CHAPTER XI

A Survey of Results in Higher Dimensions

Michael W. Davis

Department of Mathematics
Ohio State University
Columbus, Ohio

The field of compact transformation groups deals with actions of compact Lie groups on manifolds. Its origins can be traced back to the advent of the study of groups of linear transformations in the nineteenth century. The actions of such linear groups on euclidean spaces, disks, spheres, projective spaces, etc., provide a rich and natural class of basic examples. The guiding principle of the field has always been to compare arbitrary actions to linear actions, that is, to determine the extent to which arbitrary actions resemble the basic linear examples.

It is natural to begin by studying compact transformation groups of the simplest manifolds: euclidean spaces, disks, and spheres. The only examples that spring to mind are closed subgroups of the appropriate orthogonal group. Naïvely, one might conjecture that these linear actions are the only possibilities; that is, for $M^n = \mathbf{R}^n$, D^n, or S^n and for G a compact Lie group of homeomorphisms of M^n, one might conjecture that G is conjugate by a homeomorphism to a subgroup of $O(n)$ (or of $O(n+1)$ when $M^n = S^n$). We shall hereafter refer to this as the naïve conjecture. Prior to 1950 it seems that people felt this might be true, although it was clearly regarded as an

ISBN 0-12-506980-4

extremely difficult problem. The conjecture is true for $n = 2$. As stated, it is false for $n = 3$: Bing [2] produced a nonlinear involution on S^3 in 1952. However, Bing's example was essentially a local pathology in that it resulted from the nondifferentiability of the involution. There is mounting evidence (much of it in this volume) that the naïve conjecture holds for $n = 3$ provided that we stick to groups of *diffeomorphisms*. For $n \geq 4$, the conjecture is false in either category. Despite this, the determination of the exact extent to which actions of compact Lie groups on euclidean spaces, disks, or spheres resemble linear actions remains today one of the central problems in the field.

The purpose of this chapter is to survey some of the work on this problem and to describe some counterexamples to the naïve conjecture in dimensions greater than three. We shall be particularly concerned with the nature of the fixed point set. For another discussion of this subject see Bredon [4].

The earliest work in this area focused on periodic transformations, that is, on actions of finite cyclic groups. In the twenties Brouwer [5] and Kerekjarto [15] proved that any orientation-preserving periodic transformation of the 2-disk or 2-sphere was conjugate to a rotation. A gap in the proof was later filled by Eilenberg [10].

In dimension three there are substantial partial results. In the case for which the group G is compact and connected, the three-dimensional naïve conjecture was proved by Montgomery and Zippen [20]. For smooth actions of $G = \mathbf{Z}_2$ it is due to Livesay [16, 17] and to Waldhausen [31]. For smooth, orientation-preserving, periodic transformations with nonempty fixed point set the three-dimensional naïve conjecture is equivalent to the Smith conjecture solved in this volume. Rubenstein [25] has proved it for free actions of certain finite groups on S^3.

In order to attack the naïve conjecture in higher dimensions, one wants to find topological properties of linear compact transformation groups and determine the extent to which they are shared by arbitrary compact groups of homeomorphisms of euclidean spaces, disks, and spheres. Smith focused attention on one such property of linear actions: The fixed point set of any group of linear transformations of $\mathbf{R}^n$ is a linear subspace. Obviously, such a subspace intersects D^n in a linear subdisk and S^{n-1} in a linear subsphere. Thus a necessary condition for a G-action on euclidean space (respectively, disk or sphere) to be equivalent to a linear G-action is that the fixed point set must be homeomorphic to a lower-dimensional euclidean space (respectively, disk or sphere). Moreover, the embedding of the fixed point set in the ambient space must be standard.

The first positive results toward establishing this necessary condition were proved by Smith in a series of papers [26–28] published in the 1930s and 1940s. He proved that the condition holds for cyclic groups of prime power order provided that we are concerned only with homology with coefficients in $\mathbf{Z}_p$ (where p is the prime in question). For example, Smith showed that if X is a reasonably nice (e.g., compact) space with the mod p homology of a sphere, then the fixed point set of a periodic transformation of period p^m also has the mod p homology of a sphere. An easy induction can be used to extend this to actions of finite p-groups (p a prime). The results were extended to actions of tori (using integral coefficients) by Conner [6] and Floyd [12] and they can easily be extended to the case in which G is a p-group extended by a torus. Here is a precise statement.

