The Role of Advection in a Two-Species Competition Model: A Bifurcation Approach

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Contents

Abstract

Abstract: The effects of weak and strong advection on the dynamics of reaction-diffusion models have long been studied. In contrast, the role of intermediate advection remains poorly understood. For example, concentration phenomena can occur when advection is strong, providing a mechanism for the coexistence of multiple populations, in contrast with the situation of weak advection where coexistence may not be possible. The transition of the dynamics from weak to strong advection is generally difficult to determine. In this work we consider a mathematical model of two competing populations in a spatially varying but temporally constant environment, where both species have the same population dynamics but different dispersal strategies: one species adopts random dispersal, while the dispersal strategy for the other species is a combination of random dispersal and advection upward along the resource gradient. For any given diffusion rates we consider the bifurcation diagram of positive steady states by using the advection rate as the bifurcation parameter. This approach enables us to capture the change of dynamics from weak advection to strong advection. We will determine three different types of bifurcation diagrams, depending on the difference of diffusion rates. Some exact multiplicity results about bifurcation points will also be presented. Our results can unify some previous work and, as a case study about the role of advection, also contribute to our understanding of intermediate (relative to diffusion) advection in reaction-diffusion models.

CHAPTER 1

Introduction: The role of advection

For the last several decades there has been extensive study of reaction-diffusion-advection models of the form

$$u_t = d\Delta u + \alpha V \cdot \nabla u + f(x, u), \tag{1.1}$$

where the function $u(x, t)$ represents the density of a population or a substance in biological and chemical models [**12, 81, 85, 95**]. The diffusion coefficient d and and advection rate α are assumed to be positive constants. Concerning qualitative behavior of solutions of (1.1), one important and challenging question is how the advection affects the behavior of the solutions of (1.1). Such a question arises naturally in the studies of (i) the speed of biological invasions and chemical flame propagations [**4, 37, 100**]; (ii) the persistence of organisms in advective environments where individuals are exposed to unidirectional flow [**66, 72, 80, 86**]; (iii) climate change and moving ranges of species [**5, 88**], among others.

There have been many studies on the effect of incompressible drifts (i.e. V is divergence free) on reaction-diffusion equation (1.1) in bounded and unbounded domains. A large body of this work illustrates that strong mixing by an incompressible flow enhances diffusion in many contexts. Traveling front solutions of (1.1) play an important role in understanding the large-time behavior of the biological or chemical processes described by (1.1). Recent work shows that the front propagation speed in the presence of a shear flow or periodic flow is an asymptotically linear function of the amplitude of the flow [**4, 7, 36, 42, 57, 93**]. However, for cellular flows in two spatial dimensions, the speed-up is shown to be of the order of $O(\alpha^{1/4})$ [**57, 93, 102**]. Generally it is challenging to determine whether the minimal speed of traveling front solutions of (1.1) is a monotone increasing function of the advection rate; see [**4, 6**].

Another area of active research concerns the spatial population dynamics in advective environments such as streams, rivers, and lake water columns. For example, most phytoplankton species are heavier than water, so they sink (advection). On the other hand, phytoplankton all depend on light for their metabolism. How then can sinking phytoplankton persist in water columns? To address such questions, reaction-diffusion-advection models have been proposed to understand how persistence of single or multiple phytoplankton species depend upon various parameters including the sinking rate. It has been recently shown in [**50**] that critical sinking velocities (i.e. advection rates) may or may not exist. For instance, if the death rate of phytoplankton is small, phytoplankton can persist for any sinking velocity. However, if a critical sinking velocity exists (which does occur when phytoplankton death rate is suitably large), it must be unique. The effects of sinking rates on multi-species phytoplankton dynamics remain largely unpursued [**79**]. We refer to [**32, 33, 34, 35, 48, 49, 55, 58, 96, 101**] and references therein.

A different line of research in advective environments is the 'drift paradox' in river ecosystems [**98**]. Organisms in rivers and streams are always at risk of being washed out of the habitat by the advection. How can a population persist in a river system? Reaction-advection-diffusion models have been introduced as well for populations in rivers to study the persistence of single and multiple populations [**75, 76, 80, 82, 83, 84**]. An interesting finding in [**76**] is that strong advection can reverse the prediction regarding the outcome of the interaction of two competing species when there is no advection. Numerical simulations further suggest that intermediate advection can promote the coexistence of species. Nevertheless, analytical studies on the transition of dynamics from weak to strong drift in river systems are generally lacking.

In this article we will investigate the effect of active advection by considering another type of advection in population dynamics models. Our main focus is on the effect of intermediate advection, which seems to be poorly understood. This line of research started with the work of Belgacem and Cosner [**10**], where they considered the following reaction-diffusion-advection model for single species in a spatially heterogenous environment:

$$
\begin{cases} u_t = \nabla \cdot (\mu \nabla u - \alpha u \nabla m) + \lambda u(m-u) & \text{in } \Omega \times (0,\infty), \\ \mu \dfrac{\partial u}{\partial n} - \alpha u \dfrac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.2}
$$

Here $u(x,t)$ represents the density of a population at time t and location x in a bounded domain Ω with smooth boundary $\partial\Omega$. The parameter α measures the tendency to move upward along the resource gradient, where $m(x)$ represents the local growth rate of the population. We assume that there is no net flux across the boundary, where $\frac{\partial u}{\partial n} := \nabla u \cdot n$, and n is the outward unit normal vector on $\partial\Omega$.

A fascinating question was raised by C. Cosner [**10**]: Is increasing α always beneficial to the persistence of the population? It is shown [**10**] that if $\int_\Omega m \geq 0$, then for any $\lambda > 0$ and $\alpha \geq 0$, the problem (1.2) has a unique positive steady state which is globally asymptotically stable among all non-negative and non-trivial (i.e. not identically zero) initial data u_0. So the population persists, and increasing α is irrelevant in this case. However, if $\int_\Omega m < 0$, then there exists a unique positive number, denoted as $\lambda_* = \lambda_*(\alpha)$, such that the population can persist if and only if $\lambda > \lambda_*$. Therefore, Cosner's question can be rephrased as: Is $\lambda_*(\alpha)$ a monotone decreasing function of α? It turns out that the answer to Cosner's question depends on the convexity of the domain: It is proved in [**27**] that for any convex Ω, $\lambda_*(\alpha)$ is strictly decreasing for small positive α and is non-increasing for all α under suitable assumption on function m. On the other hand, it is also shown in [**27**] that there exist a (non-convex) domain Ω and function m such that $\lambda_*(\alpha)$ is strictly increasing for small positive α; i.e., increasing advection can be detrimental to the persistence of a single population for certain domains.

The above discussion suggests that increasing the advection upward along the resource gradient can be advantageous to the persistence of a single population, while the details may rely on the geometry of the underlying domain and the resource distribution. Regarding the advection upward along the resource gradient as a strategy of the population, we assess the effectiveness of such strategy by comparing it with other strategies. To this end, more recently the following reaction-diffusion-advection model has been proposed in [**13**]:

$$\begin{cases} u_t = \nabla\cdot(\mu\nabla u - \alpha u\nabla m) + u(m-u-v) & \text{in } \Omega\times(0,\infty),\\ v_t = \nu\Delta v + v(m-u-v) & \text{in } \Omega\times(0,\infty),\\ \mu\frac{\partial u}{\partial n} - \alpha u\frac{\partial m}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x) & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $u(x,t)$, $v(x,t)$ denote the densities of two populations at location x and time t, competing for the same resource in a spatially varying but temporally constant environment. It is noteworthy that the two population have the same population dynamics and they differ only in their dispersal strategies: the dispersal strategy for the species with density u is a combination of random dispersal and biased movement upward along the resource gradient, while the species with density v adopts purely random movement. The main question for (1.3) is: While the competitive ability of both populations are neutral/identical, can the advection convey some competitive advantage for the species u?

Unless otherwise stated, we assume for the rest of this work that

(M) $m\in C^2(\bar\Omega)$, $m>0$ in $\bar\Omega$, and m is non-constant.

We assume that m is non-constant to reflect the spatial heterogeneity of the environment. Provided that (M) holds, it is well known that (1.3) has two semi-trivial equilibria, denoted by $(\tilde u,0)$ and $(0,\tilde v)$, corresponding to the situations when exactly one of the populations is present. When there is no biased movement, i.e. $\alpha=0$, Hastings [**43**] showed that a species can invade when rare if only if it has the smaller random dispersal rate. The following global result was established in [**31**]:

THEOREM 1.1. *If $\alpha=0$ and $\mu<\nu$, then $(\tilde u,0)$ is globally asymptotically stable among all non-negative and nontrivial initial data.*

Theorem 1.1 shows that if both populations adopt random dispersal, then the population with the slower dispersal rate has the advantage. By a simple perturbation argument, one can extend Theorem 1.1 to weak advection as follows:

THEOREM 1.2. *If $\mu<\nu$, then there exists some positive small α^* such that for $\alpha\in[0,\alpha^*]$, $(\tilde u,0)$ is globally asymptotically stable among all non-negative and nontrivial initial data.*

We caution the readers that α^* above depends upon μ and ν. Due to the symmetry, analogous result holds when $\mu>\nu$. The following result from [**14**] addresses the case $\mu=\nu$, which turns out to be more subtle.

THEOREM 1.3. *Let Ω be convex. If $\mu=\nu$, then there exists some positive small α^* such that for $\alpha\in(0,\alpha^*]$, $(\tilde u,0)$ is globally asymptotically stable among all non-negative and nontrivial initial data.*

Theorem 1.3 suggests that the advection upward along resource gradient can indeed convey some advantage, at least for convex domains. On the other hand, Theorem 1.3 fails for some non-convex domains [**14**].

What happens if the advection is strong? At the first thought one might expect the species u with strong advection will prevail and may drive the other species to extinction, in fashion similar to the case $\mu=\nu$ and α being positive small. A bit surprisingly, we have the following result (see also [**14, 22**]):

THEOREM 1.4. *Suppose that assumption* (M) *holds and the set of critical points of m has measure zero. Then for every $\mu > 0$ and $\nu > 0$, there exists some positive constant α^{**} such that for $\alpha > \alpha^{**}$, system* (1.3) *has at least one stable positive steady state, and both semi-trivial steady states are unstable.*

Theorem 1.4 implies that if species u has strong advection, then two species will coexist. The result of Cantrell et al. [**14**] suggested that such coexistence is possible because species u concentrates on the local maxima of m, leaving resources elsewhere in the habitat for the species v to utilize. Chen and Lou [**24**] demonstrated that for a resource function m with a unique local maxima, species u with strong advection is indeed concentrated at this maximum as a Gaussian distribution. When m has multiple local maxima, Lam and Ni [**69**] and Lam [**62, 63**] completely determined the profiles of all positive steady states of (1.3) and they illustrated that species u is exactly concentrated at the set of local maximum of m where $m - \tilde{v}$ is strictly positive. These works suggested a new mechanism for the coexistence of two competing species with different dispersal strategies.

From these discussions the dynamics of (1.3) for weak and strong advection can be briefly summarized as follows:

- When the advection is weak, two species cannot coexist: One semi-trivial steady state is globally stable and the other semi-trivial steady state is unstable.
- When the advection is strong, two species coexist for any dispersal rates, and both semi-trivial steady states are unstable.

These results raise an immediate question about the role of advection: If we increase the advection from weak to strong, how do the dynamics of (1.3) change correspondingly? The goal of this paper is to address this issue, and in particular, to understand the effect of intermediate advection on the dynamics of (1.3). We consider, for any given diffusion rates, the bifurcation diagram of steady states using the advection rate α as a parameter. We will show the existence of three different types of bifurcation diagrams. Exact multiplicity results of bifurcation points will also be presented. Our results unify some previous work and, as a case study about the role of advection, also contribute to the understanding of intermediate (relative to diffusion) advection in reaction-diffusion models.

CHAPTER 2

Summary of main results

The goal of this paper is to understand the dynamics of system (1.3) and determine the structure of positive steady states by varying the parameters μ, ν and α. Our approach is to divide the μ-ν plane into three separate regions according to the stability changes of both semi-trivial steady states. By fixing (μ, ν) in each of these three regions and then varying the parameter α from 0 to ∞, we determine the bifurcation diagram of positive steady states of system (1.3) for each case, respectively. By piecing these three bifurcation diagrams together we are able to determine how the dynamics of system (1.3) change as we vary the parameters.

As it turns out, an important quantity is the ratio $\eta := \alpha/\mu$. Replacing α by the new parameter η, (1.3) becomes

$$(2.1) \qquad \begin{cases} u_t = \mu\nabla\cdot(\nabla u - \eta u\nabla m) + u(m-u-v) & \text{in } \Omega\times(0,\infty), \\ v_t = \nu\Delta v + v(m-u-v) & \text{in } \Omega\times(0,\infty), \\ \frac{\partial u}{\partial n} - \eta u\frac{\partial m}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega \end{cases}$$

for non-negative, non-trivial initial data u_0 and v_0. From now on, we denote the semi-trivial steady states of (2.1) by $(\tilde{u}, 0)$ and $(0, \tilde{v})$, where $\tilde{u} = \tilde{u}(x; \mu, \eta)$ is the unique positive solution (see [**10**]) of

$$(2.2) \qquad \begin{cases} \mu\nabla\cdot(\nabla\tilde{u} - \eta\tilde{u}\nabla m) + \tilde{u}(m-\tilde{u}) = 0 & \text{in } \Omega, \\ \frac{\partial\tilde{u}}{\partial n} - \eta\tilde{u}\frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\tilde{v} = \tilde{v}(x; \nu)$ is the unique positive solution (see, e.g. [**12**]) of

$$(2.3) \qquad \begin{cases} \nu\Delta\tilde{v} + \tilde{v}(m-\tilde{v}) = 0 & \text{in } \Omega, \\ \frac{\partial\tilde{v}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

LEMMA 2.1. *Assume that* (M) *holds.*

(i) *For $\mu > 0$ and $\eta \geq 0$,* (2.2) *has a unique positive solution, denoted by $\tilde{u}$.*
(ii) *For $\nu > 0$,* (2.3) *has a unique positive solution, denoted by $\tilde{v}$.*

The proof of (i) is due to [**10**]. The proof of (ii) is standard; See, e.g. [**12**].

2.1. Existence of positive steady states of (2.1)

Our first main result gives an explicit criterion for the existence of positive steady states of (2.1). Unless otherwise stated, we assume (M) throughout this work.

THEOREM 2.2. *If $\eta \geq \frac{1}{\min_{\bar{\Omega}} m}$, then for any $\mu > 0$ and $\nu > 0$, both semi-trivial steady states of system* (2.1) *are unstable and system* (2.1) *has at least one stable postive steady state.*

REMARK 2.3. Previously, the existence of positive steady states of (2.1) has been established in [**14, 24**] under the additional assumption that the set of critical points of m is of measure zero, and that η is sufficiently large. Our contribution here is to remove all assumptions on the set of critical points of m, and to give an explicit estimate of the coexistence region in terms of parameter η.

Concerning the uniqueness of the positive steady state of system (2.1) we have the following result for large μ, ν.

THEOREM 2.4. *For each $\Lambda > \frac{1}{\min_{\bar{\Omega}} m}$, there exists $M > 0$ such that* (2.1) *has a unique positive steady state for all $\mu, \nu \geq M$ and $\eta \in [\frac{1}{\min_{\bar{\Omega}} m}, \Lambda]$.*

It is an open problem whether (2.1) has at most one positive steady state for arbitrary diffusion coefficients and advection rates.

2.2. Local stability of semi-trivial steady states

First, we discuss, for different $\mu, \nu > 0$, the changes of stability of the semi-trivial steady states as η varies from 0 to ∞. By Theorem 2.2, it is known that $(\tilde{u}, 0)$ and $(0, \tilde{v})$ become unstable when $\eta \geq 1/\min_{\bar{\Omega}} m$. Therefore, it suffices to look at $\eta \in [0, 1/\min_{\bar{\Omega}} m]$. The stability of $(\tilde{u}, 0)$ and $(0, \tilde{v})$ determine a significant part of the bifurcation diagram of positive steady states of (2.1) with $\eta \geq 0$ as the bifurcation parameter, since each change of stability corresponds to a bifurcation of positive steady states from semi-trivial steady states. We will state the precise bifurcation results at the end of this chapter.

We now decompose the μ-ν plane into three separate regions, according to the number of stability changes of $(\tilde{u}, 0)$ as η varies from 0 to ∞.

THEOREM 2.5. *Suppose that Ω is convex. For any $\mu > 0$, there exists a unique $\bar{\nu} = \bar{\nu}(\mu) \in (0, \mu)$ such that the following hold:*

(i) *If $\nu \in (0, \bar{\nu})$, then $(\tilde{u}, 0)$ is unstable for all $\eta \geq 0$, while $(0, \tilde{v})$ is stable for small η and unstable for large η. In particular, $(0, \tilde{v})$ changes stability at least once as η varies from 0 to ∞;*

(ii) *If $\nu \in (\bar{\nu}, \mu)$, then $(\tilde{u}, 0)$ is unstable for both small and large η and it is stable for some intermediate values of η, while $(0, \tilde{v})$ is stable for small η and unstable for large η. In particular, as η varies from 0 to ∞, $(\tilde{u}, 0)$ changes stability at least twice, while $(0, \tilde{v})$ changes stability at least once;*

(iii) *If $\nu \in (\mu, \infty)$, then $(\tilde{u}, 0)$ is stable for small η and unstable for large η. In particular, as η varies from 0 to ∞, $(\tilde{u}, 0)$ changes stability at least once.*

For Case (iii) we expect that $(0, \tilde{v})$ is unstable for any η, provided that Ω is convex. For some non-convex domains it is possible that $(0, \tilde{v})$ is stable for small positive η; See Theorem 10.4.

DEFINITION 2.6. Denote the regions on the μ-ν plane identified in Theorem 2.5 as follows:

$$\begin{aligned}
\mathcal{R}_1 &:= \{(\mu, \nu) : \mu > 0, \nu \in (0, \bar{\nu}(\mu))\}; \\
\mathcal{R}_2 &:= \{(\mu, \nu) : \mu > 0, \nu \in (\bar{\nu}(\mu), \mu)\}; \\
\mathcal{R}_3 &:= \{(\mu, \nu) : \mu > 0, \nu \in (\mu, \infty)\}.
\end{aligned}$$

See Figure 1 for a graphical illustration of these three regions.

We also obtain more precise estimates of the curve $\nu = \bar{\nu}(\mu)$, which divides the region $\{(\mu, \nu) : \mu > \nu > 0\}$ into regions $\mathcal{R}_1$ and $\mathcal{R}_2$.

THEOREM 2.7. *Suppose that Ω is convex and let $\bar{\nu}(\mu)$ be given by Theorem 2.5.*

(iv) *The limit* $\lim\limits_{\mu \to 0} \dfrac{\bar{\nu}(\mu)}{\mu}$ *exists and it belongs to* $(0, 1)$.

(v) *The limit* $\lim\limits_{\mu \to \infty} \bar{\nu}(\mu)$ *exists, and it is positive and finite.*

Theorem 2.7 implies that the curve $\nu = \bar{\nu}(\mu)$ grows as a linear function for small μ and it approaches a finite positive limit for sufficiently large μ.

REMARK 2.8. For μ sufficiently small or sufficiently large, $\bar{\nu} \in (0, \mu)$ exists regardless of the convexity of Ω, and (i)-(v) of Theorems 2.5 and 2.7 hold. See Corollary 4.11.

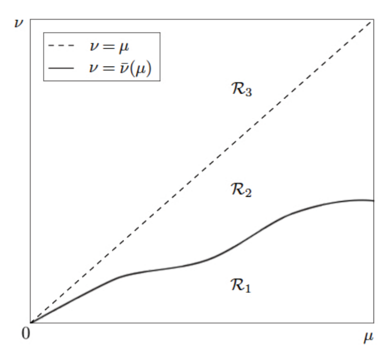

FIGURE 1. Decomposition of the μ-ν plane into three regions: $\mathcal{R}_1$ and $\mathcal{R}_2$ are separated by the curve $\nu = \bar{\nu}(\mu)$, where $\bar{\nu}(\mu)$ is determined in Theorem 2.5 and some qualitative behaviors of $\bar{\nu}(\mu)$ are given in Theorem 2.7. The regions $\mathcal{R}_2$ and $\mathcal{R}_3$ are separated by the line $\nu = \mu$. For each (μ, ν) from these regions, as the advection rate varies from 0 to ∞, the stability changes of both semi-trivial steady states are given in Theorem 2.5.

Based on Theorem 2.5, we have the following conjecture:

CONJECTURE 2.9. Suppose that Ω is convex. Then, as η varies from 0 to ∞,

(a) if $(\mu, \nu) \in \mathcal{R}_1$, then $(\tilde{u}, 0)$ is unstable for all $\eta \geq 0$, while $(0, \tilde{v})$ changes stability exactly once, from stable to unstable;

(b) if $(\mu, \nu) \in \mathcal{R}_2$, then $(\tilde{u}, 0)$ changes stability exactly twice, from unstable to stable to unstable, while $(0, \tilde{v})$ changes stability exactly once, from stable to unstable;

(c) If $(\mu, \nu) \in \mathcal{R}_3$, then $(\tilde{u}, 0)$ changes stability exactly once, from stable to unstable, while $(\tilde{v}, 0)$ remains unstable for all $\eta \geq 0$.

See Figure 2 for an illustration of the conjecture.

For each $\mu, \nu > 0$, if we denote by η^* (resp. η_*), the value(s) where $(\tilde{u}, 0)$ (resp. $(0, \tilde{v})$) changes stability, then Conjecture 2.9 describes the exact multiplicities of η^*

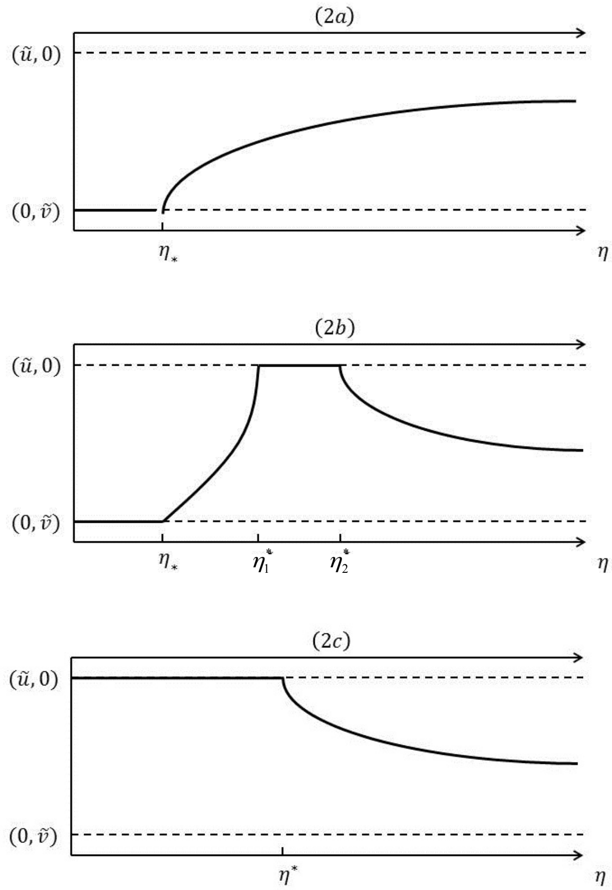

FIGURE 2. Illustration of Conjecture 2.9 and Theorem 2.14: Figures (2a), (2b), (2c) illustrate Conjecture 2.9(a), (b), (c), respectively. Figure (2a) concerns the parameter region $\nu < \bar{\nu}(\mu)$; Figure (2b) concerns the region $\bar{\nu}(\mu) < \nu < \mu$; Figure (2c) concerns the region $\nu > \mu$. Under extra assumptions on diffusion rates, Ω and m, Theorems 2.10, 2.11 and 2.12 confirm Conjecture 2.9(a), (b), (c), respectively. Furthermore, Theorem 2.14 describes the global bifurcation diagrams of positive steady states for (μ, ν) in each of the three regions $\mathcal{R}_i$, $i = 1, 2, 3$, as illustrated in Figures (2a), (2b), (2c), respectively.

and η_*. Should the conjecture hold true, a precise description of the bifurcation points from the semi-trivial branches $\{(\tilde{u}(\mu,\eta), 0; \eta) : \eta \geq 0\}$ and $\{(0, \tilde{v}(\nu); \eta) : \eta \geq 0\}$ follows immediately.

We give some biological interpretation of the conjecture:

- When $\nu < \bar{\nu}(\mu)$, i.e. ν is small relative to μ, then when the directed movement of species u is small, the slower disperser v excludes the species u. However, as η passes some critical value η_*, the species u is able to invade when rare by concentrating its effort at the local maximum points of the resource.

- We can similarly interpret the case when $\nu > \mu$. In this case, the slower disperser u excludes the species v if the advection rate η is small. When η increases beyond a critical value η^*, the species u concentrates on the local maximum points of the resource, and the species v can invade when rare by utilizing the resources elsewhere.
- It is a bit surprising that a third regime $\mathcal{R}_2$ ($\bar{\nu} < \nu < \mu$) exists. Mathematically, it connects the two different scenarios $\mathcal{R}_1$ and $\mathcal{R}_3$. Biologically, it says the following: Even though species v is the slower disperser, if the species u can adopt an appropriate advection rate η, then $(\tilde{u}, 0)$ gains stability and can somehow manage to exclude the species v. A crucial observation is that the species which is able to track the underlying resource better will ultimately outcompete the other species.

We are able to show (a), (b), (c) of Conjecture 2.9 to different degrees. In the following, we will state our results for regions $\mathcal{R}_i$ $(i = 1, 2, 3)$ separately.

THEOREM 2.10 (Results for $\mathcal{R}_1$). *Let $(\mu, \nu) \in \mathcal{R}_1$. If one of the following conditions holds:*

(a) *μ/ν is sufficiently large;*
(b) *$\Omega = (-1, 1)$, $m_x m_{xx} \neq 0$ in $\bar{\Omega}$,*

then as η varies from 0 to ∞, $(0, \tilde{v})$ changes stability exactly once, from stable to unstable at some η_*.

Theorem 2.10 implies that under suitable assumptions, there exists a critical advection rate η_* such that the faster diffuser u can invade when rare if and only if its advection rate is larger than η_*. This means that strong advection upward along the resource gradient can offset the disadvantage of being a faster diffuser. It is a bit surprising that for this region of parameters the slower random diffuser can always invade when rare, irrespective of the advection rate of the faster diffuser. We remark that Theorem 2.10 (a) holds for general smooth domains Ω that are not necessarily convex, even though $\mathcal{R}_1$ may not be globally defined.

THEOREM 2.11 (Results for $\mathcal{R}_2$). *Let Ω be convex and $(\mu, \nu) \in \mathcal{R}_2$.*

(a) *If μ is sufficiently small, then as η varies from 0 to ∞, $(\tilde{u}, 0)$ changes stability exactly twice, from unstable to stable to unstable at some η_1^* and η_2^* respectively.*
(b) *If one of the following conditions holds:*
  (i) *μ/ν is sufficiently large;*
  (ii) *$\Omega = (-1, 1)$, $m_x m_{xx} \neq 0$ in $\bar{\Omega}$;*
  (iii) *μ, ν are sufficiently large,*

  then as η varies from 0 to ∞, $(0, \tilde{v})$ changes stability exactly once, from stable to unstable at some η^*.

Theorem 2.11 implies that the invasion of the species u when rare is the same as in the previous case. However, the species v fails to invade when rare for some intermediate interval of η. This suggests that a faster diffuser with proper degree (but not too strong) of advection upward along the resource gradient can exclude the slower diffuser. In particular, an intermediate advection rate (relative to diffusion rate) can indeed convey some competitive advantage.

THEOREM 2.12 (Results for $\mathcal{R}_3$). *Let Ω be convex and $(\mu, \nu) \in \mathcal{R}_3$.*

(a) *If μ is sufficiently small, then as η varies from 0 to ∞, $(\tilde{u}, 0)$ changes stability exactly once, from stable to unstable at some η^*.*

(b) *If one of the following conditions holds:*
  (i) *μ/ν sufficiently small;*
  (ii) $\nu \leq \dfrac{4(\min_{\bar{\Omega}} m)^3}{|\nabla m|^2_{L^\infty(\Omega)}}$;
  (iii) $\Omega = (-1, 1)$, *$m_x \neq 0$ in $\bar{\Omega}$;*
  (iv) *μ, ν are sufficiently large,*
  then $(0, \tilde{v})$ is unstable for all $\eta \geq 0$.

One consequence of Theorem 2.12 is that under suitable assumptions, a rare slower diffuser with or without advection can always invade the faster diffuser with no advection. Theorem 2.12 also implies that there exists a critical advection rate η^* beyond which the slower diffuser becomes too concentrated at the locally most favorable locations so that the faster random diffuser can invade when rare.

As a corollary of Theorems 2.10, 2.11 and 2.12 we have

THEOREM 2.13. *Suppose $\Omega = (-1, 1)$ and $m_x m_{xx} \neq 0$ in $\bar{\Omega}$, then there exists $\delta_0 > 0$ such that Conjecture* 2.9 *holds true provided that $0 < \min\{\mu, \nu\} < \delta_0$.*

2.3. Global bifurcation results

Theorems 2.10, 2.11 and 2.12 addressed how the stability of two semi-trivial steady states depends on the diffusion coefficients and the advection rate. These results suggest that as the advection rate varies from 0 to ∞, one of the semi-trivial steady states loses its stability and a branch of positive steady states bifurcates from the branch of the semi-trivial steady states. The natural globally relevant question is whether this branch of positive steady states can be extended to infinity in η or whether it will be connected to the other branch of semi-trivial steady states. In this section we discuss some global bifurcation results of positive steady states of (2.1), which complement Theorems 2.10, 2.11 and 2.12.

Define $\mathcal{S} = \mathcal{S}(\mu, \nu)$ by

$$\mathcal{S} = \{(\eta, u, v) : (u, v) \text{ is a positive steady state of } (2.1)\}.$$

Let $\mathcal{C}$ be a connected component of $\mathcal{S}$. We define the projection of $\mathcal{C}$ onto the η coordinate by

$$P\mathcal{C} := \{\eta \geq 0 : (\eta, u, v) \in \mathcal{C}\}.$$

The main result of this section can be stated as follows.

THEOREM 2.14. *Let Ω be a convex domain.*

(i) *Let $(\mu, \nu) \in \mathcal{R}_1$. Suppose that hypothesis* (a) *or* (b) *of Theorem* 2.10 *holds. Then there exists a connected component $\mathcal{C}_1$ of $\mathcal{S}$ emanating from $(\eta_*, 0, \tilde{v})$, where η_* is the unique value where $(0, \tilde{v})$ changes its stability. Moreover, $\mathcal{C}_1$ is unbounded and the projection $P\mathcal{C}_1$ contains (η_*, ∞).*

(ii) *Let $(\mu, \nu) \in \mathcal{R}_2$. Suppose that hypotheses* (a) *and one of* (b)(i)-(iii) *of Theorem* 2.11 *hold. There exist $\epsilon_1 > 0$ and a function $\delta_1 : (0, \epsilon_1) \to (0, \epsilon_1)$ such that if $\mu \leq \epsilon_1$ and $\nu \in (\mu - \delta_1(\mu), \mu)$, then there exist two disjoint connected components $\mathcal{C}_{2,1}, \mathcal{C}_{2,2}$ of $\mathcal{S}$ such that*
  (1) *$\mathcal{C}_{2,1}$ connects $(\eta_*, 0, \tilde{v})$ and $(\eta_1^*, \tilde{u}, 0)$;*

(2) *$\mathcal{C}_{2,2}$ is an unbounded component of $\mathcal{S}$ emanating from $(\eta_2^*, \tilde{u}, 0)$, whose projection $P\mathcal{C}_{2,2}$ contains (η_2^*, ∞);*

(3) *For some $\epsilon_2 > 0$, $(\min\{\eta_*, \eta_1^*\}, \max\{\eta_*, \eta_1^*\}) \subset \mathcal{P}\mathcal{C}_{2,1} \subset (0, \epsilon_2)$ and $\mathcal{P}\mathcal{C}_{2,2} \subset (\epsilon_2, \infty)$.*

Here η_* is the unique value where $(0, \tilde{v})$ changes stability and $\eta_1^* < \eta_2^*$ are precisely the two distinct values where $(\tilde{u}, 0)$ changes stability.

(iii) *Let $(\mu, \nu) \in \mathcal{R}_3$. Suppose that hypotheses* (a) *and one of the conditions* (b)(i)-(b)(iv) *of Theorem* 2.12 *hold, then there exists a connected component $\mathcal{C}_3$ of $\mathcal{S}$ emanating from $(\eta^*, \tilde{u}, 0)$, where η^* is the unique value where $(\tilde{u}, 0)$ changes stability. Moreover, $\mathcal{C}_3$ is unbounded in η and the projection $P\mathcal{C}_3$ contains (η^*, ∞).*

See Figure 2 for an illustration of Theorem 2.14. Theorem 2.10 implies that $(0, \tilde{v})$ changes the stability exactly once at $\eta = \eta_*$. Part (i) of Theorem 2.14 further asserts that an unbounded branch of positive steady states of (2.1) bifurcates from the semi-trivial solution branch $\{(\eta, 0, \tilde{v})\}$ at $\eta = \eta_*$ and the branch can be extended to $\eta = \infty$. Part (iii) of Theorem 2.14 complements Theorem 2.12 in a similar fashion.

Theorem 2.11 shows that when $(\mu, \nu) \in \mathcal{R}_2$, $(0, \tilde{v})$ changes the stability exactly once at $\eta = \eta_*$ and $(\tilde{u}, 0)$ changes its stability exactly twice at $\eta = \eta_1^*$ and $\eta = \eta_2^*$, respectively. Part (ii) of Theorem 2.14 establishes that there are two disjoint branches of positive steady states of (2.1), one of which connects two semi-trivial steady state branches at $\eta = \eta_*$ and $\eta = \eta_1^*$, while the other branch bifurcates from $\{(\eta, \tilde{u}, 0)\}$ at $\eta = \eta_2^*$ and it extends to infinity in η.

How do the qualitative transitions between Figures 2(a), 2(b) and 2(c) take place? We shall fix $\mu > 0$ and vary ν for our discussion. For $\nu < \bar{\nu}(\mu)$, the branch of positive steady states $\mathcal{C}_1$ in Figure 2(a) does not connect to the branch of semi-trivial steady states $\{(\eta, \tilde{u}, 0)\}$ for any $\eta \geq 0$. As ν approaches $\bar{\nu}(\mu)$ from below, the branch $\mathcal{C}_1$ is connected to the branch $\{(\eta, \tilde{u}, 0)\}$ exactly at $\eta = \eta^*$ for some $\eta^* > 0$. As ν surpasses $\bar{\nu}(\mu)$, $\mathcal{C}_1$ is split into two disjoint branches $\mathcal{C}_{2,1}$ and $\mathcal{C}_{2,2}$ of positive steady states, where $\mathcal{C}_{2,1}$ is bounded and connected to both branches of semi-trivial steady states, and $\mathcal{C}_{2,2}$ is unbounded and connected to $\{(\eta, \tilde{u}, 0)\}$, and it can be extended to infinity in η. This gives the transition from Figure 2(a) to 2(b). As ν approaches μ from below, the bounded branch $\mathcal{C}_{2,1}$, together with both end points, will approach $\{(\eta, u, v) = (0, s\tilde{v}(\cdot; \mu), (1-s)\tilde{v}(\cdot; \mu)) : 0 \leq s \leq 1\}$. This bounded branch is removed from the picture after ν increases beyond μ, while the unbounded branch connecting $(\eta^*, \tilde{u}, 0)$ remains. This gives the transition from Figure 2(b) to 2(c).

It is natural to inquire how the bifurcation points depend on the diffusion coefficients. We refer to Chapter 8 for a summary of the asymptotic behaviors of these bifurcation points for various limits of diffusion coefficients.

The rest of the paper is organized as follows: In Chapter 3 we summarize some general statements regarding solutions of system (2.1), and give some asymptotic properties of $\tilde{u}(\cdot; \eta, \mu)$ and its derivatives for sufficiently small μ. Chapter 4 is devoted to the proofs of existence of stable positive steady states (Theorem 2.2) and concludes with the characterization of $\mathcal{R}_i$ $(i = 1, 2, 3)$ (Theorems 2.5 and 2.7). Chapters 5-7 are devoted to the proofs of Conjecture 2.9 (Theorems 2.10, 2.11 and 2.12), under various additional assumptions. In Chapter 8, a summary of the asymptotic behavior of the bifurcation points η_*, η^* is given. We study the structure of positive steady states of system (2.1) when η is positive small and ν

is slightly less than μ in Chapter 9 (for convex domains) and Chapter 10 (for non-convex domains). In Chapter 11 we complete the proof of global bifurcation results (Theorem 2.14). Some technical details in our proofs are relegated to Appendices A to C.

CHAPTER 3

Preliminaries

3.1. Abstract Theory of Monotone Dynamical Systems

We summarize some general statements regarding solutions of system (2.1) and the stability of its steady states, which will be useful in subsequent chapters. By the maximum principle for cooperative systems [**90**] and standard theory for parabolic equations [**38, 46**], if the initial conditions of (2.1) are non-negative and not identically zero, system (2.1) has a unique positive smooth solution which exists for all time. This defines a smooth dynamical system on $C(\bar{\Omega}) \times C(\bar{\Omega})$ [**12, 47, 97**]. The stability of steady states of (2.1) is understood with respect to the topology of $C(\bar{\Omega}) \times C(\bar{\Omega})$. The following result is a consequence of the maximum principle and the structure of (2.1); See Theorem 3 in [**16**].

Theorem 3.1. *System* (2.1) *is a strongly monotone dynamical system, i.e.,*

a) $u_1(x, 0) \geq u_2(x, 0)$ *and* $v_1(x, 0) \leq v_2(x, 0)$ *for all* $x \in \Omega$*, and*
b) $(u_1(x, 0), u_2(x, 0)) \not\equiv (u_2(x, 0), v_2(x, 0))$

imply that $u_1(x, t) > u_2(x, t)$ *and* $v_1(x, t) < v_2(x, t)$ *for all* $x \in \bar{\Omega}$ *and* $t > 0$*.*

The following result is consequence of Theorem 3.1 and the theory of monotone dynamical systems [**47, 97**]:

Theorem 3.2. *The following results concerning system* (2.1) *hold.*

(i) *If system* (2.1) *has no positive steady state, and one of the two semi-trivial steady states is linearly unstable, then the other one is globally asymptotically stable* [**51, 68**]*;*
(ii) *If both semi-trivial steady states are unstable, then* (2.1) *has at least one stable positive steady state* [**30, 77**]*. In addition, if every positive steady state is linearly stable, then* (2.1) *has a unique positive steady state. Furthermore, the unique positive steady state is globally asymptotically stable* [**47, 51**]*. Furthermore, the unique positive steady state is globally asymptotically stable* [**47, 51**].

The following result concerns the linear stability of semi-trivial steady states of (2.1); see, e.g., Lemma 5.5, [**22**]. Denote the principal eigenvalue of the following problem by $\lambda_u = \lambda_u(\eta, \mu, \nu)$.

$$\left\{ \begin{array}{ll} \nu\Delta\varphi + (m - \tilde{u})\varphi + \lambda\varphi = 0 & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.1}$$

Lemma 3.3. *The steady state* $(\tilde{u}, 0)$ *is linearly stable* (*resp. unstable*) *if and only if* λ_u *is positive* (*resp. negative*)*.*

Similarly, let $\lambda_v = \lambda_v(\eta, \mu, \nu)$ be the principal eigenvalue of

$$\left\{ \begin{array}{ll} \mu\nabla \cdot (\nabla\varphi - \eta\varphi\nabla m) + (m - \tilde{v})\varphi + \lambda\varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} - \eta\varphi\frac{\partial m}{\partial n} = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{3.2}$$

or equivalently, of (by the transformation $\varphi = e^{\eta m}\psi$)

$$\left\{ \begin{array}{ll} \mu\nabla \cdot (e^{\eta m}\nabla\psi) + (m - \tilde{v})e^{\eta m}\psi + \lambda e^{\eta m}\psi = 0 & \text{in } \Omega, \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.3}$$

LEMMA 3.4. *The steady state* $(0, \tilde{v})$ *is linearly stable* (*resp. unstable*) *if and only if* λ_v *is positive* (*resp. negative*).

When $\eta = 0$, the dynamics of (2.1) is completely understood.

THEOREM 3.5. [**31**] *Suppose* $\eta = 0$ *and* $\mu < \nu$, *then* $\lambda_u > 0$ *and* $\lambda_v < 0$, *i.e.* $(\tilde{u}, 0)$ *is linearly stable and* $(0, \tilde{v})$ *is unstable. Moreover,* $(\tilde{u}, 0)$ *is globally asymptotically stable among all non-negative, non-trivial solutions of* (2.1).

REMARK 3.6. Actually, $\operatorname{sign}\lambda_u(0, \mu, \nu) = -\operatorname{sign}\lambda_v(0, \mu, \nu) = \operatorname{sign}(\nu - \mu)$ follows from standard eigenvalue comparison principles. See also [**1**].

The following observation follows from the proof of Theorem 3.5 in [**14**].

LEMMA 3.7. *For each* $\mu > 0$, $\tilde{u} \rightharpoonup 0$ *weakly in* $L^2(\{x \in \Omega : |\nabla m| > 0\})$ *as* $\eta \to \infty$. *Moreover, if the set* $\{x \in \Omega : |\nabla m| = 0\}$ *is of Lebesgue measure zero, then* $\tilde{u} \to 0$ *strongly in* $L^2(\Omega)$.

PROOF. Multiplying (2.2) by $\varphi \in \mathcal{S}$, where $\mathcal{S} = \left\{\varphi \in C^2(\bar{\Omega}) : \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0\right\}$, and integrating in Ω, we have

$$-\mu\int_\Omega \nabla\tilde{u} \cdot \nabla\varphi + \mu\eta\int_\Omega \tilde{u}\nabla m \cdot \nabla\varphi = \int_\Omega \varphi\tilde{u}(\tilde{u} - m).$$

By the boundary condition of φ,

$$\int_\Omega \nabla\tilde{u} \cdot \nabla\varphi = -\int_\Omega \tilde{u}\Delta\varphi.$$

Hence,

$$\mu\int_\Omega \tilde{u}\Delta\varphi + \mu\eta\int_\Omega \tilde{u}(\nabla m \cdot \nabla\varphi) = \int_\Omega \varphi\tilde{u}(\tilde{u} - m). \tag{3.4}$$

By integrating (2.2) over Ω, we have

$$\int_\Omega \tilde{u}(m - \tilde{u}) = 0 \tag{3.5}$$

and hence

$$\|\tilde{u}\|_{L^2(\Omega)} \le \|m\|_{L^2(\Omega)}. \tag{3.6}$$

Therefore, by passing to a subsequence, we may assume that $\tilde{u} \rightharpoonup u^*$ weakly in $L^2(\Omega)$, and $u^* \ge 0$ a.e. in Ω. Dividing (3.4) by $\mu\eta$ and passing to the limit in (3.4) we have

$$\int_\Omega u^*\nabla m \cdot \nabla\varphi = 0. \tag{3.7}$$

Since (3.7) holds for any $\varphi \in \mathcal{S}$ and $\mathcal{S}$ is dense in $W^{1,2}(\Omega)$, we see that (3.7) holds for every $\varphi \in W^{1,2}(\Omega)$. In particular, we can choose $\varphi = m$ in (3.7) so that

$$\int_\Omega u^* |\nabla m|^2 = 0.$$

Hence $u^*|\nabla m|^2 = 0$ a.e. in Ω. Therefore we conclude that $\tilde{u} \rightharpoonup 0$ weakly in $L^2(\Omega_r)$, where $\Omega_r = \{x \in \Omega : |\nabla m(x)| > 0\}$. Moreover, if the set of critical points of m is of measure zero, then we see that $u^* = 0$ a.e. in Ω. Therefore $\tilde{u} \rightharpoonup 0$ weakly in $L^2(\Omega)$, which implies by (3.5) that, as $\eta \to \infty$,

$$\int_\Omega \tilde{u}^2 = \int_\Omega \tilde{u} m \to 0.$$

$\square$

REMARK 3.8. By the above argument, the conclusion actually holds under the weaker assumption $\mu\eta, \eta \to \infty$. i.e. μ need not be fixed.

