \nu}) \\
&= e^{(-\alpha+\epsilon)m(x)} \epsilon \frac{\partial m}{\partial \nu} \le 0.
\end{aligned}$$

Therefore by the maximum principle, no matter $z^* \in \partial\Omega$ or Ω, $\nabla w(z^*) = 0$ and $\Delta w(z^*) \le 0$. Hence, by (20)

$$\epsilon(\alpha - \epsilon)|\nabla m|^2 + \epsilon\Delta m + u \le m \qquad \text{at } x = z^*, \tag{22}$$

and

$$u(z^*) \le m(z^*) - \epsilon\Delta m(z^*). \tag{23}$$

Now take $\epsilon = \max_{x_0}\{\frac{m(x_0)}{\Delta m(x_0)}\}$, with the maximum taken over all positive saddle points and local minimum points x_0 of $m(x)$ such that $m(x_0) > 0$. (Take $\epsilon = 1$ if it is an empty set.) Notice that $\epsilon > 0$ by **(H4)**. Then by (22), we have

$$\epsilon(\alpha - \epsilon)|\nabla m|^2 \le m(z^*) - \epsilon\Delta m \le |m|_\infty + \epsilon|\Delta m|_\infty,$$

which implies that $|\nabla m(z^*)| \to 0$ as $\alpha \to \infty$. Thus,

$$\text{dist}(z^*, \{x \in \Omega : |\nabla m(x)| = 0\}) \to 0.$$

Next, we claim that in fact we have $\text{dist}(z^*, \mathfrak{M}) \to 0$.

Assume to the contrary that there exists $\alpha_k \to \infty$, such that $z^*(\alpha_k) \to x_0$ as $k \to \infty$ where x_0 is a saddle point or a minimum point. Then by (23) and the choice of ϵ,

$$0 \le u(z^*) \le m(z^*) - \epsilon\Delta m(z^*) \to m(x_0) - \epsilon\Delta m(x_0) < 0,$$

which is a contradiction. Therefore, $\text{dist}(z^*, \mathfrak{M}) \to 0$. Recalling that $m(x) \equiv m_1$ on $\mathfrak{M}$, we deduce that there exists $C > 0$ such that

$$m_1 - m(z^*) \le C|\nabla m(z^*)|^2, \text{ for all } \alpha \text{ large},$$

since the inequality holds in a neighborhood of $\mathfrak{M}$, where z^* eventually enters. Hence by (22) again,

$$(\alpha - \epsilon)(m_1 - m(z^*)) \le C(\alpha - \epsilon)|\nabla m(z^*)|^2 \le C\big(\frac{m(z^*)}{\epsilon} - \Delta m(z^*)\big).$$

Therefore,

$$(\alpha - \epsilon)(m_1 - m(z^*)) \le C\big(\frac{m_1}{\epsilon} + \|\Delta m\|_\infty\big) \tag{24}$$

And for every $x \in \Omega$, from (21),

$$\begin{aligned} e^{-\alpha(m(x)-m_1)}u(x) &\le e^{-\alpha(m(x)-m_1)}u(z^*)e^{(\alpha-\epsilon)[m(x)-m(z^*)]} \\ &= u(z^*)e^{\epsilon(m_1-m(x))+(\alpha-\epsilon)(m_1-m(z^*))} \\ &\le (m_1 + \epsilon\|\Delta m\|_\infty)e^{2\epsilon|m|_\infty + C(\frac{m_1}{\epsilon}+\|\Delta m\|_\infty)}, \end{aligned}$$

by (23) and (24). Since the right hand side is a constant independent of x and α, (19) is proved.

$\square$

Proof of Theorem 1.6. From (19), we see that for all $p \geq 1$, $u \to 0$ in L^p as $\alpha \to \infty$. For each $x_0 \in \mathfrak{M}$, fix a neighborhood $\mathfrak{U}(x_0)$ of x_0, by (19),

$$u(x) \leq Ce^{\alpha(m(x)-m^*)} \leq Ce^{\alpha(\frac{1}{2}(x-x_0)^T D^2 m(x_0)(x-x_0)+C_1|x-x_0|^3)},$$

where $C_1 = \|D^3 m\|_\infty/6$. Denote $M(x_0,\alpha) = \sup_{\mathfrak{U}}(x_0)u$, which is attained in $B_{R/\sqrt{\alpha}}(x_0)$ for R sufficiently large, and all large α (by Theorem 1.4 and Lemma 2.10). Define