THEOREM (Smith). *Suppose that G is a p-group and that X is either a finite dimensional or a compact G-space that has the mod p Cech cohomology of an n-sphere. The fixed point set X^G has the mod p Cech cohomology of an r-sphere for some $-1 \leq r \leq n$.*

A similar result holds for pair (X, A) with the mod p Cech cohomology of (D^n, S^{r-1}). For further information see [3, Chapter III].

Initially, Smith believed that it was merely a defect in his methods that forced him to restrict to p-groups and to homology with $\mathbf{Z}_p$-coefficients. He clearly believed that these results should extend to coefficients in $\mathbf{Z}$, and at least to all finite cyclic groups. We shall give a simple counterexample to this conjecture in Section 2.

It has now become clear that Smith's results, together with some conditions on the Euler characteristic, are essentially the only homological conditions that must be satisfied by fixed point sets of group actions on euclidean spaces, disks, and spheres. As an illustration of this principle, we shall now state some results that precisely give the homotopy types that can occur as fixed point sets of smooth G-actions on disks. Let us first remark that it follows from the differentiable slice theorem that any such fixed point set is a smooth submanifold and hence has the homotopy type of a finite complex. For finite groups there is the following result.

THEOREM. *Let F be a finite complex.* (1) *If G is a finite group of p-power order, p a prime, then F is homotopy equivalent to the fixed point set of a smooth G-action on some disk if and only if F is $\mathbf{Z}_p$-acyclic.*

(2) *If G is a finite group not of prime power order, then there is an integer n_G such that F is homotopy equivalent to the fixed point set of some smooth G-action on a disk if and only if $\chi(F) \equiv 1 \pmod{n_G}$.*

The integer n_G can be explicitly computed (see Section 7). The answer is also known when G is a compact Lie group of positive dimension. If the identity component G_0 is not a torus, then there is no condition on F: Any finite complex (including the empty set) can occur. If G_0 is a torus, then F

must satisfy either condition (1) or (2) of the above theorem, depending on whether G/G_0 has prime power order. Statement (1) in this theorem is essentially due to Jones [14]. In all other cases it is the work of Oliver [21].

One consequence of this result is that most groups can act smoothly on a disk with empty fixed point set. The existence of such actions runs contrary to one's intuition. It becomes even more counterintuitive when we combine this fact with Mostow's embedding theorem. This theorem states that we can embed any compact, smooth G-manifold in some linear action on some large dimensional euclidean space. Thus we can embed a G-action on a disk in a linear action and take an invariant regular neighborhood to obtain a smooth G-action on another disk (of larger dimension) with homotopy equivalent fixed point set. Therefore, for any sufficiently complicated G, we can find a linear action of G on $\mathbf{R}^N$ and a smoothly embedded G-invariant disk $D^N \subset \mathbf{R}^N$ such that G fixes no points of D^N.

1. The Montgomery–Samelson Example[1]

Suppose that B^m is a compact, contractible m-manifold with boundary. The boundary of B is, of course, a homology $(m - 1)$-sphere. However, it is well known that for $m \geq 4$, ∂B can be nonsimply connected. Let us suppose this. Let G be a closed subgroup of $\mathrm{O}(n)$ such that G fixes only the origin of $\mathbf{R}^n$. Consider the manifold $M^{n+m} = D^n \times B^m$ with G-action defined by $g(x, b) = (gx, b)$. If $m + n \geq 6$, then by the h-cobordism theorem, ∂M is a sphere. The G-action on ∂M is nonlinear since its fixed point set ∂B is not simply connected, hence not a sphere. The point of this example is that at most we can expect to prove results about the homology of the fixed point set: We cannot hope to control its fundamental group.

2. G-Complexes

A *G-complex* is a G-space built by successively adjoining "equivariant cells" of the form $G/H \times D^n$. The following general principle is important in the construction of examples: every finite contractible G-complex X gives rise to a smooth G-action on a disk with fixed point set homotopy equivalent to X^G. Similarly, every finite dimensional contractible G-complex gives rise to a smooth G-action on some euclidean space. The point is that provided we are willing to be generous about dimensions, we may replace our equivariant cells by equivariant handles of the form $(G \times_H D(V)) \times D^n$, where

[1] See [19].

