WHEN IS A COXETER SYSTEM DETERMINED BY ITS COXETER GROUP?

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§0. Introduction.

A *Coxeter system* is a pair (W, S) where W is a group and where S is a set of involutions in W such that W has a presentation of the form

$$W = \langle S \mid (st)^{m(s,t)} \rangle.$$

Here $m(s,t)$ denotes the order of st in W and in the presentation for W, (s,t) ranges over all pairs in $S \times S$ such that $m(s,t) \neq \infty$. We further require the set S to be finite. W is a *Coxeter group* and S is a *fundamental set of generators* for W.

Obviously, if S is a fundamental set of generators, then so is wSw^{-1}, for any $w \in W$. Our main result is that, under certain circumstances, this is the only way in which two fundamental sets of generators can differ. In Section 3, we will prove the following result as Theorem 3.1.

Main Theorem. *Suppose that the Coxeter group W is capable of acting effectively, properly and cocompactly on some contractible manifold. If S and S' are two fundamental sets of generators for W, then there is a unique element $w \in W$ such that $S' = wSw^{-1}$.*

As we shall see in Theorem 2.3, there are other conditions which are equivalent to the hypothesis of the Main Theorem. One of these is that W is "type HM", the

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definition of which will be given in Section 2. ("HM" stands for "homology manifold") A more general result will be proved in Section 5: the Main Theorem holds for Coxeter groups of "type PM" (where "PM" stands for "pseudo-manifold').

A *diagram automorphism* of (W, S) is an automorphism $\alpha : W \to W$ such that $\alpha(S) = S$. If φ denotes the restriction of α to S, then $\varphi : S \to S$ is a bijection and $m(\varphi(s), \varphi(t)) = m(s,t)$, for all $(s,t) \in S \times S$. Conversely, since W has a presentation of the prescribed form, any such φ extends to a diagram automorphism. Hence, the group $\mathrm{Diag}\,(W,S)$ of diagram automorphisms can be identified with the group of such permutations φ of S. Let $\mathrm{Aut}\,(W)$ denote the automorphism group of W and $\mathrm{Inn}\,(W)$ the normal subgroup of inner automorphisms. As an immediate corollary to the Main Theorem we have the following.

Corollary. *Under the same hypothesis as in the Main Theorem, the group $Aut\,(W)$ is the semidirect product of $Inn\,(W)$ with $Diag\,(W,S)$.*

For example, the Main Theorem and its corollary apply to (a) affine Weyl groups (otherwise known as cocompact Euclidean reflection groups) and (b) cocompact hyperbolic reflection groups (i.e., discrete groups generated by the reflections across of faces of a convex polytope in $\mathbb{H}^n$). In fact, as was shown in [5], there is a rich class of Coxeter groups for which the hypothesis of the Main Theorem holds (see also [6] and [7]).

The next two examples illustrate that without some hypothesis on the Coxeter group W, the Main Theorem is false.

Example 0.1. Consider a regular m-gon in $\mathbb{R}^2$. It has m lines of symmetry. The angle between two such lines is a multiple of π/m. For each line of symmetry L, let r_L denote the orthogonal reflection across L; it is a symmetry of the regular m-gon. The *dihedral group* D_m of order $2m$ is the group generated by these reflections. The usual fundamental set of generators S for D_m consists of two reflections r_{L_1} and r_{L_2} where L_1 and L_2 are two lines making an angle of π/m. Then (D_m, S) is a Coxeter system. (For example, see pp. 10-12 of [1].) Choose a set of two other generators, say $S' = \{r_{L'_1}, r_{L'_2}\}$ where L'_1 and L'_2 make an angle of $k\pi/m$. Then, provided k and m are relatively prime, S' will also be a fundamental set of generators for D_m.

But S and S' are conjugate if and only if $k = \pm 1$. Indeed, S and S' are conjugate if and only if there is an element w in D_m which takes $\{L_1, L_2\}$ to $\{L'_1, L'_2\}$ and since w is orthogonal, the angles must be equal. So, the Main Theorem is false for finite Coxeter groups.

Example 0.2. Suppose $W = \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$, the free product of three cyclic groups of order two. Let r, s, t denote the respective generators of the three factors and let $S = \{r, s, t\}$. Then (W, S) is a Coxeter system. Let w be any element of the subgroup generated by s and t (an infinite dihedral group). Then $S' = \{wrw^{-1}, s, t\}$ is another fundamental set of generators for W and it is easy to see that S' is not conjugate to S unless $w = 1$. (A geometric argument will be given below.)

Associated to any Coxeter system (W, S) there is a contractible cell complex, denoted by Σ or by $\Sigma(S)$, on which W acts properly as a group generated by reflections. Its definition will be given in Section 1. In the geometric cases where W is a Euclidean or hyperbolic reflection group, Σ is what we expect it to be: a tessellation of $\mathbb{R}^n$ or $\mathbb{H}^n$ by convex polytopes, any one of which is a "chamber". For example, if $W = D_\infty \times D_\infty$ (the product of two infinite dihedral groups), then Σ is $\mathbb{R}^2$ with its standard tessellation by squares. In this case the Main Theorem asserts that the only possibility for S is a set of four reflections across the edges of a square.

More generally, when the Main Theorem holds, any fundamental set of generators S' must be the set of reflections across the walls of some chamber in Σ. It follows that the complex Σ depends only on the group W and not on the particular choice of a fundamental set of generators.

Example 0.2. (continued) $W = \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ and $S = \{r, s, t\}$. The complex Σ is the infinite tree pictured in Figure 1, below. A chamber is a "Y" such as the subcomplex indicated by darker lines in the figure. The infinite dihedral subgroup $\langle s, t \rangle$ stabilizes a subtree homeomorphic to the real line. Any nontrivial element w of this subgroup takes the fixed point of r to a different point, namely the fixed point of wrw^{-1}. Thus, in Figure 1, we see that $S' = \{wrw^{-1}, s, t\}$ is not conjugate to S since the corresponding fixed points do not all belong to the same chamber.

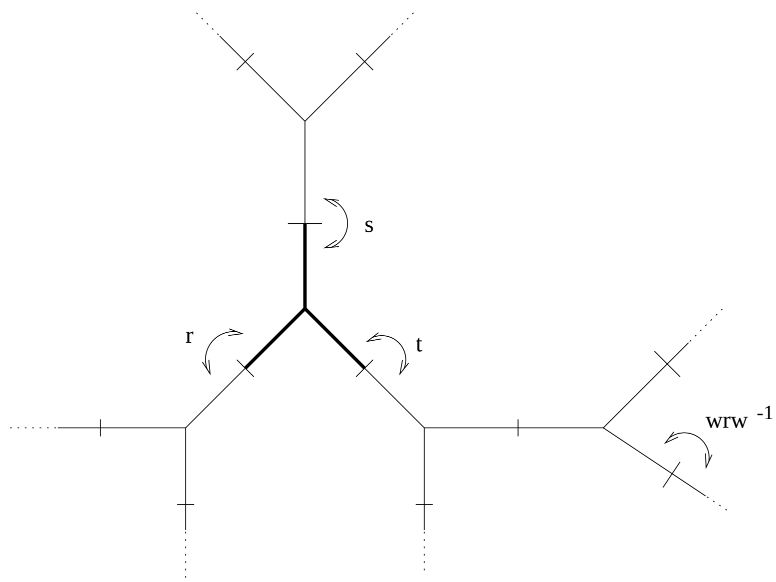

FIGURE 1.

Next we consider a situation where the Main Theorem is valid.

Example 0.3. Suppose that P is a convex k-gon in the hyperbolic plane. Suppose further that the angles at the vertices in cyclic order are $\pi/m_1, \cdots, \pi/m_k$, for some integers $m_i \geq 2$. Let S be the set of reflections across the edges of P and W the Coxeter group generated by S. Let P' be another such k-gon with angles $\pi/m'_1, \cdots, \pi/m'_k$ and let (W', S') be the corresponding Coxeter system. If $(m_1, \cdots, m_k)$ and $(m'_1, \cdots, m'_k)$ differ by a permutation, then, plausibly, the groups W and W' could be isomorphic. However, the Main Theorem implies that they are isomorphic only if there is a homeomophism $\partial P \to \partial P'$ taking each vertex with a given angle to a vertex with the same angle, that is, only if $(m_1, \cdots, m_k)$ and $(m'_1, \cdots, m'_k)$ differ by a cyclic permutation and possibly the order-reversing permutation of $\{1, \cdots, k\}$.

We began thinking about the Main Theorem after reading the papers of Rosas [12] and Prassidis and Spieler [11]. The goal of both these papers is a theorem which says something like the following. *Suppose W acts as a cocompact, locally linear, reflection group on contractible manifolds M_1 and M_2. Further suppose that for each finite subgroup G that (1) the fixed point sets M_1^G and M_2^G are contractible* *(they are automatically acyclic) and that (2) for $i = 1, 2$, if M_i^G is 3-dimensional, then it is homeomorphic to $\mathbb{R}^3$ (i.e., that it does not contain a counterexample to the Poincaré Conjecture). Then M_1 and M_2 are equivariantly homeomorphic. (Moreover, any equivariant map $M_1 \to M_2$ is equivariantly homotopic to an equivariant homeomorphism.)* Rosas' paper [12] deals only with the right-angled case. This assumption is then removed in [11]. There is a gap in the argument of both papers; a gap which is filled by our proof of the Main Theorem.

