ASYMPTOTIC BEHAVIOR OF THE PRINCIPAL EIGENVALUE FOR COOPERATIVE ELLIPTIC SYSTEMS AND APPLICATIONS

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Abstract. The asymptotic behavior of the principal eigenvalue for general linear cooperative elliptic systems with small diffusion rates is determined. As an application, we show that if a cooperative system of ordinary differential equations has a unique positive equilibrium which is globally asymptotically stable, then the corresponding reaction-diffusion system with either the Neumann boundary condition or the Robin boundary condition also has a unique positive steady state which is globally asymptotically stable, provided that the diffusion coefficients are sufficiently small. Moreover, as the diffusion coefficients approach zero, the positive steady state of the reaction-diffusion system converges uniformly to the equilibrium of the corresponding kinetic system.

1. Introduction

Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^N$ with smooth boundary, denoted as $\partial\Omega$. Given any scalar function $q \in C(\bar{\Omega})$, let $\lambda_1(d\Delta + q)$ denote the smallest eigenvalue of the linear problem

$$d\Delta\phi + q(x)\phi + \lambda\phi = 0 \quad \text{in } \Omega,$$

subject to the Dirichlet boundary condition

$$\phi = 0 \quad \text{on } \partial\Omega$$

or the Neumann boundary condition

$$\partial_\nu \phi = 0 \quad \text{on } \partial\Omega.$$

Here the diffusion coefficient d is assumed to be a positive constant, $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^N$, ν denotes the unit outer normal vector on $\partial\Omega$, and $\partial_\nu\phi := \nabla\phi \cdot \nu$. By standard variational argument, it follows readily that ([2])

$$\lim_{d\to 0+} \lambda_1(d\Delta + q) = -\max_{x\in\bar{\Omega}} q(x). \tag{1.1}$$

It is natural to ask whether (1.1) can be extended to linear elliptic systems. This question arises from the study of some reaction-diffusion systems in population biology [11]. In general, the dynamics of a system of reaction-diffusion equations is more complicated comparing to its corresponding kinetic system of ordinary differential equations. This was illustrated in [16], where the following two species Lotka-Volterra competition model was considered:

$$\begin{cases} u_t = d_1\Delta u + u(m(x) - u - cv) & \text{in } \Omega \times (0,T), \\ v_t = d_2\Delta v + v(m(x) - bu - v) & \text{in } \Omega \times (0,T), \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.2}$$

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Here u and v denote the densities of two species competing for a common resource represented by $m(x)$. The positive constants d_1 and d_2 are their diffusion coefficients, while b and c account for the interspecific competition. In the weak competition case, i.e. $b, c \in (0,1)$, the dynamics of the corresponding kinetic system

$$\begin{cases} U_t = U(m(x) - U - cV), \\ V_t = V(m(x) - bU - V) \end{cases} \tag{1.3}$$

is rather simple. Namely, for each x, there exists a unique positive equilibrium $(U^*, V^*) := (\frac{1-c}{1-bc}m(x), \frac{1-b}{1-bc}m(x))$ which is globally asymptotically stable among solutions of positive initial data. In contrast to the simple dynamics of system (1.3), it is proved in [16] that for some ranges of $b, c \in (0,1)$ and $d_1, d_2 > 0$, system (1.2) does not possess any positive steady state. Moreover, it is shown that for any initial data that are non-negative and not identically zero, one of the species is driven to extinction by its competitor. The assumption that m being non-constant plays a crucial role in such "diffusion-driven extinction" phenomenon. We refer to [1, 13, 18, 19] for related works and [6, 7, 8, 15] for recent development.

While the dynamics of (1.2) can be different from that of (1.3) for some ranges of diffusion coefficients, it is demonstrated in [11] (see also [15]) that when d_1 and d_2 are sufficiently small, then again (1.2) has a unique positive steady state, denoted by (u^*, v^*), which is globally asymptotically stable, and as $d_1, d_2 \to 0$,

$$(u^*, v^*) \to (U^*, V^*) \qquad \text{uniformly in } \bar{\Omega}. \tag{1.4}$$

In general, we have the following question:

Question. *If an ODE system has a unique equilibrium which is globally asymptotically stable, does the corresponding parabolic problem, with small diffusion rates, have a unique steady state which is globally asymptotically stable?*

An affirmative answer to this question means that sometimes the dynamics of the PDE, with small diffusion rates, is indeed fully determined by that of the corresponding kinetic system. This general question was posed by V. Hutson [10]. In [11] this question was addressed for (1.2). The approach of [11] is as follows. First, the existence and asymptotic profiles of positive steady states of (1.2), as $d_1, d_2 \to 0$, are determined. Then the authors proceed to show that any positive steady state is linearly stable, and hence locally asymptotically stable (see, e.g. Theorem 7.6.2 of [23]). Finally, the following result from the monotone dynamical system theory is invoked to yield the uniqueness and global stability of positive steady state of (1.2).

Theorem 1.1 ([9, 12, 23])**.** *Let $\underline{u}$ and $\bar{u}$ be strict sub/super solutions of a monotone dynamical system preserving the order $\preceq$, and that $\underline{u} \preceq \bar{u}$. If every steady state u such that $\underline{u} \preceq u \preceq \bar{u}$ is locally asymptotically stable, then u is unique and globally asymptotically stable in $\{v : \underline{u} \preceq v \preceq \bar{u}\}$.*

The crucial step in the proofs of [11] is to show that every positive steady state is linearly stable. After suitable transformation, it amounts to show that, for d_1, d_2 sufficiently small, the principal eigenvalue of the cooperative elliptic system

$$\begin{cases} d_1 \Delta \phi + a_{11}\phi + a_{12}\psi + \lambda\phi = 0 & \text{in } \Omega, \\ d_2 \Delta \psi + a_{21}\phi + a_{22}\psi + \lambda\psi = 0 & \text{in } \Omega, \\ \partial_\nu \phi = \partial_\nu \psi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

is positive. By the convergence result stated in (1.4), we have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} m-2u^*-cv^* & cu^* \\ bv^* & m-bu^*-2v^* \end{pmatrix} \to \begin{pmatrix} -U^* & cU^* \\ bV^* & -V^* \end{pmatrix},$$

as $d_1, d_2 \to 0$. Since both eigenvalues of the latter matrix are negative at every point $x \in \bar{\Omega}$, one expects that the principal eigenvalue of (1.5) will be positive when d_1, d_2 are small. This is indeed the case, as shown in [11] via a rescaling argument. However, the problem of determining the precise limit of the principal eigenvalue of (1.5) as $d_1, d_2 \to 0$ was left open in [11].

The first goal of this paper is to completely determine the asymptotic limit of the principal eigenvalue for general linear cooperative elliptic systems, including (1.5), when the diffusion coefficients approach zero. To this end, we consider the following eigenvalue problem in vector notation:

$$\begin{cases} D\mathcal{L}\phi + A\phi + \lambda\phi = 0 & \text{in } \Omega, \\ \mathcal{B}\phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $D = \text{diag}(d_1, ..., d_n)$, $d_i > 0$ are positive constants; $\mathcal{L} = \text{diag}(L_1, ..., L_n)$ with L_i being second-order elliptic operators of non-divergence form, i.e. for $1 \le i \le n$ and $1 \le k, l \le N$,

$$L_i u := \alpha^i_{kl}\partial^2_{x_k x_l} u + \beta^i_k \partial_{x_k} u + \gamma^i u, \tag{1.7}$$

where $\alpha^i_{kl} \in C^1(\bar{\Omega})$, $\beta^i_k, \gamma^i \in C(\bar{\Omega})$, and $\eta_0|\xi|^2 < \alpha^i_{kl}(x)\xi_k\xi_l < \eta_1|\xi|^2$ for $\xi \in \mathbb{R}^N$, $x \in \Omega$, and for some positive constants η_0, η_1; $\phi = (\phi_1, ..., \phi_n)^T \in [C^2(\bar{\Omega})]^n$; $A = (a_{ij}) \in \left(C(\bar{\Omega})\right)^{n\times n}$ satisfies $a_{ij}(x) \ge 0$ in Ω when $i \ne j$; $\mathcal{B} = (B_1, ..., B_n)$ are boundary operators satisfying for each i either the Robin boundary condition

$$B_i\phi_i := \partial_\nu \phi_i + p_i(x)\phi_i \quad \text{on } \partial\Omega, \tag{1.8}$$

where $p_i \ge 0$ and $p_i \in C(\bar{\Omega})$, or the Dirichlet boundary condition

$$B_i\phi_i := \phi_i \quad \text{on } \partial\Omega. \tag{1.9}$$

Definition 1.2. *An eigenvalue λ_1 of* (1.6) *is called the principal eigenvalue if $\lambda_1 \in \mathbb{R}$ and for any eigenvalue $\tilde{\lambda}$ such that $\tilde{\lambda} \ne \lambda_1$, we have $Re\,\tilde{\lambda} > \lambda_1$.*

Throughout this paper, λ_1 denotes the principal eigenvalue of problem (1.6). The existence of the principal eigenvalue of (1.6) is obtained by Sweers [24] via the Krein-Rutman Theorem ([14]). Nagel [17] and deFigueiredo-Mitidieri [4] also studied the principal eigenvalue problem, using semigroup theory and maximum principle, respectively. By the Krein-Rutman Theorem for positive compact operators, (1.6) has a principal eigenvalue $\lambda_1 \in \mathbb{R}$ and the corresponding eigenfunction $\phi = (\phi_1, ..., \phi_n)^T$ can be chosen to satisfy $\phi_i \ge 0$ for all i. If in addition we assume that $a_{ij} > 0$ in Ω for all $i \ne j$, then λ_1 is simple and it is the unique eigenvalue corresponding to a strictly positive eigenfunction, i.e. $\phi_i > 0$ in Ω for all i. For later purposes, we provide a proof of this fact in Section 3 (Proposition 3.1).