$D(V)$ is the unit disk in some linear H-space V (see [21]). Alternatively, in the case in which G is finite and X is a simplicial complex, we can embed X in some triangulation of a linear action on $\mathbf{R}^N$, N large, and then take a regular neighborhood. (See p. 57 in [3].)

3. The Brieskorn Examples[2]

Consider the polynomial $f: \mathbf{C}^{n+1} \to \mathbf{C}$ defined by

$$f(z) = \sum_{i=0}^{n} (z_i)^{a_i},$$

where the a_is are integers greater than one. The hypersurface $f^{-1}(0)$ has an isolated singularity at the origin. Its *link* $\Sigma^{2n-1}(a_0, \ldots, a_n)$ is defined as the intersection of the hypersurface with a sufficiently small sphere centered at the origin. Each such link is an $(n-2)$-connected smooth manifold. Formulas for computing $H_{n-1}(\Sigma^{2n-1}(a_0, \ldots, a_n))$ in terms of the exponents can be found in Milnor [18]. If this group is trivial (and if $2n - 1 > 3$), then the link is homeomorphic to S^{2n-1}. (This happens quite frequently.)

These links have many interesting symmetries. For example, for each $0 \le i \le n$, $\Sigma^{2n-1}(a_0, \ldots, a_n)$ admits a periodic diffeomorphism of period a_i defined by

$$\omega(z_0, \ldots, z_i, \ldots, z_n) = (z_0, \ldots, \omega z_i, \ldots, z_n),$$

where ω is a primitive a_ith root of unity. The fixed point set is clearly $\Sigma^{2n-3}(a_0, \ldots, \hat{a}_i, \ldots, a_n)$. More specifically, consider $\Sigma^5(3, 2, 2, 2)$. This manifold is well known to be diffeomorphic to S^5. It admits an action of $\mathbf{Z}_6 = \mathbf{Z}_3 \oplus \mathbf{Z}_2$ defined as above by acting on the first coordinate by a third root of unity and on the second coordinate by a second root of unity. The fixed point sets of $\mathbf{Z}_3$, $\mathbf{Z}_2$, and $\mathbf{Z}_6$ are, respectively, $\Sigma^3(2, 2, 2)$, $\Sigma^3(3, 2, 2)$, and $\Sigma^1(2, 2)$. These manifolds are, respectively, real projective 3-space, the lens space $L(3, 1)$, and the disjoint union of two circles. Thus S^5 admits a smooth action of $\mathbf{Z}_6$ such that (1) the fixed point set of $\mathbf{Z}_6$ is not a homology sphere with any coefficients and (2) the fixed point sets of the subgroups of order two and three are not homology spheres with integer coefficients. (Note, however, that $\mathbf{R}P^3$ is a mod 3 homology sphere and that $L(3, 1)$ is a mod 2 homology sphere.) A similar example was constructed earlier by Floyd [11] on the 41-sphere. Such examples provide rather convincing evidence that Smith's results cannot be improved.

[2] See [13].

By slight modifications of the above example one can construct similar examples in every odd dimension $2n - 1$, with $n \geq 3$. It is also easy to modify these examples to obtain, in higher dimensions, periodic transformations of the sphere such that the fixed point set is a knotted sphere of codimension two. For example, an involution on $\Sigma^{2n-1}(p, q, 2, \ldots, 2)$ will do the job for suitably chosen p and q.

4. Oliver's Example[3]

In this section we shall discuss Oliver's beautiful construction of a smooth SO(3)-action on the 8-disk with no fixed points.

Let $G = \mathrm{SO}(3)$. As a first step we shall construct a contractible finite G-complex X with no fixed points. Let X be any G-complex with orbit space as pictured in Fig. 1. Thus X is made up of five equivariant cells of the

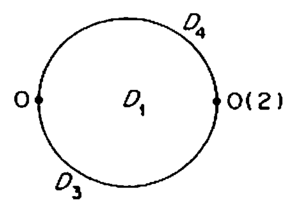

Figure 1

form $G/H \times D^n$, with $n \leq 2$. In Fig. 1 we have labeled each cell by its isotropy group H. Here D_n denotes the dihedral group of order $2n$, O(2) is the orthogonal group, and O is the octahedral group. It is easy to see that there is a G-complex with this orbit space. By construction there are no fixed points.

