# UNIVERSAL G-MANIFOLDS

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**0. Introduction.** In the mid sixties, the Hsiangs [13] and Jänich [15] defined an invariant for certain smooth $G$-manifolds with two orbit types. (See also [2], [7], and [12].) One requires that each singular isotropy group act transitively on the unit sphere of its normal representation and that the bundle of principal orbits of the $G$-manifold be (equivariantly) trivial. In this case one can define an invariant called the "twist invariant" by the Hsiangs and the "characteristic reduction" by Jänich, and Hirzebruch and Mayer. This twist invariant is the homotopy class of a map from the singular stratum of the orbit space to a certain homogeneous space. It is an invariant of the $G$-manifold together with a homotopy class of trivialization of the bundle of principal orbits. In Section 2 of this paper, this invariant is generalized to a wider class of smooth $G$-manifolds. There are two conditions which we call "admissibility" and "trivializability." Trivializability means, as before, that the bundle of principal orbits is a trivial fiber bundle. (Actually, this condition is only used to simplify the technicalities.) Admissibility is a somewhat complicated condition concerning the orbit space of the normal representation of each isotropy group. (However, there is some evidence for the conjecture that this condition is equivalent to the condition that the orbit space of the $G$-manifold be a topological manifold with boundary.) By a trivialized admissible $G$-manifold, we shall mean a pair $(M, [f])$ where $M$ is admissible and where $[f]$ is the homotopy class of a trivialization of the bundle of principal orbits of $M$. To each stratum of such a $M$ we associate a "twist invariant," that is, the homotopy class of a map from the orbit space of the stratum to a certain homogeneous space (which depends only on the type of the stratum).

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A trivialized admissible $G$-manifold $(X, [g])$ is *universal* if it has the following property. Any $(M, [f])$ with the same type of strata as $X$ must be the pullback of $X$ in an essentially unique way. That is to say, there must exist an equivariant stratified map $\lambda: M \to X$ such that $\lambda$ pulls back $[g]$ to $[f]$; moreover, $\lambda$ must be unique up to a homotopy through such maps. Our main result, Theorem 3.2.1, states that $(X, [g])$ is universal if and only if each of its twist invariants is a homotopy equivalence. That such a result might be true was first suggested by work by Bredon [4] on bi-axial actions and by its subsequent generalization to $k$-axial actions in my thesis [5]. In practice it is straightforward to check if a $G$-manifold is admissible and trivializable and if the twist invariants are homotopy equivalences. Several examples are given in Sections 4 and 5.

The main application has been to the study of multi-axial actions. These are discussed in Section 4. Here $G$ is $O(n)$, $U(n)$, or $Sp(n)$ and $X$ is the linear $G$-space consisting of $k$-tuples of vectors in $n$-space, with $k \leq n$. In 4.2 we verify that $X$ is admissible and trivializable. It is proved in [7] that each twist invariant is a homotopy equivalence. Hence, $X$ is universal. A smooth $G$-manifold with the same types of strata as $X$ is called *$k$-axial.* The fact that $X$ is both universal and linear has deep implications in the study of $k$-axial actions on homotopy spheres (for such actions the bundle of principal orbits is automatically trivial). In particular it enables one to apply the techniques of surgery theory to study such actions, as in [9] and [11].

For similar reasons it would be convenient if other admissible linear $G$-spaces happened to be universal. Unfortunately, this is not generally true. For example, consider the cases of bi-axial actions of $SO(3)$, $SU(3)$ and $G_2$. The linear models are admissible and trivializable; however, the twist invariants fail to be homotopy equivalences. In Section 5 we prove that in these cases the universal examples are certain natural actions on projective planes. Regarding $SO(3)$ as the group of automorphisms of the quaternions, we see that it acts on the quaternionic projective plane. It is verified that this action is bi-axial and trivial and, as an application of our main result, it is shown in Theorem 5.6.4 that it is universal. The automorphism group of the Cayley numbers is denoted by $G_2$. In a similar fashion $G_2$ acts on the Cayley projective plane. The subgroup of $G_2$ which stabilizes the complex numbers is isomorphic to $SU(3)$. The $G_2$-action restricts to an action of $SU(3)$ on the Cayley projective plane. It is also proved in Section 5 that these actions are bi-axial, trivializable and universal. Applications of these results to bi-axial actions of $SO(3)$, $SU(3)$ and $G_2$ on homotopy spheres will be given in a subsequent paper [8].

Some of this material appears in less detail in [7]. In particular, the main result of Section 5 is stated, without proof, on page 112 of [7]. Theorem 4.2.5, which states that the linear $k$-axial action is universal, was proved in 1974 in my thesis, by using some of the same ideas that go into the proof of the main theorem of this paper. The results are better organized here and the proof is clearer.

**1. Preliminaries.** In this section we shall recall some definitions from [6] and [19] and make some minor modifications in them.

**1.1 Normal orbit types.** Let $G$ be a compact Lie group. A $G$-orbit type is defined as the conjugacy class of a closed subgroup of $G$. We shall now consider a finer concept. Consider pairs $(H, V)$ where $H$ is a closed subgroup of $G$ and where $V$ is an orthogonal $G$-module with no non-zero invariant vector. For $i = 1, 2$, suppose that $(H_i, V_i)$ is such a pair and that $\rho_i : H_i \to O(V_i)$ is the associated representation. Let $\mathrm{Iso}(V_1, V_2)$ denote the set of linear isomorphisms from $V_1$ to $V_2$ and let $O(V_1, V_2)$ be the subset of isometries. An *equivalence* from $(H_1, V_1)$ to $(H_2, V_2)$ is a pair $(k, a) \in G \times \mathrm{Iso}(V_1, V_2)$ such that

$$kH_1k^{-1} = H_2 \tag{1.1.1}$$

and

$$a\rho_1(h)a^{-1} = \rho_2(khk^{-1}) \quad \text{for all} \quad h \in H_1.$$

If, in addition, $a \in O(V_1, V_2)$, then $(k, a)$ is called an *orthogonal equivalence.* The equivalence class of $(H, V)$, denoted by $[H, V]$, is called a *normal $G$-orbit type.* The set of all such normal $G$-orbit types is denoted by $\mathfrak{N}_G$.

