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THE HOMOLOGY OF A SPACE ON WHICH A REFLECTION GROUP ACTS

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0. Suppose that (W, S) is a Coxeter system, that X is a CW-complex, and that $(X_s)_{s\in S}$ is a family of subcomplexes indexed by S. Given the above data, there is a classical construction of a CW-complex $\mathcal{U}$ with W-action: $\mathcal{U}$ is obtained by pasting together copies of X, one for each element of W. To be more explicit, for each x in X, let W_x denote the subgroup of W generated by the set of s in S such that x belongs to X_s; let $\sim$ denote the equivalence relation on $W \times X$ defined $(w, x) \sim (v, y) \Leftrightarrow x = y$ and $w^{-1}v \in W_x$; the complex $\mathcal{U}$ is then defined as the quotient space $(W \times X)/\sim$. We identify X with the image of $1 \times X$ in $\mathcal{U}$. The subcomplexes wX, $w \in W$, are called the *chambers* of $\mathcal{U}$; while the subcomplexes wX_s, $s \in S$, are the *mirrors* of wX.

The *length* of an element w of W, denoted by $l(w)$, is the smallest integer n such that w is the product of n elements in S. Put

$$S(w) = \{s \in S \mid l(ws) < l(w)\}.$$

(If s is in S, then the element wsw^{-1} acts on $\mathcal{U}$ as a reflection across the mirror wX_s, taking the chamber wX to the adjacent chamber wsX. Therefore, the set $S(w)$ indexes the set of mirrors of wX with the property that the adjacent chamber across the mirror is one chamber closer to X.)

For each subset T of S, let X^T be the subcomplex of X defined by

$$X^T = \bigcup_{t\in T} X_t.$$

THEOREM A. *The homology of $\mathcal{U}$ is isomorphic to the following direct sum,*

$$H_*(\mathcal{U}) \cong \sum_{w\in W} H_*(X, X^{S(w)}). \tag{1}$$

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Remark. One of the principal results of [2] is that the vanishing of one side of (1) implies the vanishing of the other side.*

We introduce more notation. For each subset T of S, put

$$W_T = \text{the subgroup generated by } T,$$

$$W^T = \{w \in W | S(w) = T\},$$

$$\mathbb{Z}(W^T) = \text{the free abelian group with basis } W^T.$$

For any w in W, the subgroup $W_{S(w)}$ is finite (cf. Lemma 7.12 in [2]). (In other words, the set W^T is empty whenever W_T is infinite).

By collecting terms on the right side of (1) one can rephrase Theorem A as follows:

THEOREM A′.

$$H_*(\mathcal{U}) \simeq \sum \mathbb{Z}(W^T) \otimes H_*(X, X^T), \tag{2}$$

where the summation runs over all subsets T of S such that the subgroup W_T is finite.

Next, suppose that W is finite. Denote by Ω the rational group algebra $\mathbb{Q}(W)$. For each subset T of S, define elements ξ_T, η_T, ψ_T of Ω as follows:

$$\xi_T = |W_T|^{-1} \sum_{w \in W_T} w$$

$$\eta_T = |W_T|^{-1} \sum_{w \in W_T} \varepsilon(w) w$$

$$\psi_T = \xi_{\hat{T}} \eta_T,$$

where $\hat{T}$ denotes the complementary subset $S - T$, where $|W_T|$ denotes the order of W_T, and where $\varepsilon \colon W \to \{\pm 1\}$ is the homomorphism $w \to (-1)^{l(w)}$. (The elements ξ_T and η_T are, respectively, "symmetrization" and "alternation" over the subgroup W_T.)

In [3], L. Solomon proved the following result.