Next, we claim that

CLAIM 3.9. $\displaystyle \frac{\partial}{\partial \eta}\lambda_u\Big|_{\eta=0,\mu=\nu} = \frac{\mu\int_\Omega \tilde{u}(\mu,0)\nabla m \cdot \nabla \tilde{u}(\mu,0)}{\int_\Omega \tilde{u}(\mu,0)^2}.$

To see the claim, first differentiate (3.1) with respect to η, denoting $\varphi' = \frac{\partial}{\partial\eta}\varphi$, $\lambda_u' = \frac{\partial}{\partial\eta}\lambda_u$ and $\tilde{u}' = \frac{\partial}{\partial\eta}\tilde{u}$:

$$\left\{\begin{array}{ll} \nu\Delta\varphi' + (m-\tilde{u})\varphi' + \lambda_u\varphi' = (\tilde{u}' - \lambda_u')\varphi & \text{in } \Omega, \\ \frac{\partial\varphi'}{\partial n} = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{3.8}$$

Multiply (3.8) by φ and integrate by parts to get

$$\lambda_u' \int_\Omega \varphi^2 = \int_\Omega \tilde{u}'\varphi^2. \tag{3.9}$$

Set $\eta = 0$ and $\nu = \mu$, then $\varphi = \tilde{u}$ and we have

$$\lambda_u' \int_\Omega \tilde{u}(\mu,0)^2 = \int_\Omega \tilde{u}'(\mu,0)\tilde{u}(\mu,0)^2. \tag{3.10}$$

Next, differentiate (2.2) with respect to η,

$$\left\{\begin{array}{ll} \mu\nabla\cdot(\nabla\tilde{u}' - \eta\tilde{u}'\nabla m) + (m-2\tilde{u})\tilde{u}' = \mu\nabla\cdot(\tilde{u}\nabla m) & \text{in } \Omega, \\ \frac{\partial\tilde{u}'}{\partial n} - \eta\tilde{u}'\frac{\partial m}{\partial n} = \tilde{u}\frac{\partial m}{\partial n} & \text{on } \partial\Omega. \end{array}\right. \tag{3.11}$$

Multiply (3.11) by $-e^{-\eta m}\tilde{u}$ and integrate by parts to get

$$\int_\Omega e^{-\eta m}\tilde{u}^2\tilde{u}' = \mu\int_\Omega \tilde{u}\nabla m \cdot \nabla(e^{-\eta m}\tilde{u}). \tag{3.12}$$

Finally, Claim 3.9 follows by setting $\eta = 0$ in (3.12) and substituting the result into (3.10).

We will later need the following result, due to [**14**], in connection to convexity of the underlying domain.

THEOREM 3.10. *Define, by Claim* 3.9*,*

$$\alpha^*(\mu) := \frac{\partial}{\partial\eta}\lambda_u\Big|_{\eta=0,\mu=\nu} = \frac{\mu\int_\Omega \tilde{u}(\mu,0)\nabla m \cdot \nabla\tilde{u}(\mu,0)}{\int_\Omega \tilde{u}(\mu,0)^2}. \tag{3.13}$$

(a) *For general smooth domain Ω, $\alpha^*(\mu)$ is positive for sufficiently small or sufficiently large μ.*
(b) *Suppose in addition that Ω is convex, then $\alpha^*(\mu)$ is positive for all $\mu > 0$.*
(c) *Given any $\mu_0 > 0$, there exists a non-convex domain Ω and a smooth, sign-changing function $m(x)$ such that $\alpha^*(\mu_0) < 0$ and $\alpha^*(\mu)$ changes sign at least once in $(0, \mu_0)$.*

PROOF. Since (b) and (c) follow from [**14**], it suffices to show (a). By Theorem 3.12 (which is proved independently in Appendix A), when $\mu \to 0$, $\tilde{u}(\mu, 0) \to m$ in $H^1(\Omega)$. In particular,

$$\lim_{\mu\to 0} \frac{\alpha^*(\mu)}{\mu} = \lim_{\mu\to 0} \frac{\int_\Omega \tilde{u}(\mu,0)\nabla m \cdot \nabla \tilde{u}(\mu,0)}{\int_\Omega \tilde{u}(\mu,0)^2} = \frac{\int_\Omega m|\nabla m|^2}{\int_\Omega m^2} > 0,$$

as m is non-constant. This shows (a) in case $\mu \to 0$.

Next, for $\mu \to \infty$, we have the following lemma:

LEMMA 3.11. *Let $\bar{m} = \frac{1}{|\Omega|}\int_\Omega m$ and $w = \mu(\tilde{u}(\mu,0) - \bar{m})$, then $w - \frac{1}{|\Omega|}\int_\Omega w = w_1 + O(\mu^{-1})$, where w_1 is the unique solution to*

$$\begin{cases} -\Delta w_1 = \bar{m}(m - \bar{m}) & \text{in } \Omega, \\ \frac{\partial w_1}{\partial n} = 0 \quad \text{on } \partial\Omega, & \text{and} \quad \int_\Omega w_1 = 0. \end{cases} \tag{3.14}$$

PROOF OF LEMMA 3.11. It is easy to see that w satisfies the Neumann boundary condition, as well as the equation

$$-\Delta w = \bar{m}(m - \bar{m}) + \epsilon(m - 2\bar{m})w - \epsilon^2 w^2. \tag{3.15}$$

Let $(-\Delta)_N$ be the operator from $C_N^{2,1/2}(\Omega) = \{\psi \in C^{2,1/2}(\Omega) : \frac{\partial\psi}{\partial n}\Big|_{\partial\Omega} = \int_\Omega \psi = 0\}$ to $C^{1/2}(\Omega)$ defined by $\psi \mapsto -\Delta\psi$. It is well-known that the inverse $(-\Delta)_N^{-1}$ of $(-\Delta)_N$ exists. Let $\epsilon = 1/\mu$. We take the inverse $(-\Delta)_N^{-1}$ on both sides of (3.15) to obtain

$$w - \frac{1}{|\Omega|}\int_\Omega w = (-\Delta)_N^{-1}[\bar{m}(m-\bar{m})] + \epsilon(-\Delta)_N^{-1}[(m-2\bar{m})w] - \epsilon^2(-\Delta)_N^{-1}[w^2].$$

Hence, the Implicit Function Theorem implies that $w - \frac{1}{|\Omega|}\int_\Omega w = w_1 + O(\epsilon)$. $\square$

By Lemma 3.11, we may compute $\alpha^*(\mu)$ as follows:

$$\begin{aligned} \alpha^*(\mu) &= \frac{\mu \int_\Omega(\bar{m} + w/\mu)\nabla m \cdot \nabla(\bar{m} + w/\mu)}{\int_\Omega(\bar{m} + w/\mu)^2} \\ &= \frac{\int_\Omega(\bar{m} + w/\mu)\nabla m \cdot \nabla w}{\int_\Omega(\bar{m} + w/\mu)^2} \\ &= \frac{\int_\Omega(\bar{m} + w/\mu)\nabla m \cdot \nabla\left(w - \frac{1}{|\Omega|}\int_\Omega w\right)}{\int_\Omega(\bar{m} + w/\mu)^2} \\ &= \frac{\int_\Omega(\bar{m} + w/\mu)\nabla m \cdot \nabla(w_1 + O(1/\mu))}{\int_\Omega(\bar{m} + w/\mu)^2}. \end{aligned}$$

That is,

$$\alpha^*(\mu) = \frac{\bar{m}\int_\Omega \nabla m \cdot \nabla w_1}{\int_\Omega \bar{m}^2} + O(\mu^{-1}). \tag{3.16}$$

It suffices to show that $\int_\Omega \nabla m \cdot \nabla w_1 > 0$. This follows by multiplying (3.14) by m and integrating by parts to get

$$\int_\Omega \nabla m \cdot \nabla w_1 = \bar{m} \int_\Omega m(m - \bar{m}) = \bar{m} \int_\Omega (m - \bar{m})^2 > 0.$$

This proves (a) and completes the proof of Theorem 3.10. □

3.2. Asymptotic behavior of $\tilde{u}$ and φ as $\mu \to 0$

In this section we state some properties of $\tilde{u}$, $\tilde{v}$ and the principal eigenvalue $\lambda_u(\eta, \mu, \nu)$. The proofs of these results will be presented in Appendix A.

3.2.1. Asymptotic behavior of $\tilde{u}$. In this section we state some properties of $\tilde{u}$ (and, by setting $\eta = 0$, of $\tilde{v}$). We mainly focus on the asymptotic behaviors of $\tilde{u}$ and its derivatives as $\mu \to 0$.

THEOREM 3.12. *Let $\Lambda > 0$ be given.*

(i) *There exists a positive constant c such that for all $\mu > 0$ and $\eta \in [0, \Lambda]$,*

$$c \leq \tilde{u}(x) \leq \frac{1}{c} \quad \text{in } \Omega. \tag{3.17}$$

Moreover, $\tilde{u} \to m$ in $L^\infty(\Omega)$ as $\mu \to 0$ uniformly for $\eta \in [0, \Lambda]$.

(ii) *There exists $C > 0$ such that*

$$\int_\Omega |\nabla \tilde{u} - \nabla m|^2 \phi^2 \leq C \|\tilde{u} - m\|_{L^\infty(\Omega)} \|\phi\|^2_{H^1(\Omega)},$$

for any $\mu > 0$, $\eta \in [0, \Lambda]$ and $\phi \in H^1(\Omega)$.

(iii) *There exists $C > 0$ such that*

$$\int_\Omega |\nabla \tilde{u} - \nabla m|^2 \frac{\phi_1 \phi_2}{\tilde{u}^2} \leq C \|\tilde{u} - m\|_{L^\infty(\Omega)} \left(\|\phi_1\|^2_{H^1(\Omega)} + \|\phi_2\|^2_{H^1(\Omega)} \right),$$

for all $\mu > 0$, $\eta \in [0, \Lambda]$ and $\phi_1, \phi_2 \in H^1(\Omega)$.

(iv) *$\tilde{u} \to m$ in $L^\infty(\Omega) \cap H^1(\Omega)$ as $\mu \to 0$.*

(v) *For all $\epsilon > 0$, there exist $C = C(\epsilon, \Lambda)$ and $\delta = \delta(\epsilon, \Lambda)$ such that*

$$\left| \int_\Omega (\nabla \tilde{u} - \eta \tilde{u} \nabla m) \cdot \nabla \left(\frac{\phi^2}{\tilde{u}} \right) \right| \leq \epsilon \int_\Omega |\nabla \phi|^2 + C \int_\Omega \phi^2$$

for all $\eta \in [0, \Lambda]$, $\phi \in H^1(\Omega)$ and $\mu \in (0, \delta)$.

As an application of the maximum principle, we actually have a more precise result for $\tilde{v}$.

THEOREM 3.13. *For each $\nu > 0$, $\min_{\bar{\Omega}} m < \tilde{v} < \max_{\bar{\Omega}} m$ holds in $\bar{\Omega}$.*

Our next result concerns the limit of $\frac{\partial \tilde{u}}{\partial \eta}$ as $\mu \to 0$.

THEOREM 3.14. *Suppose that $\mu = \mu_k \to 0$ and $\phi = \phi_k \rightharpoonup \tilde{\phi}$ (weakly) in $H^1(\Omega)$, then for any $\Lambda > 0$ (denoting $\tilde{u} = \tilde{u}(\mu_k, \eta)$)*

$$\frac{1}{\mu} \int_\Omega \frac{\partial \tilde{u}}{\partial \eta} \phi \to \int_\Omega m \nabla m \cdot \nabla \left(\frac{\tilde{\phi}}{m} \right),$$

as $k \to \infty$, uniformly for $\eta \in [0, \Lambda]$.

Finally, we have the following result for $\frac{\partial^2 \tilde{u}}{\partial \eta^2}$.

THEOREM 3.15. *For each* $\Lambda > 0$,

$$\int_\Omega \left(\frac{\partial^2 \tilde{u}}{\partial \eta^2}\right)_+^2 = o(\mu^2)$$

as $\mu \to 0$ *uniformly for* $\eta \in [0, \Lambda]$*, where* f_+ *denotes the positive part of function* f.

3.2.2. Asymptotic behavior of λ_u. Recall that λ_u is the principal eigenvalue of (3.1) with corresponding eigenfunction φ. By (2.2), we rewrite (3.1) as follows.

$$\begin{cases} \frac{\nu}{\mu}\Delta\varphi - \frac{\nabla\cdot(\nabla\tilde{u} - \eta\tilde{u}\nabla m)}{\tilde{u}}\varphi + \frac{\lambda}{\mu}\varphi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

Denote the k-th eigenvalue of (3.18), counting multiplicity, by $\lambda_{u,k}/\mu$ and denote the corresponding eigenfunction by φ_k. In particular, $\lambda_{u,1} = \lambda_u(\eta, \mu, \nu)$ and $\varphi_1 = \varphi$.

We define the following limiting eigenvalue problem associated with (3.18):

$$\begin{cases} d\Delta\tilde{\varphi} - \frac{\nabla\cdot[(1-\eta m)\nabla m]}{m}\tilde{\varphi} + \sigma\tilde{\varphi} = 0 & \text{in } \Omega, \\ d\frac{\partial\tilde{\varphi}}{\partial n} - \frac{1-\eta m}{m}\frac{\partial m}{\partial n}\tilde{\varphi} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

Denote the k-th eigenvalue of (3.19), counting multiplicity, by $\sigma_k(\eta; d)$. The connection between $\sigma_k(\eta; d)$ and $\lambda_{u,k}$ as $\mu \to 0$ and $\nu/\mu \to d$ is given by

THEOREM 3.16. *For each* $d > 0$*, and* $\Lambda > 0$,

$$\lim_{\frac{\nu}{\mu}\to d, \mu\to 0} \frac{\lambda_{u,k}}{\mu} = \sigma_k(\eta; d)$$

uniformly for $\eta \in [0, \Lambda]$.

For $k = 1$ we establish some estimates on $\frac{\partial\lambda_u}{\partial\eta}$.

THEOREM 3.17. *For each* $\Lambda > 0$,

$$\frac{1}{\mu}\frac{\partial\lambda_u}{\partial\eta}(\cdot; \mu, \nu) \to \frac{\partial\sigma}{\partial\eta}(\cdot; d)$$

as $\mu \to 0$ *and* $\frac{\nu}{\mu} \to d > 0$*, uniformly for* $\eta \in [0, \Lambda]$*. Here* $\sigma(\eta; d) = \sigma_1(\eta; d)$ *is given in* (3.19).

As a consequence of Theorems 3.16 and 3.17, we have

COROLLARY 3.18. *For each* $\Lambda > 0$*, as* $\mu \to 0$ *and* $\frac{\nu}{\mu} \to d > 0$,

$$\frac{\lambda_u}{\mu}(\cdot, \mu, \nu) \to \sigma(\cdot; d)$$

in $C^1([0, \Lambda])$*. Here* $\sigma(\eta; d) = \sigma_1(\eta; d)$ *is given in* (3.19).

CHAPTER 4

Coexistence and classification of μ-ν plane

In Section 4.1, we establish a sufficient condition for coexistence of two species (Theorem 2.2) for general domains Ω. In Section 4.2, we classify the μ-ν plane into three separate regions according to the local dynamics of the semi-trivial steady states (Theorem 2.5) for convex domains Ω. We observe that the domain convexity assumption is not needed to prove results in Section 4.2, if μ is sufficiently small or sufficiently large (Corollary 4.11). In Section 4.3, we obtain addition results when μ is sufficiently small or sufficiently large (Theorem 2.7).

4.1. Coexistence: Proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2. We first show the instabilities of the semi-trivial steady states. The existence of a stable positive steady state follows from the standard theory of monotone dynamical systems (Theorem 3.2(ii)). First, we prove the following lemma:

LEMMA 4.1. *If $\eta \geq 1/\min_{\bar{\Omega}} m$, then $\int_\Omega \tilde{u} < \int_\Omega m$. Moreover,*

$$\limsup_{\mu\eta\to\infty,\eta\to\infty} \int_\Omega (\tilde{u} - m) \leq -\int_{\{x\in\Omega:|\nabla m|\neq 0\}} m < 0. \tag{4.1}$$

PROOF. Define $f(y) = ye^{-\eta y}$ for $\eta^{-1} \leq y < \infty$. Since $f'(y) < 0$ for $\eta^{-1} < y < \infty$, f has an inverse function, denoted by g. Since f assumes its maximum at $y = \eta^{-1}$ and $f(\eta^{-1}) = (e\eta)^{-1}$, g is defined in $(0, (e\eta)^{-1}]$ and $g' < 0$ in $(0, (e\eta)^{-1})$. Set

$$w = e^{-\eta m}\tilde{u}.$$

Then w satisfies

$$\begin{cases} \mu\nabla \cdot [e^{\eta m}\nabla w] + \tilde{u}(m - \tilde{u}) = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

We first check that $g(w)$ is well defined, i.e., $w(x) \leq (e\eta)^{-1}$ for every $x \in \bar{\Omega}$. Let $w(x_0) = \max_{\bar{\Omega}} w$ for some $x_0 \in \bar{\Omega}$. If $x_0 \in \Omega$, it is well known that $\Delta w(x_0) \leq 0$. We claim that if $x_0 \in \partial\Omega$, then $\Delta w(x_0) \leq 0$. To see this, we argue by contradiction: Suppose that $\Delta w(x_0) > 0$. Then there exists a small open ball B such that $B \subset \Omega$ and $\partial B \cap \partial\Omega = \{x_0\}$ and $\Delta w > 0$ in $\bar{B}$. This in particular implies, by the strong maximum principle, that $w(x) < w(x_0)$ for all $x \in \bar{B}$ and $x \neq x_0$. By the Hopf Boundary Lemma [**90**], $\frac{\partial w}{\partial n}(x_0) > 0$. However, this contradicts the boundary condition of w on $\partial\Omega$. This contradiction shows that $\Delta w(x_0) \leq 0$ always holds. Hence, by the equation of w we have $\tilde{u}(x_0) \leq m(x_0)$. Therefore,

$$\max_{\bar{\Omega}} w = w(x_0) = e^{-\eta m(x_0)}\tilde{u}(x_0) \leq e^{-\eta m(x_0)}m(x_0) \leq (e\eta)^{-1},$$

where the last inequality follows from $ye^{-\eta y} \le (e\eta)^{-1}$ for any $y > 0$. Hence, $g(w)$ is well defined. Dividing (4.2) by $g(w)$ and integrating in Ω,

$$\mu \int_\Omega \frac{e^{\eta m} g'(w)|\nabla w|^2}{g^2(w)} + \int_\Omega \frac{\tilde{u}}{g(w)}(m - \tilde{u}) = 0, \tag{4.3}$$

which can be written as

$$\int_\Omega (\tilde{u} - m) = \mu \int_\Omega \frac{e^{\eta m} g'(w)|\nabla w|^2}{g^2(w)} + \int_\Omega \frac{\tilde{u} - g(w)}{g(w)}(m - \tilde{u}). \tag{4.4}$$

We claim that

$$[\tilde{u} - g(w)]\,(m - \tilde{u}) \le 0 \quad \text{in } \Omega. \tag{4.5}$$

To establish this assertion, we consider two different cases:

(i) $\tilde{u}(x) > m(x)$. For this case, we have $f(\tilde{u}(x)) < w(x)$. As we assume that $\min_{\bar\Omega} m > \eta^{-1}$, we have $\tilde{u}(x) \ge \min_{\bar\Omega} m > \eta^{-1}$. Since g is monotone decreasing, by $f(\tilde{u}(x)) < w(x)$ we have $\tilde{u}(x) > g(w(x))$, i.e., (4.5) holds.

(ii) $\tilde{u}(x) < m(x)$. Suppose $\tilde{u}(x) \ge 1/\eta$, then $f(\tilde{u}(x))$ is defined, and $f(\tilde{u}(x)) > w(x)$. Then the monotone decreasing property of f implies that $\tilde{u}(x) > g(w(x))$. Suppose instead that $\tilde{u}(x) < 1/\eta$, then $\tilde{u}(x) > g(w(x))$ follows from the fact that $g(w(x))$ lies within the domain of f, $\mathrm{Dom}(f) = [1/\eta, \infty)$.

By $g' < 0$, (4.4) and (4.5) we have $\int_\Omega \tilde{u} < \int_\Omega m$. This proves the first part of the lemma.

For the second part, we fix $\delta \in (0, \min\{1, 1/\min_{\bar\Omega} m\})$, and consider η sufficiently large so that $\eta > 1/\delta$. Eventually, we will let $\eta \nearrow \infty$ and then $\delta \searrow 0$.

Claim 4.2. *$\dfrac{\tilde{u} - g(e^{-\eta m}\tilde{u})}{g(e^{-\eta m}\tilde{u})} < \delta - 1$ for $x \in \left\{x \in \Omega : m(x) > \delta \text{ and } \tilde{u}(x) < \delta^2\right\}$.*

Let $A = \left\{x \in \Omega : m(x) > \delta \text{ and } \tilde{u}(x) < \delta^2\right\}$. To see the claim, we note first that for $x \in A$, $\tilde{u} < m$ and hence $\tilde{u} \le g(e^{-\eta m}\tilde{u})$ by (4.5). Then, $\frac{g(e^{-\eta m}\tilde{u}) - \tilde{u}}{g(e^{-\eta m}\tilde{u})}$ is positive in A, and

$$\begin{aligned}
\frac{g(e^{-\eta m}\tilde{u}) - \tilde{u}}{g(e^{-\eta m}\tilde{u})} &\ge \frac{\inf_A g(e^{-\eta m}\tilde{u}) - \sup_A \tilde{u}}{\inf_A g(e^{-\eta m}\tilde{u})} \\
&= \frac{g(\sup_A (e^{-\eta m}\tilde{u})) - \delta^2}{g(\sup_A (e^{-\eta m}\tilde{u}))} \\
&\ge \frac{g(e^{-\eta\delta}\delta^2) - \delta^2}{g(e^{-\eta\delta}\delta^2)} \\
&\ge \frac{g(e^{-\eta\delta}\delta) - \delta^2}{g(e^{-\eta\delta}\delta)} \\
&= \frac{\delta - \delta^2}{\delta} = 1 - \delta
\end{aligned}$$

for all $x \in A$. Hence

$$\begin{aligned}\int_\Omega (\tilde{u} - m) &\le \int_\Omega \frac{\tilde{u} - g(e^{-\eta m}\tilde{u})}{g(e^{-\eta m}\tilde{u})}(m - \tilde{u}) \\ &\le \int_{\{x\in\Omega: m>\delta, \tilde{u}<\delta^2\}} \frac{\tilde{u} - g(e^{-\eta m}\tilde{u})}{g(e^{-\eta m}\tilde{u})}(m - \tilde{u}) \quad \text{(by (4.5))} \\ &\le (\delta - 1)\int_{\{x\in\Omega: m>\delta, \tilde{u}<\delta^2\}} (m - \tilde{u}) \\ &\le (\delta - 1)\int_{\{x\in\Omega: m>\delta, \tilde{u}<\delta^2\}} (m - \delta^2).\end{aligned}$$

Since, as $\eta \to \infty$, $\tilde{u} \to 0$ a.e. in the set of regular points of m (Lemma 3.7), we deduce that

$$\limsup_{\eta\to\infty} \int_\Omega (\tilde{u} - m) \le (\delta - 1)\int_{\{x\in\Omega: m>\delta, |\nabla m|>0\}} (m - \delta^2).$$

Letting $\delta \searrow 0$, we have

$$\limsup_{\eta\to\infty} \int_\Omega (\tilde{u} - m) \le -\int_{\{x\in\Omega: m>0, |\nabla m|>0\}} m.$$

This completes the proof. $\square$

REMARK 4.3. By a similar argument, we can show that if $\eta \le 1/\max_{\bar{\Omega}} m$, then $\int_\Omega m < \int_\Omega \tilde{u}$.

REMARK 4.4. (4.1) holds for non-negative m as well: Suppose $m \in C^2(\bar{\Omega})$ is non-negative and non-constant, then for all $\delta > 0$, $m+\delta$ satisfies (M). Now consider the unique positive solution $\tilde{u}_\delta$ of (2.2) with m being replaced by $m + \delta$. By the maximum principle, we have $\tilde{u}_\delta \ge \tilde{u}$. By applying the previous argument, we have

$$\limsup_{\mu\eta,\eta\to\infty} \int_\Omega (\tilde{u} - m) \le \limsup_{\mu\eta,\eta\to\infty} \int_\Omega [\tilde{u}_\delta - (m+\delta) + \delta] \le -\int_{\{x\in\Omega: |\nabla m|>0\}} (m+\delta) + \delta|\Omega|.$$

Letting $\delta \to 0$, we have (4.1).

Next, we consider the stability of $(\tilde{u}, 0)$, which is determined by the sign of the principal eigenvalue $\lambda_u = \lambda_u(\eta, \mu, \nu)$ of (3.1). Recall that $(\tilde{u}, 0)$ is linearly stable (resp. unstable) if λ_u is positive (resp. negative) (Lemma 3.3).

LEMMA 4.5. *If $\eta \ge \frac{1}{\min_{\bar{\Omega}} m}$, then $(\tilde{u}, 0)$ is unstable.*

PROOF. Let φ be a positive eigenfunction corresponding to λ_u. Dividing (3.1) by φ, (recalling that $\lambda = \lambda_u$ there), and integrating the result in Ω, we have

$$\nu \int_\Omega \frac{|\nabla \varphi|^2}{\varphi^2} + \int_\Omega (m - \tilde{u}) = -\lambda_u |\Omega|.$$

By Lemma 4.1, we see that $\lambda_u < 0$. $\square$

Next, we discuss the stability of $(0, \tilde{v})$ for $\eta \ge \frac{1}{\min_{\bar{\Omega}} m}$. Similar as before, the steady state $(0, \tilde{v})$ is linearly stable (resp. unstable) if the principal eigenvalue $\lambda_v = \lambda_v(\eta, \mu, \nu)$ is positive (resp. negative).

LEMMA 4.6. *If $\eta \ge \frac{1}{\min_{\bar{\Omega}} m}$, then $(0, \tilde{v})$ is unstable.*

PROOF. Dividing (3.3) by ψ and integrating by parts in Ω, we have

$$\mu\int_\Omega \frac{e^{\eta m}|\nabla\psi|^2}{\psi^2}+\int_\Omega e^{\eta m}(m-\tilde v)=-\lambda_v\int_\Omega e^{\eta m}. \tag{4.6}$$

Note that $\tilde v$ satisfies

$$\begin{cases} \nu\Delta\tilde v+\tilde v(m-\tilde v)=0 & \text{in } \Omega,\\ \frac{\partial \tilde v}{\partial n}=0 & \text{on } \partial\Omega. \end{cases}$$

Dividing the equation for $\tilde v$ by $\tilde v e^{-\eta\tilde v}$, integrating the resulting equation in Ω, we have

$$\begin{aligned} 0&=\nu\int_\Omega \Delta\tilde v\frac{e^{\eta\tilde v}}{\tilde v}+\int_\Omega e^{\eta\tilde v}(m-\tilde v)\\ &=-\nu\int_\Omega \nabla\tilde v\cdot\nabla\left(\frac{e^{\eta\tilde v}}{\tilde v}\right)+\int_\Omega e^{\eta\tilde v}(m-\tilde v)\\ &=-\nu\int_\Omega |\nabla\tilde v|^2\frac{e^{\eta\tilde v}(\eta\tilde v-1)}{\tilde v^2}+\int_\Omega e^{\eta\tilde v}(m-\tilde v). \end{aligned}$$

Taking (4.6) into account, we obtain

$$-\lambda_v\int_\Omega e^{\eta m}=\mu\int_\Omega\frac{e^{\eta m}|\nabla\psi|^2}{\psi^2}+\nu\int_\Omega|\nabla\tilde v|^2\frac{e^{\eta\tilde v}(\eta\tilde v-1)}{\tilde v^2}+\int_\Omega\left(e^{\eta m}-e^{\eta\tilde v}\right)(m-\tilde v). \tag{4.7}$$

As the first and third terms on the right-hand side of (4.7) are positive, we have

$$-\lambda_v\int_\Omega e^{\eta m}>\nu\int_\Omega|\nabla\tilde v|^2\frac{e^{\eta\tilde v}(\eta\tilde v-1)}{\tilde v^2}.$$

By the maximum principle, $\min_{\bar\Omega}\tilde v\ge\min_{\bar\Omega}m$ (Theorem 3.13). Hence,

$$-\lambda_v\int_\Omega e^{\eta m}>\nu\int_\Omega|\nabla\tilde v|^2\frac{e^{\eta\tilde v}(\eta\min_{\bar\Omega}m-1)}{\tilde v^2}\ge 0,$$

provided that $\eta\ge\frac{1}{\min_{\bar\Omega}m}$. Therefore, if $\eta\ge\frac{1}{\min_{\bar\Omega}m}$, then $\lambda_v<0$, i.e. the semi-trivial steady state $(0,\tilde v)$ is unstable. □

Finally, Theorem 2.2 follows from the instabilities of the semi-trivial steady states $(\tilde u,0)$ and $(0,\tilde v)$ (established in Lemmas 4.5 and 4.6) and the standard theory of monotone dynamical systems (Theorem 3.2(ii)).

4.2. Classification of μ-ν plane: Proof of Theorem 2.5

LEMMA 4.7. *For each $\mu>0$,*

$$\limsup_{\eta\to\infty}\lambda_u(\eta,\mu,\nu)\le-\frac{1}{|\Omega|}\int_{\{x\in\Omega:|\nabla m|>0\}}m<0$$

uniformly in $\nu>0$.

PROOF. Let φ denote a positive eigenfunction of $\lambda_u(\eta,\mu,\nu)$, i.e.

$$-\nu\Delta\varphi+(\tilde u-m)\varphi=\lambda_u\varphi\quad\text{in }\Omega,\quad \frac{\partial\varphi}{\partial n}|_{\partial\Omega}=0.$$

Dividing the above equation by φ and integrating by parts,

$$\lambda_u|\Omega|=-\nu\frac{\int_\Omega|\nabla\varphi|^2}{\int_\Omega\varphi^2}+\int_\Omega(\tilde u-m).$$

The claim follows from Lemma 4.1 by taking lim sup as $\eta \to \infty$. □

Next, for each $\mu > 0$ and $\nu > 0$, define $F(\mu, \nu) := \sup_{0 \le \eta < \infty} \lambda_u(\eta, \mu, \nu)$.

LEMMA 4.8. *For each $\mu > 0$, $F(\mu, \nu) < 0$ for all ν sufficiently small.*

PROOF. Fix $\mu > 0$. Suppose to the contrary that there exists $\nu_i \to 0^+$, and $\eta_i \in [0, \infty)$ such that $\lambda_{u,i} := \lambda_u(\eta_i, \mu, \nu_i) \ge -1/i$. By Lemma 4.7, η_i must be uniformly bounded. Thus we may assume without loss of generality that $\eta_i \to \eta_0 \in [0, \infty)$. Set $\tilde{u}_i := \tilde{u}(\mu, \eta_i)$. By standard elliptic regularity, $\tilde{u}_i \to \tilde{u}_0 := \tilde{u}(\mu, \eta_0)$ in $C(\bar{\Omega})$. By the definition of $\lambda_{u,i}$, there exists $\varphi_i > 0$ such that

$$\begin{cases} -\nu_i \Delta \varphi_i + (\tilde{u}_i - m)\varphi_i = \lambda_{u,i} \varphi_i & \text{in } \Omega, \\ \frac{\partial \varphi_i}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, for a given $\epsilon > 0$, denote by λ_i the principal eigenvalue of

$$\begin{cases} -\nu_i \Delta \varphi + (\tilde{u}_0 + \epsilon - m)\varphi = \lambda \varphi & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well-known that (see, e.g., [**53**]), as $\nu_i \to 0$,

$$\lim_{i \to \infty} \lambda_i = \min_{\bar{\Omega}} (\tilde{u}_0 + \epsilon - m).$$

Since $\tilde{u}_i \to \tilde{u}_0$ in $C(\bar{\Omega})$, there exists i_0 such that $\tilde{u}_i \le \tilde{u}_0 + \epsilon$ for all $i \ge i_0$. By the comparison principle of eigenvalues (see, e.g. [**81**]), $\lambda_i \ge \lambda_{u,i}$ for $i \ge i_0$. This implies that for each $\epsilon > 0$,

$$\min_{\bar{\Omega}} (\tilde{u}_0 + \epsilon - m) = \lim_{i \to \infty} \lambda_i \ge \limsup_{i \to \infty} \lambda_{u,i} \ge \limsup_{i \to \infty} \frac{-1}{i} = 0.$$

Letting $\epsilon \to 0$, we see that $\tilde{u}_0 - m \ge 0$ in $\bar{\Omega}$. By the integral identity $\int_\Omega \tilde{u}_0 (m - \tilde{u}_0) = 0$ obtained by integrating the equation of $\tilde{u}_0 = \tilde{u}(\mu, \eta_0)$, we deduce that $\tilde{u}_0 = m$, which is a contradiction, as m does not satisfy the equation of $\tilde{u}_0 = \tilde{u}(\mu, \eta_0)$ for any $\eta_0 \ge 0$. □

LEMMA 4.9. *Suppose that Ω is convex, then for each $\mu > 0$, there exists $\bar{\nu} = \bar{\nu}(\mu) \in (0, \mu)$ such that $F(\mu, \nu) < 0$ for $\nu \in (0, \bar{\nu})$ and $F(\mu, \nu) > 0$ for $\nu \in (\bar{\nu}, \infty)$.*

PROOF. By (3.13) and Theorem 3.10 (b), we deduce that for a convex domain Ω, $\frac{\partial}{\partial \eta} \lambda_u(0, \mu, \mu) > 0$ for all $\mu > 0$.

On the other hand, it is easy to see that $\lambda_u(0, \mu, \mu) = 0$ for all $\mu > 0$. In particular $\lambda_u(\eta, \mu, \mu) > 0$ when η is small and positive. Hence $F(\mu, \mu) > 0$. Hence for each $\mu > 0$, we may define

$$\bar{\nu} = \bar{\nu}(\mu) = \sup\{\nu' > 0 : F(\mu, \nu) < 0 \text{ for all } 0 < \nu < \nu'\}.$$

By the definition of $\bar{\nu}$, we see that $\bar{\nu} < \mu$ and $\lambda_u(\eta, \mu, \nu) \le F(\mu, \nu) < 0$ for all $\nu \in (0, \bar{\nu})$. Next, from Lemma 4.7 we conclude that there exists $\eta_0 \ge 0$ such that $\lambda_u(\eta_0, \mu, \bar{\nu}) = 0$, i.e. $F(\mu, \bar{\nu})$ is attained by some η_0. Moreover, by the strict monotonicity of λ_u in ν (since $m - \tilde{u}(\mu, \eta)$ is non-constant), $\lambda_u(\eta_0, \mu, \nu) > \lambda_u(\eta_0, \mu, \bar{\nu}) = 0$ for all $\nu > \bar{\nu}$. This implies that $F(\mu, \nu) > 0$ for all $\nu > \bar{\nu}$. □

For general smooth domain (not necessarily convex), the conclusion of Lemma 4.9 still holds provided that μ is sufficiently small or sufficiently large.

COROLLARY 4.10. *For general smooth domain Ω, there exists $\epsilon_0 \in (0,1)$ such that for $\mu \in (0, \epsilon_0) \cup (1/\epsilon_0, \infty)$, there exists $\bar{\nu} \in (0, \mu)$ such that $F(\mu, \nu) < 0$ for $\nu \in (0, \bar{\nu})$ and $F(\mu, \nu) > 0$ for $\nu \in (\bar{\nu}, \infty)$.*

PROOF. The proof is analogous to the proof of Lemma 4.9, with Theorem 3.10(a) in place of Theorem 3.10(b). $\square$

PROOF OF THEOREM 2.5. We first consider the stability of $(\tilde{u}, 0)$. Fix $\mu > 0$. It follows from Lemma 4.9 that if $\nu \in (0, \bar{\nu})$, then $\lambda_u(\eta, \mu, \nu) < 0$ for all $\eta \geq 0$; i.e., if $\nu < \bar{\nu}$, then $(\tilde{u}, 0)$ is unstable for any $\eta \geq 0$. For every $\nu > \bar{\nu}$, we recall from the proof of Lemma 4.9 that there exists some $\eta_0 \geq 0$ such that $\lambda_u(\eta_0, \mu, \nu) > 0$. We also have $\lambda_u(0, \mu, \nu) = \text{sign}(\nu - \mu)$ by Theorem 3.5, and that $\lambda_u(\eta, \mu, \nu) < 0$ for $\eta > \frac{1}{\min_{\bar{\Omega}} m}$ (Lemma 4.7). Hence, depending on $\nu \in (\bar{\nu}, \mu)$ or $\nu \in (\mu, \infty)$, $\lambda_u(\eta, \mu, \nu)$ changes sign at least twice or once as η ranges from 0 to ∞.

For the stability of $(0, \tilde{v})$, by Theorem 3.5 we have $\lambda_v(0, \mu, \nu) = \text{sign}(\mu - \nu)$. Hence if $\nu < \mu$, $(0, \tilde{v})$ is stable for small positive η. Lemma 4.7 implies that if $\eta > \frac{1}{\min_{\bar{\Omega}} m}$, then $(0, \tilde{v})$ is unstable. Therefore, if $\nu < \mu$, $(0, \tilde{v})$ changes stability at least once as η varies from zero to infinity. $\square$

COROLLARY 4.11. *For general smooth domain Ω, there exists $\epsilon_0 \in (0,1)$ such that for $\mu \in (0, \epsilon_0) \cup (1/\epsilon_0, \infty)$, there exists a unique $\bar{\nu} = \bar{\nu}(\mu) \in (0, \mu)$ such that the conclusions* (i)-(iii) *of Theorem* 2.5 *hold.*

PROOF. The proof is analogous to the proof of Theorem 2.5, with Corollary 4.10 in place of Lemma 4.9. $\square$

4.3. Limiting behavior of $\bar{\nu}$

By Corollary 4.11, we have seen that domain convexity is not needed for Theorem 2.5(i) - (iii) to hold when μ is sufficiently small or sufficiently large. In this section we shall prove Theorem 2.7, which concerns the limiting behavior of $\bar{\nu}$ for small and large μ. Some technical proofs are postponed to Appendix B.

Recall that $\bar{\nu} = \bar{\nu}(\mu) > 0$ can be characterized as

$$\bar{\nu}(\mu) = \sup\{\nu' > 0 : \lambda_u(\eta, \mu, \nu) < 0 \text{ for all } \eta \geq 0 \text{ and } 0 < \nu < \nu'\}.$$

We first prove Theorem 2.7(iv).

PROOF OF THEOREM 2.7(iv). By Theorem 2.2, $\lambda_u < 0$ whenever $\eta \geq \frac{1}{\min_{\bar{\Omega}} m}$. Thus, it suffices to keep track of the sign changes of λ_u for $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$. Let $a_* \in (0,1)$ be given in Theorem B.2(iv). Suppose to the contrary that for some $\mu_k \to 0$,

$$\frac{\bar{\nu}(\mu_k)}{\mu_k} \to d, \qquad \text{for some } d \neq a_*.$$

Case (i): $\bar{\nu}(\mu_k)/\mu_k \to d$, for some $0 \leq d < a_*$.

By the definition of $\bar{\nu}$ and monotonicity of λ_u in ν, we may choose ν_k, η_k such that $(\mu_k, \nu_k) \in \mathcal{R}_2$,

$$\eta_k \in \left[0, \frac{1}{\min_{\bar{\Omega}} m}\right], \qquad \text{and} \qquad \bar{\nu}(\mu_k) < \nu_k < \mu_k$$

and

$$\frac{\nu_k}{\mu_k} \to \frac{d + a_*}{2} \in (0, a_*), \quad \lambda_u(\eta_k, \mu_k, \nu_k) \geq 0.$$

Passing to a subsequence, we may assume that $\eta_k \to \eta_0 \geq 0$. On the other hand, by Corollary 3.18 and Theorem B.4(i),

$$\frac{\lambda_u(\eta_k, \mu_k, \nu_k)}{\mu_k} \to \sigma_1\left(\eta_0; \frac{d + a_*}{2}\right) < 0,$$

where $\sigma_1(\eta; d)$ is the principal eigenvalue of (3.19). This is a contradiction to the fact that $\lambda_u(\eta_k, \mu_k, \nu_k) \geq 0$.
Case (ii): $\bar{\nu}(\mu_k)/\mu_k \to d$, for some $d \in (a_*, 1]$.
By definition of $\bar{\nu}$, for each k, $\lambda_u(\eta, \mu_k, \bar{\nu}(\mu_k)) \leq 0$ for all $\eta \geq 0$. But then Corollary 3.18 implies that

$$\frac{\lambda_u(\eta, \mu_k, \bar{\nu}(\mu_k))}{\mu_k} \to \sigma_1(\eta; d).$$

But the latter changes sign exactly twice as η varies from 0 to ∞ (Theorem B.4(ii)). We obtain a contradiction again. In conclusion, $\bar{\nu}(\mu)/\mu \to a_*$, as $\mu \to 0$. □

Next, define $\tau = \tau(\eta; \nu)$ to be the principal eigenvalue of

$$\begin{cases} \nu \Delta \varphi + \left(m - \frac{\int_\Omega m e^{\eta m}}{\int_\Omega e^{2\eta m}} e^{\eta m}\right) \varphi + \tau \varphi = 0 & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

LEMMA 4.12. *As $\mu \to \infty$, $\lambda_u(\eta, \mu, \nu) \to \tau(\eta; \nu)$ uniformly in $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$ and uniformly for ν in any compact subset of $(0, \infty)$.*

PROOF. This follows from the continuous dependence of the principal eigenvalue on coefficients. More precisely, suppose that $\nu = \nu_i \to \nu_0 \in (0, \infty)$, $\mu_i \to \infty$ and $\eta_i \to \eta_0 \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$. Let φ_i be the principal eigenfunction corresponding to $\lambda_{u,i} = \lambda_u(\eta_i, \mu_i, \nu_i)$ normalized by $\|\varphi_i\|_{L^\infty(\Omega)} = 1$, which satisfies

$$\nu_i \Delta \varphi_i + (m - \tilde{u}(\mu_i, \eta_i))\varphi_i + \lambda_{u,i} \varphi_i = 0 \quad \text{in } \Omega, \quad \frac{\partial \varphi_i}{\partial n}|_{\partial\Omega} = 0. \tag{4.9}$$

We first prove that $\lambda_{u,i}$ is bounded, which follows from the eigenvalue comparison

$$\min_{\bar{\Omega}}(\tilde{u}(\mu_i, \eta_i) - m) \leq \lambda_{u,i} \leq \max_{\bar{\Omega}}(\tilde{u}(\mu_i, \eta_i) - m)$$

and the L^∞ boundedness of $\tilde{u}(\mu_i, \eta_i)$ (Lemma A.1). Therefore, by elliptic regularity theory, passing to a subsequence, we may assume that $\varphi_i \rightharpoonup \varphi_0$ in $W^{2,p}(\Omega)$ and $\lambda_{u,i} \to \lambda_0$. Passing to the limit in (4.9) (using Lemma A.1), we deduce that φ_0 is a non-trivial, non-negative eigenfunction of

$$\nu_0 \Delta \varphi_0 + \left(m - \frac{\int_\Omega m e^{\eta_0 m}}{\int_\Omega e^{2\eta_0 m}} e^{\eta_0 m}\right)\varphi_0 + \lambda_0 \varphi_0 = 0 \quad \text{in } \Omega, \quad \frac{\partial \varphi_0}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

This implies that $\lambda_0 = \tau(\eta_0; \nu_0)$ and φ_0 is the corresponding normalized principal eigenfunction. By the uniqueness of the limit, the original full sequence also converges. This proves that $\lambda_u \to \tau$ as $\mu \to \infty$ uniformly in $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$ and ν in compact subsets of $(0, \infty)$. □

PROPOSITION 4.13. *There exists a unique $\nu^+ > 0$ such that*

(i) *if $\nu \in (0, \nu^+)$, then $\tau < 0$ for all $\eta \in [0, \infty)$;*
(ii) *if $\nu \in (\nu^+, \infty)$, then τ changes sign at least twice as η varies from 0 to ∞ (from negative to positive, back to negative).*

Here $\tau = \tau(\eta; \nu)$ is the principal eigenvalue of (4.8).