Associated to $(H, V)$ there is the twisted product $G \times_H V$, defined as the orbit space of $G \times V$ under the $H$-action $h \cdot (g, v) = (gh^{-1}, hv)$. The orbit of $(g, v)$ is denoted by $[g, v]$. The twisted product may also be regarded as the $G$-vector bundle over $G/H$ associated to the representation $\rho : H \to O(V)$. Associated to an equivalence $(k, a):(H_1, V_1) \to (H_2, V_2)$, there is an isomorphism of $G$-vector bundles $\theta_{(k,a)} : G \times_{H_1} V_1 \to$

$G \times_{H_2} V_2$ defined by $[g, v] \to [gk^{-1}, av]$. Conversely, it is easy to see that any isomorphism must be of this form. Thus, $\mathfrak{N}_G$ may also be described as the set of isomorphisms classes of $G$-vector bundles of the form $G \times_H V$.

**1.2 Polar decomposition.** We recall some elementary notions from linear algebra. If $b \in \mathrm{Iso}(V_1, V_2)$, then $b$ has a unique left polar decomposition as $b = A_b Q_b$, where $A_b \in O(V_1, V_2)$ and where $Q_b \in GL(V_1)$ is symmetric and positive definite. Similarly, $b$ has a unique right polar decomposition as $b = P_b O_b$ where $O_b \in O(V_1, V_2)$ and where $P_b \in GL(V_2)$ is symmetric and positive definite. Since $b = (A_b Q_b A_b{}^{-1}) A_b$ is another right polar decomposition, we conclude that

$$A_b = O_b \tag{1.2.1}$$

$$A_b Q_b A_b{}^{-1} = P_b. \tag{1.2.2}$$

Suppose that $(k, a)$ is an equivalence from $(H_1, V_1)$ to $(H_2, V_2)$. Set $a = P_a Q_a$ and use definition (1.1.1) to get

$$P_a(O_a \rho_1(h)(O_a{}^{-1})) = \rho_2(khk^{-1}) P_a. \tag{1.2.3}$$

The two sides of this equation are the right and left polar decompositions of the map $b = \rho_2(khk^{-1})P_a \in GL(V_2)$. Hence, applying (1.2.1) and (1.2.2) to $b$ we have

$$\rho_2(khk^{-1}) = O_a \rho_1(h) O_a{}^{-1} \tag{1.2.4}$$

$$\rho_2(khk^{-1}) P_a \rho_2(khk^{-1})^{-1} = P_a \tag{1.2.5}$$

for all $h \in H_1$. In particular, the first of these equations means that $(k, O_a)$ is an orthogonal equivalence from $(H_1, V_1)$ to $(H_2, V_2)$. Therefore, we have proved the following lemma.

1.2.6 Lemma. *To any equivalence $(k, a):(H_1, V_1) \to (H_2, V_2)$ there is canonically associated the orthogonal equivalence $(k, O_a)$.*

**1.3 The group of orthogonal self-equivalences.** A self-equivalence of $(H, V)$ corresponds to the identity map on $G \times_H V$ if and only if it has the form $(h, \rho(h)) \in H \times \rho(H) \subset G \times GL(V)$. Denote by $S_{[H, V]}$ the group of orthogonal self-equivalences of $(H, V)$ modulo those which act as the identity on $G \times_H V$, i.e.,

$$S_{[H, V]} = N_{G \times O(V)}(H)/H,$$

where $H$ is embedded diagonally in $G \times O(V)$ via $h \to (h, \rho(h))$. We next want to prove a lemma, which will be useful in computing $S_{[H, V]}$. First we need some notation. Suppose $H$ is embedded as a closed subgroup of a Lie group $K$. Let $\mathrm{Aut}(H)$ be the group of automorphisms of $H$ and let $\mathrm{Inn}(H)$ be the subgroup of inner automorphisms. Let $i_K : N_K(H) \to \mathrm{Aut}(H)$ be the natural map. Clearly, $i_K^{-1}(\mathrm{Inn}(H)) = H \cdot C_K(H)$ where $C_K(H)$ denotes the centralizer of $H$ in $K$. Thus, we may write any $x \in i_K^{-1}(\mathrm{Inn}(H))$ in the form $x = h_x c_x$, where $h_x \in H$ and $c_x \in C_K(H)$. Moreover, if $h_1 c_1 = h_2 c_2$ are two such decompositions, then $h_1 h_2^{-1} = c_2 c_1^{-1}$ and $h_1 h_2^{-1}$ belongs to $Z(H)$, the center of $H$. If $K'$ is another Lie group with $H \subset K'$, then let $C_K(H) \times_{Z(H)} C_{K'}(H)$ denote the quotient of the direct product by the diagonally embedded $Z(H)$. If $(u, v) \in C_K(H) \times C_{K'}(H)$, then let $[u, v]$ denote its image in the quotient.

1.3.1 Lemma. *Suppose every automorphism of $H$ is inner. Then*

$$S_{[H, V]} \cong C_G(H) \times_{Z(H)} C_{O(V)}(H)$$

For example the hypothesis of this lemma applies if $H$ is one of the compact classical groups, $O(n)$, $U(n)$, or $Sp(n)$.

*Proof.* Suppose $(k, a) \in G \times O(V)$ normalizes $H$. Then $k \in N_G(H)$, $a \in N_{O(V)}(H)$, and $i_G(k) = i_{O(V)}(a)$. Since every automorphism is inner, we can write $k = h_k c_k$ and $a = h_a c_a$. Set $\epsilon = h_a h_k^{-1}$. Then $\epsilon \in Z(H)$. Define $\psi : N_{G \times O(V)}(H) \to C_G(H) \times_{Z(H)} C_{O(V)}(H)$, by $(k, a) \to [\epsilon c_k, \epsilon^{-1} c_a]$. One checks routinely that $\psi$ is a well defined epimorphism with kernel $H$. The lemma follows. □

**1.4 Stratification of smooth $G$-manifolds.** Suppose that $M$ is a smooth $G$-manifold and assume (as we may) that $M$ is equipped with a $G$-invariant Riemannian metric. If $x \in M$, then the tangent space $T_x M$ is an orthogonal $G_x$-module. The slice representation $S_x$ is the $G_x$-submodule of $T_x M$ orthogonal to the orbit passing through $x$. Let $F_x$ be the fixed subspace of $S_x$ and let $N_x$ be the orthogonal complement of $F_x$

in $S_x$. The orthogonal $G_x$-module $N_x$ is called the *normal representation* at $x$ and $[G_x, N_x]$ is the *normal $G$-orbit type* of $x$. The correspondence $x \to [G_x, N_x]$ defines a map $M \to \mathfrak{N}_G$ which is constant on orbits. Its image is denoted $\mathfrak{N}_G(M)$.