We next want to show that X is contractible. Divide the orbit space into three pieces A', B', C' as shown in Fig. 2. Let $A = \pi^{-1}(A')$, $B = \pi^{-1}(B')$,

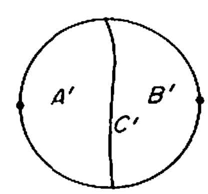

Figure 2

and $C = \pi^{-1}(C')$ be their inverse images in X. The proof that X is contractible is a straightforward exercise using van Kampen's theorem and the Mayer–Vietoris sequence for $X = A \cup_C B$. We shall sketch the details.

If H is a finite subgroup of $G = \mathrm{SO}(3)$, denote its inverse image in S^3 by $\tilde{H}$, so that $\pi_1(G/H) = \tilde{H}$. Since A deformation retracts to G/O and B deformation retracts to $G/\mathrm{O}(2) = \mathbf{R}P^2$, we have that $\pi_1(A) = \tilde{O}$ and $\pi_1(B) = \mathbf{Z}_2$.

[3] See [23].

Also, $\pi_1(C) = \tilde{D}_4 *_{\tilde{D}_1} \tilde{D}_3$. The kernel of $\pi_1(C) \to \pi_1(B)$ is clearly $\mathbf{Z}_8 *_{\mathbf{Z}_2} \mathbf{Z}_6$. Since the binary octahedral group $\tilde{O}$ is generated by the cyclic groups of order eight and six, this kernel maps onto $\pi_1(A)$. Hence, X is simply connected.

The computation of the homology of X runs as follows:

$$\bar{H}_i(A) = \bar{H}_i(G/O) = \begin{cases} \mathbf{Z}_2, & i = 1, \\ \mathbf{Z}, & i = 3, \\ 0, & \text{otherwise}. \end{cases}$$

$$\bar{H}_i(B) = \bar{H}(\mathbf{R}P^2) = \begin{cases} \mathbf{Z}_2, & i = 1, \\ 0, & \text{otherwise}. \end{cases}$$

$$\bar{H}_i(C) = \begin{cases} \mathbf{Z}_2 \oplus \mathbf{Z}_2, & i = 1, \\ \mathbf{Z}, & i = 3, \\ 0, & \text{otherwise}. \end{cases}$$

The Mayer–Victoris sequence shows that X is acyclic and hence contractible.

Next we want to "thicken" X to be an 8-disk. The idea is to thicken A and B to 8-manifolds with boundary M_A and M_B and to thicken C by a 7-manifold with boundary M_C in such a way that M_C is embedded in both ∂M_A and ∂M_B. The 8-disk will then be formed by gluing M_A and M_B along M_C. The manifolds M_A and M_B will each be unit disk bundles associated to certain G-vector bundles of the form $G \times_O V$ and $G \times_{O(2)} W$, respectively, where V and W are linear representations of O and O(2) respectively. Let V_1 be the two-dimensional representation of O defined by $O \to D_3 \subset O(2)$. Note that V_1 contains points with isotropy subgroups $D_4 \subset O$. Let V_2 be the three-dimensional representation of O defined by regarding O as the full group of symmetries of the tetrahedron. Then V_2 has points with isotropy group D_3. Let $V = V_1 \oplus V_2$.

For $m \in \mathbf{Z}_+$, let W_m be the two-dimensional representation of O(2) with kernel the cyclic group of order m. Set $W = W_3 \oplus W_4 \oplus W_k$, where $k \equiv \pm 2(12)$. Then W has points with isotropy subgroup $D_3 \subset O(2)$ and $D_4 \subset O(2)$, since W_3 and W_4 do.

Next we want to find a copy of C in both ∂M_A and ∂M_B. This is easy. For example to find it in ∂M_A, pick points x and y in $\partial M_A/G$ corresponding to orbits of type G/D_3 and G/D_4, respectively. Join x and y by an arc of orbits of type G/D. This gives a copy of C' in $\partial M_A/G$ and hence an embedding of