THEOREM (Solomon). *Suppose that W is finite. Then its rational group algebra Ω can be written as a direct sum of left ideals as follows*

$$\Omega = \sum_{T \subset S} \Omega \psi_T. \tag{3}$$

* The terminology and notation of [2] differs to some extent from that we are using above. In [2], a typical Coxeter system is denoted by (Γ, V) instead of (W, S), the word "panel" is used instead of "mirror", and the symbols $X_{\sigma(T)}$ and $B(w)$ are used in place of X^T and $S(w)$, respectively.

Moreover, the family of events $(w\psi_T)_{w \in W^T}$, *is a basis for* $\Omega\psi_T$ *as a rational vector space.*

After comparing this theorem with Theorem A′, the following result is not surprising.

THEOREM B. *Suppose that* W *is finite. Then there is an isomorphism of* Ω*-modules,*

$$H_*(\mathcal{U}) \cong \sum_{T \subset S} \Omega\psi_T \otimes H_*(X, X^T), \tag{4}$$

where homology is taken with rational coefficients.

In other words, the two sides of (4) are isomorphic as representations of W.

Remark. Theorems A and A′ are valid with arbitrary coefficients. Theorem B is valid with coefficients in any field of characteristic prime to $|W|$.

1. Preliminaries. We begin by recalling some known facts about the Coxeter system (W, S). For any subset T of S, put

$$A_T = \{a \in W | l(at) > l(a) \text{ for all } t \text{ in } T\}.$$

Obviously, $W^T \subset A_{S-T}$.

For the following result, see Exercise 3, p. 37 in [1].

LEMMA 1. *Suppose* T *is a subset of* S.

(i) *An element* a *of* W *belongs to* A_T *if and only if it is the shortest element in the coset* aW_T. *Thus,* A_T *is a complete set of representatives for* W/W_T.

(ii) *If* $a \in A_T$ *and* $u \in W_T$, *then* $l(au) = l(a) + l(u)$.

LEMMA 2. *Let* T *be a subset of* S *such that the subgroup* W_T *is finite. There is a unique element* w_T *of longest length in* W_T. *It has the following properties*:

(i) w_T *is an involution,*

(ii) w_T *is the unique element in* $W_T \cap W^T$.

Moreover, for any v *in* W_T *and* w *in* W^T,

(iii) $l(w_T v) = l(w_T) - l(v)$,

(iv) $l(wv) = l(w) - l(v)$.

Proof. For statements (i) to (iii), see Exercise 22, p. 43 in [1]. To prove (iv), first write $w = au$, where $a \in A_T$, $u \in W_T$, and $l(w) = l(a) + l(u)$. Let $t \in T$. Since $w \in W^T$, $l(w) = l(wt) + 1$. By Lemma 1 (ii), $l(wt) = l(aut) = l(a) + l(ut)$. Combining these equations, we have $l(a) + l(u) = l(a) + l(ut) + 1$; hence,

$$l(ut) = l(u) - 1.$$

By (ii), this implies that $u = w_T$, and consequently, that $a = wu^{-1} = ww_T$.

Therefore, the equation $l(w) = l(a) + l(u)$ can be rewritten as

$$l(ww_T) = l(w) - l(w_T).$$

Using this and (iii), we get

$$\begin{aligned} l(wv) &= l((ww_T)(w_Tv)) \\ &= l(ww_T) + l(w_Tv) \text{ (by Lemma 1(ii))} \\ &= l(w) - l(w_T) + l(w_T) - l(v) \\ &= l(w) - l(v), \end{aligned}$$

which proves (iv).

If T is a subset of S such that the subgroup W_T is finite, then we can define an element β_T in the group ring $\mathbb{Z}(W)$ by the formula,

$$\beta_T = \sum_{w \in W_T} \varepsilon(w) w,$$

i.e., $\beta_T = |W_T| \eta_T$. We shall also refer to β_T as "alternation over W_T." If τ is a cell in X, then put

$$S(\tau) = \{s \in S | \tau \subset X_s\}.$$

(The subgroup $W_{S(\tau)}$ is the stabilizer of τ.)