Proposition 4.13 is proved independently in Appendix C, along with results concerning other limiting eigenvalue problems. We note that the convexity of Ω is not needed for Proposition 4.13 to hold.

Proof of Theorem 2.7(v). First, we show that

$$\liminf_{\mu\to\infty} \bar{\nu} \geq \nu^+. \tag{4.10}$$

Fix $\nu < \nu^+$, then by Theorem 2.2, (regardless of $\mu, \nu > 0$)

$$\lambda_u < 0 \quad \text{ for } \eta \geq \frac{1}{\min_{\bar{\Omega}} m}. \tag{4.11}$$

It suffices to show that $\lambda_u < 0$ for $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$ when μ is sufficiently large. By Proposition 4.13(i), there exists ϵ_0 such that $\tau(\eta;\nu) < -\epsilon_0$ for all $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$. Hence by Lemma 4.12, there exists $\mu_2 > 0$ such that for all $\mu \geq \mu_2$,

$$\lambda_1(\eta,\mu,\nu) < \tau(\eta;\nu) + \epsilon_0 < 0$$

for all $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$. This implies $\liminf\limits_{\mu\to\infty} \bar{\nu} \geq \nu$. Letting $\nu \nearrow \nu^+$ proves (4.10).

Similarly, we show

$$\limsup_{\mu\to\infty} \bar{\nu} \leq \nu^+. \tag{4.12}$$

Given $\nu \in (\nu^+, \infty)$, then by Proposition 4.13(ii), $\tau(\eta;\nu)$ changes sign as η ranges from 0 to $\frac{1}{\min_{\bar{\Omega}} m}$. The same holds true for $\lambda_u(\eta,\mu,\nu)$ for μ sufficiently large, by Lemma 4.12. Hence $\limsup\limits_{\mu\to\infty} \bar{\nu} \leq \nu$. Letting $\nu \searrow \nu^+$ proves (4.12). $\square$

CHAPTER 5

Results in $\mathcal{R}_1$: Proof of Theorem 2.10

By Theorem 2.5, the semi-trivial steady state $(\tilde{u}, 0)$ is unstable for all $\eta \geq 0$ whenever $(\mu, \nu) \in \mathcal{R}_1$; i.e. the part of Conjecture 2.9(a) concerning stability of $(\tilde{u}, 0)$ holds true. The main question is whether the stability of $(0, \tilde{v})$ changes exactly once as η varies from 0 to ∞. In this chapter we will address this question and establish Theorem 2.10. Part (a) of Theorem 2.10 is proved in Section 5.1, while the proof of part (b) is given in Section 5.2.

We first observe that $(0, \tilde{v})$ changes stability at least once whenever $(\mu, \nu) \in \mathcal{R}_1$.

LEMMA 5.1. *If $\mu > \nu$, then*

$$\lambda_v(\eta, \mu, \nu) = \begin{cases} > 0 & \text{when } \eta = 0, \\ < 0 & \text{when } \eta \geq \frac{1}{\min_{\bar{\Omega}} m}. \end{cases}$$

In particular, $(0, \tilde{v})$ changes stability at least once as η varies from 0 to ∞.

PROOF. On the one hand, one can deduce by variational characterization that $\lambda_v(0, \mu, \nu)$ is strictly increasing in μ. Hence

$$\lambda_v(0, \mu, \nu) > \lambda_v(0, \nu, \nu) = 0.$$

On the other hand, Lemma 4.6 guarantees that $\lambda_v(\eta, \mu, \nu) < 0$ for all $\eta \geq 1/\min_{\bar{\Omega}} m$. Hence $\lambda_v(\eta, \mu, \nu)$ changes sign at least once. $\square$

Let us denote by $\eta_* = \eta_*(\mu, \nu)$ any point(s) where $\lambda_v(\cdot, \mu, \nu) = 0$, and let ψ be the positive eigenfunction corresponding to $\lambda_v(\eta_*, \mu, \nu) = 0$.

LEMMA 5.2. $\dfrac{\partial \lambda_v}{\partial \eta}(\eta_*, \mu, \nu) = -\mu \dfrac{\int_\Omega e^{\eta_* m} \psi \nabla m \cdot \nabla \psi}{\int_\Omega e^{\eta_* m} \psi^2}.$

PROOF. We shall prove the formula for any $\eta \geq 0$. Rewrite (3.3) equivalently as

$$\begin{cases} \mu \Delta \psi + \mu \eta \nabla m \cdot \nabla \psi + (m - \tilde{v})\psi + \lambda_v \psi = 0 & \text{in } \Omega, \\ \frac{\partial \psi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Differentiating (5.1) with respect to η and denoting this differentiation as $'$ for ease of notation, we have

$$\begin{cases} \mu \Delta \psi' + \mu \eta \nabla m \cdot \nabla \psi' + (m - \tilde{v})\psi' + \lambda_v \psi' = -\mu \nabla m \cdot \nabla \psi - \lambda_v' \psi & \text{in } \Omega, \\ \frac{\partial \psi'}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Rewriting (5.2), we have

$$e^{-\eta m} \mu \nabla \cdot (e^{\eta m} \nabla \psi') + (m - \tilde{v})\psi' + \lambda_v \psi' = -\mu \nabla m \cdot \nabla \psi - \lambda_v' \psi. \tag{5.3}$$

Multiply (5.3) by $e^{\eta m}\psi$ and integrate by parts,

$$\begin{aligned}(5.4)\qquad -\mu \textstyle\int_\Omega e^{\eta m}\nabla\psi'\cdot\nabla\psi + \int_\Omega e^{\eta m}(m-\tilde{v})\psi'\psi + \lambda_v \int_\Omega \psi'\psi e^{\eta m} \\ = -\mu \textstyle\int_\Omega e^{\eta m}\psi\nabla m\cdot\nabla\psi - \lambda_v' \int_\Omega e^{\eta m}\psi^2.\end{aligned}$$

Next, we multiply (5.1) by ψ' and integrate by parts, we have

$$-\mu\int_\Omega e^{\eta m}\nabla\psi\cdot\nabla\psi' + \int_\Omega e^{\eta m}(m-\tilde{v})\psi\psi' + \lambda_v\int_\Omega \psi\psi' e^{\eta m} = 0.$$

Hence the left hand side of (5.4) is zero and we have

$$-\mu\int_\Omega e^{\eta m}\psi\nabla m\cdot\nabla\psi - \lambda_v'\int_\Omega e^{\eta m}\psi^2 = 0.$$

This proves Lemma 5.2. □

5.1. The case when $(\mu,\nu)\in\mathcal{R}_1$ and $\frac{\mu}{\nu}$ is sufficiently large

In this section we prove Theorem 2.10(a).

REMARK 5.3. For a general smooth domain Ω, $\bar{\nu}(\mu)$ and hence $\mathcal{R}_1$ may not be well-defined. Nonetheless, by Theorem 2.7, $\bar{\nu}(\mu)$ is defined for μ sufficiently small, or sufficiently large. In fact, the convexity of domain Ω is used only to define $\bar{\nu}(\mu)$ for intermediate μ (and hence the region $\mathcal{R}_1$). And Theorem 2.10(a) holds, provided that we rephrase the statement carefully.

By the fact that the $\bar{\nu}(\mu)$ satisfies, by Theorem 2.7,

$$\lim_{\mu\to 0}\frac{\bar{\nu}(\mu)}{\mu} = a_*, \quad \text{and} \quad \lim_{\mu\to\infty}\bar{\nu}(\mu) = \nu^+ \quad \text{for some } 0<a_*<1 \text{ and } \nu^+>0,$$

since $\mathcal{R}_1 = \{(\mu,\nu) : \mu>0,\ 0<\nu<\bar{\nu}(\mu)\}$, it suffices to consider two cases: (i) $\nu\not\to 0,\infty$, $\mu\to\infty$; (ii) $\nu,\ \nu/\mu\to 0$.

5.1.1. Case (i): $\nu\not\to 0,\infty$, and $\mu\to\infty$.

PROPOSITION 5.4. *For each fixed $\epsilon\in(0,1)$, there exists $\underline{\mu}$ such that if $\nu\in[\epsilon,1/\epsilon]$ and $\mu>\underline{\mu}$, then there exists a unique positive number η_* such that $\lambda_v(\eta,\mu,\nu)>0$ for $\eta\in[0,\eta_*)$ and $\lambda_v(\eta,\mu,\nu)<0$ for $\eta\in(\eta_*,\infty)$. Moreover,*

$$\lim_{\nu\to\nu_0,\mu\to\infty}\eta_* = \eta_2,$$

where $\eta_2=\eta_2(\nu_0)$ is the unique positive root of (denoting $\tilde{v}=\tilde{v}(\nu_0)$)

$$F_2(\eta) = \int_\Omega e^{\eta m}(m-\tilde{v}).$$

LEMMA 5.5. *For each ν, $\frac{dF_2}{d\eta}>0$ for any $\eta>0$.*

PROOF.

$$\begin{aligned}\frac{dF_2}{d\eta} &= \int_\Omega e^{\eta m} m(m-\tilde{v})\\ &= \int_\Omega [e^{\eta m}m - e^{\eta\tilde{v}}\tilde{v}](m-\tilde{v}) + \int_\Omega e^{\eta\tilde{v}}\tilde{v}(m-\tilde{v})\\ &> \int_\Omega e^{\eta\tilde{v}}\tilde{v}(m-\tilde{v})\\ &= -\nu\int_\Omega e^{\eta\tilde{v}}\Delta\tilde{v}\\ &= \nu\eta\int_\Omega e^{\eta\tilde{v}}|\nabla\tilde{v}|^2\\ &> 0,\end{aligned}$$

where the strict inequalities follow from m and hence $\tilde{v}$ being non-constant. □

LEMMA 5.6. *For each ν, there exists a unique $\eta_2>0$ such that $F_2(\eta)<0$ for $\eta\in[0,\eta_2)$ and $F_2(\eta)>0$ for $\eta\in(\eta_2,\infty)$. Moreover, $\eta_2\in(0,\frac{1}{\min_{\bar\Omega} m})$.*

PROOF. Note that

$$F_2(0) = \int_\Omega (m-\tilde{v}) = -\nu\int_\Omega \frac{\Delta\tilde{v}}{\tilde{v}} = -\nu\int_\Omega \frac{|\nabla\tilde{v}|^2}{\tilde{v}^2} < 0,$$

and

$$\begin{aligned}F_2(\eta) &= \int_\Omega [e^{\eta m} - e^{\eta\tilde{v}}](m-\tilde{v}) + \int_\Omega e^{\eta\tilde{v}}(m-\tilde{v})\\ &> \int_\Omega e^{\eta\tilde{v}}(m-\tilde{v})\\ &= -\nu\int_\Omega \frac{e^{\eta\tilde{v}}}{\tilde{v}}\Delta\tilde{v}\\ &= \nu\int_\Omega \frac{e^{\eta\tilde{v}}|\nabla\tilde{v}|^2}{\tilde{v}^2}(\eta\tilde{v}-1).\end{aligned}$$

As $\tilde{v}(x)\ge\min_{\bar\Omega} m$ in Ω (Theorem 3.13), if we have $\eta\ge\frac{1}{\min_{\bar\Omega} m}$, then

$$F_2(\eta) > \nu\int_\Omega \frac{e^{\eta\tilde{v}}|\nabla\tilde{v}|^2}{\tilde{v}^2}(\eta\min_{\bar\Omega} m - 1) > 0.$$

Since F_2 is strictly monotonically increasing, we see that there exists $\eta_2\in(0,\frac{1}{\min_{\bar\Omega} m})$ such that $F_2(\eta)<0$ for $\eta<\eta_2$ and $F_2(\eta)>0$ for $\eta>\eta_2$. □

Next, we prove the main result of the subsection.

PROOF OF PROPOSITION 5.4. Recall that the principal eigenvalue of (3.3), denoted by $\lambda_v=\lambda_v(\eta,\mu,\nu)$, determines the local stability of $(0,\tilde{v})$. We normalize the associated positive eigenfunction $\psi=\psi(\eta,\mu,\nu)$ by $\int_\Omega\psi^2=|\Omega|$.

Firstly, if $\mu>\nu$, then $\lambda_v(0,\mu,\nu)>0$ (Theorem 3.5). Secondly, given any μ,ν, if $\eta\ge\frac{1}{\min_{\bar\Omega} m}$, then $\lambda_v(\eta,\mu,\nu)<0$ (Theorem 2.5). These two results imply that if $\mu>\nu$, then $\eta\mapsto\lambda_v$ has at least one positive root. To establish the theorem, it suffices to show that for sufficiently large μ, $\lambda_v(\eta,\mu,\nu)$, as a function of η, has at most one positive root. To this end, we argue by contradiction: Suppose that there exist $\{\mu_i,\nu_i\}_{i=1}^\infty$ with $\mu_i\to\infty$ and $\nu_i\to\nu_0>0$, as well as $0<\hat\eta_i<\tilde\eta_i$ such that $\lambda_v(\hat{\eta}_i,\mu_i,\nu_i)=\lambda_v(\tilde{\eta}_i,\mu_i,\nu_i)=0$. Note that $\hat{\eta}_i,\tilde{\eta}_i\le\frac{1}{\min_\Omega m}$. Passing to a subsequence if necessary, we may assume that $\hat{\eta}_i\to\hat{\eta}$, $\tilde{\eta}_i\to\tilde{\eta}$ as $i\to\infty$.

We first show that $\hat{\eta}=\tilde{\eta}=\eta_2=\eta_2(\nu_0)$, i.e. $\lim_{i\to\infty}\hat{\eta}_i=\lim_{i\to\infty}\tilde{\eta}_i=\eta_2$. Set $\psi_i=\psi(\hat{\eta}_i,\mu_i,\nu_i)$. Then ψ_i satisfies, with $\tilde{v}=\tilde{v}(\nu_i)$,

$$\mu_i\nabla\cdot[e^{\hat{\eta}_i m}\nabla\psi_i]+(m-\tilde{v})e^{\hat{\eta}_i m}\psi_i=0\quad\text{ in }\Omega,\quad \left.\frac{\partial\psi_i}{\partial n}\right|_{\partial\Omega}=0.$$

Since ψ_i is uniformly bounded in $L^2(\Omega)$, by elliptic regularity we deduce the uniform boundedness of $\|\psi_i\|_{W^{2,2}(\Omega)}$. By the Sobolev embedding theorem, passing to a subsequence if necessary, $\psi_i\to\hat{\psi}$ weakly in $W^{2,2}(\Omega)$ and strongly in $W^{1,2}(\Omega)$ for some $\hat{\psi}\in W^{2,2}(\Omega)$, which is a weak solution of

$$\nabla\cdot[e^{\hat{\eta}m}\nabla\hat{\psi}]=0\quad\text{ in }\Omega,\quad \left.\frac{\partial\hat{\psi}}{\partial n}\right|_{\partial\Omega}=0.$$

Hence, $\hat{\psi}$ is equal to some constant, denoted as C_1. Since $\int_\Omega\hat{\psi}^2=|\Omega|$, we see that $C_1=1$. Now integrating the equation of ψ_i in Ω, we have (denoting $\tilde{v}=\tilde{v}(\nu_i)$)

$$\int_\Omega(m-\tilde{v})e^{\hat{\eta}_i m}\psi_i=0.$$

Passing to the limit, we find that (here $\tilde{v}=\tilde{v}(\nu_0)$)

$$\int_\Omega(m-\tilde{v})e^{\hat{\eta}m}=0.$$

Lemma 5.5 implies that $\hat{\eta}=\eta_2$. Hence, this shows that $\hat{\eta}_i\to\eta_2$ as $i\to\infty$. Similarly, $\tilde{\eta}_i\to\eta_2$ as $i\to\infty$.

Since $\hat{\eta}_i<\tilde{\eta}_i$ and $\lambda_v(\hat{\eta}_i,\mu_i,\nu_i)=\lambda_v(\tilde{\eta}_i,\mu_i,\nu_i)=0$, there exists some $\bar{\eta}_i\in(\hat{\eta}_i,\tilde{\eta}_i)$ such that $\frac{\partial\lambda_v}{\partial\eta}(\bar{\eta}_i,\mu_i,\nu_i)=0$. Since $\hat{\eta}_i,\tilde{\eta}_i\to\eta_2$, we have $\bar{\eta}_i\to\eta_2$. Therefore, setting $\bar{\psi}_i=\psi(\bar{\eta}_i,\mu_i,\nu_i)$, one can show in much of the same manner as before that $\bar{\psi}_1\to 1$ weakly in $W^{2,2}$, and

$$\lambda_v(\bar{\eta}_i,\mu_i,\nu_i)\int_\Omega\bar{\psi}_i e^{\bar{\eta}_i m}=\int_\Omega(m-\tilde{v})e^{\bar{\eta}_i m}\bar{\psi}_i\to\int_\Omega(m-\tilde{v})e^{\eta_2 m}=0$$

and hence $\lambda_v(\bar{\eta}_i,\mu_i,\nu_i)\to 0$ as $i\to\infty$. Set $\bar{\psi}_{i,\eta}=\frac{\partial\bar{\psi}_i}{\partial\eta}(\bar{\eta}_i,\mu_i,\nu_i)$. Then, as $\frac{\partial\lambda_v}{\partial\eta}(\bar{\eta}_i,\mu_i,\nu_i)=0$, $\bar{\psi}_{i,\eta}$ satisfies

$$\begin{cases}\mu_i[\Delta\bar{\psi}_{i,\eta}+\nabla m\cdot\nabla\bar{\psi}_i+\bar{\eta}_i\nabla m\cdot\nabla\bar{\psi}_{i,\eta}]+(m-\tilde{v})\bar{\psi}_{i,\eta}=-\lambda_v(\bar{\eta}_i,\mu_i,\nu_i)\bar{\psi}_{i,\eta} & \text{in }\Omega,\\ \frac{\partial\bar{\psi}_{i,\eta}}{\partial n}=0 & \text{on }\partial\Omega.\end{cases}$$

Define the operator $L_i:W_N^{2,2}(\Omega)\equiv\{\phi\in W^{2,2}(\Omega),\left.\frac{\partial\phi}{\partial n}\right|_{\partial\Omega}=0\}\to L^2(\Omega)$ by

$$L_i\phi=\Delta\phi+\bar{\eta}_i\nabla m\cdot\nabla\phi+\frac{1}{\mu_i}\phi(m-\tilde{v})+\frac{1}{\mu_i}\lambda_v(\bar{\eta}_i,\mu_i,\nu_i)\phi.$$

It is easy to see that all eigenvalues of L_i are real, zero is the smallest eigenvalue and $\bar{\psi}_i$ is an eigenfunction of zero. This implies, by the Fredholm alternative, that if we restrict the domain of L_i to $W_N^{2,2}(\Omega)\cap\{\phi:\int_\Omega\phi\bar{\psi}_i=0\}$, then L_i^{-1} defined on the range of L_1 exists, and is uniformly bounded for all i. Differentiating the constraint $\int_\Omega\psi_i(\eta,\mu,\nu)^2=|\Omega|$ with respect to η and evaluating the result at $(\eta,\mu,\nu)=(\bar{\eta}_i,\mu_i,\nu_i)$, we have $\int_\Omega\bar{\psi}_i\bar{\psi}_{i,\eta}=0$. Since $\|\bar{\psi}_i\|_{W^{2,2}(\Omega)}$ is bounded, $\bar{\psi}_{i,\eta}=L_i^{-1}(\nabla m\cdot\nabla\bar{\psi}_i)$ is uniformly bounded in $W^{2,2}(\Omega)$. Passing to a subsequence if necessary, we may assume that $\bar\psi_{i,\eta}\to\bar\psi_\eta$ weakly in $W^{2,2}(\Omega)$ and strongly in $W^{1,2}(\Omega)$, where $\bar\psi_\eta$ satisfies (note that $\mu_i\to\infty$, $\nu_i\to\nu_0$, $\bar\eta_i\to\eta_2=\eta_v(\nu_0)$ and $\bar\psi_i\to 1$)

$$\begin{cases}\Delta\bar\psi_\eta+\eta_2\nabla m\cdot\nabla\bar\psi_\eta=0 & \text{in } \Omega,\\ \frac{\partial\bar\psi_\eta}{\partial n}=0 \quad \text{on } \partial\Omega, & \int_\Omega\bar\psi_\eta=0.\end{cases}$$

Therefore $\bar\psi_\eta=0$. This implies that $\bar\psi_{i,\eta}\to 0$ in $W^{1,2}(\Omega)$.

Integrating (3.3) in Ω, we have

$$\int_\Omega e^{\eta m}(m-\tilde v)\psi(\eta,\mu,\nu)=-\lambda_v(\eta,\mu,\nu)\int_\Omega e^{\eta m}\psi(\eta,\mu,\nu).$$

Differentiating the above equation with respect to η, and evaluating the result at $(\eta,\mu,\nu)=(\bar\eta_i,\mu_i,\nu_i)$ while using $\frac{\partial\lambda_v}{\partial\eta}(\bar\eta_i,\mu_i,\nu_i)=0$, we have

$$\int_\Omega me^{\bar\eta_i m}(m-\tilde v)\bar\psi_i+\int_\Omega e^{\bar\eta_i m}(m-\tilde v)\bar\psi_{i,\eta}=-\lambda_v(\bar\eta_i,\mu_i,\nu_i)\int_\Omega e^{\bar\eta_i m}[\bar\psi_{i,\eta}+m\bar\psi_i].$$

Since $\bar\eta_i\to\eta_2$, $\bar\psi_i\to 1$, $\bar\psi_{i,\eta}\to 0$ in $W^{1,2}(\Omega)$ and $\lambda_v(\bar\eta_i,\mu_i,\nu_i)\to 0$ as $i\to\infty$, passing to the limit in the above equation we have

$$\int_\Omega me^{\eta_2 m}(m-\tilde v)=0,$$

which is equivalent to $\frac{dF_2}{d\eta}(\eta_2)=0$, a contradiction to Lemma 5.5. This shows that if μ is sufficiently large, $(0,\tilde v)$ changes stability exactly once in $(0,\frac{1}{\min_\Omega m})$ as η varies from zero to ∞. □

5.1.2. Case (ii): $\nu,\ \nu/\mu\to 0$. In this subsection, we take up the case $\nu,\ \nu/\mu\to 0$. The main result of this section is

PROPOSITION 5.7. *There exists $\epsilon_1>0$ such that if $\max\left\{\nu,\frac{\nu}{\mu}\right\}<\epsilon_1$, then there exists a unique positive number η_* such that $\lambda_v(\eta,\mu,\nu)>0$ for $\eta\in[0,\eta_*)$ and $\lambda_v(\eta,\mu,\nu)<0$ for $\eta\in(\eta_*,\infty)$. Moreover,*

$$\lim_{\nu,\ \nu/\mu\to 0}\eta_*=\eta_3,$$

where η_3 is the unique positive root of

$$F_3(\eta)=\int_\Omega\frac{|\nabla m|^2}{m^2}e^{\eta m}(\eta m-1). \tag{5.5}$$

We shall prove Proposition 5.7 in a series of lemmas.

LEMMA 5.8. *For any $p\geq 1$, there exists $C>0$ such that whenever $\eta=\eta_*(\mu,\nu)$,*

$$\int_\Omega|\nabla\psi^p|^2\leq C\frac{\nu}{\mu}\int_\Omega\psi^{2p}$$

for all $\frac{\mu}{\nu}>\frac{4p^2}{2p-1}$. In particular, if we take $p=2$ and normalize $\int_\Omega\psi^4=|\Omega|$, then as $\frac{\mu}{\nu}\to\infty$, $\psi\to 1$ in $L^4(\Omega)$, and $\psi\nabla\psi,\nabla\psi\to 0$ in $L^2(\Omega)$.

PROOF. Multiply (3.3) by ψ^{2p-1} and integrate by parts, we have

$$(2p-1)\int_\Omega e^{\eta_* m}\psi^{2p-2}|\nabla\psi|^2=\frac{\nu}{\mu}\int_\Omega\frac{m-\tilde v}{\nu}\psi^{2p}e^{\eta_* m}.$$

Then we estimate in the following manner, making use of $|\nabla\psi^p|^2 = p^2\psi^{2p-2}|\nabla\psi|^2$ and (2.3).

$$\begin{aligned}
&\frac{2p-1}{p^2}\int_\Omega e^{\eta_* m}|\nabla\psi^p|^2\\
&= \frac{\nu}{\mu}\int_\Omega \frac{-\Delta\tilde{v}}{\tilde{v}}\psi^{2p}e^{\eta_* m}\\
&= \frac{\nu}{\mu}\int_\Omega \nabla\tilde{v}\cdot\nabla\left(\frac{\psi^{2p}e^{\eta_* m}}{\tilde{v}}\right)\\
&= \frac{\nu}{\mu}\int_\Omega e^{\eta_* m}\left[-\frac{|\nabla\tilde{v}|^2}{\tilde{v}^2}\psi^{2p} + \frac{2p\psi^{2p-1}}{\tilde{v}}\nabla\tilde{v}\cdot\nabla\psi + \eta_*\frac{\psi^{2p}}{\tilde{v}}\nabla\tilde{v}\cdot\nabla m\right]\\
&\le \frac{\nu}{\mu}\int_\Omega e^{\eta_* m}\left[-\frac{|\nabla\tilde{v}|^2}{\tilde{v}^2}\psi^{2p} + \left(\frac{1}{2}\frac{|\nabla\tilde{v}|^2}{\tilde{v}^2}\psi^{2p} + 2p^2\psi^{2p-2}|\nabla\psi|^2\right)\right.\\
&\qquad\qquad \left.+\left(\frac{1}{2}\frac{|\nabla\tilde{v}|^2}{\tilde{v}^2}\psi^{2p} + \frac{1}{2}\psi^{2p}\eta_*^2|\nabla m|^2\right)\right]\\
&= \frac{\nu}{\mu}\int_\Omega e^{\eta_* m}\left[2|\nabla\psi^p|^2 + \frac{1}{2}\eta_*^2|\nabla m|^2\psi^{2p}\right].
\end{aligned}$$

If $\frac{\mu}{\nu} > \frac{4p^2}{2p-1}$, then subtracting both sides by $2\frac{\nu}{\mu}\int_\Omega e^{\eta_* m}|\nabla\psi^p|^2$, we have

$$\frac{2p-1}{2p^2}\int_\Omega e^{\eta_* m}|\nabla\psi^p|^2 \le \frac{\eta_*^2}{2}|\nabla m|^2_{L^\infty(\Omega)}\frac{\nu}{\mu}\int_\Omega e^{\eta_* m}\psi^{2p}.$$

Since $\eta_* \in [0, \frac{1}{\min_\Omega m}]$ (Lemma 4.6), it follows that

$$\int_\Omega |\nabla\psi^p|^2 \le C\left(\frac{\nu}{\mu}\right)\int_\Omega |\psi|^{2p}.$$

Take $p = 2$ and normalize by $\int_\Omega \psi^4 = |\Omega|$, then we see that $\int_\Omega |\nabla\psi^2|^2 \to 0$ as $\frac{\mu}{\nu} \to \infty$. Hence $\psi^2 \to 1$ in $H^1(\Omega)$, which implies that $\psi\nabla\psi \to 0$ in $L^2(\Omega)$ and $\psi \to 1$ in $L^4(\Omega)$ (as $(\psi-1)^4 \le (\psi^2-1)^2$). Taking $p = 1$, we get $\nabla\psi \to 0$ in $L^2(\Omega)$. This proves the lemma. $\square$

By Lemma 5.1, $\lambda_v(\cdot,\mu,\nu)$ has at least one root. Denote any such root by η_*, we have the following asymptotic result.

Lemma 5.9. $\lim\limits_{\nu,\frac{\nu}{\mu}\to 0}\eta_* = \eta_3$, *where η_3 is the unique positive root of* (5.5).

Proof. Dividing (3.3) by ν and integrating over Ω,

$$\int_\Omega \frac{m-\tilde{v}}{\nu}e^{\eta_* m}\psi = 0.$$

Substituting (2.3), we have

$$\int_\Omega \frac{-\Delta\tilde{v}}{\tilde{v}}e^{\eta_* m}\psi = 0.$$

Integrate by parts over Ω, then

$$\int_\Omega e^{\eta_* m}\left(-\frac{|\nabla\tilde{v}|^2}{\tilde{v}^2}\psi + \eta_*\frac{\nabla\tilde{v}\cdot\nabla m}{\tilde{v}}\psi + \frac{\nabla\tilde{v}\cdot\nabla\psi}{\tilde{v}}\right) = 0.$$

Using the identity $-|\nabla\tilde v|^2 = -|\nabla\tilde v-\nabla m|^2+|\nabla m|^2-2\nabla\tilde v\cdot\nabla m$, we have

$$\int_\Omega e^{\eta_* m}\left[\frac{\psi}{\tilde v^2}\left(-|\nabla\tilde v-\nabla m|^2+|\nabla m|^2-2\nabla\tilde v\cdot\nabla m\right)+\eta_*\frac{\nabla\tilde v\cdot\nabla m}{\tilde v}\psi+\frac{\nabla\tilde v\cdot\nabla\psi}{\tilde v}\right]=0. \tag{5.6}$$

By Theorem 3.12(iii), and the fact that $\tilde v\geq\min_{\bar\Omega} m$,

$$\int_\Omega\frac{e^{\eta_* m}\psi}{\tilde v^2}|\nabla\tilde v-\nabla m|^2\leq C\|\tilde v-m\|_{L^\infty(\Omega)}\left(\|e^{\eta_* m}\|^2_{H^1(\Omega)}+\|\psi\|^2_{H^1(\Omega)}\right).$$

Using $\tilde v\to m$ in $L^\infty(\Omega)\cap H^1(\Omega)$ (Theorem 3.12(iv)) and that $\psi\to 1$ in $H^1(\Omega)$ (Lemma 5.8), we deduce that

$$\lim_{\nu,\frac{\nu}{\mu}\to 0}\int_\Omega\frac{e^{\eta_* m}\psi}{\tilde v^2}|\nabla\tilde v-\nabla m|^2=0.$$

Take any convergent subsequence so that $\eta_*\to\hat\eta$, we may pass to the limit in (5.6).

$$\int_\Omega e^{\hat\eta m}\left[-\frac{|\nabla m|^2}{m^2}+\hat\eta\frac{|\nabla m|^2}{m}\right]=0.$$

i.e. $F_3(\hat\eta)=0$. Thus $\hat\eta=\eta_3$, the unique positive root of F_3. Since this is true for all convergent subsequences, we have proved the lemma. $\square$

LEMMA 5.10. $\displaystyle\lim_{\nu,\frac{\nu}{\mu}\to 0}\frac{\mu}{\nu}\int_\Omega e^{\eta_* m}|\nabla\psi|^2=0.$

PROOF. Multiply (3.3) by ψ/ν and integrate by parts, we have

$$\frac{\mu}{\nu}\int_\Omega e^{\eta_* m}|\nabla\psi|^2=\int_\Omega e^{\eta_* m}\frac{m-\tilde v}{\nu}\psi^2.$$

By (2.3), we deduce

$$\frac{\mu}{\nu}\int_\Omega e^{\eta_* m}|\nabla\psi|^2=-\int_\Omega e^{\eta_* m}\frac{\Delta\tilde v}{\tilde v}\psi^2.$$

Integrating by parts, we have

$$\begin{aligned}&\frac{\mu}{\nu}\int_\Omega e^{\eta_* m}|\nabla\psi|^2\\ &=\int_\Omega e^{\eta_* m}\left[-\frac{|\nabla\tilde v|^2}{\tilde v^2}\psi^2+\eta_*\frac{\psi^2}{\tilde v}\nabla\tilde v\cdot\nabla m+\frac{2\psi}{\tilde v}\nabla\tilde v\cdot\nabla\psi\right]\\ &=\int_\Omega e^{\eta_* m}\left[\frac{\psi^2}{\tilde v^2}\left(-|\nabla\tilde v-\nabla m|^2+|\nabla m|^2-2\nabla\tilde v\cdot\nabla m\right)+\frac{\psi}{\tilde v}\nabla\tilde v\cdot(\eta_*\psi\nabla m+2\nabla\psi)\right].\end{aligned}$$

Arguing in the same way as before, by $\tilde v\to m$ in $L^\infty(\Omega)\cap H^1(\Omega)$ (Lemma 5.8) and that $\psi,\psi^2\to 1$ in $H^1(\Omega)$, $\psi\nabla\psi\to 0$ in $L^2(\Omega)$ (Lemma 5.8), and the previous lemma, we pass to the limit and deduce that

$$\limsup_{\nu,\frac{\nu}{\mu}\to 0}\frac{\mu}{\nu}\int_\Omega e^{\eta_* m}|\psi|^2=\int_\Omega e^{\eta_3 m}\left[-\frac{|\nabla m|^2}{m^2}+\eta_3\frac{|\nabla m|^2}{m}\right]=0.$$

Note that the last equality follows from the definition of η_3 being the unique positive root of F_3. $\square$

REMARK 5.11. In the proof of Lemma 5.10, we have actually proved the following identity that will be useful later.

$$\lim_{\nu,\frac{\nu}{\mu}\to 0}\int_\Omega e^{\eta_* m}\frac{|\nabla\tilde{v}|^2}{\tilde{v}^2}\psi^2 = \int_\Omega e^{\eta_3 m}\frac{|\nabla m|^2}{m^2}. \tag{5.7}$$

LEMMA 5.12. $\lim_{\nu,\frac{\nu}{\mu}\to 0}\int_\Omega \frac{m-\tilde{v}}{\nu}e^{\eta_* m}m\psi^2 = \int_\Omega e^{\eta_3 m}|\nabla m|^2\eta_3 > 0$, *where* η_3 *is the unique positive root of*

$$F_3(\eta) = \int_\Omega \frac{|\nabla m|^2}{m^2}e^{\eta m}(\eta m - 1).$$

PROOF. By (2.3),

$$\begin{aligned}
&\int_\Omega \frac{m-\tilde{v}}{\nu}e^{\eta_* m}m\psi^2\\
&= -\int_\Omega \frac{\Delta\tilde{v}}{\tilde{v}}e^{\eta_* m}m\psi^2\\
&= \int_\Omega \nabla\tilde{v}\cdot\nabla\left(\frac{e^{\eta_* m}m\psi^2}{\tilde{v}}\right)\\
&= \int_\Omega e^{\eta_* m}\left(-\frac{|\nabla\tilde{v}|^2}{\tilde{v}^2}m\psi^2 + \nabla\tilde{v}\cdot\nabla m\frac{\psi^2}{\tilde{v}} + \eta_*\nabla\tilde{v}\cdot\nabla m\frac{m\psi^2}{\tilde{v}} + 2\frac{m\psi}{\tilde{v}}\nabla\tilde{v}\cdot\nabla\psi\right).
\end{aligned}$$

By (5.7) and the generalized version of Lebesgue's Dominated Convergence Theorem [**92**],

$$\lim_{\nu,\frac{\nu}{\mu}\to 0}\int_\Omega e^{\eta_* m}\frac{|\nabla\tilde{v}|^2}{\tilde{v}^2}m\psi^2 = \int_\Omega e^{\eta_3 m}\frac{|\nabla m|^2}{m}.$$

Hence, by applying the convergence results (Theorem 3.12(iv) and Lemma 5.8) as before,

$$\begin{aligned}
\lim_{\nu,\frac{\nu}{\mu}\to 0}\int_\Omega \frac{m-\tilde{v}}{\nu}e^{\eta_* m}\psi^2 &= \int_\Omega e^{\eta_3 m}\left(-\frac{|\nabla m|^2}{m} + \frac{|\nabla m|^2}{m} + \eta_3|\nabla m|^2\right)\\
&= \eta_3\int_\Omega e^{\eta_3 m}|\nabla m|^2 > 0.
\end{aligned}$$

This completes the proof of Lemma 5.12. □

Now we prove Proposition 5.7.

PROOF OF PROPOSITION 5.7. By Lemma 5.1, $\lambda_v(\cdot,\mu,\nu)$ has at least one positive root. To show that $\lambda_v(\cdot,\mu,\nu)$ has in fact a unique positive root, it suffices to show that

$$\frac{\partial}{\partial\eta}\lambda_v(\eta_*,\mu,\nu) < 0 \quad \text{for any root } \eta_* \text{ of } \lambda_v(\cdot,\mu,\nu). \tag{5.8}$$

Now multiply (3.3) by $m\psi$ and integrate by parts to obtain

$$\frac{\mu}{\nu}\int_\Omega e^{\eta_* m}m|\nabla\psi|^2 + \frac{\mu}{\nu}\int_\Omega e^{\eta_* m}\psi\nabla m\cdot\nabla\psi = \int_\Omega \frac{m-\tilde{v}}{\nu}e^{\eta_* m}m\psi^2.$$

By Lemma 5.10,

$$0 \le \frac{\mu}{\nu}\int_\Omega e^{\eta_* m}m|\nabla\psi|^2 \le \frac{\mu}{\nu}\|m\|_{L^\infty(\Omega)}\int e^{\eta_* m}|\nabla\psi|^2 \to 0,$$

as $\nu, \nu/\mu \to 0$. Also, by Lemma 5.12,

$$\int_\Omega \frac{m-\tilde{v}}{\nu} e^{\eta_* m} m\psi^2 \to \int_\Omega e^{\eta_3 m} |\nabla m|^2 \eta_3 > 0.$$

Hence,

$$\frac{\mu}{\nu} \int_\Omega e^{\eta_* m} \psi \nabla m \cdot \nabla \psi \to \int_\Omega e^{\eta_3 m} |\nabla m|^2 \eta_3 > 0.$$

And (5.8) follows from Lemma 5.2. □

5.1.3. Proof of Theorem 2.10(a). Here we show Theorem 2.10(a) by combining Propositions 5.4 and 5.7.

Proof of Theorem 2.10(a). Suppose Ω is convex, then $\mathcal{R}_1$ (Theorem 2.5) is defined by

$$\mathcal{R}_1 = \{(\mu, \nu) : \mu > 0,\ 0 < \nu < \bar{\nu}(\mu)\},$$

with $\bar{\nu}(\mu)$ defined for all $\mu > 0$, satisfying (by Theorem 2.7)

$$0 < \bar{\nu}(\mu) < \mu, \quad \lim_{\mu \to 0} \frac{\bar{\nu}(\mu)}{\mu} = a_*, \quad \lim_{\mu \to \infty} \bar{\nu}(\mu) = \nu^+,$$

for some $0 < a_* < 1$ and $\nu^+ > 0$. By Theorem 2.5(i), for each $(\mu, \nu) \in \mathcal{R}_1$, $(\tilde{u}, 0)$ remains unstable for all $\eta \geq 0$. It suffices to show that $(0, \tilde{v})$ changes stability exactly once, as η varies from 0 to ∞. Suppose not, then there exists $(\mu_i, \nu_i) \in \mathcal{R}_1$ such that $\mu_i/\nu_i \to \infty$ and $\lambda_v(\cdot, \mu_i, \nu_i)$ changes sign more than once in $[0, \infty)$. Now, $\nu_i \not\to 0$ by Proposition 5.7.

Hence we may assume, by passing to a subsequence, that $\nu_i \to \nu_\infty \in (0, v^+]$. But this contradicts Proposition 5.4. This proves Theorem 2.10(a) for convex domains.

For a general domain Ω, by Corollary 4.11, there exists $\epsilon_0 > 0$ such that $\bar{\nu}(\mu) \in (0, \mu)$ is defined for $\mu \in (0, \epsilon_0] \cup [1/\epsilon, \infty)$, and Theorem 2.5, when restricted to $\mu \in (0, \epsilon_0] \cup [1/\epsilon_0, \infty)$, holds. Moreover, by Lemma 4.8, for all $\mu \in [\epsilon_0, 1/\epsilon_0]$, there exists $\nu_1 > 0$ such that $(\tilde{u}, 0)$ is unstable (i.e. $\lambda_u < 0$) whenever $\nu \in (0, \nu_1)$. Hence, if we take

$$\frac{\nu}{\mu} < \min\left\{ \inf_{\mu \in (0,\epsilon)} \frac{\bar{\nu}(\mu)}{\mu}, \epsilon_0 \nu_1, \epsilon_0 \bar{\nu}\left(\frac{1}{\epsilon_0}\right) \right\},$$

then the statement of Theorem 2.10(a) makes sense, provided $\mathcal{R}_1$ is being understood as the set of (μ, ν) such that $\lambda_u(\mu, \nu, \eta) < 0$ for all $\eta \geq 0$. And the argument above applies to the case of general domains to show that Conjecture 2.9 holds for ν/μ sufficiently small, and $\frac{\nu}{\mu} \in \mathcal{R}_1$. □

5.2. The one-dimensional case

Next, we prove part (b) of Theorem 2.10. For the rest of this section, we assume that $\Omega = (0, 1)$. The following lemma is a direct consequence of Theorem 3.13.