$$W_\alpha(y) = \frac{u(x_0 + \frac{y}{\sqrt{\alpha}})}{M(x_0,\alpha)}$$

Then $\sup W_\alpha = 1$ in $\sqrt{\alpha}\big(\mathfrak{U}(x_0) - x_0\big)$, and

$$W_\alpha(y) \leq Ce^{\frac{1}{2}y^T D^2 m(x_0)y + \frac{C_1}{\sqrt{\alpha}}|y|^3} \leq Ce^{\frac{1}{3}y^T D^2 m(x_0)y}$$

for all α large and in $\{y \in \mathbb{R}_N : x_0 + y/\sqrt{\alpha} \in \Omega, |y| \leq \frac{-\lambda_N\sqrt{\alpha}}{6C}\}$, where $\lambda_1 \leq \cdots \leq \lambda_N < 0$ are the eigenvalues of $D^2 m(x_0)$.

To prove (4), by Lemma 2.10 and the fact that $M(x_0,\alpha)$ is bounded, it suffices to show that for each $x_0 \in \mathfrak{M}$

$$\begin{cases} W_\alpha(y) \to e^{\frac{1}{2}y^T D^2 m(x_0)y} \text{ in every compact subset of } \mathbb{R}^N, \text{ and} \\ M(x_0,\alpha) \to 2^{N/2} m(x_0), \end{cases} \tag{25}$$

as $\alpha \to \infty$. W_α satisfies $\Delta_y W_\alpha + \overrightarrow{P} \cdot \nabla_y W_\alpha + QW_\alpha = 0$, where

$$\overrightarrow{P} = \overrightarrow{P}(\alpha, y) = -\sqrt{\alpha} \cdot \nabla_x m\big(x_0 + \frac{y}{\sqrt{\alpha}}\big),$$

and

$$Q(\alpha,y) = -\Delta_x m\big(x_0 + \frac{y}{\sqrt{\alpha}}\big) - \frac{u(x_0 + \frac{y}{\sqrt{\alpha}}) - m(x_0 + \frac{y}{\sqrt{\alpha}})}{\alpha}.$$

The boundedness of u (by (19)) implies that

$$\lim_{\alpha\to\infty} \overrightarrow{P}(\alpha,y) = -y^T D^2 m(x_0), \quad \lim_{\alpha\to\infty} Q(\alpha,y) = -\Delta_x m(x_0),$$

uniformly in any compact subset of $\mathbb{R}^2$. Hence by elliptic estimates (see [16]), using the fact that for each compact subset K in $\mathbb{R}^N$, W_α is bounded in $L^p(K)$ for $p \in (1,\infty]$ and all large α, after passing to a subsequence if necessary, as $\alpha \to \infty$, W_α converges to some function W^* uniformly in any compact subset of $\mathbb{R}^N$, and W^* must satisfy

$$\begin{cases} \Delta_y W^* - yD^2m(x_0)\nabla_y W^* - \Delta m(x_0)W^* = 0 & \text{in } \mathbb{R}^N, \\ \sup_{\mathbb{R}^N} W^*(y) = 1, \;\; 0 \le W^*(y) \le Ce^{\frac{1}{3}y^T D^2 m(x_0) y} & \forall y \in \mathbb{R}^N. \end{cases} \tag{26}$$

Now we invoke the following lemma, the proof of which makes use of a Liouville-type result due to [17] which is formulated differently in [15], and will be included in Appendix C for completeness.

Lemma 2.11. *If $W^* \in W^{1,2}_{loc}(\mathbb{R}^N)$ satisfies* (26), *then* $W^* = e^{\frac{1}{2}y^T D^2 m(x_0) y}$.

The uniqueness of the limit implies that

$$\lim_{\alpha\to\infty} W_\alpha(y) = e^{\frac{1}{2}y^T D^2 m(x_0) y} \text{ uniformly in any compact subset of } \mathbb{R}^N. \tag{27}$$

That W^* attains its strict maximum at the origin and (19) implies that

$$\lim_{\alpha\to\infty} \frac{u(x_0)}{M(x_0,\alpha)} = W^*(0) = 1. \tag{28}$$

To show the second part of (25), it remains to calculate $\lim\limits_{\alpha\to\infty} u(x_0)$. In [11] it was accomplished when m as a single peak via a "global" argument. Here we devise a "local" argument near each $x_0 \in \mathfrak{M}$.

Lemma 2.12. *For each $x_0 \in \mathfrak{M}$,* $\liminf\limits_{\alpha\to\infty} u(x_0) \ge 2^{N/2} m_1$.