Let $C_*(X)$ and $C_*(\mathcal{U})$ denote the cellular chain complexes of X and $\mathcal{U}$ and let $H_*(X)$ and $H_*(\mathcal{U})$ be their respective homology groups. Since W acts cellularly on $\mathcal{U}$, $C_*(\mathcal{U})$ is a $\mathbb{Z}(W)$-module.

LEMMA 3. *Let T be a subset of S such that W_T is finite. Then the kernel of β_T: $C_*(X) \to C_*(\mathcal{U})$ is $C_*(X^T)$.*

Proof. First suppose that $t \in T$ and that τ is a cell in X_t. Put $B = A_{\{t\}} \cap W_T$, so that B is a set of coset representatives for $W_T / W_{\{t\}}$. Writing β_T in the form

$$\beta_T = \sum_{v \in B} \varepsilon(v)(v - vt)$$

we have that $\beta_T \tau = \sum \varepsilon(v)(v - vt)\tau = \sum \varepsilon(v)(v\tau - v\tau) = 0$. Thus, $C_*(X^T)$ is contained in kernel β_T. Next, suppose that the cell τ is not contained in X^T, i.e., that $S(\tau) \cap T = \phi$. Then $W_{S(\tau)} \cap W_T = \{1\}$. Hence, $(v\tau)_{v \in W_T}$ is a family of pairwise distinct cells in $W_T X$; consequently, the chain $\beta_T \tau$ cannot be zero. It follows that the kernel of β_T is precisely $C_*(X^T)$.

Since β_T vanishes on $C_*(X^T)$, it induces a map $C_*(X, X^T) \to C_*(\mathscr{U})$, which we shall also denote by β_T. For each $w \in W$, we then define a map f^w: $C_*(X, X^{S(w)}) \to C_*(wW_{S(w)}X) \subset C_*(\mathscr{U})$ by $f^w = w\beta_{S(w)}$: $C_*(X, X^{S(w)}) \to C_*(\mathscr{U})$. This gives a map on homology

$$f^w_*: H_*(X, X^{S(w)}) \to H_*(\mathscr{U}).$$

The map f^w takes a relative cycle in $C_*(X, X^{S(w)})$, alternates it over $W_{S(w)}$ obtaining an absolute cycle in $C_*(W_{S(w)}X)$ and then translates it by w into $C_*(wW_{S(w)}X)$. (Alternatively, we could have defined f^w by first translating by w and then alternating over the subgroup $wW_{S(w)}w^{-1}$.)

2. The proof of Theorem A. The argument uses ideas from [2]. First, order the elements of W,

$$w_1, w_2, \ldots$$

so that $l(w_i) \leqslant l(w_{i+1})$. For each integer $n \geqslant 1$, put

$$P_n = \bigcup_{i=1}^{i=n} w_i X.$$

To simplify notation, set $w = w_n$. In Lemma 8.2 of [2], we showed that the intersection of P_{n-1} with the chamber wX is $wX^{S(w)}$. Hence, excision gives an isomorphism,

$$H_*(P_n, P_{n-1}) \xrightarrow{\cong} H_*(wX, wX^{S(w)}). \tag{5}$$

Next, consider the exact sequence of the pair (P_n, P_{n-1})

$$\to H_*(P_{n-1}) \xrightarrow{j_*} H_*(P_n) \xrightarrow{k_*} H_*(P_n, P_{n-1}) \to \tag{6}$$