Lemma 5.13. *Suppose that $m_x > 0$ in $[0, 1]$. Then $m(0) < \tilde{v}(0)$ and $m(1) > \tilde{v}(1)$. Similarly, if $m_x < 0$ in $[0, 1]$, then $m(0) > \tilde{v}(0)$ and $m(1) < \tilde{v}(1)$.*

Lemma 5.14. *Suppose that $m_x m_{xx} \neq 0$ in $[0, 1]$. Then $m(x) - \tilde{v}(x)$ changes sign exactly once in $[0, 1]$.*

PROOF. Consider the case $m_x > 0$ and $m_{xx} > 0$ in $[0,1]$. Assume that $g := m - \tilde{v}$ changes sign at least twice. By Lemma 5.13, $g(0) < 0 < g(1)$. Then there must be a local maximum point, x_M, between two roots of g, where $g(x_M) > 0$, $g'(x_M) = 0$ and $g''(x_M) \leq 0$. Hence at x_M,

$$m_{xx}(x_M) \leq \tilde{v}_{xx}(x_M) = \tilde{v}(x_M)(\tilde{v}(x_M) - m(x_M))/\nu = \tilde{v}(x_M)(-g(x_M))/\nu < 0.$$

This contradicts the assumption that $m_{xx} > 0$ in $\bar{\Omega}$. Thus g changes sign exactly once in $[0,1]$. The proofs of the other cases are similar. □

LEMMA 5.15. *Suppose that $\varphi > 0$ satisfies*

$$\begin{cases} \varphi_{xx} + p(x)\varphi_x + q(x)\varphi = 0 & \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{5.9}$$

If $q(x)$ has a single sign change in $(0,1)$, then φ is strictly monotone in $(0,1)$. Specifically, if the sign change is from (i) negative to positive, then $\varphi_x > 0$; (ii) positive to negative, then $\varphi_x < 0$.

PROOF. Beginning with (i), we assume the $q(x)$ sign change is that of negative to positive. Rewrite (5.9) as

$$\begin{cases} \left(e^{\int_0^x p}\varphi_x\right)_x + e^{\int_0^x p} q\varphi = 0 & \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{5.10}$$

Integrating (5.10) in $(0,1)$ we have

$$\int_0^1 e^{\int_0^x p} q\varphi = 0.$$

While q changes sign once, it may be zero in some interval. That is, it may be that $q(x) = 0$ in $[x_*, x^*]$ for some $0 < x_* \leq x^* < 1$, $q < 0$ in $[0, x^*)$ and $q > 0$ in $(x^*, 1]$.

Fix any $x_0 \in (0, x^*]$. Then we integrate (5.10) on $[0, x_0)$ and apply the boundary conditions on φ_x to obtain

$$e^{\int_0^{x_0} p}\varphi_x(x_0) = \int_0^{x_0} \left(e^{\int_0^{x_0} p}\varphi_x\right)_x dx = -\int_0^{x_0} e^{\int_0^{x_0} p} q\varphi \, dx > 0.$$

We may argue similarly for $x_0 \in [x_*, 1)$ over $(x_0, 1]$:

$$-e^{\int_0^{x_0} p}\varphi_x(x_0) = \int_{x_0}^1 \left(e^{\int_0^x p}\varphi_x\right)_x dx = -\int_{x_0}^1 e^{\int_0^x p} q\varphi \, dx < 0.$$

Therefore, $\varphi_x > 0$ in $(0,1)$. A similar argument beginning with the opposite sign change on $q(x)$ shows that $\varphi_x < 0$ in $(0,1)$ under that assumption. □

LEMMA 5.16. *Suppose that $m_x m_{xx} \neq 0$ in $[0,1]$. If $\lambda_v(\eta_*, \mu, \nu) = 0$ for some η_*, then $\frac{\partial}{\partial \eta}\lambda_v(\eta_*, \mu, \nu) < 0$.*

PROOF. We consider the case $m_x > 0$ in $[0,1]$. Fix $\mu, \nu > 0$. Recall that λ_v is the smallest eigenvalue of (3.3). Set $\psi_* := \psi(x; \eta_*)$, where $\psi(x; \eta_*)$ is a positive eigenfunction of $\lambda_v(\eta_*, \mu, \nu)$. Since $\lambda_v(\eta_*, \mu, \nu) = 0$, ψ_* satisfies

$$\begin{cases} \mu(\psi_*)_{xx} + \mu\eta_* m_x(\psi_*)_x + (m - \tilde{v})\psi_* = 0 & \text{in } \Omega, \\ (\psi_*)_x(0) = (\psi_*)_x(1) = 0. \end{cases}$$

By Lemmas 5.13 and 5.14, $m-\tilde{v}$ changes sign exactly once and is negative at $x=0$. By Lemma 5.15, $(\psi_*)_x > 0$ in $(0,1)$. By Lemma 5.2,

$$\frac{\partial \lambda_v}{\partial \eta}(\eta_*, \mu, \nu) = -\mu \frac{\int_0^1 e^{\eta_* m} \psi_* m_x (\psi_*)_x}{\int_0^1 e^{\eta_* m} \psi_*^2} < 0.$$

The proofs of the remaining cases are similar and are omitted. □

We are going to show the following theorem, which implies Theorem 2.10(b).

THEOREM 5.17. *Suppose that $m_x m_{xx} \neq 0$ in $[0,1]$. If $\mu > \nu$, then λ_v changes sign exactly once as η varies from 0 to ∞.*

PROOF. We consider the case $m_x > 0$ in $[0,1]$. For the case $\mu > \nu$, by Theorem 2.5 we see that λ_v changes sign at least once as η varies from 0 to ∞. Let $\eta_* = \eta_*(\mu,\nu)$ denote any positive root of $\eta \mapsto \lambda_v(\eta,\mu,\nu)$. We show that $\eta \mapsto \lambda_v(\eta,\mu,\nu)$ has exactly one root. To see that, suppose η_{**} is another positive root of $\eta \mapsto \lambda_v$ and, without loss of generality, assume that $\eta_{**} > \eta_*$ and $\lambda_v(\eta,\mu,\nu) < 0$ for all $\eta \in (\eta_*, \eta_{**})$. This implies that $\frac{\partial}{\partial \eta}\lambda_v(\eta_{**},\mu,\nu) \geq 0$, which contradicts Lemma 5.16. This proves part (b) of Theorem 2.10. □

5.3. Open problems

Here we state some open questions concerning the dynamics of (2.1) when $(\mu,\nu) \in \mathcal{R}_1$.

CONJECTURE 5.18. For any smooth domain $\Omega \subseteq \mathbb{R}^N$, there exists $\delta_0 > 0$ such that for any $(\mu,\nu) \in \mathcal{R}_1$ with $0 < \nu < \delta_0$, then $(0,\tilde{v})$ changes stability exactly once as η varies from zero to ∞, i.e. Conjecture 2.9(a) is true provided ν is sufficiently small, regardless of the convexity of Ω.

CONJECTURE 5.19. For any $\epsilon > 0$, there exists $\underline{\mu} > 0$ such that for all $\mu \geq \underline{\mu}$ and $\nu \in (\epsilon, \nu^+ - \epsilon)$, any positive steady state of (2.1), if it exists, is globally asymptotically stable. In particular, the branch $\mathcal{C}_1$ of positive steady states emanating from $(\eta_*, 0, \tilde{v})$ does not possess any secondary bifurcation points.

CHAPTER 6

Results in $\mathcal{R}_2$: Proof of Theorem 2.11

By Theorem 2.5, if $(\mu, \nu) \in \mathcal{R}_2$, as η varies from zero to infinity, the semi-trivial steady state $(\tilde{u}, 0)$ changes stability at least twice and $(0, \tilde{v})$ changes stability at least once. The main question is whether the stability of $(\tilde{u}, 0)$ and $(0, \tilde{v})$ change exactly twice and once, respectively. The goal of this chapter is to establish Theorem 2.11 which determines the stability changes of both semi-trivial steady states.

Proof of Theorem 2.11(a). By Theorem 2.5(ii), $\eta \mapsto \lambda_u(\eta, \mu, \nu)$ changes sign at least twice if $(\mu, \nu) \in \mathcal{R}_2$. Hence, it suffices to show that for $(\mu, \nu) \in \mathcal{R}_2$, if μ is sufficiently small, then $\eta \mapsto \lambda_u$ has at most two roots. Suppose to the contrary that there exists $(\mu_k, \nu_k) \in \mathcal{R}_2$ such that $\mu_k \to 0$ and that $\eta \mapsto \lambda_u(\eta, \mu_k, \nu_k)$ has at least three roots. By the estimate in Theorem 2.7 (iv), and the definition of $\mathcal{R}_2$, we have for all k,

$$a_* \leq \liminf_{k\to\infty} \frac{\nu_k}{\mu_k} \leq \limsup_{k\to\infty} \frac{\nu_k}{\mu_k} \leq 1.$$

Here a_* is given in Theorem B.2(iv). Hence we may pass to a subsequence and assume that

$$\lim_{k\to\infty} \frac{\nu_k}{\mu_k} = d \in [a_*, 1].$$

Then, by Corollary 3.18,

$$\lim_{k\to\infty} \frac{\lambda_u(\cdot, \mu_k, \nu_k)}{\mu_k} = \sigma(\cdot; d) \quad \text{ in } C^1\left(\left[0, \frac{1}{\min_{\bar{\Omega}} m}\right]\right). \tag{6.1}$$

If $\lim_{k\to\infty} \frac{\nu_k}{\mu_k} = d > a_*$, then by Theorem B.4(ii), $\sigma(\cdot; d)$ has exactly two roots $\bar{\eta}_1, \bar{\eta}_2$. Moreover, by the concavity of $\sigma(\cdot; d)$ (Theorem B.1(iv)), the two roots are necessarily non-degenerate, i.e. $\frac{\partial}{\partial \eta}\sigma(\bar{\eta}_i; d) \neq 0$ $(i = 1, 2)$. Hence by (6.1), $\lambda_u(\cdot, \mu_k, \nu_k)$ has exactly two roots for k sufficiently large. This is a contradiction.

Therefore, it must be the case that $\lim_{k\to\infty} \frac{\nu_k}{\mu_k} = a_*$. The following is an easy consequence of (6.1):

Lemma 6.1. *Suppose that η_k is a root of $\lambda_u(\cdot, \mu_k, \nu_k)$, then $\eta_k \to \tilde{\eta}$, where $\tilde{\eta}$ is the unique root of $\sigma(\cdot; a_*)$.*

We claim that $\lambda_u(\cdot, \mu_k, \nu_k)$ is strictly concave in $[0, \frac{1}{\min_{\bar{\Omega}} m}]$ for all large k.

Proposition 6.2. *$\frac{\partial^2}{\partial \eta^2}\lambda_u(\eta, \mu_k, \nu_k) < 0$ in $[0, \frac{1}{\min_{\bar{\Omega}} m}]$ for all large k.*

Proof. We denote as before the derivative with respect to η by $'$. Differentiating (A.19), the equation of λ_u', with respect to η, we have

$$\frac{\lambda_u''}{\mu} \int_\Omega \varphi^2 + \frac{\lambda_u'}{\mu} \int_\Omega \varphi\varphi' = \frac{1}{\mu} \int_\Omega \left[\tilde{u}''\varphi^2 + 2\tilde{u}'\varphi\varphi'\right].$$

By the normalization $\int_\Omega \varphi^2 = |\Omega|$, we have $\int_\Omega \varphi\varphi' = 0$ and hence

$$\frac{\lambda_u''}{\mu}\int_\Omega \varphi^2 = \frac{1}{\mu}\int_\Omega \left[\tilde{u}''\varphi^2 + 2\tilde{u}'\varphi\varphi'\right] \le \frac{1}{\mu}\int_\Omega \left[\tilde{u}''_+\varphi^2 + 2\tilde{u}'\varphi\varphi'\right],$$

where $\tilde{u}''_+ = \max\{\frac{\partial^2 \tilde{u}}{\partial \eta^2}, 0\}$. Taking lim sup on both sides, by the normalization $\int_\Omega \varphi^2 = |\Omega|$, we may apply Lemma A.8 to get

$$\limsup_{k\to\infty} \frac{\lambda_u''}{\mu} \le \limsup_{k\to\infty} \frac{2}{\mu|\Omega|}\int_\Omega \tilde{u}'\varphi\varphi', \tag{6.2}$$

with the understanding that $\mu = \mu_k$, $\nu = \nu_k$, $\tilde{u} = \tilde{u}(x;\eta,\mu_k)$, $\varphi = \varphi(x;\mu_k,\nu_k)$, etc.

LEMMA 6.3. $\limsup\limits_{k\to\infty} \dfrac{1}{\mu}\displaystyle\int_\Omega \tilde{u}'\varphi\varphi' < 0.$

PROOF OF LEMMA 6.3. Suppose Lemma 6.3 is false, then by Proposition A.9,

$$0 \le \limsup_{k\to\infty} \frac{1}{\mu}\int_\Omega \tilde{u}'\varphi\varphi' \le \limsup_{k\to\infty} -\epsilon_0 \int_\Omega |\varphi'|^2 \le 0,$$

for some positive constant ϵ_0. Hence by passing to a subsequence, we may assume that

$$\frac{1}{\mu}\int_\Omega \tilde{u}'\varphi\varphi' \to 0 \quad \text{and} \quad \int_\Omega |\varphi'|^2 \to 0. \tag{6.3}$$

Now by (2.2), we may rewrite (A.18) as

$$\begin{cases} \nu\Delta\varphi' - \frac{\mu\nabla\cdot(\nabla\tilde{u} - \eta\tilde{u}\nabla m)}{\tilde{u}}\varphi' + \lambda_u\varphi' = (\tilde{u}' - \lambda_u')\varphi & \text{in } \Omega, \\ \frac{\partial\varphi'}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.4}$$

Multiply (6.4) by φ'/μ and integrate by parts, then (using again $\int_\Omega \varphi\varphi' = 0$)

$$\int_\Omega \left\{\frac{\nu}{\mu}|\nabla\varphi'|^2 - (\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\nabla\left[\frac{(\varphi')^2}{\tilde{u}}\right] - \frac{\lambda_u}{\mu}(\varphi')^2\right\} = \frac{-1}{\mu}\int_\Omega \tilde{u}'\varphi\varphi' = o(1).$$

By Theorem 3.12(v) and the boundedness of λ_u/μ, there is a constant $C > 0$ such that

$$\frac{a_*}{2}\int_\Omega |\nabla\varphi'|^2 \le C\int_\Omega |\varphi'|^2 + o(1) = o(1) \quad \text{by (6.3)},$$

where a_* is given in Theorem B.2(iv). Hence

$$\varphi' \to 0 \quad \text{strongly in } H^1(\Omega). \tag{6.5}$$

Multiply (6.4) by a test function $\rho \in C^1(\bar{\Omega})$, divide by μ and integrate by parts, so that

$$\int_\Omega \left[-\frac{\nu}{\mu}\nabla\varphi'\cdot\nabla\rho + (\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\nabla\left(\frac{\varphi'\rho}{\tilde{u}}\right) + \frac{\lambda_u}{\mu}\varphi'\rho\right] = \frac{1}{\mu}\int_\Omega (\tilde{u}' - \lambda_u')\varphi\rho. \tag{6.6}$$

Note that all terms except the second one can be easily seen to converge. More precisely, the first and third terms on the left converge to zero, and the right hand side converges to

$$\int_\Omega m\nabla m\cdot\nabla\left(\frac{\tilde{\varphi}\rho}{m}\right) - \sigma'\int_\Omega \tilde{\varphi}\rho \tag{6.7}$$

by Theorem 3.14 and Theorem 3.17, where $\sigma' = \frac{\partial\sigma}{\partial\eta}(\tilde{\eta}; a_*)$ and $\sigma = \sigma(\eta; d)$ is the principal eigenvalue of (B.1), and $\tilde{\varphi} = \lim\varphi$.

CLAIM 6.4. $\displaystyle\int_\Omega (\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\nabla\left(\frac{\varphi'\rho}{\tilde{u}}\right) \to 0.$

To see the claim, we compute

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega (\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\nabla\left(\frac{\varphi'\rho}{\tilde{u}}\right)\\
&= \lim_{k\to\infty}\int_\Omega (\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\left[\frac{\nabla(\varphi'\rho)}{\tilde{u}} - \frac{\varphi'\rho\nabla\tilde{u}}{\tilde{u}^2}\right]\\
&= -\lim_{k\to\infty}\int_\Omega \frac{\varphi'\rho}{\tilde{u}^2}|\nabla\tilde{u}|^2,
\end{aligned}$$

as $\varphi' \to 0$ in H^1. Since φ' is not necessarily bounded in L^∞, we rewrite using parallelogram identity

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega (\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\nabla\left(\frac{\varphi'\rho}{\tilde{u}}\right)\\
&= -\lim_{k\to\infty}\int_\Omega \frac{\varphi'\rho}{\tilde{u}^2}\left[|\nabla\tilde{u} - \nabla m|^2 + |\nabla m|^2 + 2\nabla m\cdot(\nabla\tilde{u} - \nabla m)\right]\\
&= 0
\end{aligned}$$

by Theorem 3.12(iii). By Claim 6.4, we deduce that every term of the left hand side of (6.6) tends to zero, while those on the right tend to the limit given in (6.7). Hence

$$0 = \int_\Omega m\nabla m\cdot\nabla\left(\frac{\tilde{\varphi}\rho}{m}\right) - \sigma'\int_\Omega \tilde{\varphi}\rho \quad \text{for all } \rho\in C^1(\bar{\Omega}).$$

Now if $\sigma' \le 0$, choose $\rho\in C^1(\bar{\Omega})$ such that $\rho \to m^2/\tilde{\varphi}$ in $H^1(\Omega)$, then

$$0 = \int_\Omega m|\nabla m|^2 - \sigma'\int_\Omega m^2 > 0$$

and we have a contradiction. If $\sigma' > 0$, then we can take $\rho \in C^1(\Omega)$ such that $\rho \to 1/\tilde{\varphi}$ in $H^1(\Omega)$ and similarly obtain a contradiction. Hence Lemma 6.3 holds. $\square$

Finally, Proposition 6.2 follows from (6.2) and Lemma 6.3. $\square$

Therefore for all k sufficiently large, $\lambda_u(\cdot,\mu_k,\nu_k)$ must be strictly concave in $[0, \frac{1}{\min_{\bar{\Omega}} m}]$ and can have at most two roots in $[0, \frac{1}{\min_{\bar{\Omega}} m}]$ and hence in $[0,\infty)$ (Theorem 2.2). This is a contradiction to the assumption that it has at least three roots. This proves Theorem 2.11(a). $\square$

REMARK 6.5. It can be observed from the proof of Theorem 2.11(a) that for $(\mu,\nu)\in\mathcal{R}_2$ (i.e. $\bar{\nu}(\mu) < \nu < \mu$), if μ is sufficiently small, then $\lambda_u(\cdot,\mu,\nu)$ has two distinct positive roots $\eta_1^* < \eta_2^*$. Moreover,

(i) if $\mu \to 0$ and $\nu/\mu \to d \in (a_*, 1]$, then both η_i^* $(i = 1,2)$ converge respectively to the two non-negative roots of $\sigma(\cdot; d)$. Here a_* is given by Theorem B.2(iv) and $\sigma(\eta; d)$ is the principal eigenvalue of (3.19).

(ii) if $\mu \to 0$ and $\nu/\mu \to a_*$, then both η_i^* $(i = 1,2)$ converge to the unique positive root of $\sigma(\cdot; a_*)$.

6.1. Proof of Theorem 2.11(b)

PROPOSITION 6.6. *There exists $M > 0$ such that if $\mu > \nu \geq M$, then there exists a unique positive number η_* such that $\lambda_v(\eta, \mu, \nu) > 0$ for $\eta \in [0, \eta_*)$ and $\lambda_v(\eta, \mu, \nu) < 0$ for $\eta \in (\eta_*, \infty)$. Moreover, $\lim_{\nu,\ \mu/\nu \to \infty} \eta_* = 0$.*

PROOF. By Lemma 5.1, for $\mu > \nu$, $\lambda_v(\cdot, \mu, \nu)$ has at least one positive root. We proceed to show the uniqueness. Now, on the one hand, by Corollary A.14 there exist of positive constants ϵ_0 (small) and M_1 (large) such that

$$\frac{\partial}{\partial \eta} \lambda_v(\eta, \mu, \nu) < 0 \quad \text{for } 0 \leq \eta \leq \epsilon_0, \text{ and } \mu, \nu \geq M_1. \tag{6.8}$$

On the other hand, by Corollary A.15, (letting $\epsilon = \epsilon_0$ and M_1 possibly larger)

$$\lambda_v(\eta, \mu, \nu) < 0 \quad \text{for } \eta \geq \epsilon_0, \text{ and } \mu, \nu \geq M_1.$$

Hence, for $\mu, \nu \geq M_1$, any positive root η_* of $\lambda_v(\cdot, \mu, \nu)$ must lie in $[0, \epsilon_0]$, which must be unique in view of (6.8). Since $\epsilon_0 > 0$ is arbitrarily small, $\lim_{\nu, \mu/\nu \to \infty} \eta_* = 0$ and the proof of Proposition 6.6 is completed. $\square$

Now, we are in position to show a result that combines Propositions 5.4, 5.7 and 6.6 and contains Theorem 2.11(b)(i) as a corollary.

THEOREM 6.7. *There exists $M > 1$ such that if $\mu/\nu \geq M$, then there exists a unique positive number η_* such that*

$$\lambda_v(\eta, \mu, \nu) = \begin{cases} > 0 & \textit{when } 0 \leq \eta < \eta_*, \\ = 0 & \textit{when } \eta = \eta_*, \\ < 0 & \textit{when } \eta > \eta_*. \end{cases} \tag{6.9}$$

PROOF. By Proposition 5.7, there exists $M_1 > 1$ such that for $\nu \leq 1/M_1$ and $\mu/\nu \geq M_1$, there exists $\eta_* > 0$ such that (6.9) holds. By Proposition 6.6, there exists $M_2 > 1$ such that for $\mu > \nu \geq M_2$, there exists $\eta_* > 0$ such that (6.9) holds as well. By Proposition 5.4, there exists $M_3 = M_3(M_1, M_2) > 1$ such that for $1/M_1 \leq \nu \leq M_2$ and $\mu \geq M_3$, there exists $\eta_* > 0$ such that (6.9) holds. Now, take $M > \max\{M_1, 1, M_1 M_3\}$. For $\mu/\nu \geq M$, either

(i) $\nu \leq 1/M_1$ and $\mu/\nu \geq M \geq M_1$;
(ii) $\nu \geq M_2$ and $\mu \geq M\nu > \nu$;
(iii) $1/M_1 \leq \nu \leq M_2$ and $\mu \geq M\nu \geq (M_1 M_3) \cdot \frac{1}{M_1} = M_3$,

so that one of Propositions 5.4, 5.7 or 6.6 gives the desired result. $\square$

PROOF OF THEOREM 2.11(b)(ii) and (iii). Theorem 2.11(b)(ii) and Theorem 2.11(b)(iii) follow from Theorem 5.17 and Proposition 6.6 respectively. $\square$

6.2. Open problems

CONJECTURE 6.8. Suppose Ω is convex, then there exists $\delta_0 > 0$ such that for any (μ, ν) satisfying $\nu < \mu$ and $0 < \mu, \nu < \delta_0$, then $(0, \tilde{v})$ changes stability exactly once.

CHAPTER 7

Results in $\mathcal{R}_3$: Proof of Theorem 2.12

By Theorem 2.5, if $(\mu,\nu)\in\mathcal{R}_3$, the semi-trivial steady state $(\tilde{u},0)$ changes stability at least once as η varies from zero to infinity. The main question is whether $(\tilde{u},0)$ changes stability exactly once and $(0,\tilde{v})$ is always unstable as η varies from 0 to ∞. In this chapter we will address these questions and establish Theorem 2.12. Part (a) of Theorem 2.12 is proved in Section 7.1, and the proof of Theorem 2.12(b) is given in Section 7.2.

7.1. Stability result of $(\tilde{u},0)$ for small μ

In this section, we prove Theorem 2.12(a). First we consider the case $\frac{\nu}{\mu}\to\infty$ separately.

PROPOSITION 7.1. *Assume $\mu\to 0$ and $\nu/\mu\to\infty$, then*

$$\frac{\lambda_u}{\mu}(\eta,\mu,\nu)\to\frac{1}{|\Omega|}\int_\Omega(1-\eta m)\frac{|\nabla m|^2}{m^2}\qquad \text{in } C^1([0,\frac{1}{\min_{\bar\Omega} m}]).$$

In particular, the asymptotic behavior of η^* is determined.

COROLLARY 7.2. *There exists $\delta>0$ such that if $\frac{\mu}{\nu}<\delta$ and $\mu<\delta$, then $\lambda_u(\cdot,\mu,\nu)$ has a unique root η^*. Moreover,*

$$\lim_{\mu\to 0,\,\frac{\nu}{\mu}\to\infty}\eta^*=\frac{\int_\Omega|\nabla m|^2/m^2}{\int_\Omega|\nabla m|^2/m}.$$

PROOF OF PROPOSITION 7.1. Substituting (2.2) into (3.1), we have

$$\nu\Delta\varphi-\frac{\mu\nabla\cdot(\nabla\tilde{u}-\eta\tilde{u}\nabla m)}{\tilde{u}}\varphi+\lambda_u\varphi=0\quad\text{ and }\quad\frac{\partial\varphi}{\partial n}\Big|_{\partial\Omega}=0.$$

Divide by μ and integrate by parts to obtain

$$\int_\Omega(\nabla\tilde{u}-\eta\tilde{u}\nabla m)\cdot\nabla\left(\frac{\varphi}{\tilde{u}}\right)+\frac{\lambda_u}{\mu}\int_\Omega\varphi=0. \tag{7.1}$$

If we normalize $\int_\Omega\varphi^2=|\Omega|$, then by Lemma A.3 and Corollary A.4, $\varphi,\varphi^2\to 1$ in $H^1(\Omega)$ as $\mu,\mu/\nu\to 0$. Using also Theorem 3.12(ii),

$$\int_\Omega(\nabla\tilde{u}-\eta\tilde{u}\nabla m)\cdot\nabla\left(\frac{\varphi}{\tilde{u}}\right)\to\int_\Omega(1-\eta m)\nabla m\cdot\nabla\left(\frac{1}{m}\right).$$

Hence by passing to the limit in (7.1), $\lambda_u(\eta,\mu,\nu)/\mu\to\frac{1}{|\Omega|}\int_\Omega(1-\eta m)\frac{|\nabla m|^2}{m^2}$ uniformly in $[0,\frac{1}{\min_{\bar\Omega}m}]$. Next, recall that by (A.19), we have

$$\frac{\lambda_u'}{\mu}=\frac{\int_\Omega\tilde{u}'\varphi^2}{\mu\int_\Omega\varphi^2},\quad\text{where}\quad\tilde{u}'=\frac{\partial\tilde{u}}{\partial\eta}.$$

Passing to the limit, using Theorem 3.14, we have

$$\frac{\lambda_u'}{\mu} \to \frac{1}{|\Omega|}\int_\Omega m\nabla m \cdot \nabla\left(\frac{1}{m}\right) = \frac{\partial}{\partial \eta}\left[\frac{1}{|\Omega|}\int_\Omega (1-\eta m)\frac{|\nabla m|^2}{m^2}\right].$$

This proves Proposition 7.1. □

Now we prove the remaining cases of Theorem 2.12(a).

PROPOSITION 7.3. *For each $0 < \delta < 1$, if $\mu/\nu \in (\delta, 1)$, then for μ, ν sufficiently small,*

$$\lambda_u(\eta,\mu,\nu) = \begin{cases} > 0 & \text{when } 0 \le \eta < \eta^*, \\ = 0 & \text{when } \eta = \eta^*, \\ < 0 & \text{when } \eta > \eta^*, \end{cases}$$

for some positive number η^*.

PROOF. By Theorem 2.5(iii), $\lambda_u(\cdot,\mu,\nu)$ changes sign at least once in $(0, 1/\min_{\bar\Omega} m)$. Suppose to the contrary that for some sequence (μ_k, ν_k) such that $\mu_k \to 0$ and $\mu_k/\nu_k \in (\delta, 1)$, $\lambda_u(\cdot,\mu_k,\nu_k)$ has at least two roots $\eta_k, \hat\eta_k \in (0, 1/\min_{\bar\Omega} m)$.

CLAIM 7.4. *There exists $\epsilon_0 > 0$ such that*

$$\eta_k, \hat\eta_k \in (\epsilon_0, 1/\min_{\bar\Omega} m)$$

for all k large.

To see the claim, we begin with the following lemma.

LEMMA 7.5. *There exists $\epsilon_0 > 0$ such that for all μ sufficiently small,*

$$\frac{\partial}{\partial\eta}\lambda_u(\eta,\mu,\mu) > \mu\epsilon_0\eta \quad \text{for all } \eta \in [0,\epsilon_0].$$

PROOF. By Corollary 3.18, as $\mu \to 0$, $\lambda_u(\cdot,\mu,\mu)/\mu \to \sigma(\cdot;1)$ in $C^1([0, 1/\min_{\bar\Omega} m])$. The result follows from $\sigma'(0;1) > 0$ (Corollary A.7). □

Now, if we observe that $\lambda_u(0,\mu,\mu) = 0$ for all $\mu > 0$ with positive eigenfunction $\varphi = \tilde v$, we deduce from Lemma 7.5 that $\lambda_u(\eta,\mu,\mu) \ge 0$ for all $\eta \in [0,\epsilon_0]$ and hence $\lambda_u(\eta,\mu,\nu) > \lambda_u(\eta,\mu,\mu) \ge 0$ for all $\eta \in [0,\epsilon_0]$. Hence any positive root of $\lambda_u(\cdot,\mu,\nu)$ is greater than ϵ_0 and also less than $\frac{1}{\min_{\bar\Omega} m}$ (Theorem 2.2), and Claim 7.4 follows.

Passing to a subsequence, we may assume that $\nu_k/\mu_k \to d \in [1, 1/\delta]$, then by Corollary 3.18, $\lambda_u(\cdot,\mu_k,\nu_k)/\mu_k \to \sigma(\cdot;d)$ in $C^1([\epsilon_0, 1/\min_{\bar\Omega} m])$. By Theorem B.4(iii) and Remark B.5, $\sigma(\cdot;d)$ has exactly one root in $[\epsilon_0, 1/\min_{\bar\Omega} m]$, which is also non-degenerate. This contradicts the existence of two distinct roots $\eta_k, \hat\eta_k$ of $\lambda(\cdot,\mu_k,\nu_k)$ in $(\epsilon_0, 1/\min_{\bar\Omega} m)$. □

REMARK 7.6. It can be observed from the above arguments that, for any $d \in [1,\infty)$, the unique positive root η^* of $\lambda_u(\cdot,\mu,\nu)$ converges, as $\nu/\mu \to d$ and $\mu \to 0$, to the unique positive root of $\sigma(\cdot;d)$. In fact, as $d \to \infty$, the unique positive root of $\sigma(\cdot;d)$ tends to $\frac{\int_\Omega |\nabla m|^2/m^2}{\int_\Omega |\nabla m|^2/m}$ (see, e.g. Claim B.3 and Theorem B.4). This provides the connection to the asymptotic results proved separately in Corollary 7.2.

Theorem 2.12(a) follows from Corollary 7.2 and Proposition 7.3.

7.2. Stability result of $(0, \tilde{v})$

We first prove the following lemma for convex domains.

LEMMA 7.7. *Suppose that Ω is convex, then for any η, μ, ν, the unique positive solution $\tilde{v}$ of* (2.3) *satisfies*

$$\int_\Omega \tilde{v} e^{\eta m} \nabla m \cdot \nabla \tilde{v} \geq \int_\Omega e^{\eta m} |\nabla \tilde{v}|^2 \left(\tilde{v} - \frac{\eta^2 \nu}{4} |\nabla m|^2 \right).$$

PROOF. Differentiate (2.3) with respect to x_i, we have

$$m_{x_i} \tilde{v} = -\nu \Delta \tilde{v}_{x_i} - (m - 2\tilde{v}) \tilde{v}_{x_i}.$$

Multiply the equation by $e^{\eta m} \tilde{v}_{x_i}$ and integrate by parts,

$$\begin{aligned}
&\int_\Omega e^{\eta m} \tilde{v} m_{x_i} \tilde{v}_{x_i} \\
&= -\nu \int_\Omega e^{\eta m} \tilde{v}_{x_i} \Delta \tilde{v}_{x_i} - \int_\Omega (m - 2\tilde{v}) e^{\eta m} \tilde{v}_{x_i}^2 \\
&= \nu \int_\Omega \nabla (e^{\eta m} \tilde{v}_{x_i}) \cdot \nabla \tilde{v}_{x_i} - \int_\Omega (m - 2\tilde{v}) e^{\eta m} \tilde{v}_{x_i}^2 - \nu \int_{\partial\Omega} e^{\eta m} \tilde{v}_{x_i} \frac{\partial \tilde{v}_{x_i}}{\partial n} \\
&= \nu \int_\Omega e^{\eta m} \left[|\nabla \tilde{v}_{x_i}|^2 + \eta \tilde{v}_{x_i} \nabla \tilde{v}_{x_i} \cdot \nabla m \right] - \int_\Omega (m - 2\tilde{v}) e^{\eta m} \tilde{v}_{x_i}^2 - \nu \int_{\partial\Omega} e^{\eta m} \tilde{v}_{x_i} \frac{\partial \tilde{v}_{x_i}}{\partial n} \\
&= \nu \int_\Omega e^{\eta m} \left| \nabla \tilde{v}_{x_i} + \frac{\eta}{2} \tilde{v}_{x_i} \nabla m \right|^2 - \frac{\eta^2 \nu}{4} \int_\Omega e^{\eta m} \tilde{v}_{x_i}^2 |\nabla m|^2 - \int_\Omega (m - 2\tilde{v}) e^{\eta m} \tilde{v}_{x_i}^2 \\
&\qquad - \frac{\nu}{2} \int_{\partial\Omega} e^{\eta m} \frac{\partial}{\partial n} \left(\tilde{v}_{x_i}^2 \right) \\
&= \int_\Omega \left[\nu |\nabla (e^{\eta m/2} \tilde{v}_{x_i})|^2 - (m - 2\tilde{v}) (e^{\eta m/2} \tilde{v}_{x_i})^2 \right] - \frac{\eta^2 \nu}{4} \int_\Omega e^{\eta m} \tilde{v}_{x_i}^2 |\nabla m|^2 \\
&\qquad - \frac{\nu}{2} \int_{\partial\Omega} e^{\eta m} \frac{\partial}{\partial n} \left(\tilde{v}_{x_i}^2 \right) \\
&\geq \int_\Omega \tilde{v} (e^{\eta m/2} \tilde{v}_{x_i})^2 - \frac{\eta^2 \nu}{4} \int_\Omega e^{\eta m} \tilde{v}_{x_i}^2 |\nabla m|^2 - \frac{\nu}{2} \int_{\partial\Omega} e^{\eta m} \frac{\partial}{\partial n} \left(\tilde{v}_{x_i}^2 \right) \\
&= \int_\Omega e^{\eta m} \tilde{v}_{x_i}^2 \left(\tilde{v} - \frac{\eta^2 \nu}{4} |\nabla m|^2 \right) - \frac{\nu}{2} \int_{\partial\Omega} e^{\eta m} \frac{\partial}{\partial n} \left(\tilde{v}_{x_i}^2 \right).
\end{aligned}$$

The inequality on the second last line is due to the following lemma.

LEMMA 7.8. *For all $\phi \in H^1(\Omega)$,*

$$\int_\Omega \left[\nu |\nabla \phi|^2 + (\tilde{v} - m) \phi^2 \right] \geq 0. \tag{7.2}$$

PROOF. By (2.3), we see that 0 is the principal eigenvalue of the following problem with eigenfunction $\tilde{v}$.

$$\begin{cases} \nu \Delta \phi + (m - \tilde{v}) \phi + \gamma \phi = 0 & \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence (7.2) follows from the variational characterization of the principal eigenvalue.

□

By the above argument, for any $i = 1, ..., n$, we have

$$\int_\Omega e^{\eta m}\tilde{v} m_{x_i}\tilde{v}_{x_i} \geq \int_\Omega e^{\eta m}\tilde{v}_{x_i}^2\left(\tilde{v} - \frac{\eta^2\nu}{4}|\nabla m|^2\right) - \frac{\nu}{2}\int_{\partial\Omega} e^{\eta m}\frac{\partial}{\partial n}\left(\tilde{v}_{x_i}^2\right).$$

Summing $i = 1, ..., n$, we have

$$\int_\Omega e^{\eta m}\tilde{v}\nabla m\cdot\nabla\tilde{v} \geq \int_\Omega e^{\eta m}|\nabla\tilde{v}|^2\left(\tilde{v} - \frac{\eta^2\nu}{4}|\nabla m|^2\right) - \frac{\nu}{2}\int_{\partial\Omega} e^{\eta m}\frac{\partial}{\partial n}\left(|\nabla\tilde{v}|^2\right).$$

Lemma 7.7 thus follows from the following well-known lemma for convex domains due to [**20, 77**].

LEMMA 7.9. *If Ω is convex, and $\frac{\partial\tilde{v}}{\partial n}\big|_{\partial\Omega} = 0$, then $\frac{\partial}{\partial n}|\nabla\tilde{v}|^2\big|_{\partial\Omega} \leq 0$.*

This proves Lemma 7.7. □

7.2.1. Proof of Theorem 2.12(b)(ii).

THEOREM 7.10. *If $\mu < \nu \leq \frac{4(\min_{\bar{\Omega}} m)^3}{\|\nabla m\|^2_{L^\infty(\Omega)}}$, then $(0, \tilde{v})$ is unstable for all $\eta \in [0, \infty)$.*

PROOF. By Lemma 4.6, $(0, \tilde{v})$ is unstable whenever $\eta \geq \frac{1}{\min_{\bar{\Omega}} m}$. Therefore it suffices to show the instability of $(0, \tilde{v})$ for all $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$.

CLAIM 7.11. *Under the assumption of Theorem 7.10,*

$$\frac{\eta^2\nu}{4}|\nabla m|^2 < \tilde{v} \quad \text{in } \bar{\Omega}$$

for all $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$.

To see the claim, we calculate

$$\frac{1}{4}\eta^2\nu|\nabla m|^2 \leq \frac{1}{4}\left(\frac{1}{\min_{\bar{\Omega}} m}\right)^2\left[\frac{4(\min_{\bar{\Omega}} m)^3}{\|\nabla m\|^2_{L^\infty(\Omega)}}\right]|\nabla m|^2 \leq \min_{\bar{\Omega}} m < \tilde{v},$$

where the last strict inequality follows from the maximum principle (Theorem 3.13). This proves the claim.

Taking $\tilde{v}$ as the test function in the variational characterization

$$\lambda_v = \inf_{\phi\in H^1(\Omega),\,\phi\neq 0}\left\{\frac{\int_\Omega e^{\eta m}[\mu|\nabla\phi|^2 + (\tilde{v} - m)\phi^2]}{\int_\Omega e^{\eta m}\phi^2}\right\},$$

we deduce that

$$\lambda_v\int_\Omega e^{\eta m}\tilde{v}^2 \leq \int_\Omega e^{\eta m}[\mu|\nabla\tilde{v}|^2 + (\tilde{v} - m)\tilde{v}^2]. \tag{7.3}$$

Now by (2.3),

$$\begin{aligned}\int_\Omega e^{\eta m}(\tilde{v} - m)\tilde{v}^2 &= \nu\int_\Omega e^{\eta m}\frac{\Delta\tilde{v}}{\tilde{v}}\tilde{v}^2\\ &= \nu\int_\Omega e^{\eta m}\tilde{v}\Delta\tilde{v}\\ &= -\nu\int_\Omega \nabla(e^{\eta m}\tilde{v})\cdot\nabla\tilde{v}\\ &= -\nu\int_\Omega e^{\eta m}\left(|\nabla\tilde{v}|^2 + \eta\tilde{v}\nabla m\cdot\nabla\tilde{v}\right).\end{aligned}$$

Substituting into (7.3),

$$\lambda_v \int_\Omega e^{\eta m}\tilde{v}^2 \le (\mu - \nu)\int_\Omega e^{\eta m}|\nabla \tilde{v}|^2 - \nu\eta \int_\Omega e^{\eta m}\tilde{v}\nabla m \cdot \nabla \tilde{v}.$$

By Lemma 7.7 and Claim 7.11, (and that $\mu < \nu$)

$$\lambda_v \int_\Omega e^{\eta m}\tilde{v}^2 \le (\mu - \nu)\int_\Omega e^{\eta m}|\nabla \tilde{v}|^2 - \nu\eta \int_\Omega e^{\eta m}|\nabla \tilde{v}|^2 \left(\tilde{v} - \frac{\eta^2\nu}{4}|\nabla m|^2\right) < 0.$$

This proves Theorem 7.10. $\square$

Remark 7.12. If $\mu = \nu \le \frac{4(\min_{\bar{\Omega}} m)^3}{\|\nabla m\|^2_{L^\infty(\Omega)}}$, then $(0, \tilde{v})$ is unstable for all $\eta > 0$.

Note also that we have actually proved the following result, which will be useful in the proof of our next result.

Remark 7.13. If $\mu \le \nu$ and $0 \le \eta \le \sqrt{\frac{4\min_{\bar{\Omega}} m}{\nu\|\nabla m\|^2_{L^\infty(\Omega)}}}$, and either $\eta > 0$ or $\mu < \nu$, then $(0, \tilde{v})$ is unstable.

7.2.2. Proof of Theorem 2.12(b)(i).

Theorem 7.14. *Let Ω be convex. If μ/ν is sufficiently small, then $(0, \tilde{v})$ is unstable for all $\eta \in [0, \infty)$.*

Theorem 7.14 follows from the following slightly stronger result.

Proposition 7.15. *Let Ω be convex. There exists $\epsilon_0 > 0$ such that if $(\mu, \nu) \in \mathcal{R}_3$ and either*

$$\text{(i) } \nu < \frac{4(\min_{\bar{\Omega}} m)^3}{4\|\nabla m\|_{L^\infty(\Omega)}}, \quad \text{(ii) } \nu > 1/\epsilon_0, \quad \textit{or} \quad \text{(iii) } \nu \in [\epsilon_0, 1/\epsilon_0] \textit{ and } \mu < \epsilon_0,$$

then $(0, \tilde{v})$ is unstable for all $\eta \in [0, \infty)$.

Proof. By Lemma 4.6, it suffices to show the instability of $(0, \tilde{v})$ for all $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$. We consider the following cases.

(A) $\nu \to \nu_0 > 0$ and $\mu \to 0$;
(B) $\nu \to \infty$ and $\eta \ge c_0$ for some constant $c_0 > 0$;
(C) $\nu \to \infty$, $\eta \to 0$ and $\eta \ge \sqrt{\frac{c_0}{\nu}}$ for some constant $c_0 > 0$;
(D) $\mu \le \nu$ and $\eta \le \sqrt{\frac{4\min_{\bar{\Omega}} m}{\nu\|\nabla m\|^2_{L^\infty(\Omega)}}}$.

Note that (A)-(D) covers all possibilities: If (i) holds, then

$$\sqrt{\frac{4\min_{\bar{\Omega}} m}{\nu\|\nabla m\|^2_{L^\infty(\Omega)}}} \ge \frac{1}{\min_{\bar{\Omega}} m},$$

then (D) suffices. Secondly, the case (ii) is covered by (B), (C) and (D). Finally, the case (iii) is covered by (A).