The corollary to the Main Theorem is related to previous work studying the automorphism group of a Coxeter group. For example, in [14] and [9] it is proved that for certain Coxeter groups, the outer automorphism group $\mathrm{Out}(W)$ is finite. (Of course, given our hypothesis, this is a consequence of the corollary to the Main Theorem.) In [14], Tits analyzes $\mathrm{Aut}(W)$ for a class of right-angled Coxeter groups and proves that $\mathrm{Out}(W)$ is finite for W in this class. (Basically, his class excludes the group in Example 0.2 from occurring as a special subgroup of W). The principal result of [9] is that if $m(s,t)$ is never equal to ∞, then $\mathrm{Out}(W)$ is finite.

§1. The basic geometry of reflection groups.

Suppose $\mathcal{P}$ is a poset. $\mathcal{P}$ is an *abstract simplicial complex* if it is isomorphic to the poset of simplices in a simplicial complex. Given a poset $\mathcal{P}$, its *order complex*, $\mathrm{Ord}(\mathcal{P})$, (also called its "derived complex") is the poset of all finite chains in $\mathcal{P}$, partially ordered by inclusion. It is easily checked that $\mathrm{Ord}(\mathcal{P})$ is an abstract simplicial complex. The corresponding topological space will be called the *geometric realization* of $\mathcal{P}$ and denoted by $\mathrm{geom}(\mathcal{P})$. In other words, there is a k-simplex in $\mathrm{geom}(\mathcal{P})$ for each chain $x_0 < x_1 < \cdots < x_k$ of length $k+1$. Given an element $x \in \mathcal{P}$, let $\mathcal{P}_{>x}$ denote the subposet of $\mathcal{P}$ consisting of all elements greater than x. Subposets $\mathcal{P}_{<x}$, $\mathcal{P}_{\geq x}$ and $\mathcal{P}_{\leq x}$ are defined similarly.

There are two different decompositions of the polyhedron $\mathrm{geom}(\mathcal{P})$ into subcomplexes. Both decompositions are indexed by $\mathcal{P}$. Given an element $x \in \mathcal{P}$, the subcomplex $\mathrm{geom}(\mathcal{P}_{\geq x})$ will be called (in this paper) a *face* of $\mathrm{geom}(\mathcal{P})$ while the subcomplex $\mathrm{geom}(\mathcal{P}_{\leq x})$ will be a *dual face*. Faces and dual faces are topologically cones; however, in general, they need not be homeomorphic to cells.

Let (W, S) be a Coxeter system. For any subset T of S, denote by W_T, the subgroup generated by T. (If T is empty, then W_T is the trivial subgroup.) A *special subgroup of* (W, S) is any subgroup of the form W_T, for some $T \subseteq S$. A special subgroup is *spherical* if it is finite. Similarly, a subset T of S is *spherical* if W_T is finite. A *parabolic subgroup of* (W, S) is a subgroup G of W of the form $G = wW_Tw^{-1}$, where $w \in W$ and $T \subseteq S$. The *rank* of G, denoted $rk_S(G)$, is the cardinality of T. G is *spherical* if W_T is.

A *reflection* of (W, S) is an element of W which is conjugate to an element of S. Denote by R_S the set of reflections of (W, S).

We will be particularly interested in $\mathcal{S}(S)$, the poset of spherical subsets of S. The partial order is inclusion. Let $P(S)$ be the set of spherical parabolic subgroups of (W, S). Also, for a given nonnegative integer k, denote by $\mathcal{S}_k(S)$ and $P_k(S)$ the set of spherical subsets of cardinality k and the set of spherical parabolic subgroups of rank k, respectively.

The *poset of spherical special cosets* is the poset

$$W\mathcal{S}(S) = \coprod_{T \in \mathcal{S}(S)} W/W_T$$

The partial order is inclusion.

The geometric realizations of the posets $\mathcal{S}(S)$ and $W\mathcal{S}(S)$ are denoted by $K(S)$ and $\Sigma(S)$, respectively, or when there is no ambiguity, simply by K and Σ. Σ is called the *complex associated to* (W, S) and K is a *chamber* of Σ. Thus, there is a k-simplex in Σ for each chain in $W\mathcal{S}(S)$ of the form $w_0W_{T_0} < w_1W_{T_1} < \cdots < w_kW_{T_k}$.

The minimal elements of $W\mathcal{S}(S)$ correspond to the elements of W $(= W/W_\phi)$. Hence, each maximal face of Σ has the form $\mathrm{geom}(W\mathcal{S}(S)_{\geq wW_\phi})$ for some $w \in W$. If we identify the maximal face $\mathrm{geom}(W\mathcal{S}(S)_{\geq W_\phi})$ with K, then any other maximal face can be written as wK for some unique $w \in W$. The maximal faces are called the *chambers* of Σ and K is the *fundamental chamber* corresponding to S.

As we shall see in Proposition 1.9, below, the dual faces of Σ are always cells, in the sense that the subposet corresponding to such a dual face is always isomorphic to the poset of faces of a convex polytope.

The group W acts on Σ and the stabilizer of any simplex is a spherical parabolic subgroup. It follows that the W-action is proper. If G is a finite subgroup of W, then Σ^G denotes the fixed point set of G on Σ.

In our first proposition we collect a few standard facts about Σ.

Proposition 1.1.

(i) Σ is contractible. (In fact, it has a natural piecewise Euclidean metric which is $CAT(0)$.)

(ii) For each $T \in \mathcal{S}(S), \Sigma^{W_T}$ is contractible.

(iii) If H is any finite subgroup of W, then there is a parabolic subgroup G such that $\Sigma^H = \Sigma^G$.

Proof. The first sentence of (i) is proved as Theorem 13.5 in [5]; the sentence in parenthesis is the main result of [10]. There are two methods which can be used to prove (ii) and (iii). The first is due to Tits. He showed that any Coxeter system (W, S) admits a representation as a linear reflection group on $\mathbb{R}^n, n =$ Card(S), so that the elements of S act as reflections. Moreover, there is a W-stable, open, convex subset I of $\mathbb{R}^n$ (the interior of the "Tits cone") such that each isotropy subgroup of W on I is a spherical parabolic and such that the space I W-equivariantly retracts onto Σ. Since W acts on I via affine maps, it follows that for any finite subgroup H, I^H is a nonempty convex subset of I; hence, it is contractible. Since each isotropy subgroup of W on I is a spherical parabolic, it follows that I^H coincides with I^G for some $G \in P(S)$. Statements (ii) and (iii) follow, since Σ is equivariantly homotopy equivalent to I. The alternative approach is to use Moussong's result that Σ admits a $CAT(0)$ metric with W acting via isometries (see [10] or [7].) The fixed point set of any finite group of isometries on a $CAT(0)$ space is nonempty and geodesically convex; hence contractible. □

Let Γ be a discrete group. A CW-complex E with a proper Γ-action is called a *classifying space for proper Γ-actions* if for each finite subgroup G of Γ (including the trivial subgroup), the fixed point set E^G is contractible. If Y is any other CW complex with a proper Γ-action, then there is a Γ-equivariant map $Y \to E$ which is unique up to a Γ-equivariant homotopy. (See Lemma 2.1 in [4].) It follows that

E is unique up to equivariant homotopy equivalence. (For any Γ it is not difficult to construct such a classifying space E. If Γ is torsion-free, then E is the universal cover of a $K(\Gamma, 1)$ complex.) Thus, we have the following corollary to Proposition 1.1.

Corollary 1.2. *Σ is a classifying space for proper W-actions.*

For each reflection r in R_S, its fixed point set is denoted by Σ^r and called the *wall* associated to r. The complement of the wall Σ^r in Σ has two components; the closure of either one is called a *half-space* bounded by Σ^r. By a *subspace* of Σ we shall mean the fixed point set of a parabolic subgroup.

Next we prove some lemmas concerning spherical parabolic subgroups.

Given a finite subgroup G of W, let $R_S(G)$ denote the set of reflections r such that $\Sigma^G \subset \Sigma^r$. Define $\widehat{rk}_S(G)$ to be the least integer k such that there are reflections $r_1, \cdots, r_k$ in $R_S(G)$ with $\Sigma^G = \Sigma^{r_1} \cap \cdots \cap \Sigma^{r_k}$.