Proof. By following the proof of Theorem 1.4, with the same choice of test function ψ and open sets U_i, we have for each $\eta > 0$,

$$
\begin{aligned}
0 \leq & \liminf_{\alpha\to\infty} \frac{\int_{U_3} e^{\alpha[m-m_1]}(u-m)dx}{\int_{U_2} e^{\alpha[m-m_1]}\,dx} \\
\leq & \liminf_{\alpha\to\infty} \left[\frac{\int_{B_{R/\sqrt{\alpha}}(x_0)} e^{\alpha[m-m_1]}u\,dx}{\int_{U_2} e^{\alpha[m-m_1]}\,dx} + \frac{\int_{U_3\setminus B_{R/\sqrt{\alpha}}(x_0)} e^{\alpha[m-m_1]}u\,dx}{\int_{U_2} e^{\alpha[m-m_1]}\,dx} \right] \\
& - \lim_{\alpha\to\infty} \frac{\int_{U_3} e^{\alpha[m-m_1]}m\,dx}{\int_{U_2} e^{\alpha[m-m_1]}\,dx} \\
\leq & \liminf_{\alpha\to\infty} \left[\frac{\int_{B_{R/\sqrt{\alpha}}(x_0)} (1+\eta)u(x_0)e^{\alpha[m(x)-m_1]+\frac{\alpha}{2}(x-x_0)^T D^2 m(x_0)(x-x_0)}\,dx}{\int_{B_{R/\sqrt{\alpha}}(x_0)} e^{\alpha[m-m_1]}\,dx} \right. \\
& \left. + \frac{\int_{U_3\setminus B_{R/\sqrt{\alpha}}} e^{\alpha[m-m_1]}u\,dx}{\int_{U_2} e^{\alpha[m-m_1]}\,dx} \right] - m(x_0) \\
\leq & \liminf_{\alpha\to\infty} \left[(1+\eta)u(x_0) \frac{\int_{B_R(0)} e^{\alpha[m(x_0+\frac{y}{\sqrt{\alpha}})-m_1]+\frac{1}{2}y^T D^2 m(x_0)y}\,dy}{\int_{B_R(0)} e^{\alpha[m(x_0+\frac{y}{\sqrt{\alpha}})-m_1]}\,dy} \right. \\
& \left. \frac{\int_{\mathbb{R}^N\setminus B_R(0)} e^{-c_1|y|^2}\,dy}{\int_{B_R(0)} e^{-c_2|y|^2}\,dy} \right] - m(x_0) \\
\leq & (1+\eta)\left[\liminf_{\alpha\to\infty} u(x_0)\right](2^{-\frac{N}{2}}+\eta) + \eta - m(x_0)
\end{aligned}
$$

The third inequality follows from (27), (28) and the Lebesgue Dominated Convergence. In the fourth inequality, we applied the change of coordinates $x = x_0 + \frac{y}{\sqrt{\alpha}}$ and that there exist $c_1, c_2 > 0$ such that $c_1|y|^2 \leq m_1 - m(x) \leq c_2|y|^2$ (which are consequences of the non-degeneracy of m). The last line follows by taking $R > 0$ sufficiently large and that

$$
\lim_{\alpha\to\infty} \alpha[m(x_0+\frac{y}{\sqrt{\alpha}}) - m_1] = \frac{1}{2}y^T D^2 m(x_0)y
$$

uniformly in compact subsets of $\mathbb{R}^N$. Finally, the lemma is proved by letting $\eta \to 0^+$ $\square$

Next, we claim that

Claim 2.13. $\lim_{\alpha\to\infty} \sum_{x_0\in\mathfrak{M}} \int_{\mathbb{R}^N} e^{\frac{1}{2}y^T D^2 m(x_0)y}\,dy\left[u(x_0)^2 - 2^{N/2}m_1 u(x_0)\right] = 0$

Proof of Claim 2.13. Integrate (10) over Ω, we have

$$
\begin{aligned}
0 &= \int_\Omega (u^2 - um)\, dx \\
&= \left\{ \int_{\cup_{\mathfrak{M}} B_{R/\sqrt{\alpha}}(x_0)} + \int_{\cup_{\mathfrak{M}} B_{r_0}(x_0)\setminus B_{R/\sqrt{\alpha}}(x_0)} + \int_{\Omega\setminus \cup_{\mathfrak{M}} B_{r_0}(x_0)} \right\} (u^2 - um)\, dx \\
&= \sum_{x_0\in\mathfrak{M}} \left[\int_{B_{R/\sqrt{\alpha}}(x_0)} (u^2 - um)\, dx + C \int_{B_{r_0}(x_0)\setminus B_{R/\sqrt{\alpha}}(x_0)} e^{\alpha[m(x)-m_1]}\, dx \right] \\
&\quad + O(e^{-\gamma\alpha}).
\end{aligned}
$$