First we consider Case (A). We use the variational characterization of λ_v.

$$\lambda_v = \inf_{\phi \in H^1(\Omega), \phi \ne 0} \left\{ \frac{\int_\Omega e^{\eta m}[\mu|\nabla\phi|^2 + (\tilde{v} - m)\phi^2]}{\int_\Omega e^{\eta m}\phi^2} \right\}. \tag{7.4}$$

Therefore, letting $\tilde{v}_0 = \tilde{v}\big|_{\nu=\nu_0}$, we have

$$\liminf_{\mu \to 0, \nu \to \nu_0} \lambda_v \ge \min_{\bar{\Omega}}(\tilde{v}_0 - m). \tag{7.5}$$

Given $\epsilon > 0$, choose a positive test function $\phi \in C^1(\bar{\Omega})$ such that

$$\overline{\operatorname{supp}\phi} \subset \{x \in \Omega : \tilde{v}_0(x) - m(x) < \min_{\bar{\Omega}}(\tilde{v}_0 - m) + \epsilon/2\}.$$

Then for all ν close to ν_0 such that $\|\tilde{v} - \tilde{v}_0\|_{L^\infty(\Omega)} < \epsilon/2$, it is easy to see that

$$\begin{aligned}\lambda_v &\le \frac{\int_\Omega e^{\eta m}[\mu|\nabla\phi|^2 + (\tilde{v} - m)\phi^2]}{\int_\Omega e^{\eta m}\phi^2} \\ &\le \mu\frac{\int_\Omega e^{\eta m}|\nabla\phi|^2}{\int_\Omega e^{\eta m}\phi^2} + \sup_{\operatorname{supp}\phi}(\tilde{v} - m) \\ &\le \mu\frac{\int_\Omega e^{\eta m}|\nabla\phi|^2}{\int_\Omega e^{\eta m}\phi^2} + \sup_{\operatorname{supp}\phi}(\tilde{v}_0 - m) + \epsilon/2 \\ &\le \mu\frac{\int_\Omega e^{\eta m}|\nabla\phi|^2}{\int_\Omega e^{\eta m}\phi^2} + \min_{\bar{\Omega}}(\tilde{v}_0 - m) + \epsilon.\end{aligned}$$

Passing to the limit $\mu \to 0$, we deduce that $\limsup_{\mu\to 0,\nu\to\nu_0} \lambda_v \le \min_{\bar{\Omega}}(\tilde{v}_0 - m) + \epsilon$. Here $\tilde{v}_0$ is the unique positive solution of (2.3) with $\nu = \nu_0$. Letting $\epsilon \to 0$, we obtain

$$\limsup_{\mu\to 0,\nu\to\nu_0} \lambda_v \le \min_{\bar{\Omega}}(\tilde{v}_0 - m).$$

This, together with (7.5) gives

$$\lim_{\mu\to 0,\nu\to\nu_0} \lambda_v = \min_{\bar{\Omega}}(\tilde{v}_0 - m).$$

Since $\tilde{v}_0 \not\equiv m$ and that $\int_\Omega \tilde{v}_0(m - \tilde{v}_0) = 0$ by integrating (2.3), we have $\min_{\bar{\Omega}}(\tilde{v}_0 - m) < 0$. This proves the instability of $(0, \tilde{v})$ in Case (A).

For Case (B), assume to the contrary that for some sequence $\nu \to \infty$ and $\eta \in (0, \frac{1}{\min_{\bar{\Omega}} m}]$ bounded away from 0, we have $\lambda_v \ge 0$. Without loss of generality, assume that $\eta \to \tilde{\eta} \in (0, \frac{1}{\min_{\bar{\Omega}} m}]$. Setting $\phi = 1$ in (7.4), we have

$$0 \le \lambda_v \le \frac{\int_\Omega e^{\eta m}(\tilde{v} - m)}{\int_\Omega e^{\eta m}}. \tag{7.6}$$

Passing to the limit, then

$$0 \le \limsup_{\nu\to\infty} \lambda_v \le \frac{\int_\Omega e^{\tilde{\eta} m}(\bar{m} - m)}{\int_\Omega e^{\tilde{\eta} m}} = -\frac{\int_\Omega e^{\tilde{\eta}(m-\bar{m})}(m - \bar{m})}{\int_\Omega e^{\tilde{\eta}(m-\bar{m})}},$$

where $\bar{m} = \frac{1}{|\Omega|}\int_\Omega m$. Since the last weighted average of $m - \bar{m}$ is strictly negative, this gives the contradiction.

Next, we take up Case (C). Assume that for some positive constant c_0 we have $\eta^2\nu \ge c_0$. First we begin with an asymptotic expansion of $\tilde{v}$ due to X. Chen [**21**].

LEMMA 7.16. *There exists $\tilde{v}_i \in W^{2,p}(\Omega)$, $i = 1, 2$, independent of ν, such that*

$$\tilde{v} \le \bar{m} + \frac{\tilde{v}_1}{\nu} + \frac{\tilde{v}_2}{\nu^2}$$

for all ν sufficiently large.

PROOF. Define

$$\bar{v} = \bar{m} + \frac{1}{\nu}\left(v_1^* + \frac{\int_\Omega v_1^* m}{\int_\Omega m}\right) + \frac{1}{\nu^2}v_2^* + \frac{1}{\nu}\left(1 + \frac{Q}{\nu}\right),$$

where v_1^* is the unique solution to

$$\Delta v_1^* + \bar{m}(m - \bar{m}) = 0, \quad \frac{\partial v_1^*}{\partial n}\bigg|_{\partial\Omega} = 0, \quad \text{and} \quad \int_\Omega v_1^* = 0;$$

v_2^* is the unique solution to

$$\Delta v_2^* + (m - 2\bar{m})\left(v_1^* + \frac{\int_\Omega v_1^* m}{\int_\Omega m}\right) = 0, \quad \frac{\partial v_2^*}{\partial n}\bigg|_{\partial\Omega} = 0 \quad \text{and} \quad \int_\Omega v_2^* = 0;$$

and Q is the unique solution to

$$\Delta Q = \bar{m} - m, \quad \frac{\partial Q}{\partial n}\bigg|_{\partial\Omega} = 0, \quad \text{and} \quad \int_\Omega Q = 0.$$

Then one may compute that

$$\nu\Delta\bar{v} + \bar{v}(m - \bar{v}) = \nu\Delta\left(\frac{Q}{\nu^2}\right) + \frac{m - 2\bar{m}}{\nu} + O\left(\frac{1}{\nu^2}\right) = -\frac{\bar{m}}{\nu} + O\left(\frac{1}{\nu^2}\right) < 0$$

for ν sufficiently large. Hence $\bar{v}$ is a strict upper solution of (2.3), hence the result follows by the comparison principle and $\tilde{v}$ being the unique positive solution of (2.3). □

As before, we take the test function $\phi = 1$ in (7.4), we have

$$\begin{aligned}
\lambda_v \int_\Omega e^{\eta m} &\le \int_\Omega e^{\eta m}(\tilde{v} - m) \\
&\le \int_\Omega \left(1 + \eta m + \frac{1}{2}(\eta m)^2 + ...\right)\left(\bar{m} - m + \frac{1}{\nu}\tilde{v}_1 + \frac{1}{\nu^2}\tilde{v}_2\right) \\
&\le \int_\Omega \left(1 + \eta m + \frac{1}{2}(\eta m)^2 + ...\right)\left(\bar{m} - m + \eta^2\frac{\|\tilde{v}_1\|_{L^\infty}}{c_0} + \eta^4\frac{\|\tilde{v}_2\|_{L^\infty}}{c_0}\right) \\
&= \eta\int_\Omega m(\bar{m} - m) + O(\eta^2),
\end{aligned}$$

where the last line is negative for all η sufficiently small. This proves Case (C). Finally, (D) follows from Remark 7.13. □

7.2.3. Proof of Theorem 2.12(b)(iii).

THEOREM 7.17. *Let $\Omega = (0, 1)$, then $(0, \tilde{v})$ is unstable when*

$$\frac{\nu}{\mu} - 1 > -\eta\frac{\int_\Omega e^{\eta m}\tilde{v}^{2\nu/\mu - 1}\tilde{v}_x m_x}{\int_\Omega e^{\eta m}\tilde{v}^{2\nu/\mu - 2}\tilde{v}_x^2}. \tag{7.7}$$

COROLLARY 7.18. *Let $\Omega = (0, 1)$ and $m_x \ne 0$ in $[0, 1]$. If $\mu < \nu$, then $(0, \tilde{v})$ is unstable for all $\eta \in [0, \infty)$.*

REMARK 7.19. For $\mu < \nu$, i.e. $(\mu, \nu) \in \mathcal{R}_3$, then Theorem 7.17 improves Lemma 5.16 by removing the assumption $m_{xx} \ne 0$ in $[0, 1]$.

PROOF OF COROLLARY 7.18. Assuming Theorem 7.17 for the moment, we start with a claim.

CLAIM 7.20. *If $m_x \ne 0$ in $[0, 1]$, then $\tilde{v}_x m_x > 0$ in $(0, 1)$.*

Consider the case $m_x > 0$ in $[0,1]$. Differentiate (2.3) with respect to x, then we have

$$\begin{cases} \nu(\tilde{v}_x)_{xx} + (m - 2\tilde{v})\tilde{v}_x = -m_x\tilde{v} < 0 & \text{in } (0,1), \\ \tilde{v}_x(0) = \tilde{v}_x(1) = 0. \end{cases}$$

Now since the operator $[-\nu\frac{d^2}{dx^2} + (2\tilde{v} - m)]_{\mathcal{N}}^{-1}$ with Neumann boundary conditions is invertible and positive, by the eigenvalue comparison principle, the same holds true for $[-\nu\frac{d^2}{dx^2} + (2\tilde{v} - m)]_{\mathcal{D}}^{-1}$, the operator under zero Dirichlet boundary conditions. Hence

$$v_x = \left[-\nu\frac{d^2}{dx^2} + (2\tilde{v} - m)\right]_{\mathcal{D}}^{-1} [m_x\tilde{v}] > 0 \quad \text{in } (0,1).$$

Hence, $\tilde{v}_x m_x > 0$ in $(0,1)$. Therefore, for $(\mu,\nu) \in \mathcal{R}_3$,

$$\frac{\mu}{\nu} - 1 > 0 > -\eta\frac{\int_\Omega e^{\eta m}\tilde{v}^{2\nu/\mu-1}\tilde{v}_x m_x}{\int_\Omega e^{\eta m}\tilde{v}^{2\nu/\mu-2}\tilde{v}_x^2},$$

i.e. (7.7) holds. The corollary thus follows from Theorem 7.17. □

Proof of Theorem 7.17. Let ψ be a positive eigenfunction of (3.3), then upon substituting (2.3),

$$\begin{cases} \mu\psi_{xx} + \mu\eta m_x\psi_x - \frac{\nu\tilde{v}_{xx}}{\tilde{v}}\psi + \lambda_v\psi = 0 & \text{in } (0,1), \\ \psi_x(0) = \psi_x(1) = 0. \end{cases} \tag{7.8}$$

Write $\psi = \tilde{v}^\ell\omega$, where $\ell = \frac{\nu}{\mu}$, we compute the first and second derivatives of ψ.

$$\psi_x = \ell\tilde{v}^{\ell-1}\tilde{v}_x\omega + \tilde{v}^\ell\omega_x. \tag{7.9}$$

$$\psi_{xx} = \ell(\ell-1)\tilde{v}^{\ell-2}\tilde{v}_x^2\omega + \ell\tilde{v}^{\ell-1}\tilde{v}_{xx}\omega + 2\ell\tilde{v}^{\ell-1}\tilde{v}_x\omega_x + \tilde{v}^\ell\omega_{xx}. \tag{7.10}$$

Substituting (7.9) and (7.10) into (7.8), we have (using $w_x(0) = w_x(1) = 0$)

$$\begin{aligned} -\lambda_v\tilde{v}^\ell\omega = {} & \mu\ell(\ell-1)\tilde{v}^{\ell-2}\tilde{v}_x^2\omega + \mu\ell\tilde{v}^{\ell-1}\tilde{v}_{xx}\omega + 2\mu\ell\tilde{v}^{\ell-1}\tilde{v}_x\omega_x + \mu\tilde{v}^\ell\omega_{xx} \\ & + \mu\eta m_x\ell\tilde{v}^{\ell-1}\tilde{v}_x\omega + \mu\eta m_x\tilde{v}^\ell\omega_x - \frac{\nu\tilde{v}_{xx}}{\tilde{v}}\tilde{v}^\ell\omega. \end{aligned}$$

The two terms involving second derivative of $\tilde{v}$ cancel. Divide the equation by $\tilde{v}^\ell$ to obtain

$$-\lambda_v\omega = \mu\omega_{xx} + \mu\left(2\ell\frac{\tilde{v}_x}{\tilde{v}} + \eta m_x\right)\omega_x + \mu\left[\ell(\ell-1)\frac{\tilde{v}_x^2}{\tilde{v}^2} + \eta m_x\ell\frac{\tilde{v}_x}{\tilde{v}}\right]\omega.$$

Multiplying the equation by $e^{\eta m}\tilde{v}^{2\ell}$, we can write the equation in the variational form

$$-\lambda_v e^{\eta m}\tilde{v}^{2\ell}\omega = \mu\left(e^{\eta m}\tilde{v}^{2\ell}\omega_x\right)_x + \mu\left[\ell(\ell-1)\frac{\tilde{v}_x^2}{\tilde{v}^2} + \eta m_x\ell\frac{\tilde{v}_x}{\tilde{v}}\right]e^{\eta m}\tilde{v}^{2\ell}\omega.$$

Divide by $-\omega$ and integrate by parts. We have (using $w_x(0) = w_x(1) = 0$)

$$\begin{aligned} -\lambda_v\int_\Omega e^{\eta m}\tilde{v}^{2\ell} &= \mu\int_\Omega \frac{e^{\eta m}\tilde{v}^{2\ell}\omega_x^2}{\omega^2} + \mu\int_\Omega e^{\eta m}\tilde{v}^{2\ell}\ell\left[(\ell-1)\frac{\tilde{v}_x^2}{\tilde{v}^2} + \eta m_x\frac{\tilde{v}_x}{\tilde{v}}\right] \\ &\geq \nu\int_\Omega e^{\eta m}\tilde{v}^{2\nu/\mu-2}\tilde{v}_x^2\left[\left(\frac{\nu}{\mu} - 1\right) + \eta\frac{\int_\Omega e^{\eta m}\tilde{v}^{2\nu/\mu-1}\tilde{v}_x m_x}{\int_\Omega e^{\eta m}\tilde{v}^{2\nu/\mu-2}\tilde{v}_x^2}\right]. \end{aligned}$$

Since the last line is positive, by (7.7), we have $\lambda_v < 0$ and Theorem 7.17 follows. □

7.2.4. Proof of Theorem 2.12(b)(iv).

Proof of Theorem 2.12(b)(iv). It suffices to show

Claim 7.21. *There exists $M > 0$ such that if $\nu > \mu > M$, then $\lambda_v(\eta, \mu, \nu) < 0$ for all $\eta \geq 0$.*

This follows from $\lambda_v(0, \mu, \nu) < 0$ and Corollaries A.14 and A.15. □

7.3. Open problems

Conjecture 7.22. Suppose Ω is convex. If $\mu < \nu$, then $(0, \tilde{v})$ is unstable for all $\eta \in [0, \infty)$.

CHAPTER 8

Summary of asymptotic behaviors of η_* and η^*

Let $\eta^* = \eta^*(\mu, \nu)$ and $\eta_* = \eta_*(\mu, \nu)$ denote values of η at which the semi-trivial steady states $(\tilde{u}, 0)$ and $(0, \tilde{v})$ change their stability, respectively. In this chapter, we summarize our results concerning the uniqueness and asymptotic limits of η_* and η^* as the diffusion rates or their ratio tend to zero or infinity.

8.1. Asymptotic behavior of η^*

We consider the asymptotic behavior of η^* for three different cases: (i) sufficiently small μ and μ/ν; (ii) sufficiently small μ, with μ/ν bounded away from 0; (iii) sufficently large ν.

If μ/ν and μ are sufficiently small, we have the following result for the uniqueness of η^* and its limit.

PROPOSITION 8.1. *For $(\mu, \nu) \in \mathcal{R}_3$ (i.e. $\mu < \nu$), if $0 < \mu \ll 1$, then $\lambda_u(\cdot, \mu, \nu)$ has a unique positive root η^*. Moreover, for $d \in [1, \infty]$, as $\mu \to 0$ and $\nu/\mu \to d$, then η^* converges to the unique positive root of $\sigma(\cdot; d)$, where $\sigma(\eta; d)$ is the principal eigenvalue of* (3.19)*. In particular, taking $d = \infty$, we obtain*

$$\lim_{\mu \to 0, \frac{\nu}{\mu} \to \infty} \eta^* = \frac{\int_\Omega |\nabla m|^2 / m^2}{\int_\Omega |\nabla m|^2 / m},$$

PROOF. The case when $\mu \to 0$ and $\nu/\mu \to \infty$ is contained in Corollary 7.2, while the remaining case is mentioned in Remark 7.6. $\square$

PROPOSITION 8.2. *For $(\mu, \nu) \in \mathcal{R}_2$, if $0 < \mu \ll 1$, then $\lambda_u(\cdot, \mu, \nu)$ has exactly two positive roots $\eta_1^* < \eta_2^*$. Moreover,*

(i) *if $\mu \to 0$ and $\nu/\mu \to d \in (a_*, 1]$, then η_i^* $(i = 1, 2)$ converge respectively to the two different non-negative roots of $\sigma(\cdot; d)$. Here a_* is given by Theorem* B.2(iv) *and $\sigma(\eta; d)$ is the principal eigenvalue of* (3.19).

(ii) *if $\mu \to 0$ and $\nu/\mu \to a_*$, then both η_i^* $(i = 1, 2)$ converge to the unique positive root of $\sigma(\cdot; a_*)$.*

PROOF. By Remark 6.5, Proposition 8.2 follows from the proof of Theorem 2.11(a). $\square$

For fixed $\mu > 0$ and sufficiently large ν, we do not expect the uniqueness for η^*. Nonetheless, its limit can be characterized as follows by considering (3.1):

PROPOSITION 8.3. *Let η^* be a positive root of $\lambda_u(\cdot, \mu, \nu)$, then as $\nu \to \infty$, passing to a sequence, η^* converges to a positive root of $\eta \mapsto \int_\Omega (m - \tilde{u}(\eta))$.*

For any fixed $\nu > 0$ and sufficiently large μ, we can determine the asymptotic behavior of η^*.

PROPOSITION 8.4. *For each $\nu > \nu^+$, passing to a sequence if necessary,* $\lim_{\mu\to\infty} \eta^* = \eta_1$*, where $\nu^+ = \lim_{\mu\to\infty} \bar{\nu}(\mu)$ is given by Theorem* 2.7(v) *and η_1 is a positive number such that the following problem has a solution:*

$$
\begin{cases}
\nu\Delta\varphi + \left(m - \frac{\int_\Omega m e^{\eta_1 m}}{\int_\Omega e^{2\eta_1 m}} e^{\eta_1 m}\right)\varphi = 0 & \text{in } \Omega, \\
\varphi > 0 & \text{in } \Omega, \\
\frac{\partial\varphi}{\partial n} = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{8.1}
$$

We do not know whether such η_1 is unique. We refer to Lemma 4.12 for the proof of Proposition 8.4.

8.2. Asymptotic behavior of η_*

If ν/μ is sufficiently small, the following result establishes the uniqueness of η_* and determines its limit.

PROPOSITION 8.5. *If ν/μ is sufficiently small, $\lambda_v(\cdot, \mu, \nu)$ has a unique positive root η_*. Moreover,*

(i) *if $\nu \to 0$ and $\nu/\mu \to 0$, then η_* tends to the unique positive root of*
$$\eta \mapsto \int_\Omega \frac{|\nabla m|^2}{m^2} e^{\eta m}(\eta m - 1), \quad \eta > 0;$$

(ii) *if $\nu \to \nu_0$ and $\mu \to \infty$, for some $\nu_0 > 0$, then η_* tends to the unique positive root of*
$$\eta \mapsto \int_\Omega e^{\eta m}\left(m - \tilde{v}|_{\nu=\nu_0}\right), \quad \eta > 0;$$

(iii) *if $\nu \to \infty$ and $\nu/\mu \to 0$, then η_* tends to* 0.

PROOF. The uniqueness of η_*, when μ/ν is sufficiently large, is proved in Theorem 6.7, and the limiting results (i), (ii) and (iii) of η_* are contained in Propositions 5.7, 5.4 and 6.6 respectively. □

CHAPTER 9

Structure of positive steady states via Lyapunov-Schmidt procedure

In this chapter we investigate the structure of positive steady states of system (2.1) when $0 < \eta \ll 1$ and ν is close to μ. Let θ_μ denote the unique positive solution of the scalar equation

$$\mu\Delta\theta + \theta(m-\theta) = 0 \quad \text{in } \Omega, \quad \frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{9.1}$$

The main result of this chapter can be stated as follows.

THEOREM 9.1. *Suppose that $\Omega \subseteq \mathbb{R}$ or $\Omega \subseteq \mathbb{R}^N$ is convex. Fix $\mu > 0$. There exist $\delta_1 > 0$, $\delta_2 > 0$ and functions*

$$\tilde{\eta} = \tilde{\eta}(\nu, s) : (\mu - \delta_2, \mu + \delta_2) \times (0,1) \to (-\delta_1, \delta_1),$$

$$\tilde{y} = \tilde{y}(\cdot;\nu, s),\ \tilde{z} = \tilde{z}(\cdot;\nu, s) : (\mu - \delta_2, \mu + \delta_2) \times (0,1) \to C^2(\bar{\Omega}),$$

such that system (2.1) has a positive steady state, denoted by (u, v), for $\eta \in (-\delta_1, \delta_1)$ and $\nu \in (\mu - \delta_2, \mu + \delta_2)$ if and only if for some $0 < s < 1$,

$$\begin{cases} \eta = \tilde{\eta}(\nu, s), \\ u = se^{\tilde{\eta}(\nu,s)m}[\theta_\mu + \tilde{y}], \\ v = (1-s)[\theta_\mu + \tilde{z}]. \end{cases}$$

Moreover, when $\nu = \mu$, $\tilde{y}(x;\mu, s) = \tilde{z}(x;\mu, s) \equiv 0$, $\tilde{\eta}(\mu, s) \equiv 0$ and

$$\lim_{\nu\to\mu} \frac{\tilde{\eta}(\nu, s)}{\mu - \nu} = \frac{\int_\Omega |\nabla\theta_\mu|^2}{\int_\Omega \theta_\mu \nabla\theta_\mu \cdot \nabla m} > 0.$$

REMARK 9.2. If $\nu = \mu$, $\tilde{\eta}(\mu, s) \equiv 0$ and $(u, v) = (s\theta_\mu, (1-s)\theta_\mu)$. That is, if $\nu = \mu$, a positive steady state of system (2.1) exists for η close to zero if and only if $\eta = 0$.

REMARK 9.3. Under the assumption of Theorem 9.1, and making use of Theorem 3.10(b) (which gives the sign of $\int_\Omega \theta_\mu \nabla\theta_\mu \cdot \nabla m$), the following are two immediate consequences:

(i) If $\nu \in (\mu, \mu + \delta_2)$, then system (2.1) has no positive steady state for $\eta \in [0, \delta_1)$. It can be further shown that the semi-trivial steady state $(\tilde{u}, 0)$ is globally asymptotically stable among all positive initial data.

(ii) If $\nu \in (\mu - \delta_2, \mu)$, then for $\eta \in (0, \delta_1)$, system (2.1) has a positive steady state if and only if

$$\inf_{0<s<1} \tilde{\eta}(\nu, s) < \eta < \sup_{0<s<1} \tilde{\eta}(\nu, s),$$

and

$$\lim_{\nu\to\mu}\frac{\inf_{0<s<1}\tilde{\eta}(\nu,s)}{\mu-\nu}=\lim_{\nu\to\mu}\frac{\sup_{0<s<1}\tilde{\eta}(\nu,s)}{\mu-\nu}=\frac{\int_\Omega|\nabla\theta_\mu|^2}{\int_\Omega\theta_\mu\nabla\tilde{\theta}_\mu\cdot\nabla m}>0.$$

In particular, it implies that for $\nu\in(\mu-\delta_2,\mu)$, the range of (small) η in which system (2.1) has a positive steady state is at most of order $\mu-\nu$. It can be further shown that $(\tilde{u},0)$ is unstable for $\eta\in[0,\tilde{\eta}(\nu,0))$ and stable for $\eta\in(\tilde{\eta}(\nu,0),\delta_1)$. It is an open problem whether system (2.1) has exactly one positive steady state when $\inf_{0<s<1}\tilde{\eta}(\nu,s)<\eta<\sup_{0<s<1}\tilde{\eta}(\nu,s)$.

We use a Lyapunov-Schmidt procedure to classify positive steady states of system (2.1) for small positive η and $\nu=\mu-\epsilon$ for ϵ small. First, we have the following system satisfied by steady states of (2.1):

$$\begin{cases}\mu\nabla\cdot(\nabla u-\eta u\nabla m)+u(m-u-v)=0 & \text{in } \Omega,\\ (\mu-\epsilon)\Delta v+v(m-u-v)=0 & \text{in } \Omega,\\ \frac{\partial u}{\partial n}-\eta u\frac{\partial m}{\partial n}=\frac{\partial v}{\partial n}=0 & \text{on } \partial\Omega.\end{cases}\tag{9.2}$$

Set $w=e^{-\eta m}u$. Then w and v satisfy

$$\begin{cases}\mu\Delta w+\mu\eta\nabla m\cdot\nabla w+w(m-e^{\eta m}w-v)=0 & \text{in } \Omega,\\ (\mu-\epsilon)\Delta v+v(m-e^{\eta m}w-v)=0 & \text{in } \Omega,\\ \frac{\partial w}{\partial n}=\frac{\partial v}{\partial n}=0 & \text{on } \partial\Omega.\end{cases}\tag{9.3}$$

When $\eta=\epsilon=0$, the set of positive solutions of (9.2) is given by

$$\Sigma=\{(s\theta_\mu,(1-s)\theta_\mu):s\in(0,1)\}.$$

We first show that the set of positive solutions of (9.2) is close to Σ for sufficiently small $\eta>0$ and ϵ.

LEMMA 9.4. *Let (W,V) denote any positive solution of (9.3). Then, after passing to some subsequence if necessary, we have $(W,V)\to(s\theta_\mu,(1-s)\theta_\mu)$ in $C^2(\bar{\Omega})$ for some $s\in[0,1]$ as $(\eta,\epsilon)\to(0,0)$.*

PROOF. By the maximum principle [**90**] it is easy to show that

$$\|W\|_{L^\infty(\Omega)}\le\|me^{-\eta m}\|_{L^\infty(\Omega)}\quad\text{and}\quad\|V\|_{L^\infty(\Omega)}\le\|m\|_{L^\infty(\Omega)}.$$

This implies that both W and V are uniformly bounded for small η and ϵ. By elliptic regularity and Sobolev embedding theorems [**40**] we see that both W and V are uniformly bounded in $C^{2,\tau}(\bar{\Omega})$ for some $\tau\in(0,1)$ and for all small η and ϵ. Hence, passing to some subsequence if necessary, we may assume that $W\to W^*$ and $V\to V^*$ in $C^2(\bar{\Omega})$, and W^*,V^* satisfy

$$\begin{cases}\mu\Delta W^*+W^*(m-W^*-V^*)=0 & \text{in } \Omega,\\ \mu\Delta V^*+V^*(m-W^*-V^*)=0 & \text{in } \Omega,\\ \frac{\partial W^*}{\partial n}=\frac{\partial V^*}{\partial n}=0 & \text{on } \partial\Omega.\end{cases}\tag{9.4}$$

Since W,V are positive, we have $W^*\ge0$ and $V^*\ge0$. We claim that $(W^*,V^*)\neq(0,0)$. Suppose to the contrary that $W\to0$ and $V\to0$ in $L^\infty(\Omega)$ as $\eta,\epsilon\to0$. Integrate the equation of V, we have

$$\int_\Omega V(m-e^{\eta m}W-V)=0.$$

Since $m > 0$ in $\bar{\Omega}$ and $W, V \to 0$ uniformly in Ω as $\eta, \epsilon \to 0$, we have $m - e^{\eta m} W - V > 0$ in Ω for sufficiently small η, ϵ, which is a contradiction. Therefore, we have either $W^* \neq 0$ or $V^* \neq 0$; i.e. $W^* + V^* \geq 0$ and $W^* + V^* \neq 0$. Adding the equations of W^* and V^* we see that $W^* + V^*$ is a non-negative, non-trivial solution of (9.1). By the uniqueness of θ_μ, we have $W^* + V^* = \theta_\mu$. Hence, W^* and V^* satisfy

$$\begin{cases} \mu\Delta W^* + W^*(m - \theta_\mu) = 0 & \text{in } \Omega, \\ \mu\Delta V^* + V^*(m - \theta_\mu) = 0 & \text{in } \Omega, \\ \frac{\partial W^*}{\partial n} = \frac{\partial V^*}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{9.5}$$

By (9.1), we see that zero is the principal eigenvalue of the operator $\mu\Delta + (m - \theta_\mu)$ with zero Neumann boundary condition. As a consequence, the kernel of the eigenfunction space corresponding to zero is one dimensional, spanned by θ_μ. Hence, W^* and V^* must be scalar multiples of θ_μ, i.e., $W^* = s\theta_\mu$ and $V^* = \tilde{s}\theta_\mu$ for some $s, \tilde{s} \geq 0$. As $W^* + V^* = \theta_\mu$, we see that $s + \tilde{s} = 1$. This implies that $(W^*, V^*) = (s\theta_\mu, (1-s)\theta_\mu)$ for some $s \in [0, 1]$. □

Proof of Theorem 9.1. From here on we fix $\mu > 0$ and write $\theta = \theta_\mu$. By Lemma 9.4, all positive solutions of system (9.3) are close to $\Sigma = \{(s\theta, (1-s)\theta) : s \in (0,1)\}$ for sufficiently small η and ϵ. Hence, it suffices to determine the structure of the set of positive solutions of system (9.3) near Σ for $0 < \eta, |\epsilon| \ll 1$. To this end, we apply the Lyapunov-Schmidt procedure. Set $X = W^{2,p}_N(\Omega) \times W^{2,p}_N(\Omega)$ with $p > N$, $Y = L^p(\Omega) \times L^p(\Omega)$, where $W^{2,p}_N(\Omega) = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial n}\big|_{\partial\Omega} = 0\}$. We rewrite solutions (w, v) of (9.3) as $(w, v) = (s(\theta + y), (1-s)(\theta + z))$, where $s \in \mathbb{R}$ and

$$(y, z) \in X_1 := \left\{ (y, z) \in X : \int_\Omega (y - z)\theta = 0 \right\}.$$

For $\delta > 0$, define the mapping $F : X_1 \times (-\delta, \delta) \times (-\delta, \delta) \times (-\delta, 1 + \delta) \to Y$ by

$$F(y, z, \eta, \epsilon, s) = L_s \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix},$$

where

$$L_s \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \mu\Delta y + \theta[-sy - (1-s)z] + y(m - \theta) \\ \mu\Delta z + \theta[-sy - (1-s)z] + z(m - \theta) \end{pmatrix},$$

and

$$\begin{aligned} f &= \mu\eta\nabla m \cdot \nabla(\theta + y) + [1 - e^{\eta m}]s(\theta + y)^2 + y[-sy - (1-s)z], \\ g &= -\epsilon\Delta(\theta + z) + [1 - e^{\eta m}]s(\theta + y)(\theta + z) + z[-sy - (1-s)z]. \end{aligned}$$

Also, define the operator P_s by

$$P_s(y, z) = \frac{\int_\Omega (y - z)\theta}{\int_\Omega \theta^2} \begin{pmatrix} (1-s)\theta \\ -s\theta \end{pmatrix}.$$

The operator P_s satisfies $P_s^2 = P_s$ and $P_s L_s = 0$, and the range of P_s is spanned by $((1-s)\theta, -s\theta)$.

Following the Lyapunov-Schmidt procedure, it remains to solve

$$P_s F(y, z, \eta, \epsilon, s) = 0 \tag{9.6}$$

together with

$$(I - P_s) F(y, z, \eta, \epsilon, s) = 0. \tag{9.7}$$

Since $D_{(y,z)} F(0, 0, 0, 0, s) = L_s$ and $P_s L_s = 0$, we have $D_{(y,z)}(I - P_s) F(0, 0, 0, 0, s) = (I - P_s) L_s = L_s$.

CLAIM 9.5. *The kernel of L_s is spanned by $((1-s)\theta, -s\theta)$.*

This can be seen by showing that $y - z \in \text{span}\{\theta\}$, and that $sy + (1-s)z = 0$. By the above claim, we have

$$\text{Ker}(L_s) \cap X_1 = \{(0,0)\}.$$

By the property of Fredholm operators of index zero, $D_{(y,z)}(I-P_s)F(0,0,0,0,s)$ is invertible from X_1 to Y. By the implicit function theorem, there exist some neighborhood V_0 of $(0,0)$ in X_1, $\delta_1 > 0$, and scalar functions $y_1(\eta,\epsilon,s)$, $z_1(\eta,\epsilon,s)$ with $(y_1(0,0,s), z_1(0,0,s)) = (0,0)$ such that $(I-P_s)F(y,z,\eta,\epsilon,s) = 0$ for $(y,z,\eta,\epsilon,s) \in V_0 \times (-\delta_1,\delta_1) \times (-\delta_1,\delta_1) \times (-\delta_1, 1+\delta_1)$ if and only if $(y,z) = (y_1(\eta,\epsilon,s), z_1(\eta,\epsilon,s))$. That is, $F(y,z,\eta,\epsilon,s) = 0$ for $(y,z,\eta,\epsilon,s) \in V_0 \times (-\delta_1,\delta_1) \times (-\delta_1,\delta_1) \times (-\delta_1, 1+\delta_1)$ if and only if $(y,z) = (y_1(\eta,\epsilon,s), z_1(\eta,\epsilon,s))$ solves (9.6).

Define $\chi(\eta,\epsilon,s)$ by

$$P_s F(y_1(\eta,\epsilon,s), z_1(\eta,\epsilon,s), \eta, \epsilon, s) = \frac{\chi(\eta,\epsilon,s)}{\int_\Omega \theta^2} \begin{pmatrix} (1-s)\theta \\ -s\theta \end{pmatrix},$$

where $\chi(\eta,\epsilon,s)$ is given by

(9.8)
$$\begin{aligned}\chi(\eta,\epsilon,s) = \quad & \textstyle\int_\Omega \theta\{\mu\eta\nabla m \cdot \nabla(\theta + y_1) + \epsilon\Delta(\theta + z_1) + s[1 - e^{\eta m}](\theta + y_1)(y_1 - z_1) \\ & +(y_1 - z_1)[-sy_1 - (1-s)z_1]\},\end{aligned}$$

and $(y_1, z_1) = (y_1(\eta,\epsilon,s), z_1(\eta,\epsilon,s))$. Since $y_1(0,0,s) = z_1(0,0,s) = 0$,

$$\frac{\partial\chi}{\partial\eta}(0,0,s) = \mu\int_\Omega \theta\nabla m \cdot \nabla\theta. \tag{9.9}$$

Now, by Theorem 3.10(b), the last quantity in (9.9) is positive. As $\chi(0,0,s) = 0$, by the implicit function theorem, there exist some $\delta \in (0,\delta_1)$ and $s_0 > 0$ such that all solutions of $\chi(\eta,\epsilon,s) = 0$ in $(-\delta,\delta) \times (-\delta,\delta) \times (-s_0, 1+s_0)$ can be represented by some function $\eta = \eta_1(\epsilon,s)$, i.e., $\chi(\eta,\epsilon,s) = 0$ if and only if $\eta = \eta_1(\epsilon,s)$. It is easy to see that

$$\begin{aligned}\eta_1(\epsilon,s) = & \frac{1}{\mu\int_\Omega \theta\nabla m \cdot \nabla(\theta + y_1)} \left\{-\epsilon\int_\Omega \theta\Delta(\theta + z_1) - s\int_\Omega [1 - e^{\eta m}]\theta(\theta + y_1)(y_1 - z_1)\right. \\ & \left. - \int_\Omega \theta(y_1 - z_1)[-sy_1 - (1-s)z_1]\right\},\end{aligned}$$

where $(y_1, z_1) = (y_1(\eta_1(\epsilon,s),\epsilon,s), z_1(\eta_1(\epsilon,s),\epsilon,s))$. Since $y_1(\eta,\epsilon,s), z_1(\eta,\epsilon,s) = O(\eta + |\epsilon|) = O(|\epsilon|)$ for sufficiently small η and ϵ, we have

$$\eta_1(\epsilon,s) = \epsilon\left[\frac{\int_\Omega |\nabla\theta|^2}{\mu\int_\Omega \theta\nabla m \cdot \nabla\theta} + O(|\eta| + |\epsilon|)\right]. \tag{9.10}$$

Set

$$\begin{cases} \tilde{\eta}(\nu,s) := \eta_1(\mu - \nu, s), \\ \tilde{y}(x;\nu,s) := y_1(x;\eta_1(\mu - \nu,s), \mu - \nu, s), \\ \tilde{z}(x;\nu,s) := z_1(x;\eta_1(\mu - \nu,s), \mu - \nu, s), \end{cases}$$

where $\nu \in (\mu - \delta, \mu + \delta)$ and $s \in (-s_0, 1+s_0)$. Then we see that for $\nu \in (\mu - \delta, \mu + \delta)$, system (9.3) has a positive steady state, denoted by (w,v), for $\eta \in (-\delta_1,\delta_1)$ if and only if for some $s \in (0,1)$,

$$\begin{cases} \eta = \tilde{\eta}(\nu, s), \\ w = s[\theta_\mu(x) + \tilde{y}], \\ v = (1-s)[\theta_\mu(x) + \tilde{z}]. \end{cases}$$

Since $\eta_1(0,s) = 0$ and $y_1(x;0,0,s) = z_1(x;0,0,s) \equiv 0$, it is easy to check that $\tilde{y}(x;\mu,s) = \tilde{z}(x;\mu,s) \equiv 0$ and $\tilde{\eta}(\mu,s) \equiv 0$. By (9.10), we see that

$$\lim_{\nu \to \mu} \frac{\tilde{\eta}(\nu,s)}{1-\nu/\mu} = \frac{\int_\Omega |\nabla\theta_\mu|^2}{\int_\Omega \theta_\mu \nabla m \cdot \nabla\theta_\mu} > 0.$$

This completes the proof. $\square$

CHAPTER 10

Non-convex domains

The geometry of domain seems to play an important role in determining the dynamics of system (2.1). In this chapter we show that the convexity assumption is necessary in many of our preceding results and in particular in the Conjecture 2.9. We first recall the following result from [**14**]:

THEOREM 10.1. *There exist a non-convex smooth domain* Ω *and* $m \in C^2(\bar{\Omega})$ *such that for some* $\mu > 0$, $\int_\Omega \theta_\mu \nabla\theta_\mu \cdot \nabla m < 0$.

The main result of this chapter can be stated as follows.

THEOREM 10.2. *There exist some non-convex smooth domain* Ω *and some* $m \in C^2(\bar{\Omega})$ *such that for some* $\mu > 0$, *there exist* $\delta_1 > 0$, $\delta_2 > 0$ *and functions*

$$\tilde{\eta} = \tilde{\eta}(\nu, s) : (\mu - \delta_2, \mu + \delta_2) \times (0, 1) \to (-\delta_1, \delta_1),$$

$$\tilde{y} = \tilde{y}(\cdot; \nu, s),\ \tilde{z} = \tilde{z}(\cdot; \nu, s) : (\mu - \delta_2, \mu + \delta_2) \times (0, 1) \to C^1(\bar{\Omega})$$

such that system (2.1) *has a positive steady state, denoted by* (u, v) *for* $\eta \in (-\delta_1, \delta_1)$ *and* $\nu \in (\mu - \delta_2, \mu + \delta_2)$ *if only if for some* $0 < s < 1$,

$$\begin{cases} \eta = \tilde{\eta}(\nu, s), \\ u = se^{\tilde{\eta}(\nu,s)m}[\theta_\mu + \tilde{y}], \\ v = (1 - s)[\theta_\mu + \tilde{z}], \end{cases}$$

where when $\nu = \mu$, $\tilde{y}(x; \mu, s) = \tilde{z}(x; \mu, s) \equiv 0$, $\tilde{\eta}(\mu, s) \equiv 0$ *and*

$$\lim_{\nu\to\mu} \frac{\tilde{\eta}(\nu, s)}{\mu - \nu} = \frac{\int_\Omega |\nabla\theta_\mu|^2}{\int_\Omega \theta_\mu \nabla\theta_\mu \cdot \nabla m} < 0.$$

Note that Theorem 10.2 follows from the proof of Theorem 9.1 by using Theorem 10.1 instead of Theorem 3.10. (Note that the only place in the proof of Theorem 9.1 that uses the convexity of Ω is the application of Theorem 3.10.)

REMARK 10.3. Under the assumption of Theorem 10.2, for $\nu \in (\mu - \delta_2, \mu)$, (2.1) has no positive steady states for $\eta \in [0, \delta_1)$. It can be further shown that the semi-trivial steady state $(0, \tilde{v})$ is globally asymptotically stable. On the other hand, if $\nu \in (\mu, \mu + \delta_2)$, then for $\eta \in (0, \delta_2)$, $\tilde{\eta}(\nu, s) > 0$ and system (2.1) has a positive steady state for some $\eta \in [0, \delta_1)$ if and only if

$$\inf_{0<s<1} \tilde{\eta}(\nu, s) < \eta < \sup_{0<s<1} \tilde{\eta}(\nu, s),$$

and

$$\lim_{\nu\to\mu} \frac{\inf_{0<s<1} \tilde{\eta}(\nu, s)}{\mu - \nu} = \lim_{\nu\to\mu} \frac{\sup_{0<s<1} \tilde{\eta}(\nu, s)}{\mu - \nu} = \frac{\int_\Omega |\nabla\theta_\mu|^2}{\int_\Omega \theta_\mu \nabla\theta_\mu \cdot \nabla m} < 0.$$

In contrast with the case when Ω is convex, we have the following result for some non-convex domains.