We recall the existence of the principal eigenvalue for any non-negative matrix, as guaranteed by the Perron-Frobenius Theorem ([5]).

Theorem 1.3. *Given a real-valued square matrix $A = (a_{ij})$, whose off-diagonal terms are non-negative, (i.e. $a_{ij} \ge 0$ if $i \ne j$), there exists a real eigenvalue $\bar{\lambda}(A)$, corresponding to a non-negative eigenvector, with the greatest real part (for any eigenvalue $\lambda' \ne \bar{\lambda}(A)$, $\bar{\lambda}(A) > Re\,\lambda'$). Moreover, if $a_{ij} > 0$ for any $i \ne j$, then*

$\bar\lambda(A)$ is simple with strictly positive eigenvector, and it can be characterized as the unique eigenvalue corresponding to a non-negative vector.

The first main result of this paper is

Theorem 1.4. *Let λ_1 be the principal eigenvalue of (1.6) with boundary condition (1.8) or (1.9). Then*

$$\lim_{\max_{1\leq i\leq n}\{d_i\}\to 0} \lambda_1 = -\max_{x\in\bar\Omega} \bar\lambda(A(x)). \tag{1.10}$$

Theorem 1.4 extends an earlier result of Dancer [3], where it is proved that if d_i go to zero at the same rates, i.e. for each i, $d_i = \epsilon^2 \bar{d}_i$ for some constant $\bar{d}_i > 0$, then (1.10) holds as $\epsilon \to 0$. The proof in [3] is based on solving some limiting eigenvalue problem with constant coefficient in $\mathbb{R}^N$ or in a half space, which is derived by exploiting the same scale of the diffusion rates. Under our assumptions, d_i can approach zero with different rates and hence there might be no limiting eigenvalue problem. A critical ingredient in our proof is a boundary Lipschitz estimate (Theorem 2.2), which seems to be of self interest.

As an application of Theorem 1.4, we consider

$$\begin{cases} \partial_t w = D\mathcal{L}w + F(x,w) & \text{in } \Omega\times(0,T),\\ \mathcal{B}w = 0 & \text{on } \partial\Omega\times(0,T),\\ w(x,0) = w_0(x) & \text{in } \Omega, \end{cases} \tag{1.11}$$

where D, $\mathcal{L}$ are defined as before, $w = (w_1, ..., w_n)^T$ and $\mathcal{B} = (B_1, ..., B_n)$ where B_i satisfies the Neumann or the Robin boundary conditions as given in (1.8). For the reaction term F we assume that $F(x, s_1, ..., s_n) = (F_1, ..., F_n)(x, s_1, ..., s_n) \in C^1(\bar\Omega \times [0,\infty)^n; \mathbb{R}^n)$ and satisfies the following assumptions:

(A1) (cooperativity) $\partial_{s_i} F_j(x, s_1, ..., s_n) \geq 0$ if $i \neq j$.

(A2) (kinetic dynamics) For each $x_0 \in \bar\Omega$, the ODE system

$$\Phi'(t) = F(x_0, \Phi(t)), \quad \Phi(0) \in (0,\infty)^n \tag{1.12}$$

has a unique, globally asymptotically stable equilibrium, denoted by

$$\alpha(x_0) = (\alpha_1(x_0), ..., \alpha_n(x_0)).$$

Moreover, α is continuous in $\bar\Omega$ and as an equilibrium of (1.12), α is linearly stable for each $x_0 \in \bar\Omega$, i.e.,

$$\bar\lambda(D_s F(x_0, \alpha(x_0))) < 0.$$

(A3) (positivity of growth) There exists $\delta_0 > 0$ such that for all $j = 1, ..., n$, $F_j(x, s_1, ..., s_n)/s_j > \delta_0$ for all $x \in \bar\Omega$, and $0 < s_i \leq \delta_0$ for all $i = 1, ..., n$.

(A4) (dissipativity) There exist positive constants $M, \delta_0' > 0$ such that for all $j = 1, ..., n$, $F_j(x, s_1, ..., s_n)/s_j < -\delta_0'$ if $x \in \bar\Omega$ and $\min_{1\leq i\leq n}\{s_i\} \geq M$.

Assumption (A1) means that the system is cooperative, i.e. the growth of any species will help the increase of other species. (A2) says that the kinetic system has a unique equilibrium which attracts all solutions with positive initial data. (A3) ensures that at any location in the habitat the intrinsic growth rate is positive for each species. (A4) guarantees that the solutions of (1.11) will remain uniformly bounded for all time.

Our second main result is

Theorem 1.5. *Assume that* (A1)-(A4) *hold. If* $\max_{1\le i\le n}\{d_i\}$ *is sufficiently small,* (1.11) *has a unique positive steady state, denoted as* $\tilde{w}(x)$*. The positive steady state* $\tilde{w}(x)$ *is globally asymptotically stable among solutions with non-negative, non-trivial initial data. Moreover,* $\tilde{w}(x)\to\alpha(x)$ *uniformly as* $\max_{1\le i\le n}\{d_i\}\to 0$.

This paper is organized as follows: In Section 2 we establish a boundary Lipchitz estimate. For later purposes, the existence of the principal eigenvalue is proved in Section 3, together with some well-known eigenvalue comparison theorem for cooperative elliptic systems. Theorems 1.4 and 1.5 are established in Sections 4 and 5, respectively.

2. A boundary Lipchitz estimate

In this section we establish a boundary Lipchitz estimate for solutions of some linear inhomogeneous second order elliptic equations. To this end, let $f\in C(\bar{\Omega})$. For each $d>0$, let u_d be the solution to the problem

$$\begin{cases} -dL_1 u + u = f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where L_1 is given in (1.7). It is well-known that u_d exists for all $d>0$ sufficiently small, e.g. if $d\|\gamma^1\|_{L^\infty(\Omega)}<1$. We also state the following fact:

Lemma 2.1. *For each compact subset* $K\subset\Omega$, $\|u_d-f\|_{L^\infty(K)}\to 0$ *as* $d\to 0$*. If we assume in addition that* $f|_{\partial\Omega}=0$*, then* $\|u_d-f\|_{L^\infty(\Omega)}\to 0$ *as* $d\to 0$.

For later purposes, we shall prove a boundary Lipschitz estimate of u_d, which implies Lemma 2.1 for $f\in C^1(\bar{\Omega})$.

Theorem 2.2. *If* $f\in C^1(\bar{\Omega})$ *and* $f|_{\partial\Omega}=0$*, then*

$$\sup_{\Omega}\frac{|u_d-f|}{dist(x,\partial\Omega)}\to 0 \qquad \text{as } d\to 0.$$

The following result is a direct consequence of Theorem 2.2.

Corollary 2.3. *If* $f\in C^1(\bar{\Omega})$ *satisfies* $f>0$ *in* Ω, $f|_{\partial\Omega}=0$ *and* $\partial_\nu f|_{\partial\Omega}<0$*, then for any* $\epsilon>0$*, we have*

$$(1-\epsilon)f(x)<u_d(x)<(1+\epsilon)f(x) \qquad \text{in } \Omega,$$

for all sufficiently small $d>0$.

Assume in addition that $f>0$ in Ω and $\partial_\nu f|_{\partial\Omega}<0$. Theorem 2.2 follows from the next two propositions.

Proposition 2.4. *Suppose* $f>0$ *in* Ω *and* $\partial_\nu f|_{\partial\Omega}<0$*. Then for each* $\epsilon\in(0,\frac{1}{3}\inf_{\partial\Omega}|\partial_\nu f|)$*, we have*

$$u_d<f+3\epsilon\, dist(x,\partial\Omega) \qquad \text{in } \Omega, \tag{2.2}$$

for all sufficiently small $d>0$.

Proposition 2.5. *Suppose* $f>0$ *in* Ω *and* $\partial_\nu f|_{\partial\Omega}<0$*. Then for each* $\epsilon\in(0,\frac{1}{3}\inf_{\partial\Omega}|\partial_\nu f|)$*, we have*

$$u_d>f-3\epsilon\, dist(x,\partial\Omega) \qquad \text{in } \Omega, \tag{2.3}$$

for all sufficiently small $d>0$.

Assume that Propositions 2.4 and 2.5 hold.

Proof of Theorem 2.2. By Propositions 2.4 and 2.5, Theorem 2.2 is proved with the extra assumptions $f > 0$ in Ω and $\partial_\nu f|_{\partial\Omega} < 0$. To remove these extra assumptions, choose $\varphi \in C^1(\bar{\Omega})$ such that $\varphi|_{\partial\Omega} = 0$, $\varphi > \max\{0, -f\}$ in Ω and $\partial_\nu \varphi < \min\{0, -\partial_\nu f\}$ on $\partial\Omega$. Let v_d be the unique solution to

$$-dLv + v = \varphi > 0 \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0.$$

Then $u_d + v_d$ is the unique solution to

$$-dLz + z = f + \varphi > 0 \quad \text{in } \Omega, \quad z|_{\partial\Omega} = 0.$$

As previous arguments are applicable, we have

$$0 \leq \frac{|u_d - f|}{\text{dist}(x, \partial\Omega)} \leq \frac{|v_d - \varphi|}{\text{dist}(x, \partial\Omega)} + \frac{|(u_d + v_d) - (f + \varphi)|}{\text{dist}(x, \partial\Omega)} \to 0$$

uniformly in Ω, as $d \to 0$. This proves Theorem 2.2. □

We now proceed to prove Propositions 2.4 and 2.5 via a careful argument via barrier functions. Given $\epsilon > 0$, choose $R > 0$ small such that

(B1) For all $x_0 \in \partial\Omega$, there exists $B_R \subset \Omega$ such that $\partial B_R \cap \partial\Omega = \{x_0\}$.
(B2) For all $x_0 \in \partial\Omega$, there exists $\tilde{B}_R \subset \mathbb{R}^N \setminus \bar{\Omega}$ such that $\partial \tilde{B}_R \cap \bar{\Omega} = \{x_0\}$.
(B3) $|\nabla f(x_1) - \nabla f(x_2)| < \epsilon/2$ for all $x_1, x_2 \in \bar{\Omega}$ such that $|x_1 - x_2| \leq 2R$.