We claim that the map k_* is a split epimorphism. This assertion is clearly equivalent to the assertion that the map k_*^w: $H_*(P_n) \to H_*(X, X^{S(w)})$ is a split epimorphism, where k_*^w denotes the composition of k_* with the excision isomorphism (5) and left translation by w^{-1}. Consider the map f_*^w on $H_*(X, X^{S(w)})$. Its image is contained in $H_*(wW_{S(w)}X)$. By Lemma 2(iv), for every $v \neq 1$ in $W_{S(w)}$, we have $l(wv) < l(w)$; hence, $wW_{S(w)}X \subset P_n$. Therefore, the image of f_*^w is contained in $H_*(P_n)$. Moreover, it is obvious that $k_*^w \circ f_*^w$ is the identity on $H_*(X, X^{S(w)}$. Hence, the sequence (6) is split short exact, as is the sequence,

$$0 \to H_*(P_{n-1}) \xrightarrow{j_*} H_*(P_n) \xrightarrow{k_*^w} H_*(X, X^{S(w)}) \to 0. \tag{7}$$

This implies that $H_*(P_n) \cong H_*(P_{n-1}) \oplus H_*(X, X^{S(w)})$ and therefore, that

$$(7) \qquad 0 \to H_*(P_{n-1}) \xrightarrow{j_*} H_*(P_n) \xrightarrow{k_*^w} X^{S(w_i)}).$$

Since $\mathscr{U}$ is the increasing union of the P_n, we have $H_*(\mathscr{U}) = \lim_{n\to\infty} H_*(P_n)$. Furthermore, since the map j_* in (7) is a monomorphism, the inclusion $P_n \subset \mathscr{U}$ induces a monomorphism on homology. Equation (1) follows immediately. This completes the proof of Theorem A.

Remark. The proof shows that the map

$$f = \sum f_*^w : \sum H_*(X, X^{S(w)}) \to H_*(\mathscr{U})$$

is an isomorphism. We shall now define an explicit inverse. For any $T \subset S$, put $B_T = (A_T)^{-1}$. Then $B_T = \{a \in W | (l(ta) > l(a), \text{ for all } t \in T\}$. Moreover, B_T is a set of representatives for the right cosets $W_T \backslash W$, i.e., any $w \in W$ can be written uniquely in the form $w = uw'$, with $u \in W_T$ and $w' \in B_T$. (Compare Lemma 1.) Put

$$Q_T = \bigcup_{v \in B_T} vX, \qquad \mathring{Q}_T = \bigcup_{\substack{v \in B_T \\ v \neq 1}} vX.$$

Then Q_T is the chamber for W_T on $\mathscr{U}$ which contains X. The map $w \to w'$ from W to B_T induces a projection π_T: $\mathscr{U} \to Q_T$, which can be identified with the orbit map for W_T on $\mathscr{U}$. For any w in W, put $T(w) = S - S(w)$, and let π_w: $\mathscr{U} \to wQ_{T(w)}$ be the map defined by the diagram,

$$\begin{array}{ccc} \mathscr{U} & \xrightarrow{\ w\ } & \mathscr{U} \\ {\scriptstyle \pi_{T(w)}}\downarrow & & \downarrow{\scriptstyle \pi_w} \\ Q_{T(w)} & \xrightarrow{\ w\ } & wQ_{T(w)} \end{array}$$

Then $wQ_{T(w)}$ is the chamber for $wW_{T(w)}w^{-1}$ on $\mathscr{U}$ which contains wX, and π_w is the corresponding orbit projection. We have an excision isomorphism $H_*(wQ_{T(w)}, w\mathring{Q}_{T(w)}) \cong H_*(wX, wX^{S(w)})$. Let g_*^w: $H_*(\mathscr{U}) \to H_*(X, X^{S(w)})$ denote the composition of $(\pi_w)_*$ with the projection onto $H_*(wX, wX^{S(w)})$ and w^{-1}. It can then be proved (and it should be clear) that the map $g = \Sigma g_*^w$: $H_*(\mathscr{U}) \to \Sigma H_*(X, X^{S(w)})$ is an inverse for f.

3. The proof of Theorem B. In this section, the group W is finite. Also, all chain complexes and homology groups have coefficients in $\mathbf{Q}$. The proof of the following lemma has been extracted from [3].