THEOREM 10.4. *There exist some non-convex smooth domain Ω, some $m \in C^2(\bar{\Omega})$ and $\mu < \nu$ such that $(0, \tilde{v})$ changes its stability at least twice as η varies from zero to infinity.*

PROOF. Let $\mu > 0$, Ω and m be chosen as in Theorem 10.1. By the eigenvalue comparison principle, $\lambda_v(0, \mu, \nu) < 0$ for all $\nu > \mu$. Next, by Lemma 5.2, we have

$$\frac{\partial \lambda_v}{\partial \eta}(\eta, \mu, \nu) = -\mu \frac{\int_\Omega e^{\eta m} \psi \nabla m \cdot \nabla \psi}{\int_\Omega e^{\eta m} \psi^2}.$$

Setting $\eta = 0$ and $\nu = \mu$, we have $\psi = \theta_\mu$ and

$$\frac{\partial \lambda_v}{\partial \eta}(0, \mu, \mu) = -\mu \frac{\int_\Omega \theta_\mu \nabla m \cdot \theta_\mu}{\int_\Omega \theta_\mu^2} > 0.$$

Together with $\lambda_v(0, \mu, \mu) = 0$, we deduce by implicit function theorem that there exist constants $\delta_1, \delta_2 > 0$ and a function $\tilde{\eta} : (\mu - \delta_1, \mu + \delta_1) \to (-\delta_2, \delta_2)$ such that $\lambda_v(\eta, \mu, \nu) = 0$ for some $\eta \in (-\delta_2, \delta_2)$ and $\nu \in (\mu - \delta_1, \mu + \delta_1)$ if and only if $\eta = \tilde{\eta}(\nu)$. Moreover, for each $\nu \in (\mu, \mu + \delta_1)$, $\lambda_v(\eta, \mu, \nu) < 0$ for all $\eta \in (0, \tilde{\eta})$ and $\lambda_v(\eta, \mu, \nu) > 0$ for all $\eta \in (\tilde{\eta}, \delta_2)$. This, and the fact that $\limsup_{\eta \to \infty} \lambda_v < 0$ (in Theorem 2.2, m is allowed to change sign), yields the theorem. $\square$

CHAPTER 11

Global bifurcation results

The main goal of this chapter is to prove Theorems 2.14 and 2.4. Sections 11.1-11.4 is devoted the proof of Theorem 2.14. Theorem 2.4 is established in Section 11.5.

11.1. General bifurcation theorems

By the substitution $u = e^{\eta m} w$, we rewrite the steady state system of (2.1) as

$$
\begin{cases} \mu\Delta w + \mu\eta\nabla m \cdot \nabla w + w(m - e^{\eta m}w - v) = 0 & \text{in } \Omega, \\ \nu\Delta v + v(m - e^{\eta m}w - v) = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.1}
$$

The global version of the Crandall-Rabinowitz bifurcation theorem from a simple eigenvalue is ubiquitous in applications. See, e.g. [**25, 71, 91**]. We shall apply the global bifurcation theory for a C^1 Fredholm mapping initially developed by [**87**]. First we state the condition of simple bifurcation due to [**25**].

Let X and Y be Banach spaces, V be an open connected subset of $\mathbb{R} \times X$ and $(\eta_0, u_0) \in V$, and let F be a continuously differentiable mapping from V into Y.

(H1) $F(\eta, u_0) = 0$ for $(\eta, u_0) \in V$,

(H2) $D_u F, D_\eta F, D_{\eta u} F$ exist and are continuous in V,

(H3) $D_u F(\eta_0, u_0)$ is a Fredholm operator with index 0, and for some $w_0 \in X$,

$$\mathrm{Null}(D_u F(\eta_0, u_0)) = \mathrm{span}\{w_0\},$$

(H4) $D_{\eta u} F(\lambda_0, u_0)[w_0] \notin \mathrm{Range}(D_u F(\eta_0, u_0))$.

Theorem 11.1. [**87**] *Suppose* (H1)-(H4) *are satisfied. Let Z be any complement of* $\mathrm{Null}(D_u F(\eta_0, u_0))$ *in X. Then there exist an open interval $I_1 = (-\epsilon, \epsilon)$ and continuous functions $\eta : I_1 \to \mathbb{R}$, $\phi : I_1 \to Z$, such that $\eta(0) = \eta_0$, $\phi(0) = 0$, and, if $u(s) = u_0 + sw_0 + s\phi(s)$ for $s \in I_1$, then $F(\eta(s), u(s)) = 0$. Moreover, $F^{-1}(\{0\})$ near (η_0, u_0) consists precisely of the curves $u = u_0$ and $\Gamma = \{(\eta(s), u(s)) : s \in I_1\}$. If in addition, $D_u F(\eta, u)$ is a Fredholm operator for all $(\eta, u) \in V$, then the curve Γ is contained in $\mathcal{C}$, which is a connected component of $\bar{S}$ where $S = \{(\eta, u) \in V : F(\eta, u) = 0, u \neq u_0\}$; and either $\mathcal{C}$ is not compact in V, or $\mathcal{C}$ contains a point (η', u_0) with $\eta' \neq \eta_0$.*

The unilateral version below, that is suitable for dealing with positive solutions, is due to Shi and Wang (see also Chapter 6 of [**71**]).

Theorem 11.2. [**94**] *Suppose* (H1)-(H4) *are satisfied. Let $\mathcal{C}$ be defined as in Theorem* 11.1. *We define $\Gamma_+ = \{(\eta(s), u(s)) : s \in (0, \epsilon)\}$ and $\Gamma_- = \{(\eta(s), u(s)) : s \in (-\epsilon, 0)\}$. In addition we assume that*

(H5) *the norm function $u \mapsto \|u\|$ in X is continuously differentiable for any $u \neq 0$;*

(H6) *for* $k \in (0,1)$, *if* (η, u_0) *and* (η, u) *are both in* V, *then* $(1-k)D_u F(\eta, u_0) + kD_u F(\eta, u)$ *is a Fredholm operator.*

Let $\mathcal{C}^+$ *(resp.* $\mathcal{C}^-$*) be the connected component of* $\mathcal{C} \setminus \Gamma_-$ *which contains* Γ_+ *(resp. the connected component of* $\mathcal{C} \setminus \Gamma_+$ *which contains* Γ_-*). Then each of the sets* $\mathcal{C}^+$ *and* $\mathcal{C}^-$ *satisfies one of the following:* (i) *it is not compact;* (ii) *it contains a point* (η', u_0) *with* $\eta' \neq \eta_0$*; or* (iii) *it contains a point* $(\eta, u_0 + z)$*, where* $z \in Z \setminus \{0\}$*.*

11.2. Bifurcation result in $\mathcal{R}_1$

Define $\mathcal{S}$ to be the set of positive solutions of (11.1), i.e.

$$\mathcal{S} = \{(u, v) \in X : (u, v) \text{ is a solution of (11.1) and } u > 0,\ v > 0 \text{ in } \bar{\Omega}\}.$$

Let us first look at the bifurcation at $(\eta_*, (0, \tilde{v}))$ for (11.1).

Proof of Theorem 2.14(i). First, we check the conditions (H1)-(H4). Fix $p > N$ and let $X = W^{2,p}_N(\Omega) \times W^{2,p}_N(\Omega)$ and $Y = L^p(\Omega) \times L^p(\Omega)$, where

$$W^{2,p}_N(\Omega) = \{\phi \in W^{2,p}(\Omega) : \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial\Omega\}.$$

Define $V = (0, \infty) \times X$ and define $F : V \to Y$ by

$$F(\eta, (w, v)) = \begin{bmatrix} \mu\Delta w + \mu\eta\nabla m \cdot \nabla w + w(m - e^{\eta m} w - v) \\ \nu\Delta v + v(m - e^{\eta m} w - v) \end{bmatrix}.$$

Then $F(\eta, (0, \tilde{v})) = 0$ for all $\eta > 0$. It is easy to see that F is smooth in η and (w, v) with $D_{(w,v)}F$ given by

$$D_{(w,v)}F(\eta, (w, v)) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{bmatrix} \mu\Delta\phi_1 + \mu\eta\nabla m \cdot \nabla\phi_1 + (m - 2e^{\eta m} w - v)\phi_1 - w\phi_2 \\ \nu\Delta\phi_2 + (m - e^{\eta m} w - 2v)\phi_2 - e^{\eta m} v\phi_1 \end{bmatrix}.$$

By writing

$$D_{(w,v)}F(\eta, (w, v)) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = L \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + K \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{11.2}$$

where

$$L \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \mu\Delta\phi_1 - \phi_1 \\ \nu\Delta\phi_2 - \phi_2 \end{pmatrix}$$

and

$$K \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} (m - 2e^{\eta m} w - v + 1)\phi_1 - w\phi_2 \\ (m - e^{\eta m} w - 2v + 1)\phi_2 - e^{\eta m} v\phi_1 \end{pmatrix},$$

we see that for all $(\eta, (w, v)) \in V$, $D_{(w,v)}F$ is a sum of an isomorphism and a compact operator. Hence it is Fredholm with index zero. Moreover, let $w_0 = (\psi, -M[\psi])$, with ψ being the principal eigenfunction of (3.3) and $\tilde{\psi} = M[\psi]$ being the unique positive solution to

$$\begin{cases} -\nu\Delta\tilde{\psi} - (m - 2\tilde{v})\tilde{\psi} = e^{\eta m}\tilde{v}\psi & \text{in } \Omega, \\ \frac{\partial \tilde{\psi}}{\partial n} = 0 & \text{on } \Omega. \end{cases}$$

Then

$$\text{Null}(D_{(w,v)}F)(\eta_*, (0, \tilde{v})) = \text{span}\{w_0\}.$$

This verifies (H1)-(H3). Define

$$Z = \left\{(y, z) \in X : \int_\Omega (y\psi - zM[\psi]) = 0\right\}, \tag{11.3}$$

then $Z + \mathrm{span}\{w_0\} = X$.

LEMMA 11.3. *If* $\int_\Omega e^{\eta m}\psi \nabla m \cdot \nabla\psi \neq 0$*, then*

$$D_{\eta,(w,v)}F(\eta_*, (0, \tilde{v}))[w_0] \notin \mathrm{Range}(D_{(w,v)}F(\eta_*, (0, \tilde{v})).$$

PROOF. Suppose

$$D_{\eta,(w,v)}F(\eta_*, (0, \tilde{v}))[w_0] = \begin{pmatrix} \mu\nabla m \cdot \nabla\psi \\ -e^{\eta m} m\tilde{v}\psi \end{pmatrix} \in \mathrm{Range}\left(D_{(w,v)}F(\eta_*, (0, \tilde{v}))\right)$$

then for some $\phi \in W^{2,p}(\Omega)$,

$$\begin{cases} \mu\nabla\cdot(e^{\eta m}\nabla\phi) + (m - \tilde{v})e^{\eta m}\phi = \mu e^{\eta m}\nabla m \cdot \nabla\psi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiply the above by ψ and integrate by parts, then

$$\mu\int_\Omega e^{\eta m}\psi\nabla m \cdot \nabla\psi = 0.$$

This proves Lemma 11.3. □

REMARK 11.4. (H4) is equivalent to (by Lemma 5.2) to $\frac{\partial}{\partial\eta}\lambda_v(\eta_*, \mu, \nu) \neq 0$.

(H5) in Theorem 11.2 is satisfied for $\|\cdot\|_X$. For (H6), it suffices to realize that for any (η, u_0) and (η, u) in V, $(1-k)D_uF(\eta, u_0) + kD_uF(\eta, u)$ has a similar form as in (11.2) and hence it is also Fredholm of index 0.

Therefore $(\mu_*, (0, \tilde{v}))$ is a bifurcation point and Theorems 11.1 and 11.2 are applicable. We are interested in the branch of positive solutions $\mathcal{C}^+$, i.e. the branch containing

$$\{(\eta(s), (0, \tilde{v}) + s(\psi, -M[\psi]) + (y(s), z(s))) : s \in (0, \epsilon)\},$$

where $(y(s), z(s)) \in Z$. We claim that $\mathcal{C}^+$ is unbounded. Suppose to the contrary, then the first alternative of Theorem 11.2 saying $\mathcal{C}^+$ contains $(\eta', (0, \tilde{v}))$ for some $\eta' \neq \eta_*$ is impossible, since $(\eta_*, (0, \tilde{v}))$ is the unique bifurcation point for positive solutions of (11.3) lying on $\{(\eta, (0, \tilde{v})) : \eta \in (0, \infty)\}$. Hence the last alternative must happen, i.e. $\mathcal{C}^+$ contains a point $(\eta, (0, \tilde{v}) + (y, z))$ for some $(y, z) \in Z$.

CLAIM 11.5. *Let* $P = \{(u, v) \in X : u > 0$ *and* $v > 0$ *in* $\bar{\Omega}\}$*. Then* $\mathcal{C}^+$ *contains a point* $(\eta_1, (u_1, v_1)) \in [(0, \infty) \times \partial P]$.

By the maximum principle, $z < 0$. And hence $y < 0$ somewhere by the definition of Z (i.e. (11.3)) and the positivity of $\psi, M[\psi]$. Therefore, Claim 11.5 follows by continuity. Hence by the strong maximum principle, $(\eta_1, (u_1, v_1)) = (\eta_1, (0, \tilde{v}))$, $(\eta_1, (\tilde{u}, 0))$ or $(\eta_1, (0, 0))$. If $(\eta_1, (u_1, v_1)) = (\eta_1, (0, \tilde{v}))$, then $\eta_1 = \eta_*$ by uniqueness and $(\eta_*, (0, \tilde{v})) \in \mathcal{C}^+$, which is impossible by the proof of the abstract result in [**94**]. (See also Theorem 6.4.3 in [**71**].) Hence, $(\eta_1, (u_1, v_1)) = (\eta_1, (\tilde{u}, 0))$ or $(\eta_1, (0, 0))$, which implies that one of them is linearly neutrally stable. This is a contradiction to the linear instability of $(\eta, (\tilde{u}, 0))$ and $(\eta, (0, 0))$ for all $\eta \geq 0$ (Theorem 2.10).

CLAIM 11.6. $\mathcal{C}^+$ *is* (i) *unbounded;* (ii) *bounded away from* $\{(\eta, (\tilde{u}, 0)) : \eta \geq 0\}$*; and* (iii) *bounded away from* $\{(0, (u, v)) : (u, v) \in X\}$.

(i) and (ii) are already derived by the above arguments. (iii) follows from the non-existence of positive steady states when $\eta = 0$ and $\mu \neq \nu$ [**31**]. By Claim 11.6, Theorem 2.14(i) is proved. □

11.3. Bifurcation result in $\mathcal{R}_3$

One can prove the existence of an unbounded connected component of $\mathcal{S}$ emanating from $(\eta^*, (\tilde{u}, 0))$ in a similar fashion.

Proof of Theorem 2.14(iii). Since $(\tilde{u}, 0)$ depends on η, we consider instead $\tilde{F} : (0, \infty) \times X \to Y$ defined as

$$\tilde{F}(\eta, z, v) = F(\eta, \tilde{u} - z, v).$$

Now all the hypotheses of Theorems 11.1 and 11.2 are satisfied at the bifurcation point $(\eta, (z, v)) = (\eta^*, (0, 0))$, and we can repeat the proof of Theorem 2.14(i) to obtain the desired conclusion. $\square$

11.4. Bifurcation result in $\mathcal{R}_2$

Proof of Theorem 2.14(ii). Suppose Ω is convex and choose $\delta_0 > 0$ so small such that Theorem 2.11(a) holds. That is, for all $(\mu, \nu) \in \mathcal{R}_2$ satisfying $0 < \nu < \mu \leq \delta_0$, there exist $0 < \eta_1^* < \eta_2^*$ such that

$$\lambda_u(\eta, \mu, \nu) = \begin{cases} - & \text{for } \eta \in [0, \eta_1^*), \\ + & \text{for } \eta \in (\eta_1^*, \eta_2^*), \\ - & \text{for } \eta \in (\eta_2^*, \infty). \end{cases}$$

Moreover, by (A.19) and the proof of Theorem 2.11,

$$\frac{\partial \lambda_u}{\partial \eta} = \frac{\int_\Omega \frac{\partial \tilde{u}}{\partial \eta} \varphi^2}{\int_\Omega \varphi^2} > 0 \tag{11.4}$$

at $\eta = \eta_1^*$. And at $\eta = \eta_2^*$,

$$\frac{\partial \lambda_u}{\partial \eta} = \frac{\int_\Omega \frac{\partial \tilde{u}}{\partial \eta} \varphi^2}{\int_\Omega \varphi^2} < 0. \tag{11.5}$$

On the other hand, by Lemma 5.2, at any root η_* of $\lambda_v(\cdot, \mu, \nu)$,

$$\frac{\partial}{\partial \eta} \lambda_v(\eta_*, \mu, \nu) = -\mu \frac{\int_\Omega e^{\eta_* m} \psi \nabla m \cdot \nabla \psi}{\int_\Omega e^{\eta_* m} \psi^2},$$

where ψ is the principal eigenfunction corresponding to $\lambda_v(\eta_*, \mu, \nu) = 0$. Let $\nu = \mu$ and $\eta_* = 0$, then $\psi = \theta_\mu$ and by Theorem 3.10,

$$\frac{\partial \lambda_v}{\partial \eta}(0, \mu, \mu) = -\mu \frac{\int_\Omega \theta_\mu \nabla m \cdot \nabla \theta_\mu}{\int_\Omega \theta_\mu^2} < 0. \tag{11.6}$$

This and $\lambda_v(0, \mu, \mu) = 0$ imply, by the Implicit Funtion Theorem, that there are $\delta_3, \delta_4 > 0$ and a function $\eta_* = \eta_*(\nu)$, so that $\lambda_v(\eta, \mu, \nu) = 0$ for some $\eta \in (-\delta_3, \delta_3)$ if and only if $\eta = \eta_*(\nu)$ for some $\nu \in (\mu - \delta_4, \mu + \delta_4)$. Moreover, $\eta_*(\mu) = 0$. Therefore by (11.6) and continuity,

$$\frac{\partial \lambda_v}{\partial \eta}(\eta_*, \mu, \nu) < 0.$$

Now, choose δ_0 possibly smaller so that

$$\delta_0 \in \left(0, \frac{4(\min_{\bar{\Omega}} m)^3}{|\nabla m|^2_{L^\infty(\Omega)}}\right).$$

Then by Theorem 2.12(b)(ii) (see Remark 7.12), $\lambda_v(\eta,\mu,\mu)<0$ for all $\eta>0$. Hence there exists some ϵ such that $\epsilon=\epsilon(\mu)<\min_{1\le i\le 4}\{\delta_i\}$, and

$$\lambda_v(\eta,\mu,\nu)<0 \quad \text{for all } \eta\in\left[\delta_3,\frac{1}{\min_{\bar\Omega} m}\right] \text{ and } \nu\in(\mu-\epsilon(\mu),\mu].$$

Combining with Theorem 2.2,

$$\lambda_v(\eta,\mu,\nu)<0 \quad \text{for all } \eta\in[\delta_3,\infty] \text{ and } \nu\in(\mu-\epsilon(\mu),\mu]. \tag{11.7}$$

And we have proved the following.

LEMMA 11.7. *There exists a function $\epsilon:(0,\delta_0)\to(0,\delta_0)$ such that for all $(\mu,\nu)\in\mathcal{R}_2$ satisfying $\mu\in(0,\delta_0)$ and $\nu\in(\mu-\epsilon(\mu),\mu)$, then $\eta\mapsto\lambda_u(\eta,\mu,\nu)$ changes sign at exactly two values $\eta=\eta_1^*,\eta_2^*$, at which*

$$\frac{\partial\lambda_u}{\partial\eta}=\frac{\int_\Omega \frac{\partial\tilde u}{\partial\eta}\varphi^2}{\int_\Omega\varphi^2}\neq 0$$

and $\eta\mapsto\lambda_v(\eta,\mu,\nu)$ changes sign exactly once at $\eta=\eta_*$, at which

$$\frac{\partial\lambda_v}{\partial\eta}(\eta_*,\mu,\nu)=-\mu\frac{\int_\Omega e^{\eta_* m}\psi\nabla m\cdot\nabla\psi}{\int_\Omega e^{\eta_* m}\psi^2}<0.$$

By Lemma 11.7, one can check the assumptions of Theorems 11.1 and 11.2, which implies that $(\eta_i^*,(\tilde u,0))$ $(i=1,2)$ and $(\eta_*,(0,\tilde v))$ are simple bifurcation points, with a half branch of positive solutions of (11.1) emanating from each of them.

Next, we are going to see that there exists a branch of positive solutions of (11.1) connecting $(\eta_*,(0,\tilde v))$ and $(\eta_1^*,(\tilde u,0))$. By the result in Chapter 9, for each $\mu>0$, there exist $\delta_i=\delta_i(\mu)\in(0,\mu)$ $(i=1,2)$ and some functions

$$\tilde\eta=\tilde\eta(\nu,s):(\mu-\delta_2,\mu+\delta_2)\times(0,1)\to(-\delta_1,\delta_1),$$

$$\tilde y=\tilde y(\cdot;\nu,s),\tilde z=\tilde z(\cdot;\nu,s):(\mu-\delta_2,\mu+\delta_2)\times(0,1)\to C^2(\bar\Omega),$$

such that for $\nu\in(\mu-\delta_2,\mu+\delta_2)$, system (11.1) has a positive solution (u,v), for $\eta\in(-\delta_1,\delta_1)$ if and only if for some $s\in(0,1)$,

$$(\eta,u,v)=\left(\tilde\eta(\nu,s),se^{\tilde\eta(\nu,s)m}[\theta_\mu+\tilde u],(1-s)[\theta_\mu+\tilde z]\right).$$

Moreover, when $\nu=\mu$, then $\tilde y(x;\mu,s)=\tilde z(x;\mu,s)\equiv 0$, $\tilde\eta(\mu,s)\equiv 0$ and

$$\lim_{\nu\to\mu}\frac{\tilde\eta(\nu,s)}{\mu-\nu}=\frac{\int_\Omega|\nabla\theta_\mu|^2}{\int_\Omega\theta_\mu\nabla\theta_\mu\cdot\nabla m}>0.$$

This shows that

LEMMA 11.8. *For each $\mu\in(0,\delta_0)$ and $\nu\in(\mu-\delta_2(\mu),\mu)$, the following hold true.*

(i) (11.1) *has a unique branch $\mathcal{C}_{2,1}$ of positive solutions connecting $(\eta_*,(0,\tilde v))$ and $(\eta_1^*,(\tilde u,0))$.*
(ii) $\eta_*,\eta_1^*\in(0,\delta_1)$.
(iii) *$\mathcal{C}_{2,1}$ undergoes no secondary bifurcations.*
(iv) *For some $\delta_1>0$, $\mathcal{S}\cap[\{\delta_1\}\times X]=\emptyset$, where $\mathcal{S}$ denotes the set of all positive solutions of* (2.1).

For each $\mu \in (0, \delta_0)$ and $\nu \in (\mu - \delta_2(\mu), \mu)$, consider the bifurcation space

$$\{(\eta, (u, v)) : \eta \geq \delta_1, (u, v) \in X\}.$$

Then there is a unique bifurcation point $(\eta_2^*, (\tilde{u}, 0))$. Hence we can repeat the arguments in the proof of Theorem 2.14(i) to obtain the desired conclusion. $\square$

11.5. Uniqueness result for large μ, ν

In the following we assume without loss of generality that $|\Omega| = 1$ by rescaling. Theorem 2.4 is a consequence of the following result and the monotone dynamical system theory.

THEOREM 11.9. *For each $\epsilon > 0$, there exists $M > 0$ such that for $\eta \in [\epsilon, \epsilon^{-1}]$ and $\mu, \nu \geq M$, every positive solution (w, v) of (11.1), if it exists, is linearly stable. Moreover, there exists $c = c(\epsilon) > 0$ independent of μ, ν such that $u \geq c$.*

PROOF. Assume to the contrary that for some sequences $\mu_k \to \infty$, $\nu_k \to \infty$, $\eta_k \to \eta > 0$, (2.1) has a positive steady state solution (u_k, v_k) which is not linearly stable. i.e. the principal eigenvalue λ_k of the following problem is non-positive.

$$\begin{cases} \mu\nabla \cdot [e^{\eta m}\nabla\varphi] + (m - 2u - v)e^{\eta m}\varphi - u\psi + \lambda e^{\eta m}\varphi = 0 & \text{in } \Omega, \\ \nu\Delta\psi - ve^{\eta m}\varphi + (m - u - 2v)\psi + \lambda\psi = 0 & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} = \frac{\partial\psi}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{11.8}$$

where $\mu = \mu_k$, $\nu = \nu_k$, $\eta = \eta_k$, $u = u_k$, $v = v_k$, $\varphi = \varphi_k$, $\psi = \psi_k$.

CLAIM 11.10. *System* (11.8) *has a principal eigenvalue $\lambda_k \in \mathbb{R}$, such that* (i) *λ_k is simple, with an eigenfunction (φ, ψ) satisfying $\varphi < 0 < \psi$ in $\bar{\Omega}$;* (ii) *any other eigenvalue $\tilde{\lambda}$ must satisfy* Re $\tilde{\lambda} > \lambda_k$.

To see the claim, it suffices to observe that $(\hat{\varphi}, \hat{\psi}) = (-\varphi, \psi)$ satsifies a cooperative system, and Claim 11.10 follows from standard theory [**99**]. For the sake of notational simplicity we drop the index k except for λ_k.

By the proof of Lemma A.1, passing to a subsequence, there are non-negative constants C_u and C_v such that

$$\tilde{u} \to \frac{\int_\Omega me^{\eta m}}{\int_\Omega e^{2\eta m}} e^{\eta m}, \quad u = u_k \to C_u e^{\eta m}, \quad v = v_k \to C_v \quad \text{in } C^{2,\alpha}(\bar{\Omega}).$$

By integrating equations of u and v over Ω, we have

$$\int_\Omega u(m - u - v) = 0 \quad \text{and} \quad \int_\Omega v(m - u - v) = 0. \tag{11.9}$$

LEMMA 11.11. *For each η, as $\mu, \nu \to \infty$,*

$$\frac{u}{\|u\|_{L^\infty(\Omega)}} \to e^{\eta(m - \max_{\bar{\Omega}} m)}, \quad \frac{v}{\|v\|_{L^\infty(\Omega)}} \to 1$$

uniformly in Ω.

PROOF. We show $\frac{v}{\|v\|_{L^\infty(\Omega)}} \to 1$. Let $\hat{v} = v/\|v\|_{L^\infty(\Omega)}$, then $\hat{v}$ satisfies

$$\Delta\hat{v} + (\omega/\nu)\hat{v} = 0 \quad \text{in } \Omega, \quad \left.\frac{\partial\hat{v}}{\partial n}\right|_{\partial\Omega} = 0, \quad \|\hat{v}\|_{L^\infty(\Omega)} = 1,$$

where $\omega \to m - C_u e^{\eta m} - C_v$ in $L^\infty(\Omega)$. In particular, $w/\nu \to 0$ is bounded in $L^\infty(\Omega)$ for all ν large. By elliptic regularity theory, (a subsequence of) $\hat{v}$ converges to some $\hat{v}_0 \in W^{2,p}(\Omega)$ (for some $p > N$, N being the dimension of the domain Ω), weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\bar{\Omega})$. Now, $\hat{v}_0$, being the unique solution of

$$\Delta \hat{v}_0 = 0 \quad \text{in } \Omega, \quad \left.\frac{\partial \hat{v}_0}{\partial n}\right|_{\partial\Omega} = 0, \quad \|\hat{v}_0\|_{L^\infty(\Omega)} = 1,$$

must satisfy $\hat{v}_0 \equiv 1$ in Ω. This proves the convergence of $\hat{v} \to 1$. The other half of the lemma is analogous, and is omitted. □

Hence we may divide the above equations by $\|u\|_{L^\infty(\Omega)}$ and $\|v\|_{L^\infty(\Omega)}$ respectively and passing to the limit in (11.9) to get

$$\begin{pmatrix} \int_\Omega e^{2\eta m} & \int_\Omega e^{\eta m} \\ \int_\Omega e^{\eta m} & 1 \end{pmatrix} \begin{pmatrix} C_u \\ C_v \end{pmatrix} = \begin{pmatrix} \int_\Omega m e^{\eta m} \\ \int_\Omega m \end{pmatrix}. \tag{11.10}$$

Since

$$\det \begin{pmatrix} \int_\Omega e^{2\eta m} & \int_\Omega e^{\eta m} \\ \int_\Omega e^{\eta m} & 1 \end{pmatrix} = \int_\Omega e^{2\eta m} - (\int_\Omega e^{\eta m})^2 > 0,$$

we can invert in (11.10) and obtain

$$\begin{pmatrix} C_u \\ C_v \end{pmatrix} = \frac{1}{\int_\Omega e^{2\eta m} - (\int_\Omega e^{\eta m})^2} \begin{pmatrix} \int_\Omega m e^{\eta m} - \int_\Omega e^{\eta m} \int_\Omega m \\ \int_\Omega e^{2\eta m} \int_\Omega m - \int_\Omega m e^{\eta m} \int_\Omega e^{\eta m} \end{pmatrix}. \tag{11.11}$$

Remark 11.12. Before we proceed, it is important to observe that the above argument does not assume apriori that C_u and/or C_v is strictly positive. Instead, the argument asserts that any *non-negative* (subsequential) limit (C_u, C_v) necessarily satisfies (11.10) and hence (11.11).

Although we will be able to rule out $C_u = 0$ solely by formula (11.11), we will need to deal with the possibility of $C_v = 0$ carefully.

From (11.11) we can deduce the following corollary.

Corollary 11.13. *Recall that* $\eta = \lim_{k\to\infty} \eta_k$.

(i) *There exists $\delta_5 > 0$, independent of k, such that $\eta \geq \delta_5$.*
(ii) *There exists $\delta_6 = \delta_6(\epsilon) > 0$, such that $C_u \geq \delta_6$. In particular, for μ,ν large enough, there is no bifurcation from $\{(\eta,(0,\tilde{v})) : \epsilon \leq \eta \leq \epsilon^{-1}\}$.*

Proof. To prove (i), in view of (11.11), it is enough to show that there exists $\delta_5 > 0$ such that $f(\eta) := \int_\Omega e^{2\eta m} \int_\Omega m - \int_\Omega m e^{\eta m} \int_\Omega e^{\eta m} < 0$ for all $\eta \in (0, \delta_5]$. This follows from $f(0) = \int_\Omega m - \int_\Omega m = 0$ ($|\Omega| = 1$) and that (by Hölder's inequality)

$$f'(0) = \left(\int_\Omega m\right)^2 - \int_\Omega m^2 < 0.$$

To prove (ii), we let $g(\eta) = \int_\Omega m e^{\eta m} - \int_\Omega e^{\eta m} \int_\Omega m$.

Claim 11.14. *For each $\eta > 0$, $\int_\Omega m e^{\eta m} - \int_\Omega e^{\eta m} \int_\Omega m > 0$ and hence by continuity,* $\inf_{[\epsilon,\epsilon^{-1}]} g(\eta) > 0$.

The claim follows easily from (here $\bar{m} = \int_\Omega m$ as $|\Omega| = 1$)

$$g(\eta) = \int_\Omega (m - \bar{m}) \left(e^{\eta m} - e^{\eta \bar{m}}\right) > 0.$$

Hence,

$$C_u \geq \inf_{[\epsilon,\epsilon^{-1}]} \frac{g(\eta)}{\int_\Omega e^{2\eta m} - (\int_\Omega e^{\eta m})^2} > 0.$$

This proves (ii). □

Next, we study (11.8).

LEMMA 11.15. *There exists $C = C(\epsilon) > 0$ independent of μ, ν and $\eta \in [0, \epsilon^{-1}]$ such that the principal eigenvalue λ of (11.8) satisfies $\lambda \geq -C$ for any positive steady states of (2.1).*

PROOF. Multiply the second equation of (11.8) by ψ and integrate by parts, we have

$$-\nu \int_\Omega |\nabla \psi|^2 + \int_\Omega (m - u - v)\psi^2 = -\lambda \int_\Omega \psi^2 + \int_\Omega v\psi^2 + \int_\Omega v e^{\eta m} \varphi \psi. \tag{11.12}$$

By a variational argument similar to Lemma 7.8, the left hand side of (11.12) is negative, hence

$$\lambda \int_\Omega \psi^2 \geq \int_\Omega v\psi^2 + \int_\Omega v e^{\eta m} \varphi \psi. \tag{11.13}$$

Similarly, one can show

$$\lambda \int_\Omega e^{\eta m} \varphi^2 \geq \int_\Omega u e^{\eta m} \varphi^2 + \int_\Omega u \varphi \psi. \tag{11.14}$$

(Note that the integral involving $\varphi\psi$ is negative.) Adding (11.13) and (11.14), we have by Hölder's inequality,

$$\lambda \int_\Omega \left(e^{\eta m} \varphi^2 + \psi^2 \right) \geq -C(\eta, \|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}) \int_\Omega \left(e^{\eta m} \varphi^2 + \psi^2 \right).$$

Since by comparison, for any positive steady states (u, v) of (2.1), $u \leq \tilde{u}$ and $v \leq \tilde{v}$, the desired conclusion follows from the L^∞ boundedness of $\tilde{u}$ and $\tilde{v}$ on compact subsets of η in Theorems 3.12 and 3.13. $\square$

By Lemma 11.15 and $\lambda_k \leq 0$ for all k, we see that $\{\lambda_k\}$ is a bounded sequence. Hence we may assume without loss of generality that $\lambda_k \to \lambda_0 \leq 0$. Integrate (11.8) and pass to the limit, provided we normalize $\|\varphi\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\Omega)} = 1$ and $\varphi < 0 < \psi$, we have (passing to a subsequence) $\varphi \to C_\varphi$ and $\psi \to C_\psi$ for some constants C_φ and C_ψ satisfying

$$C_\varphi \leq 0 \leq C_\psi \quad \text{and} \quad |C_\varphi| + |C_\psi| = 1. \tag{11.15}$$

$$\begin{aligned} &\begin{pmatrix} \int_\Omega e^{\eta m}(m - 2C_u e^{\eta m} - C_v) & -C_u \int_\Omega e^{\eta m} \\ -C_v \int_\Omega e^{\eta m} & \int_\Omega (m - e^{\eta m} C_u - 2C_v) \end{pmatrix} \begin{pmatrix} C_\varphi \\ C_\psi \end{pmatrix} \\ &= -\lambda_0 \begin{pmatrix} C_\varphi \int_\Omega e^{\eta m} \\ C_\psi \end{pmatrix}. \end{aligned}$$

And upon substituting (11.10),

$$\lambda_0 \begin{pmatrix} C_\varphi \int_\Omega e^{\eta m} \\ C_\psi \end{pmatrix} = \begin{pmatrix} C_u \int_\Omega e^{2\eta m} / \int_\Omega e^{\eta m} & C_u \int_\Omega e^{\eta m} \\ C_v & C_v \end{pmatrix} \begin{pmatrix} C_\varphi \int_\Omega e^{\eta m} \\ C_\psi \end{pmatrix}. \tag{11.16}$$

So λ_0 is the eigenvalue of (11.16) with an eigenvector with entries of opposite sign. Hence,

$$\lambda_0 = \frac{1}{2} \left[\left(C_u \frac{\int_\Omega e^{2\eta m}}{\int_\Omega e^{\eta m}} + C_v \right) - \sqrt{\left(C_u \frac{\int_\Omega e^{2\eta m}}{\int_\Omega e^{\eta m}} + C_v \right)^2 - 4 C_u C_v \left(\frac{\int_\Omega e^{2\eta m}}{\int_\Omega e^{\eta m}} - \int_\Omega e^{\eta m} \right)} \right]. \tag{11.17}$$

And $\lambda_0 > 0$ if the product $C_u C_v > 0$. So we have a contradiction when $C_u C_v > 0$.

Therefore by Corollary 11.13, we must have $C_v = 0$. In this case, it is easy to see that then $C_u = \int_\Omega m e^{\eta m} / \int_\Omega e^{2\eta m}$ and that by (11.17), $\lambda_0 = 0$. Hence by the first equation of (11.16),

$$C_\varphi \int_\Omega e^{2\eta m} + C_\psi \int_\Omega e^{\eta m} = 0 \tag{11.18}$$

and hence (11.15) can be sharpened

$$C_\varphi < 0 < C_\psi, \qquad |C_\varphi| + |C_\psi| = 1, \tag{11.19}$$

and by $(\int_\Omega e^{\eta m})^2 \leq \int_\Omega e^{2\eta m}$,

$$|C_\varphi| \int_\Omega e^{\eta m} < |C_\psi|. \tag{11.20}$$

LEMMA 11.16. $\dfrac{v}{\|v\|_{L^\infty(\Omega)}} \to 1$ *and* $z := \dfrac{\tilde{u} - u}{\|v\|_{L^\infty(\Omega)}} e^{-\eta m} \to -\dfrac{C_\varphi}{C_\psi} \geq 0.$

PROOF. The first part of the lemma follows readily by standard elliptic estimates. Next, observe that z defined above satisfies

$$\mu \nabla \cdot (e^{\eta m} \nabla z) + e^{\eta m} z (m - u - \tilde{u}) = -\frac{uv}{\|v\|_{L^\infty(\Omega)}} \quad \text{in } \Omega, \quad \left.\frac{\partial z}{\partial n}\right|_{\partial\Omega} = 0. \tag{11.21}$$

We estimate z. Multiply (11.21) by z and integrate by parts to get

$$\begin{aligned} \int_\Omega e^{\eta m} u z^2 &\leq \int_\Omega e^{\eta m} \left[\mu |\nabla z|^2 + (u + \tilde{u} - m) z^2 \right] \\ &= \int_\Omega \frac{zuv}{\|v\|_{L^\infty(\Omega)}} \\ &\leq \left(\int_\Omega u z^2 \right)^{\frac{1}{2}} \left(\int_\Omega \frac{uv^2}{\|v\|^2_{L^\infty(\Omega)}} \right)^{\frac{1}{2}}, \end{aligned}$$

where the first inequality follows from a variational argument similar to Lemma 7.8. Since $u \to (\int_\Omega m e^{\eta m} / \int_\Omega e^{2\eta m}) e^{\eta m}$, this implies that $\|z\|_{L^2(\Omega)}$ is uniformly bounded. By elliptic estimates, $\|z\|_{W^{2,2}(\Omega)}$ is uniformly bounded and we may assume that $z \to z_0$ weakly in $W^{2,2}(\Omega)$, with z_0 satisfying (since $\mu \to \infty$)

$$\nabla \cdot (e^{\eta m} \nabla z_0) = 0 \quad \text{in } \Omega, \quad \left.\frac{\partial z_0}{\partial n}\right|_{\partial\Omega} = 0.$$

Hence $z_0 = C_z$ for some constant C_z. Now integrate (11.21), we have

$$\int_\Omega e^{\eta m} z (m - u - \tilde{u}) = -\int_\Omega u \frac{v}{\|v\|_{L^\infty(\Omega)}}.$$

Then pass to the limit,

$$\int_\Omega e^{\eta m} C_z \left(m - 2 \frac{\int_\Omega m e^{\eta m}}{\int_\Omega e^{2\eta m}} e^{\eta m} \right) = -\frac{\int_\Omega m e^{\eta m} \int e^{\eta m}}{\int_\Omega e^{2\eta m}}.$$

By cancelling on the left hand side, we have

$$-C_z \int_\Omega m e^{\eta m} = -\frac{\int_\Omega m e^{\eta m} \int_\Omega e^{\eta m}}{\int_\Omega e^{2\eta m}}.$$

This and (11.18) implies $C_z = -C_\varphi / C_\psi$. And we have proved Lemma 11.16. $\square$

Now we continue the proof of Theorem 11.9. Multiply the second equation of (11.8) by v and integrate by parts. We have

$$\begin{aligned}\int_\Omega v(-\lambda\psi + v\psi + ve^{\eta m}\varphi) &= \int_\Omega v[\nu\Delta\psi + (m-u-v)\psi] \\ &= \int_\Omega \psi[\nu\Delta v + (m-u-v)v] \\ &= 0.\end{aligned}$$

Dividing by $\|v\|^2_{L^\infty(\Omega)}$ and rearranging, we have

(11.22)
$$\frac{\lambda}{\|v\|_{L^\infty(\Omega)}}\int_\Omega \frac{v}{\|v\|_{L^\infty(\Omega)}}\psi = \int_\Omega \left[\left(\frac{v}{\|v\|_{L^\infty(\Omega)}}\right)^2\psi + \left(\frac{v}{\|v\|_{L^\infty(\Omega)}}\right)^2 e^{\eta m}\varphi\right].$$

By Lemma 11.11, $v/\|v\|_{L^\infty(\Omega)} \to 1$ uniformly, hence if we pass to the limit, we see that the right hand side, and as a consequence, the left hand side of (11.22) converges.

$$\left(\lim_{k\to\infty}\frac{\lambda}{\|v\|_{L^\infty(\Omega)}}\right)C_\psi = C_\psi + C_\varphi\int_\Omega e^{\eta m}. \tag{11.23}$$

Since $|C_\varphi| + |C_\psi| = 1$, (11.23) implies $C_\psi \neq 0$. Divide by C_ψ, we have

$$\lim_{k\to\infty}\frac{\lambda}{\|v\|_{L^\infty(\Omega)}} = 1 + \frac{C_\varphi}{C_\psi}\int_\Omega e^{\eta m},$$

where the last expression is positive, by (11.20). Hence $\lim_k \lambda/\|v\|_{L^\infty(\Omega)} > 0$. This is again a contradiction to the assumption that $\lambda = \lambda_k \leq 0$ for all k. $\square$

The following is a consequence of Corollary 11.13.

Corollary 11.17. *There exists $\delta_0 > 0$ such that for each fixed $\eta \in (0, \delta_0)$, (11.1) has no positive solutions for all μ, ν sufficiently large.*

Proof. Suppose to the contrary that for some fixed $\eta > 0$ and sequences $\mu_k, \nu_k \to \infty$, (2.1) has a positive steady state (u_k, v_k), then by the proof of Theorem 11.9, there exist non-negative constants C_u, C_v so that $u_k \to C_u e^{\eta m}$ and $v \to C_v$. Moreover, C_u, C_v satisfies (11.11). Then, by Corollary 11.13(i), necessarily $\eta \geq \delta_0$. $\square$

CHAPTER 12

Discussion and future directions

We consider a two species reaction-diffusion-advection model, where both species compete for the same resource, which is distributed unevenly in the habitat. We assume that both species have the same population dynamics but different dispersal strategies: One species diffuses randomly and the other adopts a combination of random diffusion and advection upward along the resource gradient. When the advection is weak, the species with the smaller random diffusion rate will drive the other species to extinction. If the advection is strong, two species are able to coexist as the species with strong advection will concentrate at some of the locally most favorable places and the random diffusing species will utilize resources elsewhere. In this paper we aim to understand the dynamics of the system for intermediate advection.

We first determine, for each pair of diffusion rates μ, ν, the number of stability changes for each of the two semi-trivial steady states, as $\eta \to \infty$. In general, finding these numbers depends on verifying certain non-dengeneracy conditions, in the form of integrals involving the semi-trivial steady states at the bifurcation points and the coefficients of the system. These tasks can be accomplished, for instance, (i) when the underlying spatial domain is one-dimensional and the resource function $m(x)$ is convex or concave, (ii) when the diffusion rates are both small, or (iii) when the ratio of the diffusion rates are small or large. In the course of doing so, new asymptotic estimates of the positive solution to the single semi-linear reaction-diffusion-advection equation are developed. (See Appendix A.)

Based on the number of stability changes of the semi-trivial steady states, we find that the plane of two random diffusion rates μ and ν can be partitioned into three separate regions.

Furthermore, by fixing the random diffusion rates in each of these three regions and varying the advection rate η from small to large, a distinct bifurcation diagram of positive steady states of system (2.1) is discovered for each of the three regions. By piecing these three bifurcation diagrams together we obtain a global picture on the dynamics of system (2.1) as we vary the parameters μ, ν and η in the model.