By (B1), for all $t \in (0, R]$, $\Gamma_t : \partial\Omega \to \{x \in \Omega : \text{dist}(x, \partial\Omega) = R\}$ defined by

$$\Gamma_t(x_0) := x_0 - t\nu_{x_0},$$

where ν_{x_0} is the unit outer normal of $\partial\Omega$ at x_0, is a diffeomorphism. By (B3),

$$0 < f(x) < \left[-\partial_\nu f(x_0) + \frac{\epsilon}{2}\right] \text{dist}(x, \partial\Omega) \quad \text{in } \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq R\}, \tag{2.4}$$

where $x_0 \in \partial\Omega$ is the unique point on $\partial\Omega$ closest to x.

Define the barrier function $\rho \in C^2(B_{3R/2} \setminus \{0\})$ by

$$\rho(x) := \frac{R^{\sigma+1}}{\sigma}\left(|x|^{-\sigma} - R^{-\sigma}\right),$$

where the parameter $\sigma > 1$ is specified in the following result:

Lemma 2.6. *If*

$$\sigma = \frac{N\eta_1}{\eta_0} + \frac{3R|\beta_k^1|_{L^\infty(\Omega)}}{2\eta_0} + \frac{9R^2|\gamma^1|_{L^\infty(\Omega)}}{4\eta_0},$$

then $L_1\rho \geq 0$ *in* $B_{3R/2} \setminus \{0\}$, *where* $\eta_0, \eta_1, \beta_k^1, \gamma^1$ *are given after* (1.7).

Proof. Let $|x| = r$.

$$\begin{aligned}\frac{\sigma}{R^{\sigma+1}} L_1\rho &= r^{-\sigma-2}\left[\sigma(\sigma+2)\alpha_{kl}^1 x_k x_l/r^2 - \alpha_{kk}^1\sigma - \sigma\beta_k^1 x_k + \gamma^1 r^2\right] \\ &\geq r^{-\sigma-2}\left[\sigma(\sigma+2)\eta_0 - N\eta_1\sigma - \sigma|\beta_k^1|_{L^\infty(\Omega)}(3R/2) - |\gamma^1|_{L^\infty(\Omega)}(3R/2)^2\right] \\ &\geq 0.\end{aligned}$$

This completes the proof. □

We start with a crude estimate on u_d. Define

$$\delta_1 := \min\left\{R\left[\left(\frac{c_1 + 2\epsilon}{c_1 + \epsilon}\right)^{\frac{1}{\sigma+1}} - 1\right], \frac{2\epsilon}{(\sigma+1)(c_1 + 2\epsilon)}, \frac{R}{2}\right\}, \tag{2.5}$$

where $c_1 = \max_{\partial\Omega} |\partial_\nu f|$.

Lemma 2.7. *There exists* $\bar{d}_1 > 0$ *such that for all* $d \in (0, \bar{d}_1]$,

$$u_d(x) \le (-\partial_\nu f(y_x) + \epsilon/2)\delta_1 \quad \text{in } \{x \in \Omega : dist(x, \partial\Omega) \le \delta_1\},$$

where y_x *is the unique point on* $\partial\Omega$ *closest to* x.

Proof. By (2.4), there exists $\bar{f} \in C^2(\bar{\Omega})$ such that $\bar{f} > f$ in $\bar{\Omega}$, and

$$\bar{f}(x) \le (-\partial_\nu f(y_x) + \epsilon/2)\delta_1 \quad \text{in } \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \le \delta_1\}.$$

Then for $d > 0$ sufficiently small,

$$-dL\bar{f} + \bar{f} > f \quad \text{in } \Omega, \quad \bar{f}|_{\partial\Omega} > 0.$$

By the maximum principle [20], $u_d \le \bar{f}$. This proves the lemma. □

To prove Proposition 2.4, by Lemma 2.1 it suffices to establish (2.2) in a neighborhood of $\partial\Omega$. In fact, it suffices to show that there exists $\bar{d}_1 > 0$ such that for all $d \in (0, \bar{d}_1)$ and each $x_0 \in \partial\Omega$, (2.2) holds for $x = x_0 - t\nu_{x_0}$ for all $t \in (0, \delta_1]$, where ν_{x_0} is the unit outward normal vector at $x_0 \in \partial\Omega$ and δ_1 is defined in (2.5). Without loss of generality, assume $x_0 = (0, ..., 0, R)$, $\nu_{x_0} = (0, ..., 0, -1)$ and the exterior sphere at x_0 guaranteed by (B2) is $B_R(0)$. We establish the following result:

Lemma 2.8. *Define* $z \in C^2(\overline{B_{R+\delta_1}(0) \setminus B_R(0)})$ *by*

$$z(x) := (-\partial_\nu f(x_0) + 2\epsilon)(-\rho(x)).$$

Then

(i) $z(x) > f(x)$ *in* $\Omega \cap B_{R+\delta_1}(0)$;
(ii) $z(0, ..., 0, R + t) \le f(0, ..., 0, R + t) + 3\epsilon t$ *for all* $t \in [0, \delta_1]$.

Proof. To show (i), we note that for all $t \in (0, \delta_1)$,

$$\partial_{x_N} z(0, ..., 0, t + R) = (\partial_{x_N} f(x_0) + 2\epsilon)\left(\frac{R}{t + R}\right)^{\sigma+1} \ge \partial_{x_N} f(x_0) + \epsilon,$$

by our choice of δ_1 in (2.5). Since

$$z(0, ..., 0, R) = f(0, ..., 0, R) = 0, \tag{2.6}$$

we have

$$z(0, ..., 0, t + R) > t(\partial_{x_N} f(x_0) + \epsilon) \quad \text{for } t \in (0, \delta_1]. \tag{2.7}$$

For $x \in \Omega \cap B_{R+\delta_1}(0)$, if we write $x = y - t\nu_y$ (where y is the unique point of $\partial\Omega$ closest to x), then necessarily $0 < t \le |x| - R \le \delta_1$, since $\overline{B_R(0)} \cap \Omega = \emptyset$. Now $f|_{\partial\Omega} = 0$, and by (B3), $|\nabla f(x)| < \partial_{x_N} f(x_0) + \epsilon/2$ in $B_{2R}(x_0) \cap \Omega$. If we integrate it from $\partial\Omega$ along the normal direction, then $f(x) \le t(\partial_{x_N} f(x_0) + \epsilon)$. Together with (2.7), we obtain

$$f(x) \le t(\partial_{x_N} f(x_0) + \epsilon) < z(0, ..., t + R) \le z(0, ..., 0, |x|) = z(x).$$

This proves (i). Part (ii) follows from (2.6), (B3) and

$$\begin{aligned} \partial_{x_N} z(0, ..., t + R) &= (\partial_{x_N} f(x_0) + 2\epsilon)\left(\frac{R}{t + R}\right)^{\sigma+1} \\ &\le \partial_{x_N} f(x_0) + 2\epsilon \\ &< \partial_{x_N} f(0, ..., t + R) + 3\epsilon \end{aligned}$$

for $t \in [0, \delta_1]$. □

Proof of Proposition 2.4. It suffices to show that

$$u_d \le z \quad \text{in } \Omega' := B_{R+\delta_1}(0) \cap \Omega \text{ for all } d \in (0, \bar{d}_1). \tag{2.8}$$

We will proceed by comparison. Firstly, by Lemma 2.8(i),

$$-dL_1 z + z \ge z \ge f \quad \text{in } \Omega'. \tag{2.9}$$

Secondly,

$$u_d = 0 \le z \quad \text{on } \partial\Omega' \cap \partial\Omega. \tag{2.10}$$

Thirdly, for each $x \in \partial\Omega' \cap \Omega$, and all $d \in (0, \bar{d}_1)$,

$$u_d(x) \le \left(-\partial_\nu f(y_x) + \frac{\epsilon}{2}\right) \delta_1 \le (-\partial_\nu f(x_0) + \epsilon)\, \delta_1, \tag{2.11}$$

where the first and second inequalities follow from Lemma 2.7 and (B3), respectively.