Lemma 1.3. *If $G \in P(S)$, then $\widehat{rk}_S(G) = rk_S(G)$.*

Proof. Without loss of generality we can assume that $G = W_T$ for some spherical subset T. The pair (W_T, T) is a Coxeter system (Théorème 2, p. 20 [1]). The finite Coxeter group W_T has a canonical representation as a linear reflection group on $\mathbb{R}^k$, $k = \text{Card}\,(T) = rk_S(W_T)$. (See Ch. 5, §4 of [1].) Moreover, if x is a point of Σ with isotropy group equal to W_T, then x has a neighborhood of the form $U \times V$ where U is a neighborhood of x in Σ^G and V is W_T-equivariantly homeomorphic to a neighborhood of the origin in $\mathbb{R}^k$. Since the intersection of fewer than k hyperplanes in $\mathbb{R}^k$ is a nonzero subspace, we have that $\widehat{rk}_S(W_T) \geq k$. Trivially, $\widehat{rk}_S(W_T) \leq k$, so the lemma is proved. □

Lemma 1.4. *Suppose H and G are finite subgroups of W, with $H \subseteq G$ and with $H \in P(S)$. If $\widehat{rk}_S(H) = \widehat{rk}_S(G)$, then $H = G$.*

Proof. Since $H \subseteq G, \Sigma^G \subseteq \Sigma^H$. Let $k = \widehat{rk}_S(H)$. By Lemma 1.3 the codimension of Σ^H is k. Since $k = \widehat{rk}_S(G)$, we have that the codimension of Σ^G is $\leq k$. Hence, $\Sigma^G = \Sigma^H$. But the parabolic subgroup H is the maximal subgroup which fixes Σ^H. □

Lemma 1.5. *Suppose S and S' are two fundamental sets of generators for a Coxeter group W. If $P_1(S) = P_1(S')$, then $P_k(S) = P_k(S')$ for all $k \geq 1$.*

Proof. An element of $P_1(S)$ is a cyclic group of order 2 generated by a reflection in R_S. Since $P_1(S) = P_1(S')$, the two notions of reflection coincide, i.e., $R_S = R_{S'}$. Suppose that $H \in P_k(S)$. Then we can find k reflections $r_1, \cdots, r_k$ which generate H. Clearly, $r_i \in R_{S'}(H)$ and $\Sigma(S')^H = \Sigma(S')^{r_1} \cap \cdots \cap \Sigma(S')^{r_k}$. Hence, $\widehat{rk}_{S'}(H) \leq k = rk_S(H)$. Let G be a minimal S'-parabolic containing H (so that $\Sigma(S')^H = \Sigma(S')^G$). By the same reasoning as above $\widehat{rk}_S(G) \leq rk_{S'}(G)$. Therefore, $rk_S(H) \geq \widehat{rk}_{S'}(H) = rk_{S'}(G) \geq \widehat{rk}_S(G)$. Since $H \subseteq G$, $\Sigma(S)^G \subseteq \Sigma(S)^H$ and hence, $\widehat{rk}_S(G) \geq rk_S(H)$. So both of the inequalities in the previous sentence must be equalities. Thus, $rk_S(H) = \widehat{rk}_S(G)$. Lemma 1.4 then implies that $G = H$, i.e., that $H \in P_k(S')$. Therefore, $P_k(S) \subseteq P_k(S')$. Since the argument is symmetric in S and $S', P_k(S') \subseteq P_k(S)$, which proves the lemma. □

The proofs of the next two lemmas take only a moment's thought and are left to the reader.

Lemma 1.6. *The intersection of two parabolic subgroups is a parabolic subgroup.*

Lemma 1.7. *Suppose that G_1, G_2 are two spherical parabolic subgroups of W and that G is the subgroup generated by G_1 and G_2. Then $\Sigma^{G_1} \cap \Sigma^{G_2} = \Sigma^G$. Hence, $\Sigma^{G_1} \cap \Sigma^{G_2}$ is empty if and only if G is infinite.*

Definition 1.8. The poset $\mathcal{S}(S)_{>\emptyset}$ is an abstract simplicial complex. The corresponding topological space is a simplicial complex, denoted by $L(S)$ (or sometimes simply L) and called the *nerve of* (W, S). (In other words, there is a k-simplex of $L(S)$ for each spherical subset of cardinality $k+1$.)

For each $s \in S$ let v_s denote the vertex of L corresponding to s and for each $T \in \mathcal{S}(S)_{>\emptyset}$ let σ_T denote the simplex corresponding to T. Thus, if T is a subset of S, then the corresponding set of vertices $\{v_s\}_{s \in T}$ spans a simplex of L if and only if W_T is finite.

Next we analyze the fixed point set of a spherical special subgroup on $W\mathcal{S}(S)$.

Given $T \in \mathcal{S}(S)$, let $F(T)$ denote the fixed poset of W_T on $W\mathcal{S}(S)$, i.e.,

$$F(T) = \{wW_{T'} | W_T w W_{T'} = wW_{T'}\}$$
$$= \{wW_{T'} | W_T \subseteq wW_{T'}w^{-1}\}$$

Denote the set of minimal elements in $F(T)$ by $F_{\min}(T)$, i.e.,

$$F_{\min}(T) = \{wW_{T'} | W_T = wW_{T'}w^{-1}\}.$$

Proposition 1.9. *With notation as above, suppose $T \in \mathcal{S}(S)$ and that $wW_{T'}$ lies in the fixed poset $F(T)$. Let $\alpha = wW_{T'}$. Then the following statements are true.*

(i) $F(T)_{\leq \alpha}$ is isomorphic to the poset of faces of a convex cell of dimension Card(T') − Card(T).

(ii) $F(T)_{>\alpha}$ is isomorphic to $\mathcal{S}(S)_{>T'}$.

Proof. For $T = \emptyset$ the proofs of these statements can be found in [3] (Lemma 2.13) or [7]. In these references it is shown that $W\mathcal{S}(S)_{\leq \alpha}$ is isomorphic to the poset of faces of the "Coxeter cell" corresponding to $W_{T'}$. (This is defined as the convex hull of a free $W_{T'}$-orbit on the Euclidean space $\mathbb{R}^{T'}$.) Set $G = w^{-1}W_T w$. Then G is a subgroup of $W_{T'}$. It fixes a linear subspace of $\mathbb{R}^{T'}$ of codimension Card (T). The intersection of this subspace with the Coxeter cell is a convex cell of dimension Card (T') − Card (T). Its poset of faces is isomorphic to $F(T)_{\leq \alpha}$. This proves (i). If α is fixed by W_T, then so is any larger special coset. Hence, $F(T)_{>\alpha} = W\mathcal{S}(S)_{>\alpha} \cong \mathcal{S}(S)_{>T'}$, so (ii) holds. □

Suppose Λ is a cell complex. (By which we mean that each cell is isomorphic to a convex cell in some Euclidean space and that the intersection of any two cells is either empty or a common face of both.) Let $\mathcal{P}(\Lambda)$ be the poset of cells in Λ. If $\sigma \in \mathcal{P}(\Lambda)$, then $\mathcal{P}(\Lambda)_{>\sigma}$ is isomorphic to the poset of cells in another cell complex called the *link* of σ in Λ and denoted by $Lk(\sigma, \Lambda)$. (If $\tau \in \mathcal{P}(\Lambda)_{>\sigma}$, then the corresponding cell of $Lk(\sigma, \Lambda)$ has dimension $\dim(\tau) - \dim(\sigma)$.)

From Proposition 1.9 we immediately deduce the following.

Corollary 1.10. *Let $L = L(S)$ and $\Sigma = \Sigma(S)$. For any spherical special subgroup W_T, the cells $F(T)_{\leq \alpha}, \alpha \in F(T)$, give a cell structure on Σ^{W_T} in which the link of a vertex $\alpha = wW_{T'} \in F_{\min}(T)$ is isomorphic to $Lk(\sigma_{T'}, L)$.*

We finish this section by recalling a well-known construction of Tits [13] and Vinberg [15] and then using it to give an alternate description of Σ.

Suppose X is a space equipped with a distinguished family $(X_s)_{s\in S}$ of closed subspaces indexed by S. For each $x \in X$, let $S(x) = \{s \in S | x \in X_s\}$. Define an equivalence relation $\sim$ on $W \times X$ by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$. The quotient space is denoted by $\mathcal{U}(S, X)$ and the image of (w, x) by $[w, x]$. The group W acts on $\mathcal{U}(S, X)$ and the orbit space is X. Moreover, it has the universal property described in the following lemma (the proof of which is immediate).

Lemma 1.11. ([15], p. 1088). *Let Y be a space with W-action and let $f : X \to Y$ be any map such that, for each $s \in S, f(X_s)$ is contained in the fixed point set of s on Y. Then f extends to a W-equivariant map $\mathcal{U}(S, X) \to Y$ by the formula $[w, x] \to wf(x)$.*

The above construction can be used to get another picture of Σ. Recall that Σ and K are the geometric realizations of the posets $W\mathcal{S}(S)$ and $\mathcal{S}(S)$, respectively. For each $s \in S$, let K_s denote the geometric realization of $\mathcal{S}(S)_{\geq\{s\}}$; it is a subcomplex of K. The inclusion $\mathcal{S}(S) \to W\mathcal{S}(S)$ defined by $T \to W_T$, induces an inclusion $K \to \Sigma$ which satisfies the hypothesis of Lemma 1.11. Hence, there is a W-equivariant map $\mathcal{U}(S, K) \to \Sigma$ which is easily seen to be a homeomorphism. Thus, Σ can be identified with $\mathcal{U}(S, K)$.

The translates of K in Σ by elements of W are the *chambers* of Σ. The subcomplex $wK_s, w \in W$ and $s \in S$, is a *mirror* and $w\Sigma^s$ is a *wall* of the chamber wK. Thus, S indexes the set of walls of K and since $w\Sigma^s = \Sigma^{wsw^{-1}}$, we see that wSw^{-1} indexes the set of walls of wK.

§2. Coxeter groups of type HM.