by Theorem 1.5 and Lemma 2.10. Multiplying by $\alpha^{\frac{N}{2}}$ and changing coordinates $x = x_0 + \frac{y}{\sqrt{\alpha}}$, we see that

$$
0 = \sum_{x_0\in\mathfrak{M}} \int_{B_R(0)} (u^2 - um)(x_0 + \frac{y}{\sqrt{\alpha}})\, dy + O(\int_{\mathbb{R}^N\setminus B_R(0)} e^{-c_1|y|^2}\, dy) + O(\alpha^{\frac{N}{2}} e^{-\gamma\alpha}).
$$

By (27) and (28), for each $R > 0$ large, there exists α_0 such that for any $\alpha \geq \alpha_0$,

$$
\begin{aligned}
0 &= \sum_{x_0\in\mathfrak{M}} \int_{B_R(0)} \left[u^2(x_0) e^{y^T D^2 m(x_0) y} - u(x_0) m(x_0) e^{\frac{1}{2} y^T D^2 m(x_0) y} \right] dy + o(1) \\
&\quad + O(\int_{\mathbb{R}^N\setminus B_R(0)} e^{-c_1|y|^2}\, dy) \\
&= \sum_{x_0\in\mathfrak{M}} \int_{\mathbb{R}^N} \left[u^2(x_0) e^{y^T D^2 m(x_0) y} - u(x_0) m(x_0) e^{\frac{1}{2} y^T D^2 m(x_0) y} \right] dy + o(1) \\
&\quad + O(\int_{\mathbb{R}^N\setminus B_R(0)} e^{-c_3|y|^2}\, dy).
\end{aligned}
$$

where $\lim_{\alpha\to\infty} o(1) = 0$. Now taking $\alpha \to \infty$ and then $R \to \infty$, we have the desired result.

□

Lemma 2.12 and Claim 2.13 imply the second part of (25). This concludes the proof of Theorem 1.6 □

3 Proof of Theorem 1.9

As before, assume for simplicity $d_1 = 1$.

Proof of Theorem 1.9. Notice that (U_α, V_α) satisfies

$$\begin{cases} \nabla\cdot(\nabla U - \alpha U \nabla m) + U(m-U) & = UV > 0 \qquad in\ \ \Omega, \\ d_2\Delta V + V(m-V) & = UV > 0 \qquad in\ \ \Omega, \\ \frac{\partial U}{\partial \nu} - \alpha U \frac{\partial m}{\partial \nu} = \frac{\partial V}{\partial \nu} & = 0 \qquad\qquad on\ \ \partial\Omega. \end{cases} \tag{29}$$

By method of upper and lower solutions, $0 < U_\alpha \le u$ and $0 < V_\alpha \le \theta_{d_2}$. (7) follows from the same argument as in the proof of Theorem 1.4, using the inequality $V_\alpha \le \theta_{d_2}$. That U_α converges to 0 away from the positive local maximum points of $m(x)$ follows from the corresponding property of u.

Now, assume $m \equiv m_1$ on the set of its local maximum points.

Lemma 3.1. *If* **(H2)**, **(H3)** *and* **(H4)** *hold, and $m(x)$ is constant on its local maximum points, then there exists $C_2 > 0$ such that*

$$U_\alpha(x) \le C_2 e^{\alpha(m(x)-m_1)} \qquad \textit{for all } x\in\Omega \textit{ and all } \alpha \textit{ large.}$$

Lemma 3.1 follows from Lemma 2.10 and the fact that $0 < U_\alpha \le u$.

For some α_0 large, $\int_\Omega [m - C_2 e^{\alpha_0(m(x)-m_1)}] > 0$ and by a claim on P. 498 in [9], there exists a positive solution V_0 of

$$\begin{cases} d_2\Delta V_0 + V_0(m - C_2 e^{\alpha_0(m(x)-m_1)} - V_0) = 0 & in\ \ \Omega, \\ \frac{\partial V_0}{\partial \nu} = 0 & on\ \ \partial\Omega. \end{cases}$$

Then for all $\alpha \ge \alpha_0$,

$$\begin{cases} \Delta V_0 + V_0(m - U_\alpha - V_0) & \ge 0 \qquad in\ \ \Omega, \\ \frac{\partial V_0}{\partial \nu} & = 0 \qquad on\ \ \partial\Omega. \end{cases}$$