C in ∂M_A; similarly for ∂M_B. Let Y_A and Y_B be closed invariant regular neighborhoods of C in ∂M_A and ∂M_B, respectively. We want to show that Y_A is equivariantly diffeomorphic to Y_B. It clearly suffices to check this in a neighborhood of the G/D_3 and G/D_4 orbits. And this is just a matter of checking whether the tangential representations of the isotropy groups D_3 and D_4 in Y_A agree with those of Y_B. The condition that $k \equiv \pm 2(12)$ precisely guarantees this. Thus, we set $M_C = Y_A = Y_B$ and $M = M_A \cup_{M_C} M_B$. M is a compact contractible 8-manifold since it is homotopy equivalent to X. Since ∂M is the union of two sphere bundles along M_C, the calculation of $\pi_1(\partial M)$ is the same as for $\pi_1(X)$. Hence ∂M is simply connected and M is therefore an 8-disk. Thus SO(3) *can act smoothly on* D^8 *without fixed points.*

Let $I \subset$ SO(3) be the icosahedral group. Since no isotropy group of M contains I, Oliver's example gives a bonus: *There is a smooth action of* I *on* D^8 *without fixed points.*

Actually, Floyd and Richardson had earlier constructed a fixed point free action of I on a disk. Their construction starts with the action of I on the Poincaré homology sphere G/I. This action has one fixed point. Removing a neighborhood gives an action of I on a compact acyclic 3-manifold Q. The diagonal action of I on the join $I * Q$ gives a finite contractible I-complex with no fixed points. This can be thickened to a disk. For further details see [3, p. 55]

5. Local Properties: Groups of Homeomorphisms versus Groups of Diffeomorphisms

The differentiable slice theorem asserts that any orbit of a smooth G-manifold M has an equivariant tubular neighborhood, i.e., an orbit has an invariant neighborhood in M that is G-diffeomorphic to a G-vector bundle over it. In particular, a fixed point has an invariant neighborhood on which the action is linear. It follows that the fixed point set is a smoothly embedded submanifold. Therefore, fixed point sets of smooth actions are locally indistinguishable from those of linear actions.

If the action is not smooth, we can have local pathologies. Fixed point sets need not be manifolds and even if they are manifolds, they need not be embedded in a locally flat fashion. The first example of this type was Bing's [2] involution of the 3-sphere with fixed point set an Alexander's horned sphere. Examples of actions on spheres where the fixed point set is not a manifold can be obtained by suspending any of the examples in Sections 1 or 3. (There is, however, a local version of Smith's theorem which asserts that if G is a p-group acting on a manifold, then the fixed point set is a mod p homology manifold.)

6. Work of Lowell Jones[4]

In this section we shall discuss some of the work of Jones [14] on fixed point sets of certain periodic transformation of disks.

A G-action on a space X is said to be *semifree* if for each $x \in X$ the isotropy group G_x is equal to either the trivial group or to G. If G is cyclic of order n and acts semifreely on a disk, then it follows Smith's theorem that the fixed point set is $\mathbf{Z}_p$-acyclic for every prime p dividing n. Hence, the fixed point set is actually $\mathbf{Z}_n$-acyclic. Conversely, Jones [14] proved the following result.

THEOREM (Jones). *Let G be the cyclic group of order n. A finite complex F is homotopy equivalent to the fixed point set of a smooth G-action on some disk if and only if F is $\mathbf{Z}_n$-acyclic.*

As we pointed out in Section 2, this theorem is equivalent to the statement that any such F occurs as the fixed point set of a semifree G-action on some finite contractible G-complex. Thus we want to start with F and successively adjoin a finite number of cells of free orbits to kill its fundamental group and its homology, in this way obtaining a finite contractible G-complex X with $X^G = F$. If we have succeeded in building an $(m - 1)$-connected G-complex X, then there is no problem in equivariantly attaching cells to kill $H_m(X)$. Simply choose a set of generators for $H_m(X)$ as a $\mathbf{Z}[G]$-module, represent these generators by spheres, and attach a free orbit of cells for each generator. The trouble is that we will introduce new homology in dimension $m + 1$. For an arbitrary F there is no guarantee that this process will terminate after a finite number of steps. Jones's key observation is that the hypothesis that F is $\mathbf{Z}_n$-acyclic ensures that the "extra" homology we have introduced in dimension $m + 1$ is a free module over $\mathbf{Z}[G]$. Thus we can kill this extra homology at the next stage without introducing any further homology. This observation essentially comes down to the following algebraic lemma.