LEMMA 4 (Solomon). *Suppose that T is a subset of S, that $\hat{T} = S - T$, that U is a subset of $\hat{T}$ and that λ is an element of Ω. If $\lambda\psi_T\xi_U = 0$, then $\lambda\psi_T = 0$.*

Proof. If $J \subset K \subset S$, then a quick calculation shows that

$$\xi_J\xi_K = \xi_K. \tag{9}$$

In our notation, Lemma 2 of [3] states that

$$\lambda(\psi_T)^2 = 0 \Rightarrow \lambda\psi_T = 0. \tag{10}$$

Suppose that $\lambda\psi_T\xi_U = 0$. Right multiply both sides of this equation by $\xi_{\hat{T}}\eta_T$ and use (9) to obtain $\lambda\psi_T(\xi_{\hat{T}}\eta_T) = 0$, i.e.,

$$\lambda(\psi_T)^2 = 0.$$

Statement (10) now implies the result.

LEMMA 5. *Suppose τ is a cell in X and $\lambda \in \Omega$. Then $\lambda\tau = 0$ in $C_*(\mathcal{U})$ if and only if $\lambda\xi_{S(\tau)} = 0$ in Ω.*

Proof. Put $T = S(\tau)$. Since A_T is a set of representatives for W/W_T, the elements $(a\xi_T)_{a\in A_T}$ are a basis for the rational vector space $\Omega\xi_T$. Hence, there is a unique expression for $\lambda\xi_T$ in the form

$$\lambda\xi_T = \Big(\sum_{a\in A_T} c(a)a\Big)\xi_T.$$

Since $\xi_T\tau = \tau$, we have

$$\lambda\tau = \lambda\xi_T\tau = \sum_{a\in A_T} c(a)a\tau.$$

But the cells $a\tau$, $a \in A_T$, are pairwise distinct. Thus, $\lambda\tau = 0 \Leftrightarrow c(a) = 0$ for all $a \in A_T \Leftrightarrow \lambda\xi_T = 0$.

LEMMA 6. *Suppose that T is a subset of S, that τ is a cell of X, and that $\lambda \in \Omega$. Consider the statements*

(a) $\lambda\psi_T\tau = 0$

(b) $S(\tau) \cap T \neq \varnothing$

(c) $\lambda\psi_T = 0$.

Then (a) $\Leftrightarrow$ (b) *or* (c).

Proof. $\Leftarrow$. Obviously, (c) $\Rightarrow$ (a). Suppose (b) holds. Then $\tau \subset X_t$ for some t in T. The proof of Lemma 3 shows that $\eta_T\tau = |W_T|^{-1}\beta_T\tau = 0$, and hence, that $\psi_T\tau = 0$ i.e., (b) $\Rightarrow$ (a).

$\Rightarrow$. We shall show that (a) and not (b) implies (c). Suppose that $\lambda\psi_T\tau = 0$ and that $S(\tau) \subset \hat{T}$ (where $\hat{T} = S - T$). By Lemma 5, $\lambda\psi_T\xi_{S(\tau)} = 0$. By Lemma 4, this implies $\lambda\psi_T = 0$, i.e., (c) holds.

Next consider the natural map $\Omega\psi_T \otimes C_*(X) \to C_*(\mathscr{U})$ defined by $\lambda\psi_T \otimes c \to \lambda\psi_T c$. It follows from the previous lemma that the kernel of this map is precisely $\Omega\psi_T \otimes C_*(X^T)$. By (3),

$$\Omega C_*(X) = \sum_{T \subset S} \Omega\psi_T C_*(X).$$

Hence, the chain complex $C_*(\mathscr{U})(= \Omega C_*(X))$ is isomorphic to a direct sum of chain complexes as follows:

$$C_*(\mathscr{U}) \cong \sum_{T \subset S} \Omega\psi_T \otimes C_*(X, X^T) \tag{11}$$

where the isomorphism (11) is W-equivariant. Taking homology in (11) yields Theorem B.

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