Therefore, V_0 is a lower solution of the second equation of (29) for V_α, and,

$$\theta_{d_2} \ge V_\alpha \ge V_0 > 0 \qquad \text{for all } \alpha \ge \alpha_0. \tag{30}$$

By Lemma 3.1, $U_\alpha \to 0$ in L^p for any $p > 1$. By the second equation in (29), (30), and elliptic estimates and uniqueness, $V \rightharpoonup \theta_{d_2}$ weakly in $W^{2,p}(\Omega)$ in any $p > 1$ hence strongly in $C^{1,\beta}(\overline{\Omega})$ for any $\beta \in (0,1)$. This proves (8).

Fix $x_0 \in \mathfrak{M}$ and let $\widetilde{W}_\alpha(y) = \frac{U_\alpha(x_0 + y/\sqrt{\alpha})}{M(x_0,\alpha)}$, where $M(x_0,\alpha) = \sup_{B_{r_0}(x_0)} U_\alpha$ for some small $r_0 > 0$. ($M(x_0,\alpha)$ is independent of the choice of r_0 by (7) and Lemma 3.1.) As in the proof of Theorem 1.6, notice that $\widetilde{W}_\alpha(y) \to \widetilde{W}^*(y)$ as $\alpha \to \infty$ uniformly for y in compact sets in $\mathbb{R}^N$ where $\widetilde{W}^*$ satisfies

$$\Delta_y \widetilde{W}^* - yD^2 m(x_0)\nabla_y \widetilde{W}^* - \Delta m(x_0)\widetilde{W}^* = 0 \qquad in\ \ \mathbb{R}^N.$$

Also similar as in the proof of Theorem 1.6,

$$\lim_{\alpha\to\infty} \widetilde{W}_\alpha(y) = W^*(y) = e^{\frac{1}{2}y^T D^2 m(x_0) y} \text{ on compact sets in } \mathbb{R}^N \tag{31}$$

and $\lim_{\alpha\to\infty} \dfrac{U(x_i)}{M(x_i,\alpha)} = 1$. Now, by arguments in the proof of Theorem 1.4, we have

$$\liminf_{\alpha\to\infty} \frac{\int_{U_3} e^{\alpha m}(U_\alpha + V_\alpha - m)}{\int_{U_2} e^{\alpha m}} \geq 0.$$

Then Lemma 3.1, (8) and (31) imply, for each $x_0 \in \mathfrak{M}$,

$$\liminf_{\alpha\to\infty} U_\alpha(x_0) \geq 2^{N/2}(m_1 - \theta_{d_2}(x_0)) \tag{32}$$

By integrating the first equation of (29) over Ω, we have $\int_\Omega U_\alpha(m - U_\alpha - V_\alpha)dx = 0$. And by similar arguments in proving Claim 2.13, we have

$$0 = \lim_{\alpha\to\infty} \sum_{x_0\in\mathfrak{M}} \int_{\mathbb{R}^N} e^{\frac{1}{2}y^T D^2 m(x_0) y} dy [U_\alpha(x_0)^2 - 2^{N/2}(m_1 - \theta_{d_2}(x_0))U_\alpha(x_0)]. \tag{33}$$

Finally, $\lim_{\alpha\to\infty} U_\alpha(x_0) = 2^{N/2}(m_1 - \theta_{d_2}(x_0))$ follows from (32) and (33). □

4 Concluding Remarks

In this paper, the existence of concentration phenomena in the globally stable steady state $u(x)$ of (2) is proved for $m(x)$ which has finitely many local maximum points. Furthermore, the concentration set is shown to be the set of positive local maximum points of $m(x)$. The situation when $m(x)$ contains local maximums that are not strict is however, completely open. It is possible that u would concentrate on some higher dimensional sets.

In this paper, the limiting profile is obtained in the special case when the resource function m has equal peaks. Based on the estimates established in this paper, a special method is introduced to determine the limiting profile for m with peaks of different heights in [18]. However, the method only works for $N = 1$. For $N \geq 2$, very recently the limiting profile has been found by the author. This will be published in a forthcoming paper.

We learnt recently that in [19], a lower solution for (10) can be constructed at each $x_0 \in \mathfrak{M}$ which gives an alternative proof for the existence of peaks on $\mathfrak{M}$.