LEMMA. *Suppose that G is cyclic of order n and that A is a* $\mathbf{Z}[G]$*-module such that* (1) *A is finite of order prime to n and* (2) *G acts trivially on A. Let $\psi: R \to A$ be an epimorphism with R a free* $\mathbf{Z}[G]$*-module. Then* $\ker \psi$ *is a free* $\mathbf{Z}[G]$*-module.*

A proof can be found in Jones [14].

Using similar ideas it is easy to extend Jones's result to actions of finite groups of prime power order.

[4] See [14].

THEOREM. *Let G be a p-group (p a prime). A finite complex F is homotopy equivalent to the fixed point set of a smooth G-action on some disk if and only if F is $\mathbf{Z}_p$-acyclic.*

To prove this, we choose $H \lhd G$ with G/H cyclic. Attach cells of G/H-orbits to F to obtain a finite contractible G-complex X. Then attach cells of free orbits in adjacent dimensions so as to make the action effective without introducing any homology.

7. Actions on Disks

Having dealt with p-groups in the previous section, we shall in this section summarize the definitive work of Oliver [21, 22, 24] on fixed point sets of finite group actions on disks for groups not of prime power order. He shows that in this case the only restriction on the homotopy type of the fixed point set is that its Euler characteristic must sometimes satisfy a certain congruence relation.

THEOREM (Oliver [21]). *For any finite group G not of prime power order there is an integer n_G such that a finite complex F is homotopy equivalent to the fixed point set of some smooth G-action on a disk if and only if $\chi(F) \equiv 1 \pmod{n_G}$.*

Oliver then goes on to explicitly calculate n_G. In order to give this calculation, we shall first define an integer $m(G)$. Let $\mathscr{G}^1$ be the class of finite groups G with G/P cyclic for some $P \lhd G$, with P of prime power order. For each prime q, let $\mathscr{G}^q$ be the class of groups G with G/H of q-power order for $H \lhd G$ and $H \in \mathscr{G}^1$. Thus $\mathscr{G}^1$ consists of those groups that are "cyclic mod p" and $\mathscr{G}^q$ consists of those that are "q-hyperelementary mod p" for some prime p. By definition, $m(G)$ will be either 0, 1, or a product of distinct primes. It is 0 if and only if $G \in \mathscr{G}^1$, and $q \mid m(G)$ if and only if $G \in \mathscr{G}^q$.

The result of Oliver's calculation is that almost always $n_G = m(G)$. However, for a certain class of 2-hyperelementary groups, $n_G = 4$, while $m(G) = 2$. Since the definition of this class of groups is fairly complicated we shall simply refer to it as the "exceptional case" and direct the reader to Oliver [22, p. 345] for the precise definition.

THEOREM (Oliver [24]). *In the exceptional case, $n_G = 4$. Otherwise, $n_G = m(G)$.*

If $n_G = 1$, then the congruence in the theorem is automatically satisfied. Hence, for such groups G any finite complex (including the empty set!) can occur as the homotopy type of the fixed point set of a smooth G-action on a disk. Notice that it is easy for a group to satisfy the condition $m(G) = 1$ (and hence $n_G = 1$). Any nonsolvable group has this property, as does any sufficiently complicated solvable group. The smallest abelian group with $n_G = 1$ is $\mathbf{Z}_{30} \oplus \mathbf{Z}_{30}$, while the smallest solvable groups with this property have order 72. There are two such: $S_4 \oplus \mathbf{Z}_3$ and $A_4 \oplus S_3$ [21, p 175].

Before discussing the proofs of the above theorems, let us warm up by proving the following elementary result which makes the main results possible.

PROPOSITION (Oliver), *Suppose that F is the fixed point set of a smooth G-action on a disk, then $\chi(F) \equiv 1 \pmod{m(G)}$.*

Let M be any compact smooth G-manifold. The proof of this is based on the following three facts:

(1) If G is a p-group, then $\chi(M^G) \equiv \chi(M) \pmod{p}$.
(2) If G is a p-group and M is $\mathbf{Z}_p$-acyclic, then M^G is $\mathbf{Z}_p$-acyclic.
(3) If G is cyclic and M is rationally acyclic, then $\chi(M^G) = 1$.

Statement (1) is a standard result in Smith theory and can be found, for example, in Bredon [3, p. 145]. Statement (2) is Smith's theorem. Statement (3) follows from the more general fact that if G is cyclic, then $\chi(M^G)$ is equal to the Lefschetz number of a generator of G.