Next, we claim that

$$(-\partial_\nu f(x_0) + \epsilon)\, \delta_1 \le z(x) \quad \text{on } \partial\Omega' \cap \Omega. \tag{2.12}$$

To show (2.12), we note that by (2.5),

$$1 - \sigma\frac{\delta_1}{R} + \frac{\sigma(\sigma+1)}{2}\left(\frac{\delta_1}{R}\right)^2 \le 1 - \sigma\left(\frac{-\partial_\nu f(x_0) + \epsilon}{-\partial_\nu f(x_0) + 2\epsilon}\right)\frac{\delta_1}{R}.$$

In view of the inequality $(1+s)^{-\sigma} \le 1 - \sigma s + \frac{\sigma(\sigma+1)}{2}s^2$ for $s \ge 0$, we have

$$\left(\frac{1}{1+\frac{\delta_1}{R}}\right)^\sigma \le 1 - \sigma\left(\frac{-\partial_\nu f(x_0) + \epsilon}{-\partial_\nu f(x_0) + 2\epsilon}\right)\frac{\delta_1}{R},$$

which is equivalent to (2.12). Combining (2.10), (2.11) and (2.12), we have

$$u_d \le z \quad \text{on } \partial\Omega' \text{ for all } d \in (0, \bar{d}_1). \tag{2.13}$$

By (2.9) and (2.13), we can conclude (2.8) by the comparison principle. □

We now proceed to establish Proposition 2.5. To this end, define

$$\delta_2 := \min\left\{R\left[1 - \left(\frac{c_2 - 2\epsilon}{c_2 - \epsilon}\right)^{\frac{1}{\sigma+1}}\right], \delta_1, \frac{R}{2}\right\}, \quad \text{with } c_2 = \inf_{\partial\Omega}|\partial_\nu f|. \tag{2.14}$$

As before, by Lemma 2.1 it suffices to show that there exists $\bar{d}_2 > 0$ such that for all $d \in (0, \bar{d}_2)$ and for each $x_0 \in \partial\Omega$, (2.3) holds for $x = x_0 - t\nu_{x_0}$ for all $t \in (0, \delta_2]$. Without loss of generality, assume $x_0 = (0, ..., 0, -R)$, $\nu_{x_0} = (0, ..., 0, -1)$ and the interior sphere at x_0 is $B_R(0)$.

Lemma 2.9. *Define* $w \in C^2(\overline{B_R(0) \setminus B_{R-\delta_2}(0)})$ *by*

$$w(x) := (-\partial_\nu f(x_0) - 2\epsilon)\rho(x).$$

Then

(i) $w(x) < f(x)$ *in* $B_R(0) \setminus B_{R-\delta_2}(0)$;
(ii) $w(0, ..., 0, -R+t) \ge f(0, ..., 0, -R+t) - 3\epsilon t$ *for all* $t \in [0, \delta_2]$.

Proof. Part (ii) follows from $w(0,...,0,-R) = f(0,...,0,-R) = 0$, and

$$\begin{aligned}\partial_{x_N} w(0,...,0,-R+t) &\geq \partial_{x_N} w(0,...,0,-R)\\ &= \partial_{x_N} f(0,...,0,-R) - 2\epsilon\\ &> \partial_{x_N} f(0,...,0,-R+t) - 3\epsilon,\end{aligned}$$

where the last inequality follows from (B3).

Part (i) follows from $w = 0 \leq f$ on $\partial B_R(0)$, and

$$\begin{aligned}\partial_{x_N} w(x) &= (\partial_{x_N} f(x_0) - 2\epsilon)\left(\frac{R}{|x|}\right)^{\sigma+1}\frac{-x_N}{|x|}\\ &\leq (\partial_{x_N} f(x_0) - 2\epsilon)\left(\frac{R}{R-\delta_2}\right)^{\sigma+1}\\ &\leq \partial_{x_N} f(x_0) - \epsilon\\ &< \partial_{x_N} f(x) - \frac{\epsilon}{2}\end{aligned}$$

in $B_R(0) \setminus B_{R-\delta_2}(0)$. The second last inequality follows from (2.14) and the last inequality is a consequence of (B3). □

Proof of Proposition 2.5. By the proof of Lemma 2.9(i), we have actually shown

$$f(x) - w(x) \geq \frac{\delta_2 \epsilon}{2} \quad \text{in } \{x \in B_R(0) : \operatorname{dist}(x, \partial\Omega) = \delta_2\}. \tag{2.15}$$

Let $\Omega'' = \{x \in B_R(0) : \operatorname{dist}(x, \partial\Omega) < \delta_2\}$. By Lemma 2.9(ii), it suffices to show that for some $\bar{d}_2 = \bar{d}_2(\delta_2, \epsilon, \Omega, f)$,

$$u_d \geq w \quad \text{in } \Omega'' \quad \text{for all } d \in (0, \bar{d}_2). \tag{2.16}$$

We proceed by comparison method. Firstly, by Lemma 2.9(i) and $L_1 w \geq 0$,

$$-dL_1 w + w \leq w < f \quad \text{in } \Omega''. \tag{2.17}$$

Secondly,

$$w = 0 \leq u_d \quad \text{on } \partial\Omega'' \cap \partial B_R(0). \tag{2.18}$$

Thirdly, by Lemma 2.1, there exists $\bar{d}_2 = \bar{d}_2(\delta_2, \epsilon, \Omega, f)$ such that for all $d \in (0, \bar{d}_2)$,

$$w \leq f - \frac{\delta_2 \epsilon}{2} < u_d \quad \text{in } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = \delta_2\}. \tag{2.19}$$

In particular, $w < u_d$ on $\partial\Omega'' \cap B_R(0)$ for all $d \in (0, \bar{d}_2)$. Hence by (2.17), (2.18) and (2.19), we deduce (2.16) by the comparison principle. Hence (2.3) holds for $x = x_0 - t\nu_{x_0}$ and all $t \in (0, \delta_2]$. This completes the proof of Proposition 2.5. □

3. **Existence and comparison theorems for principle eigenvalues**

In this section, following the approach in [24], we demonstrate the existence of the principal eigenvalue for (1.6) via the Krein-Rutman Theorem and establish some comparison theorems for the principle eigenvalues.

Proposition 3.1. *There exists a principal eigenvalue $\lambda_1 \in \mathbb{R}$ for* (1.6)*, with either boundary condition* (1.8) *or* (1.9).

By replacing λ by $\lambda - C$ for some large constant C, we may assume that $a_{ij} \geq 0$ in Ω for all i, j. Choose any $\beta > 0$ large such that $(-d_i L_i + \beta)^{-1}$ exists for the respective boundary conditions and is positive. Let $\lambda \in [-\beta, \infty)$. Define $K_{\lambda,\beta} : [C(\bar{\Omega})]^n \to [C(\bar{\Omega})]^n$ by

$$K_{\lambda,\beta} u = (-D\mathcal{L} + \beta I)^{-1}[Au + (\lambda + \beta)u]. \tag{3.1}$$

As $K_{\lambda,\beta}$ is a positive compact operator, by the Krein-Rutman theorem, $K_{\lambda,\beta}$ has a principal eigenvalue $r(K_{\lambda,\beta}) > 0$ given by

$$r(K_{\lambda,\beta}) = \lim_{m\to\infty} \sqrt[m]{\|(K_{\lambda,\beta})^m\|}.$$

The following result is a consequence of the monotonicity of $r(K_{\lambda,\beta})$ in $\lambda \in [-\beta, \infty)$, for each $\beta > 0$. Clearly, Proposition 3.1 follows from Lemma 3.2.

Lemma 3.2. *The principal eigenvalue λ_1 of* (1.6) *exists. Furthermore, let $\beta > 0$ and $\lambda \in [-\beta, \infty)$ be given.*

(i) *If $r(K_{\lambda,\beta}) = 1$, then $\lambda = \lambda_1$;*
(ii) *If $r(K_{\lambda,\beta}) < 1$, then $\lambda < \lambda_1$;*
(iii) *If $r(K_{\lambda,\beta}) > 1$, then $\lambda > \lambda_1$.*

Proof. Without loss of generality, assume $D = I$, i.e. $d_i \equiv 1$ for all i. Firstly, part (i) can be verified in a straightforward manner. Secondly, suppose that $r(K_{\lambda_0,\beta}) < 1$ for some $\beta > 0$ and $\lambda_0 \in [-\beta, \infty)$. Define $\mathcal{Z} : [C(\bar{\Omega})]^n \to [C(\bar{\Omega})]^n$ by

$$(\mathcal{Z}u)_i := (-L_i + \beta)^{-1} u_i.$$

Since for any $\lambda \geq \lambda_0$,

$$(K_{\lambda,\beta} u)_i \geq \lambda[(-L_i + \beta)^{-1} u_i] = (\mathcal{Z}u)_i$$

holds, where $(-L_i + \beta)^{-1}$ is the inverse of $(-L_i + \beta)$ (with boundary conditions), we have $r(K_{\lambda,\beta}) \geq \lambda r(\mathcal{Z})$ for all $\lambda \geq \lambda_0$. Since $r(\mathcal{Z}) > 0$ by the positivity of $\mathcal{Z}$, by the continuity of $r(K_{\lambda,\beta})$ in λ, $r(K_{\hat{\lambda},\beta}) = 1$ for some $\hat{\lambda} > \lambda_0$. Therefore by part (i), $\lambda_0 < \hat{\lambda} = \lambda_1$, where λ_1 is the principal eigenvalue of (1.6). Finally, suppose that $r(K_{\lambda_0,\beta}) > 1$ for some $\beta > 0$ and $\lambda_0 \in [-\beta, \infty)$. Then for $\lambda > \lambda_0$, $(K_{\lambda,\beta} u)_i > (K_{\lambda_0,\beta} u)_i$ for all i and $u_i \geq 0$. So $r(K_{\lambda,\beta}) \geq r(K_{\lambda_0,\beta}) > 1$ for all $\lambda \geq \lambda_0$. If the principal eigenvalue λ_1 of (1.6) exists, then we must have $\lambda_1 < \lambda_0$. Thus it remains to show the existence of λ_1. For that purpose, we observe that $r(K_{-\beta',\beta'}) < 1$ for some $\beta' > 0$, as $\|(-L_i + \beta')^{-1}\| \to 0$ when $\beta' \to \infty$, via the maximum principle. By the previous part of the proof, there exists $\lambda' \in (-\beta', \infty)$ such that $r(K_{\lambda',\beta'}) = 1$, i.e., λ_1 exists and equals λ'. $\square$

Next, we present a well-known comparison theorem for principle eigenvalues of cooperative systems.