A partial ordering on $\mathfrak{N}_G$ is defined by $[H, V] \leq [K, W]$ if $[K, W] \in \mathfrak{N}_G(G \times_H V)$, that is, if $[K, W]$ occurs as a normal $G$-orbit type of $G$ on $G \times_H V$. If $\bar{O}$ denotes the zero dimensional $H$-module, then $[H, \bar{O}]$ is a maximal element in $\mathfrak{N}_G$. Such a maximal element is called principal and will usually be denoted by the letter "$\pi$".

Let $B(M)$ denote the orbit space of $G$ on $M$. The sheaf of germs of smooth functions on $M$ pushes forward to a sheaf (in fact, a functional structure) on $B(M)$ called the *quotient smooth structure* on $B(M)$.[1] (See [2], [6], or [18].) In this way it makes sense to discuss smooth functions on orbit spaces, smooth maps of orbit spaces, and functional structures on subsets of orbit spaces.

If $\alpha \in \mathfrak{N}_G(M)$, then the $\alpha$-stratum of $M$, denoted by $M_\alpha$, consists of those $x \in M$ with $[G_x, N_x] = \alpha$. Its image in $B(M)$ is denoted by $B_\alpha(M)$ and is called the *$\alpha$-stratum* of $B(M)$. It follows from the Differentiable Slice Theorem that $M_\alpha$ and $B_\alpha(M)$ are smooth manifolds and that $M_\alpha \to B_\alpha(M)$ is a smooth fiber bundle. According to the Principal Orbit Theorem, if $B(M)$ is connected (we shall generally assume this), then $\mathfrak{N}_G(M)$ contains a (unique) maximum $\pi$ of the form $[K, \bar{O}]$, called the *principal orbit type* of $G$ on $M$. Also, $M_\pi$ is open dense in $M$.

1.4.1 *Definition.* A smooth $G$-manifold $M'$ is said to be *stably modeled* on $M$ if $\mathfrak{N}_G(M') \subset \mathfrak{N}_G(M)$. It follows from the Slice Theorem that this condition is equivalent to the condition that every point in $M' \times \mathbf{R}^{m'}$ has a $G$-invariant open neighborhood isomorphic to an invariant open neighborhood in $M \times \mathbf{R}^m$. (Here $G$ acts trivially on the second factors and $m$ and $m'$ are chosen so that $\dim M' + m' = \dim M + m$.)

**1.5 Normal orbit bundles.** Suppose that $\alpha = [H, V]$ is a normal orbit type of $G$ on $M$. Let $N_\alpha(M)$ denote the total space of the normal bundle of $M_\alpha$ in $M$. The fiber of $N_\alpha(M)$ at $x \in M_\alpha$ is $N_x$. The composition $N_\alpha(M) \to M_\alpha \to B_\alpha(M)$ makes $N_\alpha(M)$ into a smooth fiber bundle over $B_\alpha(M)$ with fiber $G \times_H V$. The structure group can be canonically

[1] In general, $B(M)$ is *not* a smooth manifold.

reduced to $S_\alpha$, the group of orthogonal $G$-vector bundle automorphisms of $G \times_H V$. Let $P_\alpha(M)$ be the total space of the associated principal $S_\alpha$-bundle. The bundle $P_\alpha(M) \to B_\alpha(M)$ is called the *principal $\alpha$-normal orbit bundle* of $M$. These are the basic building blocks of a smooth $G$-manifold.

An equivariant map $\psi: M \to M'$ is *isovariant* if $G_x = G_{\psi(x)}$ for all $x \in M$. If $\psi$ is smooth and isovariant, then its differential induces a $G_x$-equivariant linear map $\psi_x: N_x \to N_{\psi(x)}$. If $\psi_x$ is an isomorphism, then $\psi$ is said to be *normally transverse* at $x$. (Note that $x$ and $\psi(x)$ will then have the same normal orbit type.) A smooth isovariant map $\psi$ is *stratified* if it is normally transverse at each point. If $\psi$ is stratified, then it restricts to a map $\psi_\alpha: M_\alpha \to M_\alpha'$ and $\psi_\alpha$ induces a map $B_\alpha(\psi): B_\alpha(M) \to B_\alpha(M')$ of orbit spaces. Assume now that $M$ and $M'$ are equipped with $G$-invariant Riemannian metrics. Set $a = \psi_x: N_x \to N_{\psi(x)}$. Then $a$ is $G_x$-equivariant. Let $a = P_a O_a$ be the right polar decomposition, where $O_a \in O(N_x, N_{\psi(x)})$. It follows from (1.2.4) that $O_a$ is also $G_x$-equivariant. Set $N_x(\psi) = O_a$ and let $N_\alpha(\psi): N_\alpha(M) \to N_\alpha(M')$ be the map which is $N_x(\psi)$ on $N_x$. Clearly, $N_\alpha(\psi)$ is also a bundle map when $N_\alpha(M)$ and $N_\alpha(M')$ are regarded as bundles over $B_\alpha(M)$ and $B_\alpha(M')$, respectively. Let $P_\alpha(\psi): P_\alpha(M) \to P_\alpha(M')$ be the associated map of principal $S_\alpha$-bundles. (Obviously $P_\alpha(\psi)$ covers $B_\alpha(\psi)$.)[2]

**1.6 The orbit space of a representation.** Suppose that $V$ is an orthogonal $H$-module. H. Weyl proved in [21] that the ring of invariant polynomials $\mathbf{R}[V]^H$ is a finitely generated $\mathbf{R}$-algebra. Let $p_1, p_2, \ldots, p_s$ be generators and let $p = (p_1, p_2, \ldots, p_s): V \to \mathbf{R}^s$. Then $p(V)$ inherits a functional structure as a subset of Euclidean space. Since $p$ is constant on orbits, it induces a map on the orbit space, $B(V)$. G. Schwarz proved in [18], that the induced map $\bar{p}: B(V) \to p(V)$ is a smooth isomorphism, i.e., $\bar{p}$ is a homeomorphism which induces an isomorphism

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[2] For each $\alpha \in \mathfrak{N}_G$, the correspondence $M \to B_\alpha(M)$ is a functor from the category of smooth $G$-manifolds and equivariant stratified maps to the category of smooth manifolds and smooth maps. The correspondence $M \to P_\alpha(M)$ sends each $G$-manifold to a principal $S_\alpha$-bundle and each equivariant stratified map $\psi$ to a bundle map $P_\alpha(\psi)$. Since the map $GL(V) \to O(V)$ given by polar decomposition is not a homomorphism, this correspondence is not quite a functor: the bundle map $P_\alpha(\psi \circ \psi')$ may not equal $P_\alpha(\psi) \circ P_\alpha(\psi')$. However, it is a "homotopy functor" in the sense that $P_\alpha(\psi \circ \psi')$ is homotopic to $P_\alpha(\psi) \circ P_\alpha(\psi')$ through bundle maps covering $B_\alpha(\psi)$. The reason for this is that $GL(V) \to O(V)$ is an isomorphism of $H$-spaces.

of smooth structures. In this manner we identify $B(V)$ with the semi-algebraic set $p(V)$.