A space X is a *homology m-manifold* if for each point x in X,

$$H_i(X, X-x) = \begin{cases} \mathbb{Z} & , i = m \\ 0 & , i \neq m \end{cases}$$

In particular, an m-dimensional simplicial complex Λ is a homology m-manifold if and only if for each k-simplex σ in Λ, $Lk(\sigma, \Lambda)$ has the same homology as the sphere S^{m-k-1}. We will say that Λ is a *generalized homology m-sphere* if it is a homology m-manifold and its homology is isomorphic to that of S^m.

Definition 2.1. A Coxeter system (W, S) is *type* HM_n if $L(S)$ is a generalized homology $(n-1)$-sphere. (We will sometimes drop the "n" from our notation when it need not be specified.)

The next proposition follows immediately from Corollary 1.10.

Proposition 2.2. *Suppose (W, S) is type HM_n. Then for each $T \in \mathcal{S}_k(S), \Sigma^{W_T}$ is a homology $(n-k)$-manifold. In particular (for $T = \emptyset$), Σ is a homology n-manifold.*

Theorem 2.3. ([5], [8]). *Given a Coxeter system (W, S), the following conditions are equivalent.*

(i) (W, S) is type HM_n.

(ii) There is an effective, proper, cocompact action of W on a contractible manifold such that the elements of S are the reflections across the walls of some chamber.

(iii) There is an effective, proper, cocompact action of W on a contractible manifold.

(iv) W is a virtual Poincaré duality group of dimension n (for short a "virtual PD_n group") and W does not have a nontrivial finite subgroup as a direct factor.

Proof. The equivalence of (i) and (ii) is proved in [5], the equivalence of (ii) and (iii) in Corollary 5.6 of [8]. Theorem B of [8] states that if a Coxeter group W is a virtual PD_n group, then (W, S) splits as $(W, S) = (W_1 \times W_2, S_1 \coprod S_2)$, where W_1 is finite and (W_2, S_2) is type HM_n, thus (iv) implies (i). Conversely, by Proposition 2.2, type HM_n implies that W is a virtual PD_n group. If W has a finite direct factor F, then it has normalizer $N(F) = W$. By Corollary 2.6 below, the maximal such finite subgroup is a parabolic subgroup of rank 0, or in other words, the trivial group. □

Corollary 2.4. *Suppose S and S' are two fundamental sets of generators for W. Then (W, S) is type HM_n if and only if (W, S') is type HM_n.*

Proof. Conditions (iii) and (iv) of Theorem 2.3 are independent of the fundamental set of generators. □

Lemma 2.5. *For any finite subgroup G of W, the cohomology of its normalizer $N(G)$ with group ring coefficients is given by*

$$H^*(N(G); \mathbb{Z}N(G)) \cong H_c^*(\Sigma^G).$$

Proof. $N(G)$ acts on Σ^G with compact quotient and with finite isotropy groups. It follows that if Γ is any torsion-free subgroup of finite index in $N(G)$, then Σ^G is the universal cover of a finite $K(\Gamma, 1)$-complex; hence, it is a standard result that $H^*(\Gamma; \mathbb{Z}\Gamma) \cong H_c^*(\Sigma^G)$. That the same result holds for $N(G)$ follows from Exercise 4, p. 209 in [2]. □

If Γ is virtually torsion-free , then its *virtual cohomological dimension*, denoted $vcd(\Gamma)$, is the cohomological dimension of any torsion-free subgroup of finite index.

Corollary 2.6. *Suppose (W, S) is type HM_n. If $G \in P_k(S)$, then $vcd(N(G)) = n - k$. Conversely, if a finite subgroup G is maximal among all finite subgroups H with the property that $vcd(N(H)) = n - k$, then $G \in P_k(W, S)$.*

Proof. We have that Σ^G is contractible (Proposition 1.1 (ii)) and a homology $(n-k)$-manifold (Proposition 2.2). In particular, it satisfies Poincaré duality. Hence,

$$H_c^i(\Sigma^G) = \begin{cases} \mathbb{Z} & , \ i = n - k \\ 0 & , \ i \neq n - k \end{cases}$$

By Lemma 2.5, $H^i(N(G); \mathbb{Z}N(G)) = H_c^i(\Sigma^G)$. The corollary then follows from Proposition 2.2, p. 185 in [2]. □

Corollary 2.7. *Suppose W is type HM_n and that S and S' are two fundamental sets of generators for W. Then for each positive integer k,*

$$P_k(S) = P_k(S').$$

The special case of this corollary where $k = 1$ means that S and S' define the same notion of reflection, i.e., $R_S = R_{S'}$.

§3. The Main Theorem for Coxeter groups of type HM.

Throughout this section we will assume that W is a Coxeter group of type HM_n. Our goal is to prove the following result.

Theorem 3.1. *Suppose S and S' are two fundamental sets of generators for W. Then there is a unique element $w \in W$ such that*

$$S' = wSw^{-1}.$$

Corollary 3.2. *$Aut(W) = Inn(W) \rtimes Diag(W, S)$.*

Proof of the Corollary. If two automorphisms agree on S, they are equal. If $\alpha \in \mathrm{Aut}(W)$, then $\alpha(S)$ is another fundamental set of generators; hence, by Theorem 3.1, $\alpha(S) = wSw^{-1}$. It follows that α can be written as $\alpha = \beta \circ \gamma$ where $\beta \in \mathrm{Inn}(W)$ and $\gamma \in \mathrm{Diag}(W, S)$. By the uniqueness assertion in Theorem 3.1 this decomposition is unique, so $\mathrm{Aut}(W)$ is the semidirect product. □

Set $\Sigma = \Sigma(S)$ and $K = K(S)$. To begin with we know, by Corollary 2.6, that $R_S = R_{S'}$. Hence, each element of S' acts as a reflection on Σ. The proof of Theorem 3.1 then has the following two steps.

Step 1. For each $s' \in S'$, the wall $\Sigma^{s'}$ separates Σ into two half-spaces. We will show that we can choose one of these, call it $H^{s'}$, so that

$$\bigcap_{s' \in S'} H^{s'}$$

is a nonempty union of chambers of Σ.

Denote this intersection of half-spaces by D.

Step 2. We will show that D is a single chamber of Σ.

Step 2 will basically complete the proof since $D = wK$ implies $S' = wSw^{-1}$. The assertion that w is unique is taken care of in Lemma 3.13, below.

Suppose an m-dimensional simplicial complex Λ is a homology manifold (or a pseudo-manifold, as defined in Section 5). Two m-simplices of Λ are *adjacent* if they share a codimension one face. A *gallery* in Λ is a sequence $(\sigma_1, \cdots, \sigma_\ell)$ of adjacent m-simplices. Suppose v is a vertex of Λ and that $\mathrm{Star}\,(v)$ denotes the open star of v in Λ. Two m-simplices σ and σ' in $\Lambda - \mathrm{Star}\,(v)$ are called *v-connected* if they can be connected by a gallery in $\Lambda - \mathrm{Star}\,(v)$.

Lemma 3.3. *Suppose we are given $s' \in S'$ and $T_1, T_2 \in \mathcal{S}_n(S')$ with $s' \notin T_1, T_2$, and assume that the corresponding simplices σ_{T_1} and σ_{T_2} are adjacent $(n-1)$-simplices in $L(S')$. Then $(T_1 \cap T_2) \cup \{s'\}$ generates an infinite subgroup of W.*

Proof. This lemma is just a restatement of the fact that the generalized homology sphere $L(S)$ is a pseudo-$(n-1)$-manifold. Thus, given an $(n-2)$-simplex, at most two $(n-1)$-simplices can have it as a common face. Since $\sigma_{T_1 \cap T_2}$ is a face of σ_{T_1} and of σ_{T_2}, $(T_1 \cap T_2) \cup \{s'\}$ cannot correspond to an $(n-1)$-simplex of $L(S')$, i.e., $W_{(T_1 \cap T_2) \cup \{s'\}}$ must be infinite. □

Remark 3.4. There is a dual picture to keep in mind. Let $\Sigma' = \Sigma(S')$ and $K' = K(S')$. The fact that W is of type HM means that each k-dimensional face of K' is the cone on a $(k-1)$-dimensional generalized homology sphere. In particular, each 0-dimensional face is a point and each 1-dimensional face is homeomorphic to an interval. We shall call the 0- and 1-dimensional faces of K' "vertices" and "edges", respectively. The maximal elements $T_1, T_2 \in \mathcal{S}_n(S')$ correspond to vertices x_1' and x_2' in K'. The statement that σ_{T_1} and σ_{T_2} are adjacent in $L(S')$ means that x_1' and x_2' are connected by an edge of K'. The statement that $s' \notin T_i, i = 1, 2$, means that $x_i' \notin K'_{s'}$. The edge determines a 1-dimensional subspace of Σ' namely $(\Sigma')^{W_{T_1 \cap T_2}}$. By Lemma 1.7, the meaning of the conclusion of the Lemma 3.3 is that this 1-dimensional subspace does not intersect the wall of Σ' corresponding to s'. The next lemma shows that, in view of Lemma 1.7, the corresponding assertion for

Σ is also true.

Lemma 3.5. *Given $s' \in S'$ and maximal elements T_1, T_2 in $\mathcal{S}(S')$ with $s' \notin T_1, T_2$, let x_1, x_2 be the points in Σ corresponding to W_{T_1}, W_{T_2}, respectively, i.e.,*

$$x_1 = \Sigma^{W_{T_1}}, x_2 = \Sigma^{W_{T_2}}.$$

If σ_{T_1} and σ_{T_2} are $v_{s'}$-connected in $L(S')$, then x_1 and x_2 lie on the same side of the wall $\Sigma^{s'}$.