Proof of the Proposition. Let D be a disk with smooth G-action. If $G \in \mathscr{G}^1$, then G/P is cyclic for some $P \lhd G$, with P a p-group. By (2), D^P is $\mathbf{Z}_p$-acyclic, hence rationally acyclic. Therefore, by (3), $\chi(D^G) = \chi((D^P)^{G/P}) = 1$. If $G \in \mathscr{G}^q$, then G/H is a q-group for some $H \in \mathscr{G}^1$. We have just shown that $\chi(D^H) = 1$. Hence by (1), $\chi(D^G) = \chi((D^H)^{G/H}) \equiv 1 \pmod{q}$.

A *G-resolution* of a finite complex F is an n-dimensional finite G-complex Y, such that Y is $(n-1)$-connected, $H_n(Y)$ is a projective $\mathbf{Z}[G]$-module, and $Y^G = F$. If $H_n(Y)$ is a free $\mathbf{Z}[G]$-module (or even stably free), then we can add cells of free orbits to kill it without introducing any further homology, obtaining in this way a finite contractible G-complex. Let $\gamma_G(F, Y)$ denote the class of $H_n(Y)$ in $\tilde{K}_0(\mathbf{Z}[G])$. Given F, Oliver breaks the problem of constructing a finite contractible G-complex X with $X^G = F$ into two parts: (a) building a G-resolution Y and (b) analyzing the obstruction $\gamma_G(F, Y)$. He first proves the following result.

PROPOSITION (Oliver). *Suppose that G is a finite group not of prime power order. A finite complex F has a G-resolution if and only if $\chi(F) \equiv 1 \pmod{m(G)}$.*

In order to construct a finite contractible G-complex with fixed point set F, we are free to vary the G-resolution Y. Thus the obstruction we are interested in has indeterminacy in the subgroup $\beta(G) = \{\gamma_G(\text{point}, Y)\} \subset \tilde{K}_0(\mathbf{Z}[G])$. There results an obstruction $\gamma_G(F) \in \tilde{K}_0(\mathbf{Z}[G])/\beta(G)$. Oliver [21] shows that if G is not hyperelementary, then $\gamma_G(F) = 0$ and hence $n_G = m(G)$. He also shows that even when G is hyperelementary $m(G)$ divides n_G. Finally, in a later paper [24] he analyzes the hyperelementary case and shows that $n_G = m(G)$ precisely in the nonexceptional case.

8. Actions on Spheres

We have already given examples of nonlinear actions on spheres in Sections 1 and 3. Although there is a large body of literature on this subject we shall mention only one further example. Stein [30] constructed a smooth action of the binary icosahedral group on S^7 with only one fixed point. Further work along these lines can be found in Assadi [1] and Doverman and Petrie [8].

9. Actions on Euclidean Spaces

As we pointed out in Section 2, the construction of such actions is related to the construction of finite dimensional contractible G-complexes. Since we are free to add infinitely many cells, the obstructions in the projective class group are no longer relevant. Also, since $\mathbf{R}^n$ is not compact, the Lefschetz fixed point theorem no longer imposes constraints on the Euler characteristics of fixed point sets of cyclic subgroups. Hence it is generally much easier to construct actions with exotic fixed point sets on euclidean spaces than on either disks or spheres.

For example, exotic fixed point sets occur even for actions of finite cyclic groups which are not of prime power order. The first example of this type was constructed by Connor and Floyd [7]. If p and q are relatively prime integers, then they showed that a certain linear action of the cyclic group $\mathbf{Z}_{pq}$ on S^3 admits an equivariant self-map of degree 0. By taking the infinite mapping telescope of this map they obtain a $\mathbf{Z}_{pq}$-action on a contractible 4-complex with empty fixed point set and hence, a smooth $\mathbf{Z}_{pq}$-action on euclidean space with empty fixed point set. As Bredon observes [3, p. 62], a slight modification of this construction shows that any finite complex is homotopy equivalent to the fixed point set of some $\mathbf{Z}_{pq}$-action on a euclidean space.

The fixed point set of a linear action on euclidean space is a linear subspace; hence it is either a point or noncompact. Smith [29] asked if a compact manifold other than a point could occur as fixed point set. This was answered by Edmonds and Lee [9]. They showed that a smooth closed manifold occurs as the fixed point set of some finite cyclic group action on euclidean space if and only if its tangent bundle admits a complex structure.