Definition 3.3. (i) *Let λ_1^* be the principal eigenvalue of*

$$\begin{cases} D\mathcal{L}\phi^* + A^*\phi^* + \lambda^*\phi^* = 0 & \text{in } \Omega, \\ \mathcal{B}^*\phi^* = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $A^*(x) = (a^*_{ij}(x)) \in \left(C(\bar{\Omega})\right)^{n\times n}$ satisfies $a^*_{ij} \geq 0$ in Ω when $i \neq j$; $\mathcal{B}^* = (B^*_1, ..., B^*_n)$ with $B^*_i = \partial_\nu + p^*_i$, where $p^*_i \geq 0$, $p^*_i \in C(\bar{\Omega})$.

(ii) *Let λ_1^D be the principal eigenvalue of* (1.6) *with Dirichlet boundary condition* (1.9).

(iii) *For each smooth subdomain* $\Omega_0 \subset \Omega$, *let* $\lambda_1^D(\Omega_0)$ *be the principal eigenvalue of* (1.6) *with* Ω *replaced by* Ω_0, *and Dirichlet boundary condition on* $\partial\Omega_0$.

Proposition 3.4. *Suppose that* $a_{ij}^* \geq a_{ij}$ *in* $\bar{\Omega}$ *for* $1 \leq i, j \leq n$, *and* $p_i^* \leq p_i$ *in* $\bar{\Omega}$ *for* $1 \leq i \leq n$ *if* B_i *satisfies* (1.8). *Let* λ_1 *be the principal eigenvalue of* (1.6) *with boundary condition* (1.8). *Then* $\lambda_1^* \leq \lambda_1 \leq \lambda_1^D \leq \lambda_1^D(\Omega_0)$.

Proof. Define $(-D\mathcal{L} + \beta I)^{-1}$, $(-D\mathcal{L} + \beta I)_D^{-1}$ and $[(-D\mathcal{L} + \beta I)^*]^{-1}$ as the inverse operator of $(-D\mathcal{L} + \beta I)$ on Ω with boundary conditions $\mathcal{B}[\cdot] = 0$, $\mathcal{D}[\cdot] = 0$ and $\mathcal{B}^*[\cdot] = 0$, respectively. (Here $\mathcal{D}\phi = \phi|_{\partial\Omega}$.) Let $(-D\mathcal{L} + \beta I)_{0,D}^{-1}$ denote the inverse operator of $(-D\mathcal{L} + \beta I)$ on Ω_0 with Dirichlet boundary condition on $\partial\Omega_0$. As before, by replacing λ with $\lambda - C$ for some large constant $C > 0$, and take a large positive β, we may assume that $a_{ij}^* \geq a_{ij} \geq 0$ in Ω for all i, j, and that for any $\beta > 0$, $(-D\mathcal{L} + \beta I)^{-1}$, $(-D\mathcal{L} + \beta I)_D^{-1}$, $[(-D\mathcal{L} + \beta I)^*]^{-1}$ and $(-D\mathcal{L} + \beta I)_{0,D}^{-1}$ exist and are positive.

First we show $\lambda_1^* \leq \lambda_1$. Let $K^* : [C(\bar{\Omega})]^n \to [C(\bar{\Omega})]^n$ be defined by

$$K^* u := [(-D\mathcal{L} + |\lambda_1^*| + 1)^*]^{-1}(A^* u + (\lambda_1^* + |\lambda_1^*| + 1)u).$$

Then $r(K^*) = 1$ by Lemma 3.2. It follows from the maximum principle that for all $f \in C(\bar{\Omega})$ and $f \geq 0$,

$$[(-D\mathcal{L} + (|\lambda_1^*| + 1)I)^*]^{-1} f \geq (-D\mathcal{L} + (|\lambda_1^*| + 1)I)^{-1} f.$$

So $K_{\lambda_1^*, |\lambda_1^*|+1} f \leq K^* f$ holds for all $f = (f_1, ..., f_n)^T \in [C(\bar{\Omega})]^n$ such that $f_i \geq 0$ in Ω. Therefore,

$$r(K_{\lambda_1^*, |\lambda_1^*|+1}) \leq r(K^*) = 1.$$

Hence, by Lemma 3.2, $\lambda_1 \geq \lambda_1^*$. The proof for $\lambda_1 \leq \lambda_1^D$ is similar and is omitted.

Next, we show $\lambda_1^D \leq \lambda_1^D(\Omega_0)$. Define $K^D : [C(\bar{\Omega})]^n \to [C(\bar{\Omega})]^n$ and $K_0^D : [C(\bar{\Omega})]^n \to [C(\bar{\Omega}_0)]^n$ by

$$K^D f := (-D\mathcal{L} + (|\lambda_1^D(\Omega_0)| + 1)I)_D^{-1}(Af + (\lambda_1^D(\Omega_0) + |\lambda_1^D(\Omega_0)| + 1)f),$$
$$K_0^D f := (-D\mathcal{L} + (|\lambda_1^D(\Omega_0)| + 1)I)_{0,D}^{-1} \left[(Af + (\lambda_1^D(\Omega_0) + |\lambda_1^D(\Omega_0)| + 1)f)|_{\Omega_0}\right].$$

Applying the maximum principle to Ω_0, one can show that for any $f \geq 0$, $f \in C(\bar{\Omega})$,

$$(-D\mathcal{L} + (|\lambda_1^D(\Omega_0)| + 1)I)_D^{-1} [f] \geq (-D\mathcal{L} + (|\lambda_1^D(\Omega_0)| + 1)I)_{0,D}^{-1} [f|_{\Omega_0}]$$

holds in Ω_0. Hence, for all $f = (f_1, ..., f_n) \in [C(\bar{\Omega})]^n$ such that $f_i \geq 0$ in Ω_0 and $f_i = 0$ in $\Omega \setminus \Omega_0$, we have $K^D f|_{\Omega_0} \geq K_0^D f$ in Ω_0. So $r(K^D) \geq r(K_0^D) = 1$. By Lemma 3.2, we obtain $\lambda_1^D \leq \lambda_1^D(\Omega_0)$. □

4. Proof of Theorem 1.4

In this section we establish Theorem 1.4. For the sake of clarity we divide the proofs into several lemmas. It is clear that Theorem 1.4 follows immediately from Lemmas 4.1 and 4.3.

First we consider the lower bound of the principle eigenvalue of (1.6).

Lemma 4.1. *The following estimate holds:*

$$\liminf_{\max_{1 \leq i \leq n}\{d_i\} \to 0} \lambda_1 \geq -\max_{x \in \bar{\Omega}} \bar{\lambda}(A(x)). \tag{4.1}$$

Proof. We only need to treat the Neumann boundary condition (i.e. (1.8) with $p_i \equiv 0$), since by Proposition 3.4, replacing the Neumann boundary condition by the Dirichlet boundary condition or the Robin boundary condition only increases λ_1. By Lemma 3.2, it suffices to show that given $\epsilon > 0$, if $\lambda < -\max_{x\in\bar\Omega} \bar\lambda(A(x)) - \epsilon$, then $r(K_{\lambda,|\lambda|+1}) < 1$ for $\max_{1\leq i\leq n}\{d_i\}$ sufficiently small.

Replacing λ by $\lambda - C$ for some large constant $C > 0$, we may assume $a_{ij} \geq 0$ for all i, j. For the meanwhile, we make the additional assumption that

$$a_{ij}(x) > 0 \quad \text{in } \bar\Omega \quad \text{for all } i, j. \tag{4.2}$$

For each $x \in \bar\Omega$, $A(x)$ is a cooperative matrix. By the Perron-Frobenius Theorem, for each $x \in \bar\Omega$ there exists a unique eigenvalue $\bar\lambda(A(x)) \in \mathbb{R}$ with a corresponding non-negative eigenvector $\Phi(x) = (\Phi_1(x), ..., \Phi_n(x))$, $\Phi_i(x) \geq 0$ for all i. That is, $A(x)\Phi(x) = \bar\lambda(A(x))\Phi(x)$ for all $x \in \bar\Omega$, where we normalize $\Phi(x)$ by $\sum_i \Phi_i^2(x) = 1$. Moreover, by (4.2), $\bar\lambda(A(x))$ is simple and $\Phi_i > 0$ in $\bar\Omega$ for all i.

Claim 4.2. $\Phi \in [C(\bar\Omega)]^n$.

To establish our assertion, given any $x_0 \in \bar\Omega$, let x_k be a sequence in $\bar\Omega$ such that $x_k \to x_0$, and $A(x_k)\Phi(x_k) = \bar\lambda(A(x_k))\Phi(x_k)$. By passing to a subsequence, we may assume that for some $\Phi' \in \mathbb{R}^n$, $\lambda' \in \mathbb{R}$, $\Phi(x_k) \to \Phi'$ and $\bar\lambda(A(x_k)) \to \lambda'$ as $k \to \infty$, with $\Phi'_i \geq 0$ for all i, $\sum_i(\Phi'_i)^2 = 1$, and $A(x_0)\Phi' = \lambda'\Phi'$. Since Φ' is a non-negative eigenvector of $A(x_0)$, by the simplicity of the principle eigenvalue we must have $\Phi' = \Phi(x_0)$ and $\lambda' = \bar\lambda(A(x_0))$. Since the limits λ', Φ' are independent of subsequences, the full sequence converges and the claim is proved.