Proof. Without loss of generality we may assume that σ_{T_1} and σ_{T_2} are adjacent, i.e., that $T_1 \cap T_2 \in \mathcal{S}_{n-1}(S')$. By Corollary 2.6, $W_{T_1 \cap T_2} \in P_{n-1}(S)$, so $E = \Sigma^{W_{T_1 \cap T_2}}$ is a 1-dimensional subspace of Σ. By Lemmas 1.7 and 3.3, $E \cap \Sigma^{s'} = \emptyset$. Hence, E is contained in an open half-space bounded by $\Sigma^{s'}$. Since x_1 and x_2 are both in E, this completes the proof. □

Lemma 3.6. *Suppose v is a vertex of a generalized homology $(n-1)$-sphere L. Then any two $(n-1)$-simplices in $L -$ Star(v) are v-connected.*

Proof. Set $N = L -$ Star (v). Then N is a homology $(n-1)$-manifold with boundary. Since Star (v) is contractible, N is acyclic. In particular, it is connected. It follows that any two $(n-1)$-simplices in N can be connected by a gallery. □

Thus, Lemma 3.5 can be applied to any two maximal elements of $\mathcal{S}(S')$ not containing s'.

Let $T_1, \cdots, T_\ell$ be the elements of $\mathcal{S}_n(S')$ (i.e., $\sigma_{T_1}, \cdots, \sigma_{T_\ell}$ are the $(n-1)$-simplices in $L(S')$). Denote the corresponding points in Σ by $x_i = \Sigma^{W_{T_i}}$. Suppose $s' \in S'$. If $s' \in T_i$, then $x_i \in \Sigma^{s'}$. By Lemma 3.5, $\Sigma^{s'}$ bounds a half-space $H^{s'}$ which contains all the x_i, with $s' \notin T_i$, in its interior. In any case $\{x_i\} \subset H^{s'}$. Set

$$D = \bigcap_{s' \in S'} H^{s'}.$$

The following lemma completes the proof of Step 1.

Lemma 3.7. *D is a nonempty union of chambers in Σ*

Proof. By definition, $\{x_i\} \subset D$ so D is nonempty. To prove that such an intersection of half-spaces is a union of chambers it suffices to show that it is not contained in any proper subspace of Σ (i.e., that it is not contained in the fixed point set of any nontrivial parabolic subgroup). Let E be the smallest subspace of Σ containing $\{x_1, \cdots, x_\ell\}$. Since $\{x_1, \cdots, x_\ell\} \subset D$, if D is contained in a proper subspace, then this subspace must contain E. Clearly, $E = \Sigma^{W_T}$ where $T = T_1 \cap \cdots \cap T_\ell$. If $T \neq \emptyset$, then every $(n-1)$-simplex of $L(S')$ contains σ_T. But this is not the case since $L(S')$ is a generalized homology sphere. Hence, $T = \emptyset$ and $E = \Sigma$. □

For each $s' \in S'$, set $D_{s'} = D \cap \Sigma^{s'}$. For each subset T of S', set

$$D_T = D \cap \bigcap_{s' \in T} \Sigma^{s'}.$$

Also, let

$$\partial D = \bigcup_{s' \in S'} D_{s'}.$$

Subcomplexes $K_{s'}(S')$, $K_T(S')$ and $\partial K(S')$ are defined similarly. D_T (resp. $K_T(S')$) is called a *face* of D (resp. $K(S')$).

Lemma 3.8. *(i) D_T is nonempty if and only if $T \in \mathcal{S}(S')$.*

(ii) For each $T \in \mathcal{S}(S')$, D_T is a contractible homology manifold with boundary $(\dim D_T = n - \text{ Card}(T))$.

(iii) There are face-preserving maps $\varphi : K(S') \to D$ and $\theta : D \to K(S')$ such that $\theta \circ \varphi$ and $\varphi \circ \theta$ are homotopic to the appropriate identity map and the homotopy is through face-preserving maps.

(iv) D contains only a finite number of chambers of Σ.

Proof. (i) If $T \in \mathcal{S}(S')$, then $D_T \neq \emptyset$ since it contains the point x_i for any i such that $T \leq T_i$. Conversely, since $D_T \subseteq \Sigma^{W_T}$ we see that if $D_T \neq \emptyset$, then W_T must be finite.

(ii) Let ∂D_T denote the union of the $D_{T'}$ where $T < T'$. Suppose $x \in D_T - \partial D_T$. Then, since $D_T - \partial D_T$ is an open subset of Σ^{W_T}, we see from Proposition 2.2 that x has a neighborhood in D_T which is a homology manifold of dimension $n-$ Card (T). Suppose $x \in \partial D_T$, say $x \in D_{T'} - \partial D_{T'}$. Then x has a neighborhood in D_T of the form $U_1 \times U_2$ where U_1 is open in $D_{T'} - \partial D_{T'}$ and U_2 is homeomorphic to a convex cone in some Euclidean space. It follows that $(D_T, \partial D_T)$ is a homology manifold with boundary. Since each half-space and each subspace of Σ is geodesically convex (in the Moussong metric) it follows that D_T is geodesically convex and hence, contractible

(iii) This is immediate from (i) and (ii).

(iv) By (ii), $(D, \partial D)$ is a contractible homology n-manifold with boundary. By (iii), ∂D and $\partial K(S')$ have the same homology, namely, that of S^{n-1}. Hence, $H_n(D, \partial D) \cong \mathbb{Z}$. This implies that D is compact and hence, contains only a finite number of chambers of Σ. □

We turn now to Step 2. We can apply the Tits-Vinberg construction of Section 1 to D and the family of subspaces $(D_{s'})_{s' \in S'}$ to define

$$\mathcal{U} = \mathcal{U}(S', D).$$

Lemma 3.9. *(i) $\mathcal{U}$ is W-equivariantly homotopy equivalent to $\Sigma(S')$.*

(ii) $\mathcal{U}$ is a classifying space for proper W-actions.

Proof. Using Lemma 1.11, the maps $\varphi : K(S') \to D$ and $\theta : D \to K(S')$ extend to maps $\Sigma(S') \to \mathcal{U}$ and $\mathcal{U} \to \Sigma(S')$ which are equivariant homotopy inverse to each other. By Corollary 1.2, $\Sigma(S')$ is a classifying space for proper W-actions, so (ii) is a consequence of (i). □

By Lemma 1.11 the inclusion of D into Σ extends to a W-equivariant map, which we will denote by $f : \mathcal{U} \to \Sigma$.

Lemma 3.10. *The map $f : \mathcal{U} \to \Sigma$ is a W-equivariant homotopy equivalence.*

Proof. Indeed, any equivariant map between two classifying spaces for proper W-actions is an equivariant homotopy equivalence. □

Let $\Sigma^{(2)}$ denote the union of the codimension-two subspaces of Σ, i.e.,

$$\Sigma^{(2)} = \bigcup_{G \in P_2(S)} \Sigma^G.$$

In the next lemma we will show that the map $f : \mathcal{U} \to \Sigma$ is a branched covering in an appropriate sense.

Lemma 3.11. *The restriction of f to $f^{-1}(\Sigma - \Sigma^{(2)})$ is a covering projection . Moreover, the number of sheets of this covering is equal to the number of chambers of Σ which are contained in D.*

Proof. Since D is a finite union of chambers of Σ, we have $D = w_1K \cup \cdots \cup w_pK$, where $w_1, \cdots, w_p$ are distinct elements of W. Let $x \in K - \Sigma^{(2)}$. If $[w, y] \in f^{-1}(x)$, then $wy = x$. Since $y \in D$, we must have $y = w_i x$ for some i, $1 \leq i \leq p$, and so $w \in W_x w_i^{-1}$ where W_x denotes the isotropy subgroup at x. We must show that x has a neighborhood U which is evenly covered by f. The first case to consider is where x lies in $K - \partial K$ (the interior of K). In this case, we can take $U = K - \partial K$. For each i, the subset $w_i^{-1} \times w_i U$ of $W \times D$ maps homeomorphically onto an open subset U_i of $\mathcal{U}$ and f takes U_i homeomorphically onto U. This shows that $f^{-1}(U) = \coprod U_i$ and that U is evenly covered by f.

The remaining case is where $x \in \partial K - \Sigma^{(2)}$. In other words, x lies in some mirror say $K_s, s \in S$, but not in any codimension-two face of K. Let U_+ be an open neighborhood of x in K such that $U_+ \cap K_t = \emptyset$, for any $t \neq s$. Then $U = U_+ \cup sU_+$ is an open neighborhood of x in Σ. For each i, $1 \leq i \leq p$, we shall now define an open subset U_i of $f^{-1}(U)$. There are two cases to consider.

Case 1. $w_i s w_i^{-1} \notin S'$.

In this case $w_i U \subset D$. As before, define U_i to be the image of $w_i^{-1} \times w_i U$ in $\mathcal{U}$.

Case 2. $w_i s w_i^{-1} \in S'$, say, $w_i s w_i^{-1} = s'$.