Suppose that $[H, V]$ is a normal $G$-orbit type. The inclusion $V \to G \times_H V$ defined by $v \to [1, v]$ induces a smooth isomorphism $B(V) \cong B(G \times_H V)$. (Here $B(V)$ is the orbit space of $V$ under $H$.) If $\pi$ is the principal $G$-orbit type of $G \times_H V$ and $\pi'$ is the principal $H$-orbit type of $V$, then according to [2], page 180, $B_{\pi'}(V) \cong B_\pi(G \times_H V)$. In general, however, the stratifications of $B(V)$ and $B(G \times_H V)$ may differ in that different components of a single stratum in $B(G \times_H V)$ may have distinct labels in $B(V)$. For example, if $G = U(n)$, $H = T^n$ (a maximal torus) and $V = \mathbf{C}^n$ with the standard action of $T^n$, then $B(V)$ may be identified with the quadrant in $\mathbf{R}^n$ given by

$$\{(x_1, \ldots, x_n) \mid x_i \geq 0, 1 \leq i \leq n\}.$$

Each codimension one face of this quadrant is a distinct stratum of $B(V)$; however, in $B(G \times_H V)$ they must be regarded as different components of a single stratum labelled $[T^1, \mathbf{C}^1]$.

**1.7 Equivariant fiber bundles.** Suppose that $I$ and $J$ are Lie groups and that $I$ is a differentiable transformation group of a smooth manifold $F$. By a *smooth $(I, J)$-bundle* we shall mean a smooth fiber bundle $E \to B$ with fiber $F$ and structure group $I$ together with an action of $J$ on $E$ by bundle maps. For example, a smooth $G$-vector bundle is a smooth $(GL(n), G)$-bundle. An *$(I, J)$-bundle map* means a $J$-equivariant bundle map. An *equivalence* is an $(I, J)$-bundle map which covers the identity map on the base space and which is a diffeomorphism on each fiber. If $f: A \to B$ is a smooth $J$-equivariant map of $J$-manifolds and $E \to B$ is an $(I, J)$-bundle, then the pullback $f^*(E) \to A$ is obviously an $(I, J)$-bundle over $A$. A principal $(I, J)$-bundle over a point clearly just amounts to a homomorphism $\varphi: J \to I$. The $(I, J)$-bundle $E \to B$ is *trivial* if it pulls back from an $(I, J)$-bundle over a point, i.e., if it is equivalent to $B \times F$ where $J$ acts on $B$ as before and on $F$ via some representation $\varphi: J \to I$. For $J$ compact, Bierstone proved in [22] that the Equivariant Covering Homotopy Theorem holds for $(I, J)$-bundles. Thus, if $f_t: B \to B$ is a smooth $J$-equivariant homotopy, then $f_0^*(E)$ and $f_1^*(E)$ are equivalent $(I, J)$-bundles. In particular, if $A$ has the $J$-homotopy type of a point, then any $(I, J)$-bundle over it is trivial.

If $E \to B$ is an $(I, J)$-bundle and $P \to X$ is a principal $J$-bundle, then one can construct the twisted products $E \times_J P$ and $B \times_J P$. The first twisted product is a fiber bundle over the second with structure group $I$. Also, both twisted products are bundles over $X$ with structure group $J$.

As an application of these ideas, suppose that $\alpha = [H, V] \in \mathfrak{N}_G$ and that $\beta \geq \alpha$. Since $S_\alpha$ acts on $G \times_H V$ by equivariant diffeomorphisms, it acts on $B_\beta(G \times_H V)$. If $x$ is a point in the $\beta$-stratum of $G \times_H V$, then its normal representation $N_x$ may be identified with a subspace of $V$ and we may assume that the inner product on $N_x$ is induced by restricting the one on $V$. Since $S_\alpha$ acts on $G \times_H V$ by orthogonal equivalences it acts on $N_\beta(G \times_H V) \to B_\beta(G \times_H V)$ and hence on $P_\beta(G \times_H V) \to B_\beta(G \times_H V)$ through bundle maps. Therefore, $P_\beta(G \times_H V) \to B_\beta(G \times_H V)$ is a principal $(S_\beta, S_\alpha)$-bundle. If $M$ is any smooth $G$-manifold, then there is a canonical isomorphism

$$N_\alpha(M) \cong (G \times_H V) \times_{S_\alpha} P_\alpha(M). \tag{1.7.1}$$

This induces for each $\beta \geq \alpha$ an isomorphism

$$P_\beta(N_\alpha(M)) \cong P_\beta(G \times_H V) \times_{S_\alpha} P_\alpha(M). \tag{1.7.2}$$

Thus, $P_\beta(N_\alpha(M))$ has the structure of a bundle over $B_\beta(N_\alpha(M))$ and of a bundle over $B_\alpha(M)$.

**1.8 Tubular maps.** If $\alpha \in \mathfrak{N}_G(M)$, then let $M(\alpha)$ be the complement of all the strata of $M$ of index less than $\alpha$. Then $M_\alpha$ is a properly embedded invariant submanifold of $M(\alpha)$ with normal bundle $N_\alpha(M)$. By a *tubular map* $\tau: N_\alpha(M) \to M(\alpha)$ we shall mean that

a) $\tau$ is an equivariant diffeomorphism onto its image,
b) $\tau | M_\alpha$ is the inclusion,
c) for each $x \in M_\alpha$, the map $\tau_x: N_x \to N_x$ induced by the differential is the identity. (Recall that the normal bundle of the 0-section of $N_\alpha(M)$ is canonically identified with $N_\alpha(M)$.)

By the Invariant Tubular Neighborhood Theorem (see [2]), such a $\tau$ exists and is unique up to an equivariant isotopy.