Next, we claim that for each i, if $f \in C(\bar\Omega)$, then as $d_i \to 0$,

$$(-d_iL_i + |\lambda| + 1)^{-1} f \to \frac{f}{|\lambda| + 1} \quad \text{in } L^\infty(\Omega). \tag{4.3}$$

To show (4.3), we observe that for any $\epsilon > 0$, there exists $\tilde f \in C^2(\bar\Omega)$ such that

$$f - \frac{\epsilon}{2} \leq \tilde f \leq f + \frac{\epsilon}{2} \quad \text{in } \bar\Omega \quad \text{and} \quad \partial_\nu \tilde f = 0 \quad \text{on } \partial\Omega.$$

Then for d_i small,

$$(-d_iL_i + |\lambda| + 1)\left(\frac{\tilde f}{|\lambda| + 1} + \epsilon\right) = -d_iL_i\left(\frac{\tilde f}{|\lambda| + 1} + \epsilon\right) + \tilde f + \epsilon(|\lambda| + 1) \geq f$$

in Ω. As $(-d_iL_i + |\lambda| + 1)^{-1}$ (with the Neumann boundary condition) is a positive operator for d_i small, we can conclude by the comparison principle that for d_i small,

$$(-d_iL_i + |\lambda| + 1)^{-1} f \leq \frac{\tilde f}{|\lambda| + 1} + \epsilon \leq \frac{f}{|\lambda| + 1} + \left(\frac{1}{|\lambda| + 1} + 1\right)\epsilon.$$

Similarly, for d_i small we have

$$(-d_iL_i + |\lambda| + 1)^{-1} f \geq \frac{\tilde f}{|\lambda| + 1} - \epsilon \geq \frac{f}{|\lambda| + 1} - \left(\frac{1}{|\lambda| + 1} + 1\right)\epsilon.$$

Hence (4.3) is proved.

Next, we observe that

$$A(x)\Phi(x) + (\lambda + |\lambda| + 1)\Phi(x) = [\bar\lambda(A(x)) + \lambda + |\lambda| + 1]\Phi(x).$$

Hence, as $\max_{1\le i\le n}\{d_i\}\to 0$,

$$K_{\lambda,|\lambda|+1}\Phi(x)\to \frac{\bar\lambda(A(x))+\lambda+|\lambda|+1}{|\lambda|+1}\Phi(x)$$

in $L^\infty(\Omega)$. Since we chose $\lambda < -\max_{\bar\Omega}\bar\lambda(A(x))-\epsilon$, so for d_i small,

$$K_{\lambda,|\lambda|+1}\Phi < \frac{|\lambda|+1-\epsilon/2}{|\lambda|+1}\Phi.$$

Hence, $r(K_{\lambda,|\lambda|+1})<1$. This proves (4.1) under assumption (4.2).

We now remove the extra assumption (4.2). Let $\delta>0$ be any small constant. Consider (1.6) with the Neumann boundary condition, with $A(x)=(a_{ij}(x))$ replaced by $\tilde A(x)=(\tilde a_{ij}(x)) := (a_{ij}(x)+\delta)$. Denote the corresponding principle eigenvalue by $\tilde\lambda_1$. Previous arguments apply and we have

$$\liminf_{\max_{1\le i\le n}\{d_i\}\to 0}\tilde\lambda_1 \ge -\max_{x\in\bar\Omega}\bar\lambda(\tilde A(x)).$$

By Proposition 3.4, $\lambda_1\ge\tilde\lambda_1$. Hence, $\liminf_{\max_i\{d_i\}\to 0}\lambda_1 \ge -\max_{\bar\Omega}\bar\lambda(\tilde A(x))$. On the other hand, $\bar\lambda(\tilde A(x))\to\bar\lambda(A(x))$ uniformly in $\bar\Omega$ as $\delta\to 0$. This proves (4.1). □

Next, we consider the upper bound of the principle eigenvalue of (1.6).

Lemma 4.3. *The following estimate holds:*

$$\limsup_{\max_{1\le i\le n}\{d_i\}\to 0}\lambda_1 \le -\max_{x\in\bar\Omega}\bar\lambda(A(x)). \tag{4.4}$$

Here the boundary condition of (1.6) *can be either* (1.8) *or* (1.9).

Proof. Given any $\delta'>0$, choose $B=B_r(x_0)$ with $r>0$ and $x_0\in\Omega$ so that $B\subset\Omega$ and $\bar\lambda(A')>\max_{\bar\Omega}\bar\lambda(A(x))-\delta'$, where $A'=(a'_{ij})$ is a constant $n\times n$ matrix given by $a'_{ij}=\min_{x\in\bar B}a_{ij}(x)$. Let λ'_1 be the principal eigenvalue of the problem

$$\begin{cases} D\mathcal{L}\phi + A'\phi+\lambda\phi = 0 & \text{in } B,\\ \phi=0 & \text{on } \partial B.\end{cases}$$

By Proposition 3.4, $\lambda'_1\ge\lambda_1$.

Claim 4.4. $\limsup_{\max_{1\le i\le n}\{d_i\}\to 0}\lambda'_1 \le -\bar\lambda(A')$.

To establish this assertion, choose $\varphi\in C^1(\bar B)$ so that $\varphi>0$ in B, $\varphi|_{\partial B}=0$, and $\partial_\nu\varphi|_{\partial B}<0$. Similar as before, given $\eta>0$, set $\lambda=-\bar\lambda(A')+\eta$ and

$$A'_\lambda u := (-D\mathcal{L}+|\lambda|+1)^{-1}[A'u+(\lambda+|\lambda|+1)u].$$

Set $u_i:=a_i\varphi$, where $a_i\ge 0$ are constants, not all equal to zero, which satisfy $A'_{ij}a_j=\bar\lambda(A')a_i$. Choose $\epsilon>0$ such that

$$\frac{\eta+|\lambda|+1}{|\lambda|+1}(1-\epsilon)>1.$$

Then for $\max_{1\le i\le n}\{d_i\}$ small, Corollary 2.3 implies

$$\begin{aligned}(A'_\lambda u)_i &= [\bar\lambda(A')+\lambda+|\lambda|+1](-d_iL_i+|\lambda|+1)^{-1}a_i\varphi\\ &\ge (\eta+|\lambda|+1)(1-\epsilon)\frac{a_i\varphi}{|\lambda|+1}.\end{aligned}$$

Hence, there exists some constant $c > 1$ such that

$$(A'_\lambda u)_i \geq c u_i \quad \text{if } a_i > 0;$$
$$(A'_\lambda u)_i = 0 = c u_i \quad \text{if } a_i = 0.$$

Therefore, $r(A'_{\lambda,|\lambda|+1}) > 1$ for $\max_{1\leq i\leq n}\{d_i\}$ small. By Lemma 3.2, given any $\eta > 0$, for $\max_{1\leq i\leq n}\{d_i\}$ small, we have $\lambda'_1 < \lambda = -\bar{\lambda}(A') + \eta$. Hence,

$$\limsup_{\max_{1\leq i\leq n}\{d_i\}\to 0} \lambda'_1 \leq -\bar{\lambda}(A') + \eta.$$

Our claim follows by letting $\eta \to 0$.

To finish the proof of (4.4), we notice that $\lambda_1 \leq \lambda'_1$, so

$$\limsup_{\max_i\{d_i\}\to 0} \lambda_1 \leq -\bar{\lambda}(A') < -\max_{\bar{\Omega}} \bar{\lambda}(A(x)) + \delta'.$$

Finally, (4.4) follows by letting $\delta' \to 0$. □

5. Global stability in nonlinear cooperative systems

Theorem 1.5 will be proved in this section, with (A1) - (A4) being assumed throughout the whole section. We outline the main steps in the proof here. First, it is shown that (1.11) has at least one positive steady state (Lemma 5.1). Next, it is proved in Proposition 5.2 that any positive steady state of system (1.11) converges uniformly to the unique positive equilibrium of the corresponding kinetic system as $\max_{1\leq i\leq n}\{d_i\} \to 0$. Finally, we prove (Proposition 5.9) that every positive state is linearly stable and thus asymptotically stable; this step makes use of the linear theory introduced earlier in the paper. Theorem 1.5 follows immediately from the monotonicity of system (1.11).

Lemma 5.1. *System* (1.11) *has at least one positive steady state. Furthermore, if* $\max_{1\leq i\leq n}\{d_i\}$ *is sufficiently small, then any positive steady state* $u = (u_1, ..., u_n)$ *must satisfy* $\delta_0 < u_j < M$ *in* Ω*, where* δ_0 *and* M *are given in* (A3) *and* (A4)*, respectively.*

Proof. Define $\bar{w}^0 = (\bar{w}^0_1, ..., \bar{w}^0_n)$ and $\underline{w}^0 = (\underline{w}^0_1, ..., \underline{w}^0_n)$ by $\bar{w}^0_j(x) \equiv M$ and $\underline{w}^0_j(x) \equiv \delta_0$ in $\bar{\Omega}$, $j = 1, ..., n$. Since for $\max_{1\leq i\leq n}\{d_i\}$ sufficiently small, $\bar{w}^0$ and $\underline{w}^0$ are upper and lower solutions of (1.11), respectively, and $\bar{w}^0_j \geq \underline{w}^0_j$ in $\bar{\Omega}$ for all j, (1.11) has at least one positive steady state $w = (w_1, ..., w_n)$ such that $\underline{w}^0_j \leq w_j \leq \bar{w}^0_j$.