A challenging open problem is whether system (2.1) has at most one positive steady state, which has only been partially resolved in this work. A complete, affirmative answer will yield much clearer bifurcation diagrams of system (2.1). The same question can be asked for more general competition models, such as

$$(12.1) \qquad \begin{cases} u_t = \nabla \cdot (\mu \nabla u - \alpha u \nabla m_1) + u(m_1(x) - u - bv) & \text{in } \Omega \times (0, \infty), \\ v_t = \nu \Delta v + v(m_2(x) - cu - v) & \text{in } \Omega \times (0, \infty), \\ (\mu \nabla u - \alpha u \nabla m_1) \cdot n = \nabla v \cdot n = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

For recent progress on (12.1), we refer to [**19**] for the case $m_1 \equiv m_2$, $0 < b, c < 1$ and [**44, 45**] for the case $\alpha = 0$.

Another open problem is Conjecture 2.9, i.e., the changes of stability of both semi-trivial steady states.

For spatially and temporally varying environments, i.e. $m = m(x,t)$, very little is known about (2.1). We refer to [**54**] for the case $\mu \neq \nu$, $\alpha = 0$. We are not aware of any work on system (2.1) with $m = m(x,t)$, $\alpha > 0$.

Recently, system (2.1) with sign-changing environment function $m(x)$ was studied in [**70**]. It was shown that, under strong advection, the species with density u with directed advection may competitively exclude the random disperser with density v and a sharp coexistence criterion on the environment function is given. This stands in contrast to the case of positive environment treated here, where strong advection always mediates coexistence, and we expect yet different kinds of bifurcation structure of positive steady states.

It will be of interest to assume that species with density v also adopts a combination of random diffusion and advection upward along resource gradient, i.e. to consider

$$\begin{cases} u_t = \nabla \cdot (\mu \nabla u - \alpha u \nabla m) + u(m - u - v) & \text{in } \Omega \times (0, \infty), \\ v_t = \nabla \cdot (\nu \nabla v - \beta v \nabla m) + v(m - u - v) & \text{in } \Omega \times (0, \infty), \\ (\mu \nabla u - \alpha u \nabla m) \cdot n = (\nu \nabla v - \beta v \nabla m) \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.2}$$

System (12.2) has been considered in [**22, 23, 41, 64, 65**], and the stability of two semi-trivial steady states is understood to some extent. But the global bifurcation diagram and the structure of steady states of (12.2) remain largely unpursued. Two special but interesting cases are (i) $\mu = \nu$, $\alpha \neq \beta$, and (ii) $\mu \neq \nu$, $\alpha = \beta$. The global bifurcation diagrams for these two special cases are yet to be determined.

Active movement of organisms may also be biased in other directions, e.g. moving up a fitness gradient, instead of moving up the resource gradient as considered in this work. We refer to [**17, 18, 26, 28, 29, 39, 59, 60, 61, 67, 89, 74**] for some recent development on the effect of directed movement of organisms along the fitness gradient on population dynamics.

APPENDIX A

Asymptotic behavior of $\tilde{u}$ and λ_u

In this chapter, we examine the asymptotic behavior of $\tilde{u}$ and λ_u. In particular, we supply here the complete proof of the various results summarized in Chaper 3.2. We note that the results in this chapter are independent of convexity of the underlying domain Ω.

A.1. Asymptotic behavior of $\tilde{u}$ when $\mu \to \infty$

The following lemma is used in the proofs of Lemma 4.12 and Theorem 11.9.

Lemma A.1. *As $\mu \to \infty$,*

$$\tilde{u} \to \frac{\int_\Omega m e^{\eta m}}{\int_\Omega e^{2\eta m}} e^{\eta m} \qquad \text{in } C^2(\bar{\Omega}),$$

uniformly for $\eta \in [0, \frac{1}{\min_{\bar{\Omega}} m}]$.

Proof. Consider the transformation $w = e^{-\eta m}\tilde{u}$, which satisfies

$$\begin{cases} \mu \nabla \cdot (e^{\eta m} \nabla w) + e^{\eta m} w (m - e^{\eta m} w) = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.1}$$

Maximum principle gives L^∞ boundedness of w independent of μ. Divide by μ and let $\mu \to \infty$, we see (from the limiting equation $\nabla \cdot (e^{\eta m} \nabla w) = 0$ with Neumann boundary condition) that, passing to a subsequence, w converges to C in $C^2(\bar{\Omega})$ for some non-negative constant C. We first show $C > 0$. If not, i.e. $C = 0$, then $\tilde{u} \to 0$ uniformly. In particular, $\tilde{u} < \min_{\bar{\Omega}} m$ eventually. But this contradicts the following equation obtained by integrating (2.2) over Ω, namely

$$\int_\Omega \tilde{u}(m - \tilde{u}) = 0. \tag{A.2}$$

Hence $C > 0$. Taking the limit of (A.2), we obtain

$$\int_\Omega C e^{\eta m}(m - C e^{\eta m}) = 0.$$

Hence we find that

$$C = \frac{\int_\Omega m e^{\eta m}}{\int_\Omega e^{2\eta m}}.$$

This proves the lemma. $\square$

A.2. Asymptotic behavior of $\tilde{u}$ and its derivatives as $\mu \to 0$

In this section we prove the results in Section 3.2. We first prove Theorem 3.13.

PROOF OF THEOREM 3.13. We would like to show that $\min_{\bar{\Omega}} m < \tilde{v}(x) < \max_{\bar{\Omega}} m$ in $\bar{\Omega}$. Suppose first by way of contradiction that $\inf_\Omega \tilde{v} \in (0, \min_{\bar{\Omega}} m]$, then $w := \tilde{v} - \inf_\Omega \tilde{v}$ is non-negative, and satisfies

$$\begin{cases} \nu\Delta w + w(m - \tilde{v} - \inf_\Omega \tilde{v}) = -(m - \inf_\Omega \tilde{v})(\inf_\Omega \tilde{v}) \le 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

As m is non-constant, $-(m - \inf_\Omega \tilde{v})(\inf_\Omega \tilde{v})$ is not identically zero. Therefore, w is a non-negative, strict upper solution. By strong maximum principle, $w > 0$ in Ω. Since $\inf_\Omega w = 0$ (by definition), there exists $x_0 \in \partial\Omega$ such that $w(x_0) = 0 = \inf_\Omega w$. Hence $\frac{\partial w}{\partial n}(x_0) < 0$ by the Hopf Boundary Point Lemma. This is in contradiction with the boundary condition of w. The proof for $\sup_\Omega \tilde{v} < \max_{\bar{\Omega}} m$ is similar. □

Next, we prove Theorem 3.12.

PROOF THEOREM 3.12(i). Set $w = e^{-\eta m}\tilde{u}$, then w satisfies (A.1). By the maximum principle, if $w(x_1) = \max_{\bar{\Omega}} w$, then

$$w(x) \le e^{-\eta m(x_1)} m(x_1) \le \max_{\bar{\Omega}} m \quad \text{for all } x \in \Omega.$$

Similarly, $w(x) \ge \min_{\bar{\Omega}} (e^{-\eta m} m)$ for all $x \in \Omega$. Therefore, if we take

$$c = \min\left\{ \min_{\bar{\Omega}} \left(e^{-\Lambda m} m\right), \min_{\bar{\Omega}} (m^{-1}) \right\},$$

then (3.17) holds.

Lastly, $\|\tilde{u} - m\|_{L^\infty(\Omega)} \to 0$ as $\mu \to 0$ follows by applying the arguments in the Appendix of [**52**] to (A.1). □

PROOF OF THEOREM 3.12(ii). Write (2.2) as

$$-\mu\nabla \cdot \{e^{\eta m}\nabla[e^{-\eta m}(m - \tilde{u})]\} + \tilde{u}(m - \tilde{u}) = -\mu\nabla \cdot [e^{\eta m}\nabla(e^{-\eta m} m)].$$

Multiplying the above by $e^{-\eta m}(m - \tilde{u})\phi^2$, where ϕ is a given function in $H^1(\Omega)$, and integrating by parts (applying the boundary conditions of $\tilde{u}$), we deduce

$$\begin{aligned} &\mu \int e^{\eta m} |\nabla[e^{-\eta m}(m - \tilde{u})]|^2 \phi^2 + 2\mu \int \phi(m - \tilde{u})\nabla[e^{-\eta m}(m - \tilde{u})] \cdot \nabla\phi \\ &\quad \le \mu \int_{\partial\Omega} \frac{\partial}{\partial n}[e^{-\eta m} m](m - \tilde{u})\phi^2 - \mu \int \nabla \cdot [e^{\eta m}\nabla(e^{-\eta m} m)] e^{-\eta m}(m - \tilde{u})\phi^2. \end{aligned}$$

And hence by Hölder's inequality and the Trace theorem for Sobolev spaces (see, e.g. [**40**]), and also the boundedness of $\|m - \tilde{u}\|_{L^\infty(\Omega)}$,

$$\begin{aligned} &\int e^{\eta m} |\nabla[e^{-\eta m}(m - \tilde{u})]|^2 \phi^2 \\ &\le C \left[\int e^{-\eta m} |\nabla\phi|^2 (m - \tilde{u})^2 + \int \phi^2 |m - \tilde{u}| + \int_{\partial\Omega} \phi^2 |m - \tilde{u}| \right] \\ &\le C \|m - \tilde{u}\|_{L^\infty(\Omega)} \|\phi\|^2_{H^1(\Omega)}. \end{aligned}$$

This completes the proof. □

PROOF OF THEOREM 3.12(iii) AND (iv). First we prove (iii). Let $\Lambda > 0$ be given, then by (i), there exists c_Λ such that $\tilde{u} \geq c_\Lambda$ for all μ and all $\eta \in [0, \Lambda]$. By this and Young's inequality,

$$\begin{aligned}\int_\Omega |\nabla\tilde{u} - \nabla m|^2 \frac{\phi_1\phi_2}{\tilde{u}^2} &\leq \frac{1}{c_\Lambda^2}\int_\Omega |\nabla\tilde{u} - \nabla m|^2 |\phi_1\phi_2| \\ &\leq \frac{1}{2c_\Lambda^2}\int_\Omega |\nabla\tilde{u} - \nabla m|^2 (|\phi_1|^2 + |\phi_2|^2).\end{aligned}$$

Hence the result follows from (ii). (iv) is a direct consequence of (i) and (ii). □

PROOF OF THEOREM 3.12(v). Rewrite

$$\begin{aligned}&\int_\Omega (\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\nabla\left(\frac{\phi^2}{\tilde{u}}\right) \\ &= \int_\Omega (\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\left(\frac{2\phi\nabla\phi}{\tilde{u}} - \frac{\phi^2\nabla\tilde{u}}{\tilde{u}^2}\right) \\ &= \int_\Omega (\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\frac{2\phi\nabla\phi}{\tilde{u}} + \int_\Omega \eta\frac{\phi^2}{\tilde{u}}\nabla m\cdot\nabla\tilde{u} - \int_\Omega \frac{\phi^2}{\tilde{u}^2}|\nabla\tilde{u}|^2 \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

We shall estimate I_i, $i = 1, 2, 3$ separately using Theorem 3.12(ii). C below represents some generic constant independent of ϵ and μ small.

$$\begin{aligned}I_1 &= \textstyle\int_\Omega (1-\eta\tilde{u})\nabla m\cdot\frac{2\phi\nabla\phi}{\tilde{u}} + \int_\Omega (\nabla\tilde{u} - \nabla m)\cdot\frac{2\phi\nabla\phi}{\tilde{u}} \\ &\leq \left(\tfrac{\epsilon}{9}\textstyle\int_\Omega |\nabla\phi|^2 + \frac{9}{\epsilon}\int_\Omega (1-\eta m)^2|\nabla m|^2\frac{\phi^2}{\tilde{u}}\right) + \left(\tfrac{\epsilon}{9}\textstyle\int_\Omega |\nabla\phi|^2 + \frac{9}{\epsilon}\int_\Omega |\nabla\tilde{u} - \nabla m|^2\phi^2\right) \\ &\leq \tfrac{2\epsilon}{9}\textstyle\int_\Omega |\nabla\phi|^2 + \frac{C}{\epsilon}\int_\Omega \phi^2 + \frac{C}{\epsilon}\|\tilde{u} - m\|_{L^\infty(\Omega)}\|\phi\|^2_{H^1(\Omega)} \\ &\leq \tfrac{\epsilon}{3}\textstyle\int_\Omega |\nabla\phi|^2 + \frac{C}{\epsilon}\int_\Omega \phi^2.\end{aligned}$$

$$\begin{aligned}I_2 &= \textstyle\int_\Omega \eta\frac{\phi^2}{\tilde{u}}|\nabla m|^2 + \int_\Omega \eta\frac{\phi^2}{\tilde{u}}\nabla m\cdot(\nabla\tilde{u} - \nabla m) \\ &\leq C\left[\textstyle\int_\Omega \phi^2 + \int_\Omega |\nabla\tilde{u} - \nabla m|\phi^2\right] \\ &\leq C\left[\textstyle\int_\Omega \phi^2 + \|\tilde{u} - m\|_{L^\infty(\Omega)}\|\phi\|^2_{H^1(\Omega)}\right] \\ &\leq \tfrac{\epsilon}{3}\textstyle\int_\Omega |\nabla\phi|^2 + C\int_\Omega \phi^2 \quad \text{since } \|\tilde{u} - m\|_{L^\infty} \to 0 \text{ as } \mu \to 0.\end{aligned}$$

$$\begin{aligned}I_3 &\leq C\textstyle\int_\Omega \phi^2|\nabla\tilde{u}|^2 \\ &= C\textstyle\int_\Omega \phi^2\left[|\nabla\tilde{u} - \nabla m|^2 + |\nabla m|^2 + 2\nabla m\cdot(\nabla\tilde{u} - \nabla m)\right] \\ &\leq 2C\textstyle\int_\Omega |\nabla\tilde{u} - \nabla m|\phi^2 + C\int_\Omega \phi^2 \\ &\leq C\|\tilde{u} - m\|_{L^\infty(\Omega)}\|\phi\|^2_{H^1(\Omega)} + C\textstyle\int_\Omega \phi^2 \\ &\leq \tfrac{\epsilon}{3}\textstyle\int_\Omega |\nabla\phi|^2 + C\int_\Omega \phi^2 \quad \text{since } \|\tilde{u} - m\|_{L^\infty} \to 0 \text{ as } \mu \to 0.\end{aligned}$$

Combining the above, we obtain the lemma. □

Next, we prove Theorem 3.14 concerning $\frac{\partial\tilde{u}}{\partial\eta}$.

PROOF OF THEOREM 3.14. Denote $\tilde{u}' = \frac{\partial\tilde{u}}{\partial\eta}$. Then $\tilde{u}'$ satisfies

$$\begin{cases}\mu\nabla\cdot(\nabla\tilde{u}' - \eta\tilde{u}'\nabla m) + (m - 2\tilde{u})\tilde{u}' = \mu\nabla\cdot(\tilde{u}\nabla m) & \text{in } \Omega, \\ \frac{\partial\tilde{u}'}{\partial n} - \eta\tilde{u}'\frac{\partial m}{\partial n} = \tilde{u}\frac{\partial m}{\partial n} & \text{on } \partial\Omega.\end{cases} \tag{A.3}$$

We may rewrite the equation as

$$\mu\nabla\cdot[e^{\eta m}\nabla(e^{-\eta m}\tilde{u}')] + (m - 2\tilde{u})\tilde{u}' = \mu\nabla\cdot(\tilde{u}\nabla m). \tag{A.4}$$

Multiplying by $-e^{-\eta m}\tilde{u}'$ and integrating by parts, then by the boundary condition, the boundary integrals cancel out and we obtain

$$\mu\int_\Omega e^{\eta m}|\nabla(e^{-\eta m}\tilde{u}')|^2+\int_\Omega(2\tilde{u}-m)e^{-\eta m}(\tilde{u}')^2=\mu\int_\Omega\tilde{u}\nabla m\cdot\nabla(e^{-\eta m}\tilde{u}'). \tag{A.5}$$

Applying Hölder's inequality to the right hand side, we have

$$\begin{aligned}\mu\int_\Omega e^{\eta m}|\nabla(e^{-\eta m}\tilde{u}')|^2&+\int_\Omega(2\tilde{u}-m)e^{-\eta m}(\tilde{u}')^2\\&\le\frac{\mu}{2}\int_\Omega e^{\eta m}|\nabla(e^{-\eta m}\tilde{u}')|^2+\frac{\mu}{2}\int_\Omega e^{-\eta m}\tilde{u}^2|\nabla m|^2,\end{aligned}$$

and hence

$$\frac{\mu}{2}\int_\Omega e^{\eta m}|\nabla(e^{-\eta m}\tilde{u}')|^2+\int_\Omega(2\tilde{u}-m)e^{-\eta m}(\tilde{u}')^2\le\frac{\mu}{2}\int_\Omega e^{-\eta m}\tilde{u}^2|\nabla m|^2.$$

Since $2\tilde{u}-m\to m$ uniformly by Theorem 3.12(i), both terms on the left are positive, and

$$\|\nabla(e^{-\eta m}\tilde{u}')\|_{L^2(\Omega)}=O(1)\quad\text{and}\quad\|e^{-\eta m}\tilde{u}'\|_{L^2(\Omega)}=O(\mu).$$

Therefore, $e^{-\eta m}\tilde{u}'\to 0$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Applying this to (A.5), we deduce that $e^{-\eta m}\tilde{u}'\to 0$ strongly in $H^1(\Omega)$. This can be seen from the fact that (by Theorem 3.12(i) and weak convergence of $e^{-\eta m}\tilde{u}'$ in H^1) the left hand side of (A.5) is of order $o(\mu)$, which implies that

$$\mu\int_\Omega e^{\eta m}|\nabla(e^{-\eta m}\tilde{u}')|^2+\int_\Omega(2\tilde{u}-m)e^{-\eta m}(\tilde{u}')^2=o(\mu). \tag{A.6}$$

And we see that $\|e^{-\eta m}\tilde{u}'\|_{H^1(\Omega)}\to 0$ and $\int_\Omega(\tilde{u}')^2=o(\mu)$ as $\mu\to 0$ uniformly for $\eta\in[0,\Lambda]$. In particular, given any $\phi^*=\phi^*(\mu)$ such that $\phi^*\rightharpoonup\tilde{\phi}^*$ weakly in H^1,

$$\int_\Omega(\nabla\tilde{u}'-\eta\tilde{u}'\nabla m)\cdot\nabla\phi^*=\int_\Omega e^{\eta m}\nabla(e^{-\eta m}\tilde{u}')\cdot\nabla\phi^*\to 0. \tag{A.7}$$

Next, suppose $\phi\rightharpoonup\tilde{\phi}$ weakly in H^1 as $\mu\to 0$, we claim that

$$\frac{\phi}{m-2\tilde{u}}\rightharpoonup-\frac{\tilde{\phi}}{m}\quad\text{weakly in } H^1 \text{ as } \mu\to 0. \tag{A.8}$$

To prove (A.8), firstly we observe that $\frac{\phi}{m-2\tilde{u}}\to-\frac{\tilde{\phi}}{m}$ in L^2. Secondly,

$$\begin{aligned}\nabla\left(\frac{\phi}{m-2\tilde{u}}\right)&=\frac{\nabla\phi}{m-2\tilde{u}}-\frac{\phi}{(m-2\tilde{u})^2}(\nabla m-2\nabla\tilde{u})\\&=\left[\frac{\nabla\phi}{m-2\tilde{u}}+\frac{\phi\nabla m}{(m-2\tilde{u})^2}\right]+\frac{2\phi}{(m-2\tilde{u})^2}(\nabla\tilde{u}-\nabla m)\\&\rightharpoonup\left[-\frac{\nabla\tilde{\phi}}{m}+\frac{\tilde{\phi}\nabla m}{m^2}\right]+0\quad\text{weakly in } L^2,\end{aligned}$$

where the convergence in the square bracket follows from Theorem 3.12(i) and the assumption that $\phi\rightharpoonup\tilde{\phi}$ weakly in H^1, whereas the last term converges strongly to 0 in L^2 by Theorem 3.12(i) and (ii).

Now, multiplying (A.4) by $\frac{\phi}{\mu(m-2\tilde{u})}$, and integrating, we have

$$\int_\Omega\frac{\phi}{(m-2\tilde{u})}\nabla\cdot[e^{\eta m}\nabla(e^{-\eta m}\tilde{u}')]+\frac{1}{\mu}\int_\Omega\tilde{u}'\phi=\int_\Omega\frac{\phi}{(m-2\tilde{u})}\nabla\cdot(\tilde{u}\nabla m).$$

Integrating by parts, noting again that the boundary terms cancel exactly by the boundary conditions of $\tilde{u}'$,

$$-\int_\Omega e^{\eta m}\nabla(e^{-\eta m}\tilde{u}')\cdot\nabla\left(\frac{\phi}{(m-2\tilde{u})}\right)+\frac{1}{\mu}\int_\Omega \tilde{u}'\phi = -\int_\Omega \tilde{u}\nabla m\cdot\nabla\left(\frac{\phi}{(m-2\tilde{u})}\right).$$

Passing to the limit, we have by (A.7),

$$\lim_{\mu\to 0}\frac{1}{\mu}\int_\Omega \tilde{u}'\phi = \int_\Omega m\nabla m\cdot\nabla\left(\frac{\tilde{\phi}}{m}\right)$$

uniformly for $\eta \in [0,\Lambda]$. □

Next, we prove Theorem 3.15 concerning $\frac{\partial^2\tilde{u}}{\partial\eta^2}$.

Proof of Theorem 3.15. Denote $\frac{\partial\tilde{u}}{\partial\eta}=\tilde{u}'$, $\frac{\partial^2\tilde{u}}{\partial\eta^2}=\tilde{u}''$ and

$$\left(\frac{\partial^2\tilde{u}}{\partial\eta^2}\right)_+ = \max\left\{0,\frac{\partial^2\tilde{u}}{\partial\eta^2}\right\} = \tilde{u}''_+.$$

Differentiate the equation of $\tilde{u}'$, namely (A.3), with respect to η to obtain

$$\begin{cases} \mu\nabla\cdot(\nabla\tilde{u}''-\eta\tilde{u}''\nabla m)+(m-2\tilde{u})\tilde{u}'' = 2\mu\nabla\cdot(\tilde{u}'\nabla m)+2(\tilde{u}')^2 & \text{in } \Omega,\\ \frac{\partial\tilde{u}''}{\partial n}-\eta\tilde{u}''\frac{\partial m}{\partial n} = 2\tilde{u}'\frac{\partial m}{\partial n} & \text{on } \partial\Omega. \end{cases} \tag{A.9}$$

Rewriting (A.9) in variational form, we have

$$\mu\nabla\cdot[e^{\eta m}\nabla(e^{-\eta m}\tilde{u}'')]+(m-2\tilde{u})\tilde{u}'' = 2\mu\nabla\cdot(\tilde{u}'\nabla m)+2(\tilde{u}')^2. \tag{A.10}$$

Multiply (A.10) by $e^{-\eta m}\tilde{u}''_+ \in H^1(\Omega)$. Using the boundary condition satisfied by $\tilde{u}''$, we may integrate by parts to obtain

$$\begin{aligned}
&\mu\int_\Omega e^{\eta m}|\nabla(e^{-\eta m}\tilde{u}''_+)|^2+\int_\Omega(2\tilde{u}-m)e^{-\eta m}(\tilde{u}''_+)^2\\
&=2\mu\int_\Omega \tilde{u}'\nabla m\cdot\nabla[e^{-\eta m}\tilde{u}''_+]-2\int_\Omega(\tilde{u}')^2\tilde{u}''_+e^{-\eta m}\\
&\le 2\mu\int_\Omega \tilde{u}'\nabla m\cdot\nabla[e^{-\eta m}\tilde{u}''_+].
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\mu\int_\Omega e^{\eta m}|\nabla(e^{-\eta m}\tilde{u}''_+)|^2+\int_\Omega(2\tilde{u}-m)e^{-\eta m}(\tilde{u}''_+)^2\\
&\le\frac{\mu}{2}\int_\Omega e^{\eta m}|\nabla(e^{-\eta m}\tilde{u}''_+)|^2+2\mu\int_\Omega(\tilde{u}')^2|\nabla m|^2e^{-\eta m}.
\end{aligned}$$

Hence

$$\frac{\mu}{2}\int_\Omega e^{\eta m}|\nabla(e^{-\eta m}\tilde{u}''_+)|^2+\int_\Omega(2\tilde{u}-m)e^{-\eta m}(\tilde{u}''_+)^2 \le C\mu\int_\Omega(\tilde{u}')^2 = o(\mu^2).$$

The last estimate follows from (A.6). Since $2\tilde{u}-m\to m$ in $L^\infty(\Omega)$ (Theorem 3.12(iv)), the result is proved. □

A.2.1. Asymptotic behavior of λ_u as $\mu \to 0$. To prepare for the proof of Theorem 3.16, we first prove a series of lemmas.

LEMMA A.2. *For each $k \geq 1$, $d > 0$ and $\Lambda > 0$,*

$$\limsup_{\frac{\nu}{\mu} \to d, \mu \to 0} \frac{\lambda_{u,k}}{\mu} \leq \sigma_k(\eta; d)$$

uniformly in $\eta \in [0, \Lambda]$. In particular, for each k, $\frac{\lambda_{u,k}}{\mu}$ is uniformly bounded for all small μ.

PROOF. Define

$$\begin{aligned} J_\mu(\phi) &= \frac{\int \frac{\nu}{\mu}|\nabla\phi|^2 - \int(\nabla\tilde{u} - \eta\tilde{u}\nabla m) \cdot \nabla\left(\frac{\phi^2}{\tilde{u}}\right)}{\int \phi^2} \\ &= \frac{1}{\int \phi^2}\left\{\int_\Omega \frac{\nu}{\mu}|\nabla\phi|^2 - \int_\Omega \left[(\nabla\tilde{u} - \eta\tilde{u}\nabla m) \cdot \frac{\nabla(\phi^2)}{\tilde{u}} + \eta\frac{\phi^2 \nabla\tilde{u} \cdot \nabla m}{\tilde{u}}\right.\right. \\ &\qquad \left.\left. - \frac{\phi^2}{\tilde{u}^2}\left(|\nabla\tilde{u} - \nabla m|^2 - |\nabla m|^2 + 2\nabla m \cdot \nabla\tilde{u}\right)\right]\right\}. \end{aligned}$$

Then by variational characterization,

$$\frac{\lambda_{u,k}}{\mu} = \inf \max J_\mu(\phi), \tag{A.11}$$

where the maximum is taken over a given $k-$dimensional subspace of $C^1(\bar{\Omega})$ with the infimum being taken over all such subspaces. Similarly, let

$$\begin{aligned} J_0(\phi) &= \frac{\int d|\nabla\phi|^2 - \int(1 - \eta m)\nabla m \cdot \nabla\left(\frac{\phi^2}{m}\right)}{\int \phi^2} \\ &= \frac{\int d|\nabla\phi|^2 - \int \left[(1 - \eta m)\frac{\nabla m \cdot \nabla(\phi^2)}{m} + \eta\frac{\phi^2|\nabla m|^2}{m} - \frac{\phi^2}{m^2}|\nabla m|^2\right]}{\int \phi^2}. \end{aligned}$$

Then the kth eigenvalue $\sigma_k = \sigma_k(\eta; d)$ of (3.19) satisfies the variational characterization

$$\sigma_k = \inf \max J_0(\phi), \tag{A.12}$$

where the maximum is taken over a given $k-$dimensional subspace of $C^1(\bar{\Omega})$ with the infimum being taken over all such subspaces. Note that the principal eigenvalue $\sigma = \sigma_1$ is simple, hence

$$\sigma_2 > \sigma_1 \quad \text{for all } d, \eta. \tag{A.13}$$

Further properties of $\sigma = \sigma_1$ will be proved in Appendix B.

For any (fixed) $\phi \in C^1(\bar{\Omega})$, one can show by Theorem 3.12(i) and (ii) that $J_\mu(\phi) \to J_0(\phi)$ uniformly for $\eta \in [0, \Lambda]$ as $\mu \to 0$ and $\nu/\mu \to d$. Hence fix any k, and for each k-dimensional subspace Y of $C^1(\bar{\Omega})$,

$$\limsup_{\frac{\nu}{\mu} \to d, \mu \to 0} \frac{\lambda_{u,k}}{\mu} \leq \lim_{\frac{\nu}{\mu} \to d, \mu \to 0} \max_{\phi \in Y} J_\mu(\phi) = \max_{\phi \in Y} J_0(\phi).$$

Taking infimum over all k-dimensional subspaces Y of $C^1(\bar{\Omega})$, we deduce

$$\limsup_{\frac{\nu}{\mu} \to d, \mu \to 0} \frac{\lambda_{u,k}}{\mu} \leq \sigma_k(\eta; d), \tag{A.14}$$

uniformly for $\eta \in [0, \Lambda]$. □

Next, we prove an estimate of φ_k in a space slightly stronger than $H^1(\Omega)$.

LEMMA A.3. *For any $p \geq 1$, $\Lambda > 0$ and $\epsilon_0 > 0$, there exists $C > 0$ such that*

$$\limsup_{\mu \to 0, \frac{\nu}{\mu} \geq \epsilon_0, 0 \leq \eta \leq \Lambda} \frac{\int_\Omega |\nabla \varphi_k^p|^2}{\int_\Omega |\varphi_k^p|^2} \leq \frac{C}{\epsilon_0^2}$$

for all $\eta \in [0, \Lambda]$.

PROOF. Multiplying both sides of (3.18) by φ_k^{2p-1} and integrating,

$$\begin{aligned}
&\frac{\nu}{\mu} \frac{2p-1}{p^2} \int |\nabla \varphi_k^p|^2 \\
=& - \int \frac{\varphi_k^{2p}}{\tilde{u}} \nabla \cdot (\nabla \tilde{u} - \eta \tilde{u} \nabla m) + \frac{\lambda_{u,k}}{\mu} \int \varphi_k^{2p} \\
=& \int (\nabla \tilde{u} - \eta \tilde{u} \nabla m) \cdot \nabla \left(\frac{\varphi_k^{2p}}{\tilde{u}} \right) + \frac{\lambda_{u,k}}{\mu} \int \varphi_k^{2p} \\
=& \int (\nabla \tilde{u} - \eta \tilde{u} \nabla m) \cdot \left(\frac{2\varphi_k^p \nabla \varphi_k^p}{\tilde{u}} - \frac{\varphi_k^{2p} \nabla \tilde{u}}{\tilde{u}^2} \right) + \frac{\lambda_{u,k}}{\mu} \int \varphi_k^{2p} \\
=& 2 \int \frac{\varphi_k^p \nabla \varphi_k^p \cdot \nabla \tilde{u}}{\tilde{u}} - \int \frac{\varphi_k^{2p} |\nabla \tilde{u}|^2}{\tilde{u}^2} - 2\eta \int \varphi_k^p \nabla m \cdot \nabla \varphi_k^p + \eta \int \frac{\nabla m \cdot \nabla \tilde{u}}{\tilde{u}} \varphi_k^{2p} \\
&\quad + \frac{\lambda_{u,k}}{\mu} \int \varphi_k^{2p} \\
\leq& \left(\delta \int |\nabla \varphi_k^p|^2 + \frac{1}{\delta} \int \frac{\varphi_k^{2p} |\nabla \tilde{u}|^2}{\tilde{u}^2} \right) - \int \frac{\varphi_k^{2p} |\nabla \tilde{u}|^2}{\tilde{u}^2} + \left(\delta \int |\nabla \varphi_k^p|^2 + \frac{\eta^2}{\delta} \int \varphi_k^{2p} |\nabla m|^2 \right) \\
&\quad + \left(\int \frac{\varphi_k^{2p} |\nabla \tilde{u}|^2}{\tilde{u}^2} + \frac{\eta^2}{4} \int \varphi_k^{2p} |\nabla m|^2 \right) + C \int \varphi_k^{2p}.
\end{aligned}$$

Lemma A.2 is used in the last inequality for boundedness of $\lambda_{u,k}/\mu_k$. Hence, if we take $0 < \delta < \frac{\epsilon_0}{3} \cdot \frac{2p-1}{p^2}$, then

$$\frac{2p-1}{3p^2} \frac{\nu}{\mu} \int_\Omega |\nabla \varphi_k^p|^2 \leq \left(\frac{2p-1}{p^2} \frac{\nu}{\mu} - 2\delta \right) \int |\nabla \varphi_k^p|^2 \leq C \left(\int \varphi_k^{2p} |\nabla \tilde{u}|^2 + \int \varphi_k^{2p} \right) / \epsilon.$$

Apply Theorem 3.12(ii), with $\phi = \varphi_k^p$. Then we have

$$\begin{aligned}
&\sqrt{\frac{2p-1}{3p^2} \epsilon_0} \|\nabla \varphi_k^p\|_{L^2(\Omega)} \\
\leq& \sqrt{\frac{2p-1}{3p^2} \frac{\nu}{\mu}} \|\nabla \varphi_k^p\|_{L^2(\Omega)} \\
\leq& C \left(\|\varphi_k^p \nabla \tilde{u}\|_{L^2(\Omega)} + \|\varphi_k^p\|_{L^2(\Omega)} \right) / \sqrt{\epsilon_0} \\
\leq& C \left(\|\varphi_k^p \nabla (\tilde{u} - m)\|_{L^2(\Omega)} + \|\varphi_k^p \nabla m\|_{L^2(\Omega)} + \|\varphi_k^p\|_{L^2(\Omega)} \right) / \sqrt{\epsilon_0} \\
\leq& C \left[\|m - \tilde{u}\|_{L^\infty(\Omega)} \left(\|\varphi_k^p\|_{L^2(\Omega)} + \|\nabla \varphi_k^p\|_{L^2(\Omega)} \right) + \|\varphi_k^p\|_{L^2(\Omega)} \right] / \sqrt{\epsilon_0}.
\end{aligned}$$

The lemma thus follows from Theorem 3.12(i). □

An immediate consequence is the following: For each $k \geq 1$, normalize $\int_\Omega \varphi_k^4 = |\Omega|$. By passing to a subsequence, we may assume that

$$\varphi_k \to \tilde{\varphi}_k, \quad \varphi_k^2 \to \tilde{\varphi}_k^2 \quad \text{weakly in } H^1(\Omega),$$

for some non-zero $\tilde{\varphi}_k$. In particular, the precompactness of $\{\varphi_k^2\}$ in $L^2(\Omega)$ ensures that $\int_\Omega \varphi_k^4/(\int_\Omega \varphi_k^2)^2$ remains bounded. Hence, it is sufficient to normalize either $\int_\Omega \varphi_k^2$ or $\int_\Omega \varphi_k^4$ to conclude the convergence of φ_k and φ_k^2 in $H^1(\Omega)$, and we have the following result.

COROLLARY A.4. *For each $k \geq 1$, normalize φ_k such that $\int_\Omega \varphi_k^2 = |\Omega|$. By passing to a subsequence, we may assume that φ_k and φ_k^2 converge weakly in $H^1(\Omega)$ to some nonzero limits $\tilde{\varphi}_k$ and $\tilde{\varphi}_k^2$ locally uniformly in $\eta \geq 0$.*

PROOF. Let φ_k be normalized by $\int_\Omega \varphi_k^2 = |\Omega|$. By taking $p = 1$ in Lemma A.3, it is clear that φ_k is bounded in $H^1(\Omega)$. It remains to show that φ_k^2 is also bounded in $H^1(\Omega)$. We first claim that

$$\sup_k \int_\Omega \varphi_k^4 < +\infty. \tag{A.15}$$

Suppose not, then passing to a subsequence, we may assume that $\int_\Omega \varphi_k^4 \to \infty$. Then

$$\psi_k := \left(\frac{|\Omega|}{\int_\Omega \varphi_k^4} \right)^{1/4} \varphi_k$$

satisfies $\int_\Omega \psi_k^4 = |\Omega|$, and

$$\int_\Omega \psi_k^2 = \left(\frac{|\Omega|}{\int_\Omega \varphi_k^4} \right)^{1/2} \int_\Omega \varphi_k^2 = \frac{|\Omega|^{3/2}}{(\int_\Omega \varphi_k^4)^{1/2}} \to 0. \tag{A.16}$$

However, by Lemma A.3 (taking $p = 2$), ψ_k^2 is bounded in $H^1(\Omega)$ and hence for some $\psi_0^2 \in H^1(\Omega)$, $\psi_k^2 \to \psi_0^2$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Moreover, $\int_\Omega \psi_0^4 = |\Omega|$ so $\psi_0^2 \not\equiv 0$. So, $\int_\Omega \psi_k^2 \to \int_\Omega \psi_0^2 > 0$. This contradicts (A.16). This establishes (A.15). By (A.15) and Lemma A.3 (taking $p = 2$), we deduce that φ_k^2 is bounded in $H^1(\Omega)$. This completes the proof. □

For later convenience, we show the following application of Theorem 3.12(ii).

LEMMA A.5. *For all $\Lambda > 0$ and $\rho \in C^1(\bar{\Omega})$, and $\phi \rightharpoonup \tilde{\phi}$ in $H^1(\Omega)$,*

$$\int_\Omega \frac{\phi\rho}{\tilde{u}^2} |\nabla \tilde{u}|^2 \to \int_\Omega \frac{\tilde{\phi}\rho}{m^2} |\nabla m|^2 \text{ as } \mu \to 0,$$

uniformly for $\eta \in [0, \Lambda]$.

PROOF. Write

$$\int_\Omega \frac{\phi\rho}{\tilde{u}^2} |\nabla \tilde{u}|^2 = \int_\Omega \frac{\phi\rho}{\tilde{u}^2} \left(|\nabla \tilde{u} - \nabla m|^2 - |\nabla m|^2 + 2\nabla m \cdot \nabla \tilde{u} \right).$$

And the result follows from Theorem 3.12(ii), (iv) and Corollary A.4. □

Now we are in position to prove Theorem 3.16.

Proof of Theorem 3.16. We shall pass to a the limit (via a subsequence) in (3.1) using the weak formulation. Let $\rho \in C^1(\bar{\Omega})$ be a test function and consider a subsequence (μ_k, ν_k) satisfying

$$\frac{\lambda_{u,k}}{\mu} \to \liminf_{\mu\to 0, \nu/\mu \to d} \frac{\lambda_{u,k}}{\mu}.$$

Multiplying (3.1) by ρ/μ and integrating over Ω, we have

$$\int_\Omega \frac{\rho}{\mu}\left[\nu\Delta\varphi_k - \frac{\mu\nabla\cdot(\nabla\tilde{u} - \eta\tilde{u}\nabla m)}{\tilde{u}}\varphi_k + \lambda_{u,k}\varphi_k\right] = 0.$$

Integrating by parts,

$$-\frac{\nu}{\mu}\int_\Omega \nabla\varphi_k\cdot\nabla\rho + \int_\Omega(\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\nabla\left(\frac{\varphi_k\rho}{\tilde{u}}\right) + \frac{\lambda_{u,k}}{\mu}\int_\Omega\varphi_k\rho = 0. \tag{A.17}$$

Passing to the limit as $\mu \to 0$, $\frac{\nu}{\mu} \to d > 0$ (via the normalization given in Corollary A.4), the first term obviously converges, with

$$-\frac{\nu}{\mu}\int_\Omega \nabla\varphi_k\cdot\nabla\rho \to -d\int_\Omega \nabla\tilde{\varphi}_k\cdot\nabla\rho.$$

We claim that the second term also converges, as it can be rewritten as

$$\int_\Omega\left[(\nabla\tilde{u} - \eta\tilde{u}\nabla m)\cdot\left(\frac{\rho\nabla\varphi_k + \varphi_k\nabla\rho}{\tilde{u}}\right) + \eta\frac{\varphi_k\rho}{\tilde{u}}\nabla m\cdot\nabla\tilde{u} - \frac{\varphi_k\rho}{\tilde{u}^2}|\nabla\tilde{u}|^2\right],$$

which converges to (making use of Lemma A.5)

$$\begin{aligned}&\int_\Omega\left[(1-\eta m)\nabla m\cdot\left(\frac{\rho\nabla\tilde{\varphi}_k + \tilde{\varphi}_k\nabla\rho}{m}\right) + \eta\frac{\tilde{\varphi}_k\rho}{m}|\nabla m|^2 - \frac{\tilde{\varphi}_k\rho}{m^2}|\nabla m|^2\right]\\ &\quad = \int_\Omega(1-\eta m)\nabla m\cdot\nabla\left(\frac{\tilde{\varphi}_k\rho}{m}\right).\end{aligned}$$

Hence the remaining term of (A.17) also converges. Then for all $\rho \in C^1(\bar{\Omega})$,

$$-d\int_\Omega\nabla\tilde{\varphi}_k\cdot\nabla\rho + \int_\Omega(1-\eta m)\nabla m\cdot\nabla\left(\frac{\tilde{\varphi}_k\rho}{m}\right) + \left(\liminf_{\mu\to 0,\nu/\mu\to d}\frac{\lambda_{u,k}}{\mu}\right)\int_\Omega(\tilde{\varphi}_k\rho) = 0.$$

For $k = 1$, $\tilde{\varphi}_1$ is a non-trivial non-negative solution of

$$\begin{cases} d\Delta\tilde{\varphi}_1 - \frac{\nabla\cdot[(1-\eta m)\nabla m]}{m}\tilde{\varphi}_1 + \left(\liminf_{\mu\to 0,\nu/\mu\to d}\frac{\lambda_{u,1}}{\mu}\right)\tilde{\varphi}_1 = 0 & \text{in } \Omega,\\ d\frac{\partial\tilde{\varphi}_1}{\partial n} - \frac{1-\eta m}{m}\frac{\partial m}{\partial n}\tilde{\varphi}_1 = 0 & \text{on } \partial\Omega.\end{cases}$$

This implies that

$$\liminf_{\mu\to 0,\nu/\mu\to d}\frac{\lambda_{u,1}(\eta,\mu,\nu)}{\mu} = \sigma_1(\eta; d)$$

uniformly in $\eta \in [0, \Lambda]$. Here σ_1 is the principal eigenvalue of (3.19). This and Lemma A.2 proves the lemma for $k = 1$. Next, for $k = 2$, we observe similarly that $\tilde{\varphi}_2$ is an eigenfunction of (3.19) with eigenvalue given by $\liminf_{\mu\to 0,\nu/\mu\to d}\frac{\lambda_{u,2}}{\mu}$, satisfying $\int_\Omega\tilde{\varphi}_2\tilde{\varphi}_1 = 0$. So by variational characterization, $\liminf_{\mu\to 0,\nu/\mu\to d}\frac{\lambda_{u,2}}{\mu} \geq \sigma_2$. Upon combining with Lemma A.2, we have

$$\lim_{\mu\to 0,\nu/\mu\to d}\frac{\lambda_{u,2}}{\mu} = \sigma_2(\cdot; d)$$

locally uniformly in η. The remaining cases ($k \geq 3$) can be treated inductively. $\square$

Proof of Theorem 3.17. Similar to the proof of Lemma A.6, differentiating (3.18) with respect to η, denoting $\varphi' = \frac{\partial \varphi}{\partial \eta}$ and $\lambda_u' = \frac{\partial \lambda_u}{\partial \eta}$, we have

$$\text{(A.18)}\qquad \begin{cases} \nu\Delta\varphi' + (m-\tilde{u})\varphi' + \lambda_u\varphi' = \frac{\partial \tilde{u}}{\partial \eta}\varphi - \lambda_u'\varphi & \text{in } \Omega, \\ \frac{\partial \varphi'}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying by the principal eigenfunction φ of (3.1) and integrating by parts, we have

$$0 = \int_\Omega \varphi' \left[\nu\Delta\varphi + (m-\tilde{u})\varphi + \lambda_u\varphi\right] = \int_\Omega \frac{\partial \tilde{u}}{\partial \eta}\varphi^2 - \lambda_u' \int_\Omega \varphi^2.$$

And hence we obtain

$$\text{(A.19)}\qquad \frac{\lambda_u'}{\mu} = \frac{\int_\Omega \frac{\partial \tilde{u}}{\partial \eta}\varphi^2}{\mu \int_\Omega \varphi^2}.$$

By Theorem 3.14, we can deduce that as $\mu \to 0$,

$$\text{(A.20)}\qquad \frac{\lambda_u'}{\mu} \to \frac{\int_\Omega m\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}^2}{m}\right)}{\int_\Omega \tilde{\varphi}^2},$$

uniformly for $\eta \in [0, \Lambda]$. Here $\tilde{\varphi}$ is the weak limit of φ in $H^1(\Omega)$ and is a positive eigenfunction of (3.19). Consequently, the result follows from the following representation formula of $\frac{\partial \sigma}{\partial \eta}$: (Here we denote the principal eigenvalue of (3.19) by $\sigma = \sigma(\eta; d)$ with principal eigenfunction $\tilde{\varphi}$.)