We still have $w_i U_+ \subset D$, and $w_i U_+ \cap \partial D \subset D_{s'}$. Thus, the images of $w_i^{-1} \times w_i U_+$ and of $w_i^{-1} s' \times w_i U_+$ in $\mathcal{U}$ fit together to give an open set U_i in $\mathcal{U}$. Moreover, since $w_i^{-1} \times w_i U_+$ maps onto U_+ and $w_i^{-1} s' \times w_i U_+$ maps onto $w_i^{-1} s' w_i U_+ = sU_+$, we see that f takes U_i homeomorphically onto U, that $f^{-1}(U) = \coprod U_i$, and that U is evenly covered by f.

Since any point in $\Sigma - \Sigma^{(2)}$ can be written in the form wx for some $w \in W$ and $x \in K$, and since f is equivariant, it follows that every point in $\Sigma - \Sigma^{(2)}$ has a neighborhood which is evenly covered by f. Moreover, the number of sheets is p.
□

Remark 3.12. Let us revisit, for a moment, Example 0.1 of the Introduction. Suppose that G is a finite dihedral group of order $2m$ acting on $\mathbb{R}^2$ as the group generated by a set T of reflections across two lines making an angle of π/m and bounding a sector C. Let T' be the set of two reflections across two other lines making an angle of $k\pi/m$ and bounding a sector C', where $gcd(k,m)=1$. Then (G,T) and (G,T') are both Coxeter systems. Moreover, the natural map $\mathcal{U}(T',C')\to\mathbb{R}^2$ is a k-fold cyclic branched cover branched at the origin of $\mathbb{R}^2$.

This local picture shows that if y is a point in some codimension-two face of D and if y belongs to more than one chamber in D, then f is branched over y, i.e., there is no neighborhood of y which is evenly covered by f. Conversely, if each point in each codimension-two stratum of D belongs to exactly one chamber in D, then the map $f:\mathcal{U}\to\Sigma$ is a covering projection. (These remarks hold regardless of whether or not (W,S) is type HM.)

We are now in position to complete the proof of Step 2. The argument goes roughly as follows. The spaces $\mathcal{U}$ and Σ are contractible homology n-manifolds. Hence, $H^n_c(\mathcal{U})\cong H^n_c(\Sigma)\cong\mathbb{Z}$. So, after picking orientations for $\mathcal{U}$ and Σ, it makes sense to speak of the *degree* of f, denoted $\deg(f)$: it is the integer d such that f^* takes the orientation class of Σ to d times the orientation class of $\mathcal{U}$. The fact (Lemmas 3.8 (iv) and 3.10) that f is a proper homotopy equivalence means that $\deg(f)=1$. On the other hand, a local computation using the fact that f is a branched cover (Lemma 3.11) shows that $\deg(f)=p$ where p is the number of chambers in D. Hence, $p=1$ and D consists of a single chamber. The details of these arguments follow.

First we orient K,Σ,D and $\mathcal{U}$ so that we will have distinguished generators for the infinite cyclic groups, $H^n(K,\partial K),H^n_c(\Sigma),H^n(D,\partial D)$ and $H^n_c(\mathcal{U})$. (By an "orientation " we mean a compatible choice of orientation for each n-simplex in the space.) First orient K. (This can be done since $H^1(K;\mathbb{Z}/2)=0$.) Each n-simplex in wK is then oriented by $(-1)^{\ell_S(w)}$ times the orientation of the corresponding simplex of K. (Here $\ell_S(w)$ denotes the word length of w with respect to S.) This orients Σ and D. Then orient $\mathcal{U}$ by orienting each n-simplex in wD using $(-1)^{\ell_{S'}(w)}$

times the orientation of the corresponding simplex in D.

Let $U = K - \partial K$ and $f^{-1}(U) = U_1 \cup \cdots \cup U_p$, as in the proof of Lemma 3.11. Denote the duals of the orientation classes in $H^n(K, \partial K), H^m(\overline{U}_i, \partial\overline{U}_i), H^n_c(\Sigma)$ and $H^n_c(\mathcal{U})$ by e, e_i, e_Σ and $e_{\mathcal{U}}$, respectively. By excision, $H^n(\Sigma, \Sigma - U) = H^n(K, \partial K) \cong \mathbb{Z}$. Since $H^{n-1}_c(\Sigma - U) = 0 = H^n_c(\Sigma - U)$, the exact sequence of the pair $(\Sigma, \Sigma - U)$ shows that the natural map $H^n(K, \partial K) = H^n(\Sigma, \Sigma - U) \to H^n_c(\Sigma)$ sends e to e_Σ. Similarly,

$$\begin{aligned} H^n(\mathcal{U}, \mathcal{U} - (U_1 \cup \cdots \cup U_p)) &\cong \bigoplus_{i=1}^{p} H^n(\overline{U}_i, \partial\overline{U}_i) \\ &\cong \mathbb{Z}^p \end{aligned}$$

and the natural map $\oplus H^n(\overline{U}_i, \partial\overline{U}_i) \to H^n_c(\mathcal{U})$ sends e_i to $e_{\mathcal{U}}$. Also, the map $(f|U_i)^* : H^n(K, \partial K) \to H^n(\overline{U}_i, \partial\overline{U}_i)$ sends e to e_i. Consider the commutative diagram

$$\begin{array}{ccc} H^n_c(\Sigma) & \xrightarrow{f^*} & H^n_c(\mathcal{U}) \\ \uparrow & & \uparrow \\ H^n(K, \partial K) & \longrightarrow & \oplus H^n(\overline{U}_i, \partial\overline{U}_i). \end{array}$$

It follows that $f^*(e_\Sigma)$ is the sum of the images of the e_i in $H^n_c(\mathcal{U})$, i.e., $f^*(e_\Sigma) = pe_{\mathcal{U}}$, so $\deg f = p$.

By Lemma 3.8 (iv) and Lemma 3.10, f is a proper homotopy equivalence, so $\deg(f) = 1$, and hence, $p = 1$. Thus, D is a single chamber of Σ, say, $D = wK$, for some $w \in W$. But this implies that $S' = wSw^{-1}$.

To complete the proof of Theorem 3.1 it remains to show that w is unique. This follows immediately from the next lemma.

Lemma 3.13. *If $wSw^{-1} = S$, then $w = 1$.*

Proof. The proof is similar to that of Lemma 3.7. Let $T_1, \cdots, T_\ell$ be the elements of $\mathcal{S}_n(S)$ and for $1 \le i \le \ell$, let y_i denote the unique fixed point of W_{T_i}. Then $y_i \in K$. Suppose $wSw^{-1} = S$. Since wy_i is the unique fixed point of $W_{wT_iw^{-1}}$ and $wT_iw^{-1} \subset S$, we also have that $wy_i \in K$; hence, $w \in W_{T_i}$. Since this holds for all i, $w \in W_T$, where $T = T_1 \cap \cdots \cap T_\ell$. But, as in the proof of Lemma 3.7, the fact that

$L(S)$ is a generalized homology sphere implies that $T = \emptyset$ and hence, that $w = 1$.

□

Corollary 3.14. *The center of* W *is trivial.*

§4. The cohomology of fixed point sets of parabolic subgroups.

We return to the situation of Section 1 where (W, S) is an arbitrary Coxeter system, Σ the associated complex and K a fundamental chamber. One of the main results of [8] is a formula for $H^*_c(\Sigma)$ as a direct sum of terms of the form $H^*(K, \delta_w K)$ where the $\delta_w K$ are certain subcomplexes of ∂K. In fact, the argument of [8] gives a similar calculation of $H^*_c(\Sigma^G)$ for any spherical parabolic subgroup G. In order to state this formula it is first necessary to develop some notation.

For each $T \in \mathcal{S}(S)$, define subcomplexes K_T and ∂K_T of K to be the geometric realizations of $\mathcal{S}(S)_{\geq T}$ and $\mathcal{S}(S)_{>T}$, respectively.

The vertex set of $Lk(\sigma_T, L)$ is the set of $T' \in \mathcal{S}(S)$ such that $T \subset T'$ and $\mathrm{Card}\,(T' - T) = 1$. Hence, the vertices of $Lk(\sigma_T, L)$ are indexed by the set

$$V_T = \{a \in S - T | T \cup \{a\} \subset \mathcal{S}(S)\}.$$

For each $a \in V_T$, set $\partial_a K_T = K_{T\cup\{a\}}$ and for each subset A of V_T, put

$$\partial_A K_T = \bigcup_{a \in A} \partial_a K_T.$$

Recall from Section 1 that $F(T)$ is the subposet of $W\mathcal{S}(S)$ fixed by W_T and that $F_{\min}(T)$ is the set of minimal elements in $F(T)$, i.e.,

$$F_{\min}(T) = \{wW_{T'} | W_T = wW_{T'}w^{-1}\}.$$

We need to pick a set of coset representatives for $F_{\min}(T)$. As the next lemma shows there is a natural way to do this.

Lemma 4.1. *Given a coset* $wW_{T'} \in F_{\min}(T)$, *there is a unique element* w_0 *of longest length in* $wW_{T'}$. *Moreover,* $T = w_0 T' w_0^{-1}$.

Proof. By Exercise 3, p. 37 in [1] and Lemma 1.6 in [8] the following statements are equivalent:

(i) w is an element of longest length in $W_T w$.