On the other hand, by (A1) and (A4), there exists $\delta > 0$ such that

$$d_i L_i M' + F(x, M', ..., M') < 0 \qquad \text{for all } i, \text{ if } \max_{1\leq i\leq n}\{d_i\} \leq \delta \text{ and } M' \geq M. \tag{5.1}$$

Assume that $\max_{1\leq i\leq n}\{d_i\} \leq \delta$ and $w = (w_1, ..., w_n)$ is a positive steady state of (1.11) such that $\|w_j\|_{L^\infty(\Omega)} = \max_{1\leq i\leq n} \|w_i\|_{L^\infty(\Omega)} = M' \geq M$ for some j, then by (5.1) and (A1),

$$d_j L_j M' + F_j(x, w_1, ..., w_{j-1}, M', w_{j+1}, ..., w_n) < 0.$$

So $w_j \not\equiv M'$, and $z := \|w_j\|_{L^\infty(\Omega)} - w_j$ is a non-negative function in $\bar{\Omega}$ that satisfies $z(x_0) = 0$ for some $x_0 \in \bar{\Omega}$, and

$$d_j L_j z + pz < 0 \quad \text{in } \Omega, \quad B_j z \geq 0 \quad \text{on } \partial\Omega,$$

where $p(x) = \int_0^1 \partial_{s_j} F_j(x, w_1, ..., w_{j-1}, w_j + t(\|w_j\|_{L^\infty(\Omega)} - w_j), w_{j+1}, ..., w_n)\, dt$. This contradicts the strong maximum principle if $x_0 \in \Omega$, and the Hopf Boundary Lemma if $x_0 \in \partial\Omega$. Hence, $w_i(x) < M$ in $\bar{\Omega}$ for all i. Similarly one can show that $w_i > \delta_0$ in $\bar{\Omega}$ for all i. □

Proposition 5.2. *For any positive steady state w of* (1.11)*, $w \to \alpha$ uniformly in Ω as $\max_i\{d_i\} \to 0$. Here α is given in* (A2).

We adopt a monotone iteration procedure as in [24]. Let $\bar{w}^0$ and $\underline{w}^0$ be as defined in the proof of Lemma 5.1. Let $K > 0$ be chosen so large that for all i, $K + \partial_{s_i} F_i(x, s_1, ..., s_n) > 0$ for all $x \in \bar{\Omega}$ and $0 \le s_1, ..., s_n \le M$. For any $(u_1, ..., u_n) \in \left[C(\bar{\Omega})\right]^n$, define $v = (v_1, ..., v_n) = \mathcal{T}u$ as the unique solution to

$$\begin{cases} -D\mathcal{L}v + Kv = Ku + F(x, u) & \text{in } \Omega, \\ \mathcal{B}v = 0 & \text{on } \partial\Omega. \end{cases}$$

Now for $k \in \mathbf{N}$, define $\bar{w}^k = (\bar{w}_i^k)_{i=1}^n := \mathcal{T}\bar{w}^{k-1}$ and $\underline{w}^k = (\underline{w}_i^k)_{i=1}^n := \mathcal{T}\underline{w}^{k-1}$.

Lemma 5.3. *For every $k \in \boldsymbol{N}$, $\underline{w}^k < \underline{w}^{k-1} < \bar{w}^{k-1} < \bar{w}^k$ holds in $\bar{\Omega}$.*

Proof. We proceed by induction. For $k = 1$ and $i = 1, ..., n$,

$$\begin{cases} -D\mathcal{L}(\bar{w}^1 - \bar{w}^0) + K(\bar{w}^1 - \bar{w}^0) = D\mathcal{L}\bar{w}^0 + F(x, \bar{w}^0) < 0 & \text{in } \Omega, \\ \mathcal{B}(\bar{w}^1 - \bar{w}^0) = 0 & \text{on } \partial\Omega. \end{cases}$$

By the strong maximum principle (applied to each component), $\bar{w}^1 < \bar{w}^0$ in $\bar{\Omega}$. Assume for induction that for some $k \ge 1$, $\bar{w}^k < \bar{w}^{k-1}$ in $\bar{\Omega}$. Then, as $Ks_i + F_i(x, s_1, ..., s_n)$ is increasing in $s_1, ..., s_n$ and strictly increasing in s_i, we have

$$\begin{cases} -D\mathcal{L}(\bar{w}^{k+1} - \bar{w}^k) + K(\bar{w}^{k+1} - \bar{w}^k) \\ \qquad = K(\bar{w}^k - \bar{w}^{k-1}) + F(x, \bar{w}^k) - F(x, \bar{w}^{k-1}) < 0 \quad \text{in } \Omega, \\ \mathcal{B}(\bar{w}^{k+1} - \bar{w}^k)|_{\partial\Omega} = 0. \end{cases}$$

By the maximum principle we obtain $\bar{w}^{k+1} < \bar{w}^k$ in $\bar{\Omega}$. By induction, $\bar{w}^0 > \bar{w}^1 > \bar{w}^2 > ...$ Similarly, we have $\underline{w}^0 < \underline{w}^1 < \underline{w}^2 < ...$ It remains to show that for any $k \in \mathbf{N} \cup \{0\}$,

$$\underline{w}^k < \bar{w}^k. \tag{5.2}$$

The statement obviously holds for $k = 0$. Assume (5.2) holds for some $k \ge 0$, then

$$\begin{cases} -D\mathcal{L}(\bar{w}^{k+1} - \underline{w}^{k+1}) + K(\bar{w}^{k+1} - \underline{w}^{k+1}) \\ \qquad = K(\bar{w}^k - \underline{w}^k) + F(x, \bar{w}^k) - F(x, \underline{w}^k) > 0 \quad \text{in } \Omega, \\ \mathcal{B}(\bar{w}^{k+1} - \underline{w}^{k+1})\big|_{\partial\Omega} = 0. \end{cases}$$

Hence, $\bar{w}^{k+1} > \underline{w}^{k+1}$ in $\bar{\Omega}$. By induction on k, (5.2) holds for all $k \in \mathbf{N} \cup \{0\}$. □

Lemma 5.4. *If $w = (w_1, ..., w_n)$ is a positive steady state of* (1.11)*, then for all k,*

$$\underline{w}^k \le w \le \bar{w}^k \qquad \text{in } \bar{\Omega}. \tag{5.3}$$

Proof. By Lemma 5.1, (5.3) holds for $k = 0$. Suppose that (5.3) holds for some $k \ge 0$. Then

$$-D\mathcal{L}(\bar{w}^{k+1} - w) + K(\bar{w}^{k+1} - w) = K(\bar{w}^k - w) + F(x, \bar{w}^k) - F(x, w) \ge 0$$

holds in Ω, and $\mathcal{B}(\bar{w}^{k+1} - w) = 0$ on $\partial\Omega$. By the comparison principle we have $\bar{w}^{k+1} - w \ge 0$. Similarly, $\underline{w}^{k+1} - w \le 0$. This completes the proof. □

Next, for each fixed k we investigate the convergence of $\bar{w}^k, \underline{w}^k$ as $\max_{1\leq i\leq n}\{d_i\} \to 0$. Define $\bar{W}^0(x) \equiv (M,...,M)$ and $\underline{W}^0(x) \equiv (\delta_0,...,\delta_0)$. For $k \in \mathbf{N}$ and $x \in \bar{\Omega}$, define successively

$$\bar{W}_i^k(x) := \bar{W}_i^{k-1}(x) + \frac{1}{K}F_i(x, \bar{W}_i^{k-1}(x)),$$
$$\underline{W}_i^k(x) := \underline{W}_i^{k-1}(x) + \frac{1}{K}F_i(x, \underline{W}_i^{k-1}(x)).$$

Lemma 5.5. *For each $k \in \mathbf{N}\cup\{0\}$, $i = 1,...,n$, as $\max_{1\leq i\leq n}\{d_i\} \to 0$, $\bar{w}_i^k \to \bar{W}_i^k$ and $\underline{w}_i^k \to \underline{W}_i^k$ uniformly in $\bar{\Omega}$.*

Proof. We only show $\bar{w}_i^k \to \bar{W}_i^k$, as $\underline{w}_i^k \to \underline{W}_i^k$ follows in a similar fashion. Given $\epsilon > 0$, it suffices to find, for each k, i, some positive constant δ such that $|\bar{w}_i^k - \bar{W}_i^k|_{L^\infty(\Omega)} < \epsilon$ if $\max_{1\leq i\leq n}\{d_i\} < \delta$. We proceed by induction on k. For $k = 0$ and any $i = 1,...,n$, the claim is trivially true as $\bar{w}_i^0 = \bar{W}_i^0$. Suppose that $\bar{w}_i^k \to \bar{W}_i^k$ for some $k \geq 0$ and all $i = 1,...,n$. Choose smooth functions ρ_i, $i = 1,...,n$, satisfying $\rho_i < \bar{W}_i^k$ in $\bar{\Omega}$. By the monotonicity of F_i,

$$\rho_i + \frac{1}{K}F_i(x,\rho) < \bar{W}_i^k + \frac{1}{K}F_i(x,\bar{W}_i^k) = \bar{W}_i^{k+1}.$$

Therefore, for each $\epsilon > 0$ sufficiently small, there exist smooth functions $\tilde{\rho} = (\tilde{\rho}_1,...,\tilde{\rho}_n)$ such that for each i,

$$\tilde{\rho}_i - \frac{\epsilon}{2} \leq \rho_i + \frac{1}{K}F_i(x,\rho) \leq \tilde{\rho}_i \leq \bar{W}_i^{k+1} \quad \text{in } \Omega, \qquad B_i\tilde{\rho}_i \leq 0 \quad \text{on } \partial\Omega. \tag{5.4}$$