Lemma A.6. $\displaystyle \frac{\partial \sigma}{\partial \eta} \int_\Omega \tilde{\varphi}^2 = \int_\Omega m\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}^2}{m}\right).$

Proof. Differentiating (3.19) with respect to η, denoting $\sigma' = \frac{\partial \sigma}{\partial \eta}$ and $\tilde{\varphi}' = \frac{\partial \tilde{\varphi}}{\partial \eta}$, we have

$$\text{(A.21)}\qquad \begin{cases} d\Delta\tilde{\varphi}' - \frac{\nabla\cdot[(1-\eta m)\nabla m]}{m}\tilde{\varphi}' + \sigma\tilde{\varphi}' = -\frac{\nabla\cdot(m\nabla m)}{m}\tilde{\varphi} - \sigma'\tilde{\varphi} & \text{in } \Omega, \\ d\frac{\partial \tilde{\varphi}'}{\partial n} - \frac{1-\eta m}{m}\frac{\partial m}{\partial n}\tilde{\varphi}' = -\frac{\partial m}{\partial n}\tilde{\varphi} & \text{on } \partial\Omega. \end{cases}$$

Multiplying (A.21) by the principal eigenfunction $\tilde{\varphi}$ of (3.19) and integrating by parts, we obtain

$$\begin{aligned} &\int_\Omega \left[-d\nabla\tilde{\varphi}' \cdot \nabla\tilde{\varphi} + (1-\eta m)\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}'\tilde{\varphi}}{m}\right) + \sigma\tilde{\varphi}'\tilde{\varphi}\right] \\ &= \int_\Omega \left[m\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}^2}{m}\right) - \sigma'\tilde{\varphi}^2\right]. \end{aligned}$$

Integrating by parts again on the left, we see that the left hand side is zero. This proves the lemma. □

This finishes the proof of Theorem 3.17. □

We also prove the following corollary of Lemma A.6:

Corollary A.7. $\displaystyle \frac{\partial \sigma}{\partial \eta}(0;1) = \left.\frac{\partial \sigma}{\partial \eta}\right|_{d=1,\eta=0} > 0.$

PROOF. When $d = 1$ and $\eta = 0$, it is easy to see that $\tilde{\varphi} = m|\Omega|^{1/2}/\sqrt{\int_\Omega m^2}$ corresponding to $\sigma = 0$. Hence by Lemma A.6,

$$\frac{\partial \sigma}{\partial \eta}(0;1) = \frac{\int_\Omega m|\nabla m|^2}{\int_\Omega m^2} > 0.$$

This proves Corollary A.7. □

Finally, we prove the following two results used in the proof of Proposition 6.2 and Lemma 6.3.

LEMMA A.8. *With the principal eigenfunction φ of* (3.1) *normalized as in Corollary* A.4, *we have as $\mu \to 0$,*

$$\frac{1}{\mu}\int_\Omega \tilde{u}_+''\varphi^2 \to 0.$$

PROOF. By Hölder's inequality,

$$\frac{1}{\mu}\int_\Omega \tilde{u}_+''\varphi^2 \le \left(\frac{1}{\mu^2}\int_\Omega (\tilde{u}_+'')^2\right)^{\frac{1}{2}}\left(\int_\Omega \varphi^4\right)^{\frac{1}{2}} \le C\left(\frac{1}{\mu^2}\int_\Omega (\tilde{u}_+'')^2\right)^{\frac{1}{2}} \to 0,$$

by Theorem 3.15. □

PROPOSITION A.9. *Given a compact subset $K \subset\subset (0,\infty)$, there exists $\epsilon_0 > 0$ such that if μ, ν are sufficiently small and $\frac{\nu}{\mu} \in K$, then*

$$-\frac{1}{\mu}\int_\Omega \tilde{u}'\varphi\varphi' \ge \epsilon_0 \int_\Omega (\varphi')^2$$

for all $\eta \in [0, \frac{1}{\min_\Omega m}]$.

PROOF. Multiplying (A.18) by $\varphi' = \frac{\partial \varphi}{\partial \eta}$ and integrating by parts, we have

$$\begin{aligned} -\frac{1}{\mu}\int_\Omega (\tilde{u}' - \lambda_u')\varphi\varphi' &= \int_\Omega \left[\frac{\nu}{\mu}|\nabla\varphi'|^2 + \frac{\tilde{u}-m}{\mu}(\varphi')^2 - \frac{\lambda_u}{\mu}(\varphi')^2\right] \\ &= J_\mu(\varphi') - \frac{\lambda_u}{\mu}\int_\Omega (\varphi')^2, \end{aligned} \tag{A.22}$$

recalling the definition of J_μ given in the proof of Lemma A.2, and the equivalence of (3.1) and (3.18). By the normalization $\int_\Omega \varphi^2 = |\Omega|$, we deduce that

$$\int_\Omega \varphi\varphi' = 0. \tag{A.23}$$

As a result, the term involving λ_u' in (A.22) vanishes, and by Poincaré's inequality,

$$J_\mu(\varphi') \ge \frac{\lambda_{u,2}}{\mu}\int_\Omega (\varphi')^2.$$

Hence

$$-\frac{1}{\mu}\int_\Omega \tilde{u}'\varphi\varphi' \ge \frac{\lambda_{u,2} - \lambda_u}{\mu}\int_\Omega (\varphi')^2.$$

By Lemma A.2 and equation (A.13),

$$\frac{\lambda_{u,2} - \lambda_u}{\mu} \to \sigma_2 - \sigma_1 > 0,$$

where the last inequality follows from (A.13). This proves the proposition. □

A.3. Asymptotic behavior of λ_v as $\mu, \nu \to \infty$

LEMMA A.10. *Suppose that ψ is the principal eigenfunction of* (3.3) *normalized by $\int_\Omega e^{\eta m}\psi^2 = \int_\Omega e^{\eta m}$, then for each $\Lambda > 0$, $\psi \to 1$ in $H^1(\Omega)$ as $\mu, \nu \to \infty$ uniformly for $\eta \in [0, \Lambda]$.*

PROOF. We first establish the following assertion:

CLAIM A.11. *$\lambda_v \le \max_{\bar{\Omega}}(\tilde{v} - m)$. In particular by Theorem* 3.13*, $\lambda_v \le \max_{\bar{\Omega}} m$.*

The claim follows by setting the test function $\phi \equiv 1$ in the variational characterization

$$\lambda_v = \inf_{\phi \in H^1(\Omega)} \frac{\int_\Omega e^{\eta m}[\mu|\nabla\phi|^2 + (\tilde{v} - m)\phi^2]}{\int_\Omega e^{\eta m}\phi^2}.$$

Since we have the normalization $\int_\Omega e^{\eta m}\psi^2 = \int_\Omega e^{\eta m}$, it suffices to show that $\mu\int_\Omega e^{\eta m}|\nabla\psi|^2 = O(1)$ uniformly for $\eta \in [0, \Lambda]$. We now multiply (3.3) by ψ and integrate by parts,

$$\begin{aligned}
\mu\int_\Omega e^{\eta m}|\nabla\psi|^2 &= \int_\Omega e^{\eta m}(m - \tilde{v})\psi^2 + \lambda_v\int_\Omega e^{\eta m}\psi^2 \\
&\le [\max_{\bar{\Omega}}(m - \tilde{v}) + \lambda_v]\int_\Omega e^{\eta m}\psi^2 \\
&= [\max_{\bar{\Omega}}(m - \tilde{v}) + \lambda_v]\int_\Omega e^{\eta m} \\
&\le 2(\max_{\bar{\Omega}} m)\int_\Omega e^{\eta m} \\
&\le 2(\max_{\bar{\Omega}} m)\int_\Omega e^{\Lambda m}.
\end{aligned}$$

This proves Lemma A.10. ☐

THEOREM A.12. *For each $\Lambda > 0$,*

$$\lambda_v \to \frac{\int_\Omega e^{\eta m}(\bar{m} - m)}{\int_\Omega e^{\eta m}}$$

in $C^1([0, \Lambda])$ as $\mu, \nu \to \infty$.

PROOF. Integrating (3.3), we have

$$\int_\Omega e^{\eta m}(m - \tilde{v})\psi + \lambda_v\int_\Omega e^{\eta m}\psi = 0. \tag{A.24}$$

By passing to the limit in (A.24),

$$\lambda_v \to \frac{\int_\Omega e^{\eta m}(\bar{m} - m)}{\int_\Omega e^{\eta m}} \quad \text{in } L^\infty(\Omega). \tag{A.25}$$

CLAIM A.13. *$\mu\int_\Omega e^{\eta m}|\nabla\psi|^2 \to 0$ as $\mu, \nu \to \infty$ uniformly for $\eta \in [0, \Lambda]$.*

To see the claim, multiply (3.3) by ψ, and integrate by parts, to obtain

$$\mu\int_\Omega e^{\eta m}|\nabla\psi|^2 = \int_\Omega e^{\eta m}(m - \tilde{v})\psi^2 + \lambda_v\int_\Omega e^{\eta m}\psi^2.$$

Passing to the limit and using Lemma A.10, we have

$$\lim_{\mu,\nu\to\infty} \mu \int_\Omega e^{\eta m}|\nabla\psi|^2 = \int_\Omega e^{\eta m}(m-\bar{m}) + \left(\lim_{\mu,\nu\to\infty}\lambda_v\right)\int_\Omega e^{\eta m} = 0.$$

The last equality follows from (A.25). This proves the claim.

Now, multiplying (3.3) by $m\psi$ and integrating by parts,

$$\mu\int_\Omega e^{\eta m} m|\nabla\psi|^2 + \mu\int_\Omega e^{\eta m}\psi\nabla m\cdot\nabla\psi = \int_\Omega e^{\eta m} m(m-\tilde{v})\psi^2 + \lambda_v\int_\Omega e^{\eta m} m\psi^2.$$

Passing to the limit, we see that the first term goes to zero by Claim A.13, and

$$\begin{aligned}\lim_{\mu,\nu\to\infty}\mu\int_\Omega e^{\eta m}\psi\nabla m\cdot\nabla\psi &= \int_\Omega e^{\eta m} m(m-\bar{m}) + \left(\lim_{\mu,\nu\to\infty}\lambda_v\right)\int_\Omega e^{\eta m} m \\ &= \int_\Omega e^{\eta m} m(m-\bar{m}) - \frac{\int_\Omega e^{\eta m}(m-\bar{m})}{\int_\Omega e^{\eta m}}\int_\Omega e^{\eta m} m.\end{aligned}$$

Therefore, by Lemma 5.2 and Lemma A.10,

$$\begin{aligned}\lim_{\mu,\nu\to\infty}\frac{\partial\lambda_v}{\partial\eta} &= -\frac{1}{\int_\Omega e^{\eta m}}\left[\int_\Omega e^{\eta m} m(m-\bar{m}) - \frac{\int_\Omega e^{\eta m}(m-\bar{m})}{\int_\Omega e^{\eta m}}\int_\Omega e^{\eta m} m\right] \\ &= \frac{\partial}{\partial\eta}\left[\frac{\int_\Omega e^{\eta m}(\bar{m}-m)}{\int_\Omega e^{\eta m}}\right],\end{aligned}$$

as $\mu,\nu\to\infty$ uniformly for $\eta\in[0,\Lambda]$. This proves Theorem A.12. □

Corollary A.14. *There exists $\epsilon_0 > 0$ such that for all μ,ν sufficiently large,*

$$\frac{\partial\lambda_v}{\partial\eta} < 0 \qquad \text{for } \eta\in[0,\epsilon_0].$$

Proof. It suffices to show that $\lim_{\mu,\nu\to\infty}\frac{\partial\lambda_v}{\partial\eta}(0,\mu,\nu) < 0$ as follows.

$$\lim_{\mu,\nu\to\infty}\frac{\partial\lambda_v}{\partial\eta}(0,\mu,\nu) = -\frac{1}{|\Omega|}\int_\Omega m(m-\bar{m}) = -\frac{1}{|\Omega|}\int_\Omega(m-\bar{m})^2 < 0.$$

This proves Corollary A.14. □

Corollary A.15. *For each $\epsilon > 0$, $\lambda_v(\eta,\mu,\nu) < 0$ in $[\epsilon,\infty)$ for all μ,ν sufficiently large. In particular, suppose η_* is a root of $\lambda_v(\eta,\mu,\nu)$, then $\eta_*\to 0$ as $\mu,\nu\to\infty$.*

Proof. By Theorem 2.2, $\lambda_v(\eta,\mu,\nu) < 0$ for all $\eta \geq \frac{1}{\min_\Omega m}$ and $\mu,\nu > 0$. Therefore the result follows from Theorem A.12, since

$$\lambda_v \to \frac{\int_\Omega e^{\eta m}(\bar{m}-m)}{\int_\Omega e^{\eta m}} = -\frac{\int_\Omega(e^{\eta m}-e^{\eta\bar{m}})(m-\bar{m})}{\int_\Omega e^{\eta m}} = \begin{cases} <0 & \text{if } \eta>0,\\ 0 & \text{if } \eta=0.\end{cases}$$

This proves Corollary A.15. □

APPENDIX B

Limit eigenvalue problems as $\mu, \nu \to 0$

This chapter studies two related limiting eigenvalue problems that have arisen in our analysis. Again, domain convexity is not assumed in the results contained here.

For $d > 0$, let $\sigma = \sigma(\eta; d)$ be the principal eigenvalue of

$$\text{(B.1)} \qquad \begin{cases} d\Delta\tilde{\varphi} - \frac{\nabla\cdot[(1-\eta m)\nabla m]}{m}\tilde{\varphi} + \sigma\tilde{\varphi} = 0 & \text{in } \Omega, \\ d\frac{\partial\tilde{\varphi}}{\partial n} - \frac{1-\eta m}{m}\frac{\partial m}{\partial n}\tilde{\varphi} = 0 & \text{on } \partial\Omega. \end{cases}$$

THEOREM B.1. (i) *σ can be characterized as*

$$\text{(B.2)} \qquad \sigma(\eta; d) = \inf_{C^1(\bar{\Omega})} \frac{\int_\Omega [d|\nabla\psi|^2 - (1-\eta m)\nabla m \cdot \nabla\left(\frac{\psi^2}{m}\right)}{\int_\Omega \psi^2} := \inf_{C^1(\bar{\Omega})} J_0(\psi).$$

(ii) *σ depends smoothly in d and η.*
(iii) *σ is increasing in d.*
(iv) *σ is strictly concave in η, i.e. $\frac{\partial^2}{\partial\eta^2}\sigma < 0$.*

PROOF. It follows from the boundary condition that problem (B.1) is actually variational. This proves (i). (ii) is standard (see, e.g. Theorem 3.1 in [**10**]). (iii) follows from the variational characterization in (i). For the strict concavity, normalize $\tilde{\varphi}$ by $\int_\Omega \tilde{\varphi}^2 = |\Omega|$ and differentiate (B.1) with respect to η to obtain (denoting the derivative with respect to η by $'$)

$$\text{(B.3)} \qquad \begin{cases} d\Delta\tilde{\varphi}' - \frac{\nabla\cdot[(1-\eta m)\nabla m]}{m}\tilde{\varphi}' + \sigma\tilde{\varphi}' = -\frac{\nabla\cdot(m\nabla m)}{m}\tilde{\varphi} - \sigma'\tilde{\varphi} & \text{in } \Omega, \\ d\frac{\partial\tilde{\varphi}'}{\partial n} - \frac{1-\eta m}{m}\frac{\partial m}{\partial n}\tilde{\varphi}' = -\frac{\partial m}{\partial n}\tilde{\varphi} \quad \text{on } \partial\Omega, \quad \int_\Omega \tilde{\varphi}\tilde{\varphi}' = 0. \end{cases}$$

Differentiating again, we have

(B.4)

$$\begin{cases} d\Delta\tilde{\varphi}'' - \frac{\nabla\cdot[(1-\eta m)\nabla m]}{m}\tilde{\varphi}'' + \sigma\tilde{\varphi}'' = -2\frac{\nabla\cdot(m\nabla m)}{m}\tilde{\varphi}' - 2\sigma'\tilde{\varphi}' - \sigma''\tilde{\varphi} & \text{in } \Omega, \\ d\frac{\partial\tilde{\varphi}''}{\partial n} - \frac{1-\eta m}{m}\frac{\partial m}{\partial n}\tilde{\varphi}'' = -2\frac{\partial m}{\partial n}\tilde{\varphi}' & \text{on } \partial\Omega. \end{cases}$$

Multiplying (B.4) by $\tilde{\varphi}$ and integrating by parts, we have

$$\begin{aligned} &\int_\Omega \left[-d\nabla\tilde{\varphi}'' \cdot \nabla\tilde{\varphi} + (1-\eta m)\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}''\tilde{\varphi}}{m}\right) + \sigma\tilde{\varphi}''\tilde{\varphi}\right] \\ &= 2\int_\Omega m\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}'\tilde{\varphi}}{m}\right) - 2\sigma'\int_\Omega \tilde{\varphi}'\tilde{\varphi} - \sigma''\int_\Omega \tilde{\varphi}^2. \end{aligned}$$

Integrating by parts again, we see that the left hand side vanishes. Hence,

$$\sigma''|\Omega| = \sigma''\int_\Omega \tilde{\varphi}^2 = 2\int_\Omega m\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}'\tilde{\varphi}}{m}\right) - 2\sigma'\int_\Omega \tilde{\varphi}'\tilde{\varphi} = 2\int_\Omega m\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}'\tilde{\varphi}}{m}\right).$$

The first and last equalities follow from the normalization $\int_\Omega \tilde{\varphi}^2 = |\Omega|$ and, upon differentiating, $\int_\Omega \tilde{\varphi}\tilde{\varphi}' = 0$. It remains to show that

$$\int_\Omega m\nabla m \cdot \nabla \left(\frac{\tilde{\varphi}'\tilde{\varphi}}{m}\right) < 0.$$

This follows by multiplying (B.3) by $\tilde{\varphi}'$ and integrating by parts,

$$\int_\Omega \left[-d|\nabla\tilde{\varphi}'|^2 + (1-\eta m)\nabla m \cdot \nabla\left(\frac{(\tilde{\varphi}')^2}{m}\right) + \sigma(\tilde{\varphi}')^2\right] = \int_\Omega m\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}\tilde{\varphi}'}{m}\right).$$

The term involving σ' vanishes since $\int_\Omega \tilde{\varphi}'\tilde{\varphi} = 0$, which follows from the normalization of $\tilde{\varphi}$. Recalling the functional J_0 introduced in the proof of Lemma A.2, we have

$$\text{(B.5)} \qquad -\int_\Omega m\nabla m \cdot \nabla\left(\frac{\tilde{\varphi}\tilde{\varphi}'}{m}\right) = J_0(\tilde{\varphi}') - \sigma\int_\Omega (\tilde{\varphi}')^2 \geq (\sigma_2 - \sigma)\int_\Omega (\tilde{\varphi}')^2 > 0.$$

The second last inequality follows from the fact that $\tilde{\varphi}' \perp \tilde{\varphi}$ in $L^2(\Omega)$ and the variational characterization of the second eigenvalue $\sigma_2 > \sigma$, whereas the last inequality of (B.5) follows from the fact that $\tilde{\varphi}'$ satisfying (B.3) is necessarily non-zero and (A.13). ☐

Next, we define

$$\text{(B.6)} \qquad a(\eta) = \sup_{\mathcal{S}} \left\{ \frac{\int_\Omega (1-\eta m)\nabla m \cdot \nabla \frac{\psi^2}{m}}{\int_\Omega |\nabla\psi|^2} \right\},$$

where the supremum is taken over the set

$$\mathcal{S} = \left\{ \psi \in C^1(\bar{\Omega}) : \int_\Omega (1-\eta m)\nabla m \cdot \nabla\frac{\psi^2}{m} > 0 \right\}.$$

THEOREM B.2. (i) *$a(\eta) > 0$ for all $\eta \geq 0$.*

(ii) *$a(\eta)$ is convex and depends smoothly in η (where it is finite).*

(iii) *There exists $\eta_1 \in (\frac{1}{\max_{\bar{\Omega}} m}, \frac{1}{\min_{\bar{\Omega}} m})$ such that $0 < a(\eta) < \infty$ in $[0, \eta_1)$ and $a(\eta) = \infty$ in $[\eta_1, \infty)$.*

(iv) *$a_* := \min_{\eta \geq 0} a \in (0,1)$ and is attained in $(0, \eta_1)$.*

PROOF. To show (i), it suffices to show that $e^{-\eta m/2}m \in \mathcal{S}$, i.e. $\mathcal{S}$ is non-empty. This can be verified by computing the following integral.

$$\begin{aligned}\int_\Omega (1-\eta m)\nabla m \cdot \nabla\left(\frac{(e^{-\eta m/2}m)^2}{m}\right) &= \int_\Omega (1-\eta m)\nabla m \cdot \nabla\left(e^{-\eta m}m\right)\\ &= \int_\Omega (1-\eta m)^2|\nabla m|^2 e^{-\eta m} > 0.\end{aligned}$$

For (ii), the convexity follows from the definition of $a(\eta)$ as the supremum of a family of affine functions of η. For the smoothness, we notice that $\tilde{\lambda} := 1/a$ is the principal eigenvalue of

$$\begin{cases} \Delta\phi - \tilde{\lambda}\frac{\nabla\cdot[(1-\eta m)\nabla m]}{m}\phi = 0 & \text{in } \Omega,\\ \frac{\partial\phi}{\partial n} - \tilde{\lambda}\frac{(1-\eta m)}{m}\frac{\partial m}{\partial n}\phi = 0 & \text{on } \partial\Omega, \end{cases}$$

which depends smoothly on η. We make the following claim which implies (iii).

Claim B.3. *$a = a(\eta)$ satisfies*

$$a = a(\eta) = \begin{cases} < +\infty & \text{when } \eta \in [0, \eta_1), \\ = +\infty & \text{when } \eta \in [\eta_1, \infty), \end{cases}$$

where

$$\eta_1 = \left(\int_\Omega \frac{|\nabla m|^2}{m^2} \right) / \left(\int_\Omega \frac{|\nabla m|^2}{m} \right) \in \left(\frac{1}{\max_{\bar{\Omega}} m}, \frac{1}{\min_{\bar{\Omega}} m} \right).$$

To see the claim, we first note that

$$\int_\Omega (1 - \eta m) \nabla m \cdot \nabla \left(\frac{1}{m} \right) = \begin{cases} < 0 & \text{if } \eta < \eta_1, \\ = 0 & \text{if } \eta = \eta_1, \\ > 0 & \text{if } \eta > \eta_1. \end{cases}$$

Now, if $\eta \in (\eta_1, \infty)$, then $1 \in S$ and it is clear that $a = \infty$.

Next, if $\eta = \eta_1$, then $a = \infty$ as well, by convexity of a with respect to η (by part (ii) proved above).

Now, if $\eta \in [0, \eta_1)$, then we claim that $a < \infty$. For, suppose $a = \infty$ for some fixed $\eta \in (0, \eta_1)$, then there exists a sequence $\psi_i \in C^1(\bar{\Omega})$, and $\epsilon_i \to 0$ such that $\int_\Omega (\psi_i)^2 = |\Omega|$, and

$$\int_\Omega |\nabla \psi_i|^2 \leq \epsilon_i \int_\Omega (1 - \eta m) \nabla m \cdot \nabla \left(\frac{\psi_i^2}{m} \right),$$

from which one can deduce by Young's inequality, that (for some constant C depending on $\|m\|_{C^1}$ and η)

$$\int_\Omega |\nabla \psi_i|^2 \leq C \epsilon_i \int_\Omega \psi_i^2 \leq C \epsilon_i |\Omega|.$$

Hence $\psi_i \to 1$ in $H^1(\Omega)$. Since $\psi_i \in \mathcal{S}$, we have

$$\int_\Omega (1 - \eta m) \nabla m \cdot \nabla \left(\frac{\psi_i^2}{m} \right) > 0. \tag{B.7}$$

By taking $i \to \infty$ in (B.7), we have $\int_\Omega (1 - \eta m) \nabla m \cdot \nabla \left(\frac{1}{m} \right) \geq 0$. That is, $\eta \geq \eta_1$, which is a contradiction. Hence, if $\eta \in [0, \eta_1)$, then $a < \infty$. This proves Claim B.3, from which (iii) follows.

Finally, to show (iv), it suffices to show that $a(0) = 1$ and $a'(0) = \frac{\partial}{\partial \eta} a(0) < 0$, then the rest follows from the convexity and (iii). Now $a = 1/\tilde{\lambda}$ satisfies the following equation for some positive eigenfunction φ:

$$\begin{cases} a \Delta \varphi - \frac{\nabla \cdot [(1 - \eta m) \nabla m]}{m} \varphi = 0 & \text{in } \Omega, \\ a \frac{\partial \varphi}{\partial n} - \frac{1 - \eta m}{m} \frac{\partial m}{\partial n} \varphi = 0 & \text{on } \partial \Omega. \end{cases}$$

Differentiating with respect to η, we obtain

$$\begin{cases} a \Delta \varphi' - \frac{\nabla \cdot [(1 - \eta m) \nabla m]}{m} \varphi' = -a' \Delta \varphi - \frac{\nabla \cdot (m \nabla m)}{m} \varphi & \text{in } \Omega, \\ a \frac{\partial \varphi'}{\partial n} - \frac{1 - \eta m}{m} \frac{\partial m}{\partial n} \varphi' = -\frac{\partial m}{\partial n} \varphi - a' \frac{\partial \varphi}{\partial n} & \text{on } \partial \Omega. \end{cases}$$

Multiplying by φ and integrating by parts, we can argue similarly as before to deduce that

$$a' \int_\Omega |\nabla \varphi|^2 = -\int_\Omega m \nabla m \cdot \nabla \left(\frac{\varphi^2}{m} \right).$$

Setting $\eta = 0$, then by inspection $a(0) = 1$ and $\varphi = m$ and hence

$$a'(0) = -\frac{\int_\Omega m|\nabla m|^2}{\int_\Omega |\nabla m|^2} < 0.$$

$\square$

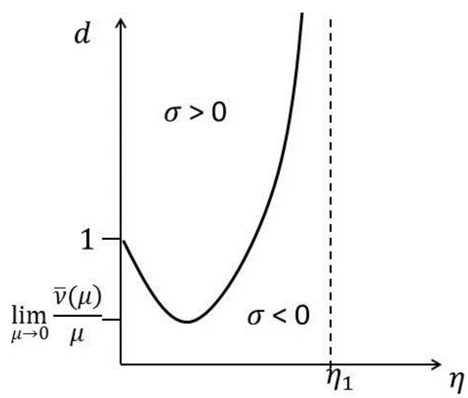

FIGURE 1. Graph of $a(\eta)$.

Recall that $a_* = \min_{\eta \geq 0} a$. The following is the main theorem of this section.

THEOREM B.4. (i) *If $d \in (0, a_*)$, then $\sigma(\eta; d) < 0$ for all $\eta \geq 0$.*
(ii) *If $d \in (a_*, 1)$, then $\sigma(\eta; d)$ changes sign exactly twice as η varies from 0 to ∞.*
(iii) *If $d \in (1, \infty)$, then $\sigma(\eta; d)$ changes sign exactly once as η varies from 0 to ∞.*

Moreover, as long as $d \neq a_*$, all the (non-negative) roots of $\sigma(\eta; d)$ are non-degenerate. i.e.

$$d \neq a_*, 1 \quad \textit{and} \quad \sigma(\eta; d) = 0 \quad \Rightarrow \quad \frac{\partial \sigma}{\partial \eta}(\eta; d) \neq 0.$$

REMARK B.5. In fact, it follows from the proof of Theorem B.2 that $a(\eta) = \infty$ for $\eta \geq \frac{1}{\min_{\bar{\Omega}} m}$, so that $\sigma(d, \eta) < 0$ for all $d > 0$ and $\eta \geq \frac{1}{\min_{\bar{\Omega}} m}$. As a result, all the sign changes occur in the interval $[0, \frac{1}{\min_{\bar{\Omega}} m})$.

PROOF. Suppose $d \in (0, a_*)$, then if $a(\eta) < \infty$ (i.e. $\eta \in [0, \eta_1)$), then

$$\sigma(\eta; d) < \sigma(\eta; a_*) \leq \sigma(\eta; a(\eta)) = 0.$$

On the other hand, if $a(\eta) = \infty$ (i.e. $\eta \in [\eta_1, \infty)$), then taking $\psi = 1$ in Theorem B.1(i), we have

$$\sigma(\eta; d) \leq -\frac{1}{|\Omega|} \int_\Omega (1 - \eta m) \nabla m \cdot \nabla \left(\frac{1}{m} \right) \leq 0. \tag{B.8}$$

The last inequality is a consequence of the fact that $a(\eta) = \infty$ (Claim B.3). In fact, the first inequality in (B.8) is strict, which follows from the fact that 1 is not an eigenfunction corresponding to $\sigma(\eta; d)$. This proves (i).

Now suppose $d \in (a_*, 1)$, then by Theorem B.2 (ii) and (iv), there exist exactly two distinct non-negative numbers $\eta' < \eta''$ such that

$$a(\eta) = \begin{cases} = d & \text{if } \eta = \eta' \text{ or } \eta''; \\ > d & \text{if } \eta \in [0, \eta') \cup (\eta'', \infty); \\ < d & \text{if } \eta \in (\eta', \eta''). \end{cases}$$

In fact, by the comparison principle of eigenvalues, we have

$$\begin{array}{ll} \sigma(\eta; d) < \sigma(\eta; a(\eta)) \leq 0 & \text{for } \eta \in [0, \eta') \cup (\eta'', \infty), \\ \sigma(\eta; d) > \sigma(\eta; a(\eta)) = 0 & \text{for } \eta \in (\eta', \eta''). \end{array}.$$

This proves (ii). (iii) can be established in a similar fashion. Finally, the non-degeneracy claim follows from the strict concavity of a in η. $\square$

REMARK B.6. We note here that $\eta \mapsto \sigma(\eta, a_*)$ has a unique root. This provides a transition from no roots (when $a < a_*$) to double roots (when $a \in (a_*, 1)$). See Figure 1 in Appendix B.

APPENDIX C

Limiting eigenvalue problem as $\mu \to \infty$

This section is devoted to the proof of Proposition 4.13 concerning the principal eigenvalue of (4.8), which we display below as (C.1) for convenience. The main steps of the proof resemble those of Theorem 2.5.

$$
\text{(C.1)} \qquad \begin{cases} \nu\Delta\varphi + \left(m - \frac{\int_\Omega m e^{\eta m}}{\int_\Omega e^{2\eta m}} e^{\eta m}\right)\varphi + \tau\varphi = 0 & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}
$$

Set

$$
G(\eta, x) := m(x) - \frac{\int_\Omega m e^{\eta m}}{\int_\Omega e^{2\eta m}} e^{\eta m(x)}.
$$

Lemma C.1. *There exists some $0 < \eta_* < \eta^*$ such that*

(i) $\int_\Omega G(\eta, x)\, dx < 0$ *for* $0 < \eta < \eta_*$;
(ii) $\int_\Omega G(\eta, x)\, dx > \frac{1}{2} \int_{\{x\in\Omega: m(x) < \max_{\bar\Omega} m\}} m > 0$ *for* $\eta > \eta^*$.

Proof. Define $g(\eta) = \int_\Omega G(\eta, x)\, dx$. Then $g(0) = 0$ and

$$
g'(0) = \frac{1}{|\Omega|}\left(\int_\Omega m\right)^2 - \int_\Omega m^2 < 0,
$$

where the last inequality follows from the Hölder inequality and m being non-constant. Hence, there exists some $\eta_* > 0$ such that $g < 0$ for $0 < \eta < \eta_*$. Let $m^* := \max_{\bar\Omega} m > 0$. We claim that

Claim C.2. $\dfrac{\left(\int_\Omega m^{\eta m}\right)\left(\int_\Omega e^{\eta m}\right)}{\int_\Omega e^{2\eta m}} \to m^*|\{x \in \Omega : m(x) = m^*\}| \geq 0$ *as* $\eta \to \infty$.

On one hand, if $|\{x \in \Omega : m(x) = \max_{\bar\Omega} m\}| > 0$, then it follows by Bounded Convergence Theorem that

$$
\begin{aligned}
\frac{(\int_\Omega m e^{\eta m})(\int_\Omega e^{\eta m})}{\int_\Omega e^{2\eta m}} &= \frac{(\int_\Omega m e^{\eta(m-m^*)})(\int_\Omega e^{\eta(m-m^*)})}{\int_\Omega e^{2\eta(m-m^*)}} \\
&\to m^*|\{x \in \Omega : m(x) = m^*\}|.
\end{aligned}
$$

Hence as $\eta \to \infty$,

$$
g(\eta) \to \int_\Omega m - m^*|\{x \in \Omega : m(x) = m^*\}| = \int_{\{x\in\Omega: m(x) < m^*\}} m > 0.
$$

On the other hand, assume $|\{x \in \Omega : m(x) = m^*\}| = 0$. Given $\epsilon > 0$, let $\delta > 0$ be chosen such that

$$
m^*|\{x \in \Omega : m(x) > m^* - \delta\}| < \frac{\epsilon}{2}.
$$

Then we estimate in the following way:

$$
\begin{aligned}
&\frac{(\int_\Omega m e^{\eta m})(\int_\Omega e^{\eta m})}{\int_\Omega e^{2\eta m}}\\
&\leq \frac{m^*\left[\int_\Omega e^{\eta m}\right]^2}{\int_\Omega e^{2\eta m}}\\
&= \frac{m^*\left[\int_{\{x\in\Omega: m(x)\leq m^*-\delta\}} e^{\eta m} + \int_{\{x\in\Omega: m(x)> m^*-\delta\}} e^{\eta m}\right]^2}{\int_\Omega e^{2\eta m}}\\
&\leq \frac{2m^*\left[\left(\int_{\{x\in\Omega: m(x)\leq m^*-\delta\}} e^{\eta m}\right)^2 + \left(\int_{\{x\in\Omega: m(x)> m^*-\delta\}} e^{\eta m}\right)^2\right]}{\int_\Omega e^{2\eta m}}\\
&\leq 2m^*\left[\frac{\left(\int_{\{x\in\Omega: m(x)\leq m^*-\delta\}} e^{\eta m}\right)^2}{\int_{\{x\in\Omega: m(x)> m^*-\delta/2\}} e^{2\eta m}} + \frac{\left(\int_{\{x\in\Omega: m(x)> m^*-\delta\}} e^{\eta m}\right)^2}{\int_{\{x\in\Omega: m(x)> m^*-\delta\}} e^{2\eta m}}\right]\\
&\leq 2m^*\left[\frac{|\{x\in\Omega: m(x)\leq m^*-\delta\}|}{|\{x\in\Omega: m(x)> m^*-\delta/2\}|}e^{-\delta\eta} + |\{x\in\Omega: m(x)> m^*-\delta\}|\right]\\
&< \epsilon
\end{aligned}
$$

for η sufficiently large. This proves Claim C.2. Applying Claim C.2, we have

$$\lim_{\eta\to\infty} g(\eta) = \int_\Omega m - m^*\,|\{x\in\Omega: m(x)=m^*\}| = \int_{\{x\in\Omega: m<m^*\}} m > 0.$$

This completes the proof. □

COROLLARY C.3. *Let η^* be as given in the previous lemma, then τ in* (C.1) *satisfies*

$$\tau \leq -\frac{1}{2}\int_{\{x\in\Omega: m(x)<\max_{\bar\Omega} m\}} m < 0$$

for every $\nu > 0$ and $\eta \geq \eta^*$.

PROOF. Divide (C.1) by φ and integrate by parts. Then by Lemma C.1,

$$\tau|\Omega| = -\nu\int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2} - \int_\Omega G \leq -\int_\Omega G \leq -\frac{1}{2}\int_{\{x\in\Omega: m<m^*\}} m.$$

This proves Corollary C.3. □

LEMMA C.4. *There exists some small positive constant C_* such that*

$$\sup_{0\leq\eta<\infty} \tau < 0$$

for every $\nu \in (0, C_*)$.

PROOF. We argue by contradiction. If not, we may assume that there exists some sequence $\nu_i \to 0^+$ such that $\sup\limits_{0\leq\eta<\infty} \tau(\eta;\nu_i) \geq 0$ for every i. For each such ν_i, by Corollary C.3 we see that $\tau(\eta,\nu_i) \leq -\frac{1}{2}\int_{\{x\in\Omega: m<m^*\}} m < 0$ for $\eta > \eta^*$. Hence for every i, there exists some $\eta_i \in [0, \eta^*]$ such that $\tau(\eta_i, \nu_i) = \sup_{0 \le \eta < \infty} \tau(\eta, \nu_i) \ge 0$. Passing to a subsequence if necessary, we may assume that $\eta_i \to \hat{\eta}$ as $i \to \infty$. Hence,

$$\lim_{i\to\infty} \tau(\eta_i, \nu_i) = \min_{\bar{\Omega}} \left(m - \frac{\int_\Omega m e^{\hat{\eta} m}}{\int_\Omega e^{2\hat{\eta} m}} e^{\hat{\eta} m} \right).$$

We claim that

$$\min_{\bar{\Omega}} \left(m - \frac{\int_\Omega m e^{\hat{\eta} m}}{\int_\Omega e^{2\hat{\eta} m}} e^{\hat{\eta} m} \right) < 0.$$

To establish this assertion, we argue by contradiction. Suppose that

$$\min_{\bar{\Omega}} \left(m - \frac{\int_\Omega m e^{\hat{\eta} m}}{\int_\Omega e^{2\hat{\eta} m}} e^{\hat{\eta} m} \right) \ge 0,$$

or equivalently

$$\min_{\bar{\Omega}} \left(m(x) e^{-\hat{\eta} m(x)} \right) \ge \frac{\int_\Omega m e^{\hat{\eta} m}}{\int_\Omega e^{2\hat{\eta} m}}.$$

But this contradicts

$$\frac{\int_\Omega m e^{\hat{\eta} m}}{\int_\Omega e^{2\hat{\eta} m}} = \frac{\int_\Omega m e^{-\hat{\eta} m} \cdot e^{2\hat{\eta} m}}{\int_\Omega e^{2\hat{\eta} m}} > \min_{\bar{\Omega}} \left(m(x) e^{-\hat{\eta} m(x)} \right).$$

Note that the last inequality is strict because m is non-constant. This proves our assertion. Hence,

$$\lim_{i\to\infty} \tau(\nu_i, \eta_i) = \min_{\bar{\Omega}} \left(m - \frac{\int_\Omega m e^{\hat{\eta} m}}{\int_\Omega e^{2\hat{\eta} m}} e^{\hat{\eta} m} \right) < 0,$$

which is a contradiction to our assumption $\tau(\eta_i; \nu_i) \ge 0$ for all i. $\square$

LEMMA C.5. *There exists some (large) positive constant C^* such that*

$$\sup_{0 \le \eta < \infty} \tau > 0$$

for every $\nu \in [C^*, \infty)$.

PROOF. We argue by contradiction. Suppose that there exists some sequence ν_i such that $\nu_i \to \infty$ and $\sup_{0 \le \eta < \infty} \tau(\eta; \nu_i) \le 0$, i.e., $\tau(\eta; \nu_i) \le 0$ for all $\eta \ge 0$ and i. For each fixed η, we claim that

$$\lim_{\nu_i \to \infty} \tau(\eta; \nu_i) = -\frac{1}{|\Omega|} \int_\Omega G(\eta, x)\, dx. \tag{C.2}$$

To show (C.2), first we realize that $\tau \le -\min_{x\in\bar{\Omega}} G(\eta, x)$ is bounded independent of ν_i large. Next, normalize $\int_\Omega \varphi^2 = |\Omega|$. Then by multiplying (C.1) by φ/ν_i and integrating by parts, we get

$$\int_\Omega |\nabla \varphi|^2 = \frac{1}{\nu_i} \left[\int_\Omega G(\eta, x) \varphi^2 + \tau \int_\Omega \varphi^2 \right] \to 0.$$

Hence $\nabla \varphi \to 0$ in $L^2(\Omega)$ and $\varphi \to 1$ in $H^1(\Omega)$. Finally, (C.2) follows by integrating (C.1) and letting $\nu_i \to \infty$.

By (C.2) and the fact that $\tau(\eta, \nu_i) \le 0$, $\int_\Omega G(\eta, x)\, dx \ge 0$ for every η, which is a contradiction to Lemma C.1. $\square$

Proof of Proposition 4.13. Define

$$\nu^+ := \sup\{\tilde{\nu} > 0 : \sup_{0 \le \eta < \infty} \tau(\eta; \nu) < 0, \qquad \text{for all } 0 < \nu < \tilde{\nu}\}.$$

By Lemmas C.4 and C.5, we see that $0 < \nu^+ < \infty$. From the definition of ν^+ we see that $\tau(\eta; \nu) < 0$ for all $\nu \in (0, \nu^+)$ and $\eta > 0$. By definition of ν^+, $\sup_{0 \le \eta < \infty} \tau(\eta; \nu^+) = 0$. Now, Corollary C.3 implies that $\tau(\eta_0; \nu^+) = 0$ for some $\eta_0 > 0$. Hence, by the fact that τ is strictly monotone increasing in ν, we deduce that $\sup_{0 \le \eta < \infty} \tau(\eta; \nu) > 0$ for all $\nu > \nu^+$. Note that for each $\nu > 0$, $\tau(0, \nu) < 0$ (divide (C.1) by φ and integrate by parts) and $\tau(\eta; \nu) < 0$ for every $\eta \ge \eta^*$ (Corollary C.3). We see that for $\nu \in (\nu^+, \infty)$, $\tau(\eta; \nu)$ must change sign at least twice as η varies from 0 to ∞. $\square$

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