(ii) For all $u \in W_T$, $\ell(uw) = \ell(w) - \ell(u)$.

(iii) For all $t \in T, \ell(tw) = \ell(w) - 1$.

Similarly, the following are equivalent:

(i)′ w is an element of longest length in $wW_{T'}$.

(ii)′ For all $u' \in W_{T'}$, $\ell(wu') = \ell(w) - \ell(u')$.

(iii)′ For all $t' \in T'$, $\ell(wt') = \ell(w) - 1$.

It follows from the equivalence of (i) and (ii) (or (i)′ and (ii)′) that an element of longest length in a coset is unique.

Since $W_T w = wW_{T'}$, (i) is equivalent to (i)′; hence, all six statements are equivalent. Now suppose that w_0 is the longest element in $wW_{T'}$. Then for all $t' \in T'$, $\ell(w_0) - 1 = \ell(w_0 t') = \ell((w_0 t' w_0^{-1}) w_0) = \ell(w_0) - \ell(w_0 t' w_0^{-1})$. Hence, $\ell(w_0 t' w_0^{-1}) = 1$, i.e., $w_0 t' w_0^{-1} \in T$. So, $T = w_0 T' w_0^{-1}$.

□

Let $\mathcal{F}(T)$ be the set of w in W such that for some $T' \in \mathcal{S}(S)$ the coset $wW_{T'} \in F_{\min}(T)$ and such that w is the longest element in $wW_{T'}$. ($\mathcal{F}(T)$ is a set of coset representatives for $F_{\min}(T)$.) For each $w \in \mathcal{F}(T)$, let T_w be the corresponding element $T' \in \mathcal{S}(S)$, i.e., $T_w = w^{-1}Tw$.

Recall from [8], that $S(w) = \{s \in S | \ell(ws) < \ell(w)\}$.

For each $w \in \mathcal{F}(T)$, set $A(w,T) = V_{T_w} - S(w)$ and

$$\delta_w K_{T_w} = \bigcup_{a \in A(w,T)} \partial_a K_{T_w}.$$

For $T = \emptyset$, the following theorem was proved as Theorem A in [8].

Theorem 4.2. *Given $T \in \mathcal{S}(S)$,*

$$H_c^*(\Sigma^{W_T}) = \bigoplus_{w \in \mathcal{F}(T)} H^*(K_{T_w}, \delta_w K_{T_w}).$$

Remarks 4.3. (i) In the above formula T_w ranges over those subsets T' of S which are conjugate to T.

(ii) The $wK_{T_w}, w \in \mathcal{F}(T)$, are the codimension-zero faces of Σ^{W_T}.

(iii) If we order the elements of $\mathcal{F}(T)$ by word length, then $\delta_w K_{T_w}$ is the union of faces $\partial_a K_{T_w}$ such that the adjacent face to wK_{T_w} across $w\partial_a K_{T_w}$ is further from the base K_T than is wK_{T_w}.

Proof of Theorem 4.2. In §4 of [8] we defined, for each $w \in W$, a map of simplicial cochain complexes $\rho_w : C^*(K, \delta_w K) \to C_c^*(\Sigma)$. If $w \in \mathcal{F}(T)$, then ρ_w restricts to a map

$$\rho_w^T : C^*(K_{T_w}, \delta_w K_{T_w}) \to C_c^*(\Sigma^{W_T}),$$

hence, we get

$$\rho^T : \bigoplus_{w \in \mathcal{F}(T)} C^*(K_{T_w}, \delta_w K_{T_w}) \to C_c^*(\Sigma^{W_T}).$$

A similar argument to the proof of Theorem 4.2 in [8] then shows that ρ^T induces an isomorphism in cohomology. □

§5. Coxeter groups of type PM.

A locally finite simplicial complex Λ is a *pseudo-m-manifold* if (a) each maximal simplex of Λ is m-dimensional and (b) each $(m-1)$-simplex is a face of precisely two m-simplices. It follows that if σ is a k-simplex in a pseudo-m-manifold Λ, then $Lk(\sigma, \Lambda)$ is a pseudo-manifold of dimension $m - k - 1$. A pseudo-m-manifold is *orientable* if one can choose orientations for the m-simplices so that their sum is an m-cycle (possibly an infinite m-cycle). An *orientation* for Λ is a choice of such an orientation for each m-simplex. If Λ is orientable, then so is $Lk(\sigma, \Lambda)$.

Definition 5.1. A Coxeter system (W, S) is *type PM_n* if $L(S)$ is an orientable pseudo-$(n-1)$-manifold and if $H_{n-1}(L(S)) \cong \mathbb{Z}$. (This last condition is equivalent to the condition that the complement of the codimension-two skeleton of $L(S)$ be connected.)

We suppose throughout this section that (W, S) is type PM_n.

We begin with the analogs of Proposition 2.2 and Corollary 2.6 for Coxeter groups of type PM_n.

Proposition 5.2. *For each $T \in \mathcal{S}_k(S), \Sigma^{W_T}$ is an orientable, pseudo-manifold of dimension $n-k$.*

Proof. As in Proposition 2.2, it follows from Corollary 1.10 that Σ^{W_T} is a pseudo-$(n-k)$-manifold. The fact that it is orientable is a consequence of the fact that it is contractible (Proposition 1.1 (ii)) and hence, that $H^1(\Sigma^{W_T}; \mathbb{Z}/2) = 0$. □

Corollary 5.3. *If $G \in P_k(S)$, then $vcd(N(G)) = n-k$.*

Corollary 5.4. *If $G \in (P_{n-1}(S))$, then Σ^G is homeomorphic to the real line.*

Proof. A pseudo-1-manifold is a 1-manifold. □

Lemma 5.5. $H^n(W; \mathbb{Z}W) \cong \mathbb{Z}$

Proof. By Lemma 2.5, this is equivalent to showing that $H^n_c(\Sigma) \cong \mathbb{Z}$. (Even though we know that Σ is a connected, orientable, pseudo-n-manifold, it could happen that $H^n_c(\Sigma)$ has rank > 1.) We apply Theorem 4.2 when $T = \emptyset$. Let $\delta_w K$ be the subcomplex of ∂K defined in Section 4 as $\delta_w K_{T_w}$ (where $T = T_w = \emptyset$). Then

$$\begin{aligned} H^n_c(\Sigma) &= \bigoplus_{w \in W} H^n(K, \delta_w K) \\ &= \bigoplus_{w \in W} \overline{H}^{n-1}(\delta_w K) \end{aligned}$$

Since L is an orientable, pseudo-$(n-1)$-manifold with $\overline{H}_{n-1}(L) \cong \mathbb{Z}$, we have that $\overline{H}^{n-1}(L) \cong \mathbb{Z}$ and for any proper subcomplex X of L, we have that $\overline{H}^{n-1}(X) = 0$. In the above formula, if $w \neq 1$, then $\delta_w K$ is a proper subcomplex of the barycentric subdivision of L; while, for $w = 1$, $\delta_1 K = \partial K = L$. Hence,

$$\overline{H}^{n-1}(\delta_w K) = \begin{cases} \mathbb{Z} & : \text{ if } w = 1 \\ 0 & ; \text{ if } w \neq 1 \end{cases}$$

Thus, $H^n_c(\Sigma) = \overline{H}^{n-1}(\delta_1 K) \cong \mathbb{Z}$. □

Our next goal is to prove the analog of Corollary 2.4: that if S' is another fundamental set of generators for W, then (W, S') is also type PM_n. The sticky point is that the analog of Theorem 2.3 is not available. To get around this we will first have to prove the analog of Corollary 2.7 (as Proposition 5.8, below) by a separate argument.

Lemma 5.6. *W does not have a nontrivial finite subgroup as a direct factor.*

Proof. Suppose $W = G \times W'$ with G finite. Then $vcd(N(G)) = vcd(W) = n$. Since Σ^G coincides with the fixed point set of a parabolic subgroup of some rank k, we must have $k = 0$ and $\Sigma^G = \Sigma$. Hence, G is trivial. $\square$

Lemma 5.7. *Let S' be another fundamental set of generators for W. Then, for any $s' \in S'$, $vcd(N(s')) = n - 1$.*

Proof. Set $\Sigma' = \Sigma(S')$ and $K' = K(S')$. By Lemma 2.5 (for G the trivial subgroup) and Lemma 5.5, $H^n_c(\Sigma') \cong \mathbb{Z}$. So, by Theorem 4.2,

$$\begin{aligned}\mathbb{Z} &\cong \bigoplus_{w \in W} H^n(K', \delta_w K') \\ &\cong \bigoplus_{w \in W} \overline{H}^{n-1}(\delta_w K').\end{aligned}$$

In [8] (Lemma 1.10) it is proved that if w is not the element of longest length in a spherical special subgroup which is a direct factor of W, then terms isomorphic to $\overline{H}^{n-1}(\delta_w K')$ occur more than once in the sum. So, by Lemma 5.6, the only possibility for a nonzero term in the sum is when $w = 1$. Hence, we must have that

$$\overline{H}^{n-1}(\delta_w K') = \begin{cases} \mathbb{Z} & , \text{ if } w = 1 \\ 0 & , \text{ if } w \neq 1 \end{cases}$$