We also remark that the assumptions on $m(x)$ in $\{x \in \Omega : m(x) < 0\}$ can be weakened substantially. In fact, instead of **(H2)**, **(H3)** and **(H4)**, we only need to assume that there exists $\delta > 0$, such that the followings hold.

(H2') $\frac{\partial m}{\partial \nu} \leq 0$ on $\{x \in \partial\Omega : m(x) \geq -\delta\}$.

(H3') $m(x)$ has finitely many local maximum points in $\{x \in \overline{\Omega} : m(x) \geq -\delta\}$, all being strict local maxima and are located in the interior of Ω.

(H4') If $x_0 \in \overline{\Omega}$ satisfies $m(x_0) \geq -\delta$ and is a local minimum or a saddle point of $m(x)$, then $\Delta m(x_0) > 0$

Finally, notice that although we have set the diffusion coefficient $d, d_1 = 1$ for simplicity, the results proved in this paper hold true for any $d, d_1 > 0$, as stated in Section 1.

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5 Appendix A

Denote θ_d to be the unique positive solution to

$$\begin{cases} d\Delta\theta + \theta(m - \theta) &= 0 \qquad in\ \Omega, \\ \qquad\qquad\quad \frac{\partial \theta}{\partial \nu} &= 0 \qquad on\ \partial\Omega. \end{cases}$$

The existence part is standard. (See, e.g. P. 498 in [9].) Also, it is known that (Prop. 3.16 of [1])

$$\lim_{d \to 0^+} \theta_d = m^+, \qquad \text{uniformly in } \Omega, \tag{A1}$$

where $m^+(x) := \max\{m(x), 0\}$.

Here we shall prove that if $x_0 \in \Omega$ is a positive strict local maximum point of m and $\Delta m(x_0) < 0$, then $m(x_0) - \theta_d(x_0) > 0$ for all $d > 0$ sufficiently small.

Remark 5.1 ([20])**.** *When d is not small, there are counter examples showing that the conclusion is not true in general for $x_0 \in \mathfrak{M}$ other than the global maximum point(s).*

First we show that $m(x_0) \geq \theta_d(x_0)$. Assume now to the contrary that for some positive strict positive local maximum point x_0 of $m(x)$, for some sequence $d_i \to 0$,

$$\theta_{d_i}(x_0) > m(x_0) > 0.$$

Now, $x_0 \in \{x \in \Omega : \theta_{d_i}(x) > m(x_0)\}$ for all i. Denote by U_i the connected component of $\{x \in \Omega : \theta_{d_i}(x) > m(x_0)\}$ that contains x_0, then $U_i \neq \emptyset$ and $\Delta\theta_{d_i} = \theta_{d_i}(\theta_{d_i} - m) \geq 0$ in U_i. i.e. θ_{d_i} is subharmonic in U_i. Now for d_i sufficiently small, by (A1), U_i is compactly contained in a neighborhood of x_0. In particular, $\theta_{d_i} > m(x_0)$ in U_i and $\theta_{d_i}(x) = m(x_0)$ on ∂U_i. This contradicts the property of subharmonic functions. Therefore, $m(x_0) \geq \theta_d(x_0)$ for all $d > 0$ sufficiently small.

Now assume there exists a sequence $d_i \to 0$ such that $\theta_{d_i}(x_0) = m(x_0)$. We claim that

Claim 5.2. $\nabla\theta_{d_i}(x_0) = 0$ *for all* i *sufficiently large.*

Otherwise there exists $x_i \to x_0$ such that $\theta_{d_i}(x_i) > m(x_0)$ and a contradiction can be reached by previous arguments by choosing a horizontal hyperplane.

Now since $\theta_{d_i}(x_0) = m(x_0)$, $\nabla\theta_{d_i}(x_0) = \nabla m(x_0)$ and $\Delta\theta_{d_i} = 0 > \nabla m(x_0)$, there exists $x_i \to x_0$ such that $\theta_{d_i}(x_i) > m(x_i)$. (Since otherwise the mean curvature of the surface defined by θ_{d_i} in $\mathbb{R}^{N+1}$ at x_0, which is a multiple of $\Delta\theta_{d_2}(x_0)$, would not be not equal to 0.) Now fix a neighborhood U_0 of x_0, and a (slightly tilted) hyperplane $\Sigma_i : L(\mathbb{R}^N, \mathbb{R})$ such that

$$\theta_{d_i}(x_i) > \Sigma_i(x_i) \text{ and } \Sigma_i(x) > m(x) \text{ in } U_0.$$

By (A1), $\theta_{d_i} \to m$ uniformly on ∂U_0 while $\min_{\partial U_0}\{\Sigma_i(x) - m(x)\} \geq c > 0$ for some constant c independent of i. This implies that there is some $U_i \neq \emptyset$ such that

$$\begin{cases} \Delta\theta_{d_i} = \theta_{d_i}(\theta_{d_i} - m) \geq 0 \text{ in } U_i \\ \theta_{d_i} > \Sigma_i \text{ in } U_i, \quad \theta_{d_i} = \Sigma_i \text{ on } \partial U_i \end{cases}$$

which again contradicts the fact that θ_{d_i} is subharmonic in U_i.