## 2. Twist invariants.

**2.1 Admissibility.** Suppose that $E \to B$ is a smooth $(I, J)$-bundle. Recall that this means that the structure group is $I$ and that $J$ acts on $E$ by bundle maps. If $B$ has the $J$-homotopy type of a point with $J$ compact, then $E$ is equivalent to the product $(I, J)$-bundle $F \times B$ where $J$ acts on $F$ via a homomorphism $\varphi: J \to I$.

Let $\alpha = [H, V]$ be a normal $G$-orbit type and let $\pi$ be the principal orbit type of $G$ on $G \times_H V$. Then $P_\pi(G \times_H V) \to B_\pi(G \times_H V)$ is an $(S_\pi, S_\alpha)$-bundle. We shall say that $\alpha$ is *admissible* if $B_\pi(G \times_H V)$ has the $S_\alpha$-homotopy type of a point and if the associated homomorphism $\varphi: S_\alpha \to S_\pi$ is injective.

A smooth $G$-manifold is called *admissible* if each of its normal orbit types is admissible.

2.1.1 *Remark.* An orthogonal $H$-module $V$ is *coregular* if its ring of invariant polynomials $\mathbf{R}[V]^H$ is regular, i.e., if $\mathbf{R}[V]^H$ is a free polynomial algebra. The following are examples of coregular $H$-modules: 1) $H$ is a finite group generated by reflections, 2) the adjoint representation of a compact Lie group $H$, 3) the natural action of $O(n)$ on the space of $k$-tuples of vectors in $\mathbf{R}^n$, with $n \geq k$. W. Y. Hsiang has tabulated coregular $H$-modules for $H$ compact and connected. More generally, a complete list of coregular representations of simple Lie groups has been published by G. Schwarz in [20].

Let $SV$ be the unit sphere in $V$. Several people have conjectured that $V$ is coregular if and only if the orbit space $B(SV)$ is homeomorphic to a disk with $B_\pi(SV)$ as its interior. This would imply that $B_\pi(V)$ is diffeomorphic to Euclidean space. Suppose that $[H, V]$ is a normal $G$-orbit type with $V$ a coregular $H$-module. The action of $S_{[H,V]}$ on $B_\pi(G \times_H V) \cong B_\pi(V)$ factors through the action of $N_{O(V)}(H)/H$ on $B_\pi(V)$ (supposedly, Euclidean space). It seems plausible to conjecture that this action of $N_{O(V)}(H)/H$ is equivalent to a linear action (provided $V$ is coregular). This would obviously imply that $B_\pi(V)$ has the $S_{[H,V]}$-homotopy type of a point. Thus we are led to the following conjecture.

2.1.2 *Conjecture.* *$[H, V] \in \mathfrak{N}_G$ is admissible if and only if $V$ is a coregular $H$-module.*

**2.2 The bundles $D_\alpha$ and $E_\alpha$.** Suppose the $\alpha = [H, V]$ is an admissible normal $G$-orbit type. Set

$$Q = B_\pi(G \times_H V)$$

$$R = P_\pi(G \times_H V).$$

Let $T$ denote the fixed point set of $S_\alpha$ on $Q$. Since $Q$ has the $S_\alpha$-homotopy type of a point, $T$ is contractible (in particular, it is non-empty and connected). Let $U$ be the restriction $R$ to $T$. Fix an $(S_\pi, S_\alpha)$-bundle isomorphism $\eta : R \to S_\pi \times Q$. Note that $\eta$ restricts to an isomorphism $U \cong S_\pi \times T$.

In (1.7.2), it was shown that for a smooth $G$-manifold $M$, both $B_\pi(N_\alpha(M))$ and $P_\pi(N_\alpha(M))$ may be regarded as bundles over $B_\alpha(M)$ via the canonical isomorphisms,

$$B_\pi(N_\alpha(M)) \cong Q \times_{S_\alpha} P_\alpha(M)$$

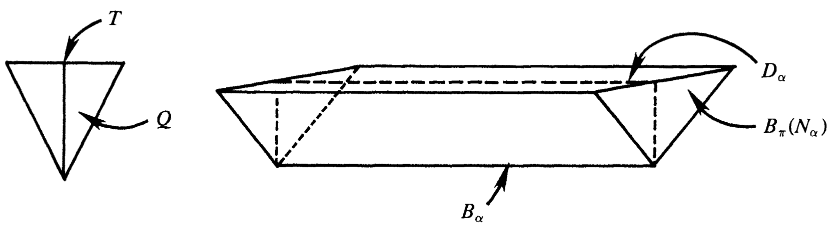
$$P_\pi(N_\alpha(M)) \cong R \times_{S_\alpha} P_\alpha(M).$$

Hence, we may define sub-bundles $D_\alpha(M) \subset B_\pi(N_\alpha(M))$ and $E_\alpha(M) \subset P_\pi(N_\alpha(M))$ via the identifications,

$$D_\alpha(M) \cong T \times_{S_\alpha} P_\alpha(M)$$

$$E_\alpha(M) \cong U \times_{S_\alpha} P_\alpha(M).$$

Note that $E_\alpha(M) \to D_\alpha(M)$ is a principal $S_\pi$-bundle. Also, note that $D_\alpha(M) \to B_\alpha(M)$ is a trivial $T$-bundle.




If $P \to P'$ is any $S_\alpha$-bundle map, then the induced map $Q \times_{S_\alpha} P \to Q \times_{S_\alpha} P'$ takes $T \times_{S_\alpha} P$ to $T \times_{S_\alpha} P'$. Similarly, $R \times_{S_\alpha} P \to R \times_{S_\alpha} P'$ takes $U \times_{S_\alpha} P$ to $U \times_{S_\alpha} P'$. Hence if $\psi : M \to M'$ is an equivariant stratified map of $G$-manifolds, then $B_\pi(N_\alpha(\psi))$ and $P_\pi(N_\alpha(\psi))$ restrict to maps $D_\alpha(\psi):D_\alpha(M) \to D_\alpha(M')$ and $E_\alpha(\psi):E_\alpha(M) \to E_\alpha(M')$. Thus, $D_\alpha(\ )$ and $E_\alpha(\ )$ are both "homotopy functors" in equivariant stratified maps in the sense discussed in the footnote following Section 1.5.