Then $P_i = \bar{w}^{k+1} - \tilde{\rho}_i + \epsilon$ satisfies

$$\begin{aligned}
&- d_iL_iP_i + KP_i \\
&= K\bar{w}_i^k + F_i(x,\bar{w}^k) - K\tilde{\rho}_i + d_iL_i(\tilde{\rho}_i - \epsilon) + K\epsilon \\
&= K\bar{w}_i^k + F_i(x,\bar{w}^k) - K\rho_i - F_i(x,\rho) - K\epsilon/2 + d_iL_i(\tilde{\rho}_i - \epsilon) + K\epsilon \\
&\geq K(\bar{w}_i^k - \bar{W}_i^k) + F_i(x,\bar{w}^k) - F_i(x,\bar{W}^k) + d_iL_i(\tilde{\rho}_i - \epsilon) + K\epsilon/2 \\
&\geq K\epsilon/2 - d_iC_1 - C_2\|\bar{w}_i^k - \bar{W}_i^k\|_{L^\infty(\Omega)} \\
&\geq 0,
\end{aligned}$$

provided that $\max_{1\leq i\leq n}\{d_i\}$ is sufficiently small. Therefore, for any $\epsilon > 0$ and $\rho_i < \bar{W}_i^k$, we have $\bar{w}_i^{k+1} \geq \tilde{\rho}_i - \epsilon$ for sufficiently small d_i. Hence,

$$\liminf_{\max_{1\leq i\leq n}\{d_i\}\to 0} \bar{w}_i^{k+1} \geq \tilde{\rho}_i$$

uniformly in $\bar{\Omega}$. By choosing $\rho_i \to \bar{W}_i^k$ for all i, we see by (5.4) that $\tilde{\rho}_i \to \bar{W}_i^{k+1}$ for all i. Hence,

$$\liminf_{\max_{1\leq i\leq n}\{d_i\}\to 0} \bar{w}_i^{k+1} \geq \bar{W}_i^{k+1}$$

uniformly in $\bar{\Omega}$. Similarly, we can show that for all i,

$$\limsup_{\max_{1\leq i\leq n}\{d_i\}\to 0} \bar{w}_i^{k+1} \leq \bar{W}_i^{k+1}$$

uniformly in $\bar{\Omega}$. Therefore for all i, $\bar{w}_i^{k+1} \to \bar{W}_i^{k+1}$ uniformly in $\bar{\Omega}$ as $\max_{1\leq i\leq n}\{d_i\} \to 0$. This completes the proof. □

Lemma 5.6. *For all $x \in \bar{\Omega}$, $i = 1, ..., n$, and $k \in \boldsymbol{N} \cup \{0\}$,*

$$\underline{W}_i^k(x) \leq \underline{W}_i^{k+1}(x) \leq \bar{W}_i^{k+1}(x) \leq \bar{W}_i^k(x) \qquad \text{in } \bar{\Omega}. \tag{5.5}$$

Proof. We proceed by induction. Consider $k = 0$. For all $x \in \bar{\Omega}$ and all i,

$$\begin{aligned}
&\bar{W}_i^1(x) = \bar{W}_i^0(x) + \tfrac{1}{K} F_i(x, M, ..., M) < \bar{W}_i^0(x), \\
&\underline{W}_i^1(x) = \underline{W}_i^0(x) + \tfrac{1}{K} F_i(x, \delta_0, ..., \delta_0) > \underline{W}_i^0(x), \\
&\bar{W}_i^1(x) - \underline{W}_i^1(x) = M + \tfrac{1}{K} F_i(x, M, ..., M) - [\delta_0 + \tfrac{1}{K} F_i(x, \delta_0, ..., \delta_0)] > 0.
\end{aligned}$$

The last line holds as for all i, $Ks_i + F_i(x, s_1, ..., s_n)$ is increasing in $s_1, ..., s_n$ and strictly increasing in s_i. Now assume (5.5) is true for some $k \geq 0$, then

$$\begin{aligned}
&\bar{W}_i^{k+2}(x) - \bar{W}_i^{k+1}(x) = \bar{W}_i^{k+1}(x) - \bar{W}_i^k(x) + \tfrac{1}{K}[F_i(x, \bar{W}^{k+1}) - F_i(x, \bar{W}^k)] < 0, \\
&\underline{W}_i^{k+2}(x) - \underline{W}_i^{k+1}(x) = \underline{W}_i^{k+1}(x) - \underline{W}_i^k(x) + \tfrac{1}{K}[F_i(x, \underline{W}^{k+1}) - F_i(x, \underline{W}^k)] > 0, \\
&\bar{W}_i^{k+2}(x) - \underline{W}_i^{k+2}(x) = \bar{W}_i^{k+1}(x) - \underline{W}_i^{k+1}(x) + \tfrac{1}{K}[F_i(x, \bar{W}^{k+1}) - F_i(x, \underline{W}^{k+1})] > 0.
\end{aligned}$$

Thus (5.5) is true for $k + 1$. The proof is complete. □

Lemma 5.7. *$\bar{W}^k \to \alpha$ and $\underline{W}^k \to \alpha$ uniformly in $\bar{\Omega}$ as $k \to \infty$.*

Proof. Notice that for each $x_0 \in \bar{\Omega}$ and each $i = 1, ..., n$, $\bar{W}_i^k(x_0)$ is decreasing in k and $\bar{W}_i^k(x_0) \geq \underline{W}_i^0(x_0) > 0$ is bounded from below. Hence, $\bar{W}_i^\infty(x_0) := \lim_{k\to\infty} \bar{W}_i^k(x_0) > 0$ exists for all $i = 1, ..., n$. In addition,

$$\bar{W}_i^\infty(x_0) = \bar{W}_i^\infty(x_0) + \frac{1}{K} F_i(x, \bar{W}^\infty(x_0)),$$

and thus $F_i(x, \bar{W}^\infty(x_0)) = 0$ for all i. By (A2), $\bar{W}_i^\infty(x_0) = \alpha_i(x_0)$ for all i.

Now for each i, $\{\bar{W}_i^k\}_{k\geq 0}$ is a sequence of continuous function decreasing in k and converges pointwise to $\alpha_i \in C(\bar{\Omega})$. By the following well-known calculus lemma (see, e.g. Theorem 7.13 in [21]), we see that as $k \to \infty$, $\bar{W}^k$ converges to α uniformly in $\bar{\Omega}$.

Theorem 5.8. *Suppose that $\mathcal{K}$ is a compact set in $\boldsymbol{R}^N$ and*

(a) *$\{f_k\}_{k=0}^\infty$ is a sequence of continuous functions on $\mathcal{K}$,*
(b) *$\{f_k\}$ converges pointwise to a continuous function f on $\mathcal{K}$,*
(c) *$f_k(x) \geq f_{k+1}$ for all $x \in \mathcal{K}$, $k = 0, 1, 2, ...$*

Then $f_k \to f$ uniformly in $\mathcal{K}$.

Similarly, $\underline{W}^k \to \alpha$ uniformly in $\bar{\Omega}$ as $k \to \infty$. The proof of Lemma 5.5 is complete. □

Proposition 5.2 follows from Lemmas 5.4, 5.5 and 5.7.

Proposition 5.9. *There exists some positive constant δ such that every positive steady state of* (1.11) *is linearly stable (hence locally asymptotically stable) if* $\max_i\{d_i\} \leq \delta$.

Proof. To consider the linear stability of a positive steady state $w = (w_1, ..., w_n)$, it suffices to show that the principal eigenvalue λ_1 of the following problem is positive:

$$\begin{cases} D\mathcal{L}\phi + \partial_s F(x, w)\phi + \lambda\phi = 0 & \text{in } \Omega, \\ \mathcal{B}\phi = 0 & \text{on } \partial\Omega. \end{cases}$$

By (A2), we have $\bar{\lambda}(D_s F(x,\alpha(x))) < 0$ for all $x \in \bar{\Omega}$. Therefore, for any $\eta > 0$ small, $\hat{F}_{ij}(x) := \partial_{s_j} F_i(x,\alpha(x)) + \eta$ satisfies

$$\bar{\lambda}(\hat{F}_{ij}(x)) < 0 \qquad \text{for all } x \in \bar{\Omega}. \tag{5.6}$$

Let $\hat{\lambda}_1$ be the principal eigenvalue of

$$\begin{cases} D\mathcal{L}\phi + \hat{F}\phi + \lambda\phi = 0 & \text{in } \Omega, \\ \mathcal{B}\phi = 0 & \text{on } \partial\Omega. \end{cases}$$

By Proposition 5.2, there exists $\delta > 0$ such that for any positive steady state w of (1.11),

$$\partial_{s_j} F_i(x, w(x)) \le \partial_{s_j} F_i(x,\alpha(x)) + \eta = \hat{F}_{ij}(x) \qquad \text{in } \bar{\Omega},$$

whenever $\max_{1\le i\le n}\{d_i\} \le \delta$. Therefore, $\lambda_1 \ge \hat{\lambda}_1$ by Proposition 3.4. While by Theorem 1.4 and (5.6), we have

$$\liminf_{\max_{1\le i\le n}\{d_i\}\to 0} \lambda_1 \ge \lim_{\max_{1\le i\le n}\{d_i\}\to 0} \hat{\lambda}_1 = -\max_{x\in\bar{\Omega}} \bar{\lambda}(\hat{F}_{ij}(x)) > 0.$$

Hence, there exists $\delta > 0$ such that whenever $\max_{1\le i\le n}\{d_i\} \le \delta$, $\lambda_1 > 0$ for any positive steady state w of (1.11). □

Proof of Theorem 1.5. By Lemma 5.1, (1.11) has at least one positive steady state. By Proposition 5.9, any positive steady state of (1.11) is linearly stable. Since (1.11) is an order-preserving system, it follows that (by Theorem 1.1) (1.11) has a unique positive steady state $\tilde{w}$, and $\tilde{w}$ is globally asymptotically stable. Moreover, by Proposition 5.2, this unique positive steady state $\tilde{w}$ converges to α uniformly in Ω as $\max_{1\le i\le n}\{d_i\} \to 0$. □

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