6 Appendix B

Here we discuss the proof of the general case of Theorem 1.5. Recall

$$\mathfrak{M} = \{ \text{ positive strict local maximum points of } m(x) \text{ in } \Omega \ \}.$$

By **(H3)**, $m(x)$ has finitely many local maximum points. Let $0 < m_1 < m_2 < \cdots < m_{n_0}$ be the distinct values of $m(x)$ on $\mathfrak{M}$. Decompose

$$\mathfrak{M} = \bigcup_{i=1}^{n_0} \mathfrak{M}_i,$$

where $\mathfrak{M}_i = \{x_0 \in \mathfrak{M} : m(x_0) = m_i\}$. And let

$$\mathfrak{M}_0 := \{\text{local maximum points } x_0 \text{ of } m(x) \text{ s.t. } m(x_0) = 0\},$$

which is possibly empty. For each $c < 1$, close to 1. Define δ_0 as in the proof of Lemma 2.8. Decompose Ω according to the value of $m(x)$:

$$\begin{aligned}
\Gamma_1 =&\{x \in \Omega : m(x) > 0\}\\
\Lambda_1 =&\text{Union of connected components of } \{x \in \Omega : m(x) > -\delta_0\}\\
&\quad \text{not intersecting } \mathfrak{M}_0,\\
\Gamma_i =&\text{Union of connected components of } \{x \in \Omega : m(x) > cm_{i-1}\}\\
&\quad \text{not intersecting } \mathfrak{M}_{i-1},\\
\Lambda_i =&\text{Union of connected components of } \{x \in \Omega : m(x) > c^2 m_{i-1}\}\\
&\quad \text{not intersecting } \mathfrak{M}_{i-1},
\end{aligned}$$

for $i = 2, ..., n_0$. Notice that $\Lambda_i \supseteq \Gamma_i \supseteq \Lambda_{i+1} \supseteq \Gamma_{i+1}$. Define

$$\overline{u}(x) = \begin{cases}
e^{\epsilon_{n_0}\alpha(m(x)-cm_{n_0})} & \text{in } \Gamma_{n_0}\\
e^{\epsilon_i\alpha(m(x)-cm_i)} & \text{in } \Gamma_i \setminus \Lambda_{i+1}\\
& \text{for } i = 1, ..., n_0 - 1\\
e^{\alpha(m(x)-k)} & \text{in } \Omega \setminus \Lambda_1\\
\min\{e^{\epsilon_i\alpha(m(x)-cm_i)}, e^{\epsilon_{i+1}\alpha(m(x)-cm_{i+1})}\} & \text{in } \Lambda_{i+1} \setminus \Gamma_{i+1}\\
& \text{for } i = 1, ..., n_0 - 1\\
\min\{e^{\alpha(m(x)-k)}, e^{\epsilon_1\alpha(m(x)-cm_1)}\} & \text{in } \Lambda_1 \setminus \Gamma_1.
\end{cases}$$

where $0 < \epsilon_i < 1, k > 0$ are constants chosen such that

$$\epsilon_1 < \frac{\delta_0}{cm_1 + \delta_0}, \quad 0 < k < \epsilon_1 cm_1, \text{ and}$$

$$0 < \epsilon_{i+1} < \min\{\frac{\epsilon_i(c^2 m_i - cm_i)}{c^2 m_i - cm_{i+1}}, 1\}, \quad \text{for } i = 1, \cdots, n_0 - 1.$$

Then, we have

Lemma 6.1. *Given $m(x)$ satisfying* **(H2)**, **(H3)** *and* **(H4)**. *For every $c < 1$ sufficiently close to 1, $\overline{u} > 0$ is an upper solution to* (10) *according to Definition* 2.1.

The proof of Lemma 6.1 is similar to that of Lemma 2.9 and Lemma 2.8 and is omitted.

Notice that the full statement of Theorem 1.5 follows from the above lemma and Lemma 2.2.