**2.3 Trivializations.** An admissible $G$-manifold $M$ is *trivializable* if $P_\pi(M) \to B_\pi(M)$ is a trivial fiber bundle. A *trivialized admissible* $G$-manifold is a pair $(M, [f])$ where $[f]$ is the $S_\pi$-homotopy class of a trivialization $f:P_\pi(M) \to S_\pi$. A *morphism* $\psi:(M, [f]) \to (M', [f'])$ of trivialized admissible $G$-manifolds is an equivariant stratified map $\psi$ such that $f$ and $f' \circ P_\pi(\psi)$ are $S_\pi$-homotopic as maps from $P_\pi(M)$ to $S_\pi$,

$$\begin{tikzcd} P_\pi(M) \arrow[rr, "P_\pi(\psi)"] \arrow[dr, "f"'] & & P_\pi(M') \arrow[dl, "f'"] \\ & S_\pi & \end{tikzcd}$$

**2.4 Twist invariants.** We shall now show how to associate to each normal orbit type $\alpha$ of a trivialized admissible $G$-manifold $(M, [f])$ a map $f_\alpha:B_\alpha(M) \to S_\pi/S_\alpha$, the homotopy class of which will be called the "$\alpha$-twist invariant" of $(M, [f])$. For each $\alpha \in \mathfrak{N}_G(M)$, we must make two choices:

a) a section $s$ of the trivial $T$-bundle $D_\alpha(M) \to B_\alpha(M)$,
b) a tubular map $\tau:N_\alpha(M) \to M(\alpha)$.

Using the identifications $\eta:U \cong S_\pi \times T$ and $E_\alpha(M) \cong U \times_{S_\alpha} P_\alpha(M)$ we obtain an identification

$$E_\alpha(M)|s(B_\alpha(M)) \cong S_\pi \times_{S_\alpha} P_\alpha(M).$$

Let $i:P_\alpha(M) \to S_\pi \times_{S_\alpha} P_\alpha(M)$ denote the inclusion $z \to [1, z]$. We regard $i$ as a reduction of the structure group of $E_\alpha(M)|s(B_\alpha(M))$ from $S_\pi$ to $S_\alpha$. Let $\theta:P_\alpha(M) \hookrightarrow E_\alpha(M)$ denote the composition of $i$ with the inclusion $E_\alpha(M)|s(B_\alpha(M)) \subset E_\alpha(M)$. The tubular map $\tau$ induces a bundle map $P_\pi(\tau):P_\pi(N_\alpha(M)) \to P_\pi(M)$. Denote the restriction of $P_\pi(\tau)$ to $E_\alpha(M)$ by $\omega:E_\alpha(M) \hookrightarrow P_\pi(M)$. Finally, let $\tilde{f}_\alpha = f \circ \omega \circ \theta:P_\alpha(M) \to S_\pi$. The map $\tilde{f}_\alpha$ is clearly $S_\alpha$-equivariant ($\theta$ is $S_\alpha$-equivariant, $\omega$ and $f$ are $S_\pi$-equivariant); hence, it covers a map $f_\alpha:B_\alpha(M) \to S_\pi/S_\alpha$. The homotopy class $[f_\alpha]$ is called the *$\alpha$-twist invariant* of $(M, [f])$.

Ostensibly, $[f_\alpha]$ depends on the choice of section and on the choice of tubular map. Any two sections of $D_\alpha(M) \to B_\alpha(M)$ are homotopic, since the fiber $T$ is connected. Hence, the effect of altering the section is to change $\theta: P_\alpha(M) \hookrightarrow E_\alpha(M)$ by an $S_\alpha$-homotopy. Since any two tubular maps are equivariantly isotopic, the effect of altering $\tau$ is to change $\omega: E_\alpha(M) \to P_\pi(M)$ by an $S_\pi$-homotopy. In either case, the twist invariant remains unchanged.

2.4.1 **Proposition.** *If $\psi:(M, [f] \to (M', [f])$ is a morphism of trivialized admissible G-manifolds, then each twist invariant of M is the pullback of the corresponding twist invariant of $M'$, i.e., for each $\alpha \in \mathfrak{N}_G(M)$, in the following diagram,*

$$\begin{tikzcd} B_\alpha(M) \arrow[rr, "B_\alpha(\psi)"] \arrow[dr, "f_\alpha"'] & & B_\alpha(M') \arrow[dl, "f_\alpha"] \\ & S_\pi/S_\alpha & \end{tikzcd}$$

*the map $f_\alpha$ is homotopic to $f_\alpha' \cdot B_\alpha(\psi)$.*

*Proof.* Choose a section $s': B_\alpha(M') \to D_\alpha(M')$. By the above remarks we are free to choose $s: B_\alpha(M) \to D_\alpha(M)$ to be the pullback of $s'$ via $D_\alpha(\psi)$. After using $s$ and $s'$ to define $\theta: P_\alpha(M) \to E_\alpha(M)$ and $\theta': P_\alpha(M') \to E_\alpha(M')$, we see that the following square commutes

$$\begin{tikzcd} P_\alpha(M) \arrow[r, "\theta"] \arrow[d, "P_\alpha(\psi)"'] & E_\alpha(M) \arrow[d, "E_\alpha(\psi)"] \\ P_\alpha(M') \arrow[r, "\theta'"] & E_\alpha(M'). \end{tikzcd}$$

Next choose tubular maps $\tau: N_\alpha(M) \to M(\alpha)$ and $\tau': N_\alpha(M') \to M'(\alpha)$. The diagram

$$\begin{tikzcd} N_\alpha(M) \arrow[r, "\tau"] \arrow[d, "N_\alpha(\psi)"'] & M(\alpha) \arrow[d, "\psi"] \\ N_\alpha(M') \arrow[r, "\tau'"] & M(\alpha') \end{tikzcd}$$

need not commute; however, recalling the proof of the uniqueness part of the Invariant Tubular Neighborhood Theorem (page 310 in [2]), we see that $\psi \circ \tau$ and $\tau' \circ N_\alpha(\psi)$ are homotopic through equivariant stratified maps. Hence, In the diagram

$$\begin{tikzcd}
E_\alpha(M) \arrow[r, "\omega"] \arrow[d, "E_\alpha(\psi)"'] & P_\pi(M) \arrow[d, "P_\pi(\psi)"] \\
E_\alpha(M') \arrow[r, "\omega'"] & P_\pi(M')
\end{tikzcd}$$

the maps $P_\pi(\psi) \circ \omega$ and $\omega' \circ E_\alpha(\psi)$ are $S_\pi$-homotopic. Consider the diagram

$$\begin{tikzcd}
P_\alpha(M) \arrow[r] \arrow[d] & E_\alpha(M) \arrow[r] \arrow[d] & P_\pi(M) \arrow[d] \arrow[dr, "f"] & \\
P_\alpha(M') \arrow[r] & E_\alpha(M') \arrow[r] & P_\pi(M') \arrow[r, "f'"'] & S_\pi
\end{tikzcd}$$