References

[1] Assadi, A., Finite group actions on simply connected manifolds and CW complexes. *Mem. Amer. Math. Soc.* **35** (257) (1982).

[2] Bing, R. H., A homeomorphism between the 3-sphere and the sum of two solid horned spheres, *Ann. of Math.* **56**, 354–362.

[3] Bredon, G., "Introduction to Compact Transformation Groups." Academic Press, New York, 1972.

[4] Bredon, G., Book review, *Bull. Amer. Math. Soc.* **83** (1977), 711–718.

[5] Brouwer, L. E. J., Über die periodischen Transformationen der Kugel, *Math. Ann.* **80** (1919), 262–280.

[6] Conner, P. E., On the action of the circle group, *Michigan Math. J.* **4** (1957), 241–247.

[7] Conner, P. E., and Floyd, E. E., On the construction of periodic maps without fixed points. *Proc. Amer. Math. Soc.* **10** (1959), 354–360.

[8] Doverman, K. H., and Petrie, T., G-surgery II. *Mem. Amer. Math. Soc.* No. 260 **37** (1982).

[9] Edmonds, A. L., and Lee, R., Fixed point sets of group actions on Euclidean space, *Topology* **14** (1975), 339–345.

[10] Eilenberg, S., Sur les transformations périodiques de la surface du sphère, *Fund. Math.* **22** (1934), 28–41.

[11] Floyd, E. E., Examples of fixed point sets of periodic maps, *Ann. of Math.* **55** (1952), 167–171.

[12] Floyd, E. E., Fixed point sets of compact abelian groups of transformations, *Ann. of Math.* **65** (1957), 30–35.

[13] Hirzebruch, F., Singularities and exotic spheres, *Sem. Bourbaki* No. 314 **19**, (1966/7).

[14] Jones, L., A converse to the fixed point theory of P. A. Smith, I, *Ann. of Math.* **94** (1971), 52–68.

[15] Kerekjarto, B., Über die periodischen Transformationen de Kreisschube and Kugelflache, *Math. Ann.* **80** (1919), 36–38.

[16] Livesay, G. R. Involutions with two fixed points of the 3-sphere, *Ann. of Math.* **78** (1963), 582–593.

[17] Livesay, G. R., Fixed point free involutions on the 3-sphere, *Ann. of Math.* **78** (1960), 603–611.

[18] Milnor, J. W., Singular points of complex hypersurfaces. *Ann. of Math. Studies*, Vol. 61. Princeton Univ. Press, Princeton, New Jersey, 1969.

[19] Montgomery, D., and Samelson, H., Examples for differentiable group actions on spheres, *Proc. Nat. Acad. Sci.* **47** (1961), 1202–1205.

[20] Montgomery, D., and Zippen, L., "Topological Transformation Groups." Wiley (Interscience), New York, 1955.

[21] Oliver, R., Fixed point sets of group actions of finite acyclic complexes, *Comment. Math. Helv.* **50**, (1975), 155–177.

[22] Oliver, R., Group actions on disks, integral permutation representations, and the Burnside ring, "Algebraic and Geometric Topology." *Proc. Sympos. Pure Math.*, Vol. 32, 339–346. Amer. Math. Soc., Providence, Rhode Island, 1978.

[23] Oliver, R., Weight systems for SO(3)-actions, *Ann. of Math.* **110** (1979), 227–241.

[24] Oliver, R., G-actions on disks and permutation representations II, *Math. Z.* **157** (1977), 237–263.

[25] Rubinstein, J. H. Free actions of some finite groups on S^3 I, *Math. Ann.* **240** (1979), 165–175.

[26] Smith, P. A., Transformations of finite period, *Ann. of Math.* **39** (1938), 127–164.

[27] Smith, P. A., Transformations of finite period, II, *Ann. of Math.* **40** (1939), 690–711.

[28] Smith, P. A., Transformations of finite period, III, *Ann. of Math.* **42** (1941), 446–458.

[29] Smith, P. A., New results and old problems in finite transformation groups, *Bull. Amer. Math. Soc.* **66** (1960), 401–415.

[30] Stein, E., Surgery on products with finite fundamental groups, *Topology* **16** (1977), 473–493.

[31] Waldhausen, F., Über Involutionen der 3-Sphäre, *Topology* **8** (1969), 81–91.