7 Appendix C

Next, we shall prove Lemma 2.11. We first state and prove the following Liouville-type theorem which is due to [17], following the formulation in [15].

Theorem 7.1. *Let $\sigma \in L^\infty_{loc}(\mathbb{R}^N)$ be a positive function. Assume that $\Phi \in W^{1,2}_{loc}(\mathbb{R}^N)$ satisfies in the weak sense*

$$\Phi \ \mathrm{div}(\sigma^2 \nabla \Phi) \geq 0 \ \text{in } \mathbb{R}^N, \tag{C1}$$

and for some $C > 0$ and every $R > 1$,

$$\int_{B_R(0)} (\sigma\Phi)^2 dx \leq CR^2. \tag{C2}$$

Then Φ is a constant.

Proof of Theorem 7.1. From (C1) we deduce, for any smooth function ψ,

$$\mathrm{div}(\Phi\psi^2\sigma^2\nabla\Phi) \geq \psi^2\sigma^2|\nabla\Phi|^2 + 2\Phi\psi\sigma^2\nabla\psi\cdot\nabla\Phi. \tag{C3}$$

Let ζ be a C^∞ function on $[0,\infty)$ with $0 \leq \zeta(t) \leq 1$ and $\zeta(t) = 1$ for $0 \leq t \leq 1$, $\zeta(t) = 0$ for $t \geq 2$. For $R > 0$ and $x \in \mathbb{R}^N$ set $\zeta_R(x) = \zeta(|x|/R)$. Taking $\psi = \zeta_R$ in (C3) and integrating over $\mathbb{R}^N$, we find, by the divergence theorem,

$$\begin{aligned} \int_{\mathbb{R}^N} \zeta_R^2\sigma^2|\nabla\Phi|^2 dx &\leq 2\left|\int_{\mathbb{R}^N} \sigma^2\zeta_R\Phi\nabla\zeta_R\cdot\nabla\Phi dx\right| \\ &\leq 2\left[\int_{R<|x|<2R} \sigma^2\zeta_R^2|\nabla\Phi|^2 dx\right]^{1/2}\left[\int_{\mathbb{R}^N} \sigma^2\Phi^2|\nabla\zeta_R|^2 dx\right]^{1/2}. \end{aligned}$$

By (C2) and the definition of ζ_R, we can find $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} \sigma^2\Phi^2|\nabla\zeta_R|^2 dx \leq C_1,$$

Therefore

$$\int_{\mathbb{R}^N} \zeta_R^2\sigma^2|\nabla\Phi|^2 dx \leq 2\sqrt{C_1}\left[\int_{R<|x|<2R} \zeta_R^2\sigma^2|\nabla\Phi|^2 dx\right]^{1/2}. \tag{C4}$$

This implies that

$$\int_{\mathbb{R}^N} \zeta_R^2\sigma^2|\nabla\Phi|^2 dx \leq 4C_1,$$

and hence, letting $R \to \infty$ in (C4) we obtain

$$\int_{\mathbb{R}^N} \sigma^2|\nabla\Phi|^2 dx = 0.$$

This implies $|\nabla\Phi| \equiv 0$ *a.e.* Hence Φ is a constant. $\square$

Proof of Lemma 2.11. Given W^* satisfying (26), we want to show that $W^* = e^{\frac{1}{2}y^T D^2 m(x_0) y}$.

First we make the transformation $W^* = e^{-\frac{1}{2}y^T D^2 m(x_0) y}\Phi$. By (26), we see that Φ satisfies

$$\begin{cases} \operatorname{div}(e^{\frac{1}{2}y^T D^2 m(x_0) y}\nabla\Phi) = 0 \quad \text{ in } \mathbb{R}^N, \\ 0 < \Phi \leq K_3 e^{-\frac{1}{6}y^T D^2 m(x_0) y}, \\ \sup_{\mathbb{R}^N} \Phi(y) e^{\frac{1}{2}y^T D^2 m(x_0) y} = 1. \end{cases}$$

It remains to show that Φ is a constant. By Theorem 7.1, it suffices to show that for some $C > 0$ and every $R > 1$,

$$\int_{B_R(0)} e^{\frac{1}{2}y^T D^2 m(x_0) y}\Phi^2 dx \leq CR^2. \tag{C5}$$

By noticing that the integrand can be dominated by

$$e^{\frac{1}{2}y^T D^2 m(x_0) y}\Phi^2 \leq K_3^2 e^{\frac{1}{6}y^T D^2 m(x_0) y},$$

we have immediately that (C5) is true. Hence the theorem is proved. □

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