The map $\tilde{f}_\alpha$ is the composition across the top; while $\tilde{f}_\alpha'$ is the composition across the bottom. Since both squares and the triangle commute up to homotopy, it follows that the maps $\tilde{f}_\alpha$ and $\tilde{f}_\alpha' \circ P_\alpha(\psi)$ are $S_\alpha$-homotopic and hence, that $[f_\alpha] = [f_\alpha' \circ B_\alpha(\psi)]$. □

## 3. The Main Theorem

**3.1 Universality.** A trivialized admissible $G$-manifold $(X, [g])$ is *universal* provided it has the following property: if $(M, [f])$ is any trivialized admissible $G$-manifold which is stably modeled on $X$, there is a morphism $\lambda: (M, [f]) \to (X, [g])$ of trivialized admissible $G$-manifolds, unique up to a homotopy through such morphisms.

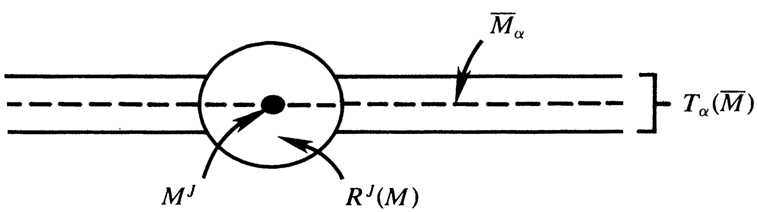
**3.2 A necessary and sufficient condition for universality.** Our main result is the following theorem.

3.2.1 THEOREM. *A necessary and sufficient condition for $(X, [g]$ to be universal is that each twist invariant $g_\alpha: B_\alpha(X) \to S_\pi/S_\alpha$ be a homotopy equivalence.*

The following observation is the key to the proof. Suppose that $g_\alpha: B_\alpha(X) \to S_\pi/S_\alpha$ is a homotopy equivalence. Then the bundle map $\tilde{g}_\alpha: P_\alpha(X) \to S_\pi$ has an $S_\alpha$-homotopy inverse $\tilde{r}_\alpha: S_\pi \to P_\alpha(X)$. Also suppose that we have a morphism $\lambda:(M, [f]) \to (X, [g])$. Then by Proposition 2.4.1, the bundle map $\tilde{f}_\alpha:P_\alpha(M) \to S_\pi$ is $S_\alpha$-homotopic to $\tilde{g}_\alpha \circ P_\alpha(\lambda)$. Hence, $P_\alpha(\lambda)$ is $S_\alpha$-homotopic to $\tilde{r}_\alpha \circ \tilde{f}_\alpha:P_\alpha(M) \to P_\alpha(X)$. Now suppose that we know that each twist invariant is a homotopy equivalence and that we are trying to construct the morphism $\lambda:(M, [f]) \to (X, [g])$. Roughly speaking, the idea is to define $\lambda$ on a tubular neighborhood of each stratum to be the map induced by

$$\tilde{r}_\alpha \circ \tilde{f}_\alpha:P_\alpha(M) \to P_\alpha(X).$$

*Proof of sufficiency.* Suppose that each $g_\alpha:B_\alpha(X) \to S_\pi/S_\alpha$ is a homotopy equivalence and that $(M, [f])$ is a trivialized admissible $G$-manifold with $\mathfrak{N}_G(M) \subset \mathfrak{N}_G(X)$. We shall construct $\lambda:(M, [f]) \to (X, [g])$ by induction on $\mathfrak{N}_G(X)$. First we need some notation. Suppose that $J$ is a subset of $\mathfrak{N}_G(X)$ which is closed from below, that is, if $\beta \in J$ and $\gamma \in \mathfrak{N}_G(X)$ is such that $\gamma < \beta$, then $\gamma \in J$. Let $M^J$ be the union of all strata of $M$ with indices in $J$ and let $R^J(M)$ be a closed invariant regular neighborhood of $M^J$ in $M$. Let $\overline{M}$ denote the closure of $M - R^J(M)$. Then $\partial\overline{M} = \partial R^J(M)$. Suppose $\alpha$ is minimal in $\mathfrak{N}_G(X) - J$ and set $J' = J \cup \{\alpha\}$. Let $T_\alpha(\overline{M})$ be a closed tubular neighborhood of $\overline{M}_\alpha$ in $\overline{M}$, i.e., $T_\alpha(\overline{M})$ is the image of the unit disk bundle of $N_\alpha(\overline{M})$ under a tubular map. Then $R^{J'}(M) = R^J(M) \cup T_\alpha(\overline{M})$ is a closed invariant regular neighborhood of $M^{J'}$. Define $X^J$, $R^J(X)$, $\bar{X}$, $T_\alpha(\bar{X})$ and $R^{J'}(X)$ in a similar fashion.




Suppose by induction that we have a morphism $\lambda:(R^J(M), [f]) \to (R^J(X), [g])$ mapping $\partial R^J(M)$ to $\partial R^J(X)$. Then it suffices to prove that we can extend $\lambda$ to a morphism

$$\lambda':(R^{J'}(M), [f]) \to (R^{J'}(X), [g])$$

mapping boundary to boundary.

Let $C\overline{M}$ be an invariant collared neighborhood of $\partial\overline{M}$ in $\overline{M}$. The isomorphism $C\overline{M} \cong M \times [0, 1]$ induces a bundle isomorphism $P_\gamma(C\overline{M}) \cong P_\gamma(\partial\overline{M}) \times [0, 1]$ for each $\gamma \in \mathfrak{N}_G(M)$. Denote by $\partial f$ and by $\partial\tilde{f}_\alpha$ the restrictions of $f: P_\pi(\overline{M}) \to S_\pi$ and $\tilde{f}_\alpha: P_\alpha(\overline{M}) \to S_\pi$ to $P_\pi(\partial\overline{M})$ and to $P_\alpha(\partial\overline{M})$, respectively. We may change $f$ by a $S_\pi$-homotopy so that $f \mid P_\pi(C\overline{M})$ is constant in the $t$-direction, i.e., so that $f(z, t) = \partial f(z)$ for all $(z, t) \in P_\pi(\partial\overline{M}) \times [0, 1]$. When defining $\tilde{f}_\alpha$ if we choose the tubular map and the section to be compatible with the collared neighborhood structure, then we will have that $\tilde{f}_\alpha \mid P_\alpha(C\overline{M})$ is also constant in the $t$-direction. Finally, we can alter $\lambda$ by a homotopy through morphisms so that it maps $T_\alpha(\partial\overline{M})$, the tubular neighborhood of $\partial\overline{M}_\alpha$ in $\partial\overline{M}$, into $T_\alpha(\partial\bar{X})$ and so that the restriction of $\lambda$ to $T_\alpha(\partial\overline{M})$ is induced by $P_\alpha(\lambda \mid \partial\overline{M})$.