Also, since $H^i_c(\Sigma') = 0$ for all $i > n$, $H^i_c(\delta_w K') = 0$ for all $i \geq n$ and all $w \in W$. Consider the special case $w = s'$. Then $\delta_{s'} K'$ is the closure of $\partial K' - K'_{s'}$. By excision,

$$H^*(\partial K', \delta_{s'} K') \cong H^*(K'_{s'}, \partial K'_{s'}).$$

Combining this with the exact sequence of the pair $(\partial K', \partial_{s'} K')$ we get

$$\begin{array}{ccccc} \to H^{n-1}(\partial K', \delta_{s'} K') & \to \overline{H}^{n-1}(\partial K') & \to \overline{H}^{n-1}(\delta_{s'} K') \\ \cong\downarrow & \cong\downarrow & \cong\downarrow \\ H^{n-1}(K'_{s'}, \partial K'_{s'}) \quad \to & \mathbb{Z} \quad \to & 0 \end{array}$$

where the calculations $H^{n-1}(\partial K') \cong \mathbb{Z}$ and $\overline{H}^{n-1}(\delta_{s'} K') = 0$ are given above. Thus, $H^{n-1}(K'_{s'}, \partial K'_{s'})$ has rank at least one. Similarly, $H^i(K'_{s'}, \partial K'_{s'}) = 0$ for $i \geq n$. Theorem 4.2 then implies that $\overline{H}^{n-1}((\Sigma')^{s'})$ has rank at least one and that $H^i((\Sigma')^{s'}) = 0$ for $i \geq n$. So, by Lemma 2.5, $vcd(N(s')) = n - 1$. $\square$

Proposition 5.8. *Let S' be another fundamental set of generators for W. Then, for all positive integers k, $P_k(S) = P_k(S')$.*

Proof. A maximal spherical parabolic subgroup (with respect to either S or S') is just a maximal finite subgroup of W. Thus, the maximal elements of $P(S)$ and $P(S')$ coincide. Since $L(S)$ is a pseudo-manifold, any T in $\mathcal{S}(S)$ is an intersection of maximal elements (i.e., of elements of $\mathcal{S}_n(S)$). It follows that any spherical S-parabolic subgroup is an intersection of maximal parabolic subgroups. Therefore, by Lemma 1.6, any spherical S-parabolic is also an S'-parabolic, i.e., $P(S) \subseteq P(S')$. By Lemma 1.5, it suffices to prove that $P_1(S) = P_1(S')$. A parabolic subgroup lies in $P_1(S)$ (or $P_1(S')$) if and only if it is of order 2. So, $P_1(S) \subseteq P_1(S')$. Let $s' \in S'$. By Lemma 5.7, $vcd(N(s')) = n - 1$. Hence, the fixed point set of s' on Σ must have dimension $n - 1$, i.e. $s' \in R_S$. Thus, $P_1(S') \subseteq P_1(S)$ and consequently, $P_1(S) = P_1(S')$. □

Proposition 5.9. *Let S' be another fundamental set of generators for W. Then (W, S') is also type PM_n.*

Proof. We must show that $L(S')$ satisfies the conditions of Definition 5.1. Since the maximal elements of $P(S)$ and $P(S')$ are equal and since $P_n(S) = P_n(S')$, it follows that $P_n(S')$ is the set of maximal elements of $P(S')$. In other words, every maximal simplex of $L(S')$ has dimension $n-1$. Let $T \in \mathcal{S}_{n-1}(S')$ (so that σ_T is an $(n-2)$-simplex). By Proposition 5.8, $W_T \in P_{n-1}(S)$ and by Corollary 5.4, Σ^{W_T} is homeomorphic to $\mathbb{R}$. Thus,

$$H^i_c(\Sigma^{W_T}) = \begin{cases} \mathbb{Z} & , \text{ if } i = 1 \\ 0 & , \text{ if } i \neq 1 \end{cases} \tag{1}$$

By Lemma 2.5, this is also the formula for $H^i(N(W_T); \mathbb{Z}N(W_T))$. Hence, (1) holds for $\Sigma(S')^{W_T}$ as well. For any T, with $\dim \sigma_T = n - 2$, we have that K'_T is 1-dimensional and that $\partial K'_T$ is a finite set of points. By Theorem 4.2,

$$H^1_c(\Sigma(S')^{W_T}) = \bigoplus_{w \in \mathcal{F}(T)} H^1(K'_{T_w}, \delta_w K'_{T_w}). \tag{2}$$

All the terms on the right hand side of (2) occur an infinite number of times except the one for w the element of longest length in W_T. Hence, $H^1(K'_T, \partial K'_T) \cong \mathbb{Z}$, i.e., $\partial K'_T = S^0$. In other words, $Lk(\sigma_T, L(S')) = S^0$. So, $L(S')$ is a pseudo-$(n-1)$-manifold. By Lemmas 2.5 and 5.5,

$$\mathbb{Z} \cong H^n_c(\Sigma(S')) = \bigoplus_{w \in W} H^n(K', \delta_w K').$$

A similar argument to the above shows that $H^n(K', \partial K') \cong \mathbb{Z}$ and hence, that $H_{n-1}(L(S')) \cong \mathbb{Z}$. This together with the calculation $H_{n-1}(L(S'); \mathbb{Z}/2) \cong \mathbb{Z}/2$, shows that $L(S')$ is orientable. Thus, (W, S') is also type PM_n. □

We are now in position to prove the Main Theorem for groups of type PM. We state it again.

Theorem 5.10. *Suppose that W is type PM and that S and S' are two fundamental sets of generators for W. Then there is a unique element $w \in W$ such that*

$$S' = wSw^{-1}.$$

The proof is substantially the same as the one given in Section 3. One point that requires some further amplification is the analog of Lemma 3.6, which was needed in the proof of Step 1. This is the following lemma.

Lemma 5.11. *Suppose that L is an orientable, pseudo-$(n-1)$-manifold and that $\overline{H}_{n-1}(L) \cong \mathbb{Z}$. If v is a vertex of L, then any two $(n-1)$-simplices in $L-$ Star(v) are v-connected.*

Before giving the proof, we remark that the lemma does not follow simply from the fact that the orientable pseudo-manifold L is connected (if $n \geq 1$). An example to keep in mind is where L is the suspension of two disjoint circles and v is one of the suspension points. The complement of Star(v) is then a wedge of two 2-disks. Two 2-simplices in this complement can be connected by a gallery if and only if they lie in the same 2-disk. (This is not a counterexample to Lemma 5.11 because $H_2(L) \cong \mathbb{Z} \oplus \mathbb{Z}$.)

Proof of Lemma 5.11. Let $C_1, \cdots, C_k$ be the v-connected classes of $(n-1)$-simplices in $L-$ Star (v). Each $(n-2)$-simplex in $Lk(v, L)$ is a face of exactly two $(n-1)$-simplices one of which belongs to Star (v) and the other to some C_i. Extend each equivalence class C_i to C_i' by adjoining the appropriate simplices in Star (v). Choose an orientation for L and use it to orient each $(n-1)$-simplex. Define cycles ζ_i, by

$$\zeta_i = \sum_{\sigma \in C_i'} \sigma.$$

Then $\zeta_1, \cdots, \zeta_k$ are $(n-1)$-cycles which represent linearly independent classes in $\overline{H}_{n-1}(L)$. By hypothesis $\overline{H}_{n-1}(L) \cong \mathbb{Z}$, so $k=1$. □

Using this lemma in place of Lemma 3.6, the proof of Step 1 in Section 3 goes through without change. The proof of Step 2 also goes through. Basically the only change which is needed is to replace the phrase "homology manifold" by "orientable pseudo-manifold" whenever it occurs.

Example 5.12. Here is a generalization of the idea in Example 0.2. First, suppose that the nerve of (W, S) is the disjoint union of two $(n-1)$-spheres. In other words, $S = S_1 \coprod S_2$, $W = W_1 * W_2$, and $L(S) = L(S_1) \coprod L(S_2)$, where $L(S_1)$ and $L(S_2)$ are triangulations of the $(n-1)$-sphere. Then although $L(S)$ is an orientable pseudo-manifold, (W, S) is not type PM since the second condition in Definition 5.1 (that $H_{n-1}(L(S)) \cong \mathbb{Z}$) fails to hold. In fact, the Main Theorem also fails for W. To see this, first note that $\Sigma(S)$ is the union of an infinite number of copies of the contractible manifolds $\Sigma(S_1)$ and $\Sigma(S_2)$. The intersection of two such copies is either empty or a single point. Choose a nontrivial element $w_1 \in W_1$ and set $S_2' = w_1 S_2 w_1^{-1}$ and $S' = S_1 \coprod S_2'$. S' is clearly a fundamental set of generators for W; however, S and S' are not conjugate since S' is not the set of walls of a chamber of $\Sigma(S)$.

As a slight modification of this example, replace W by $W \times D_\infty$ and S by $S \coprod S_0$, where S_0 consists of two fundamental generators for the infinite dihedral group. In this case the nerve is the suspension of $L(S)$, i.e., it is the suspension of two $(n-1)$-spheres. Thus, the nerve is a *connected* pseudo-manifold (but the second condition in Definition 5.1 still fails). As before, we see that $S \coprod S_0$ and $S' \coprod S_0$ are fundamental sets of generators for $W \times D_\infty$ and that they are not conjugate. These examples show that the second condition in Definition 5.1 is necessary for the main theorem to hold.

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