Since $\lambda$ is a morphism, $P_\alpha(\lambda \mid \partial\overline{M}): P_\alpha(\partial\overline{M}) \to P_\alpha(\bar{X})$ is $S_\alpha$-homotopic to $\tilde{r}_\alpha \circ \partial\tilde{f}_\alpha$. Let $h: P_\alpha(\partial\overline{M}) \times [0, 1] \to P_\alpha(\bar{X})$ be such a homotopy. Let $\overline{\overline{M}}$ be the closure of $\overline{M} - C\overline{M}$. Define $k: P_\alpha(\overline{M}) \to P_\alpha(\bar{X})$ to be $h$ on $P_\alpha(C\overline{M})$ and to be $\tilde{r}_\alpha \circ \tilde{f}_\alpha$ on $P_\alpha(\overline{\overline{M}})$. Since $T_\alpha(\overline{M})$ is a fiber bundle associated to $P_\alpha(\overline{M})$, we may use $k$ to define a map $\hat{k}: T_\alpha(\overline{M}) \to T_\alpha(\bar{X})$ so that $\hat{k}$ agrees with $\lambda$ on $T_\alpha(\partial\overline{M})$. Define $\lambda': R^{J'}(M) \to R^{J'}(X)$ to be

$$\lambda \cup \hat{k}: R^J(M) \cup T_\alpha(\overline{M}) \to R^J(X) \cup T_\alpha(\bar{X}).$$

It remains to check that $\lambda'$ is a morphism, that is, we must show that in the following diagram

$$\begin{tikzcd}[column sep=small] P_\pi(R^{J'}(M)) \arrow[rr, "P_\pi(\lambda')"] \arrow[dr, "f"'] & & P_\pi(R^{J'}(X)) \arrow[dl, "g"] \\ & S_\pi & \end{tikzcd} \tag{1}$$

the map $f$ is $S_\pi$-homotopic to $g \circ P_\pi(\lambda')$. By the induction hypothesis, the restrictions of these maps to $P_\pi(R^J(M))$ are $S_\pi$-homotopic. Also, we have arranged that they are equal on $P_\pi(\partial R^J(M)) = P_\pi(\partial\overline{M})$. Therefore, we are reduced to proving that in the following diagram

$$\begin{tikzcd}[column sep=small] P_\pi(T_\alpha(\overline{M})) \arrow[rr, "P_\pi(\hat{k})"] \arrow[dr, "f"'] & & P_\pi(T_\alpha(\bar{X})) \arrow[dl, "g"] \\ & S_\pi & \end{tikzcd} \tag{2}$$

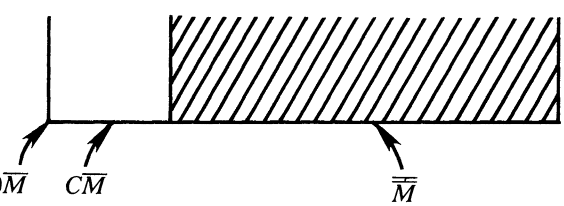
the map $f$ is $S_\pi$-homotopic to $g \circ P_\pi(\hat{k})$ rel $P_\pi(T_\alpha(\partial\overline{M}))$. In Section 2.4 we showed how to use a tubular map and a section of $D_\alpha(\overline{M}) \to B_\alpha(\overline{M})$ to define an embedding $\omega \circ \theta : P_\alpha(\overline{M}) \to P_\pi(T_\alpha(\overline{M}))$. The map $\tilde{f}_\alpha$ was defined as the composition of $f$ and $\omega \circ \theta$. We shall identify $P_\alpha(\overline{M})$ with its image under $\omega \circ \theta$ (and similarly for $P_\alpha(\bar{X})$). Since $\alpha$ is admissible, $B_\pi(G \times_H V)$ $S_\alpha$-deformation retracts to a point; hence, there is a $(S_\pi, S_\alpha)$-bundle deformation retraction of $P_\pi(G \times_H V)$ onto $S_\pi$. This defines a fiberwise $S_\pi$-deformation retraction of $P_\pi(T_\alpha(\overline{M}))$ onto $S_\pi \times_{S_\alpha} P_\alpha(\overline{M})$. Therefore, we may replace diagram (2) by the following diagram

$$\begin{tikzcd} S_\pi \times_{S_\alpha} P_\alpha(\overline{M}) \arrow[rr, "\times k"] \arrow[dr, "f"'] & & S_\pi \times_{S_\alpha} P_\alpha(\bar{X}) \arrow[dl, "g"] \\ & S_\pi & \end{tikzcd} \tag{3}$$

After restricting these maps to $P_\alpha(\overline{M})$ and $P_\alpha(\bar{X})$, we obtain the following

$$\begin{tikzcd} P_\alpha(\overline{M}) \arrow[rr, "k"] \arrow[dr, "\tilde{f}_\alpha"'] & & P_\alpha(\bar{X}) \arrow[dl, "\tilde{g}_\alpha"] \\ & S_\pi & \end{tikzcd} \tag{4}$$

So it now suffices to find an $S_\alpha$-homotopy from $\tilde{f}_\alpha$ to $\tilde{g}_\alpha \circ k$ rel $P_\alpha(\partial\overline{M})$, since such a homotopy will induce an $S_\pi$-homotopy in diagram (3). On $P_\alpha(\overline{\overline{M}})$, $k = \tilde{r}_\alpha \circ \tilde{f}_\alpha$ where $\tilde{r}_\alpha$ is an $S_\alpha$-homotopy inverse for $\tilde{g}_\alpha$. Hence, we can construct the homotopy on $P_\alpha(\overline{\overline{M}})$. It remains to extend it to $P_\alpha(C\overline{M})$ rel $\partial C\overline{M} = \partial\overline{M} \times \{0, 1\}$.




This is trivial, since $C\overline{M} \times [0, 1]$ equivariantly deformation retracts onto $C\overline{M} \times \{0\} \cup \partial C\overline{M} \times [0, 1]$. This completes the verification of the induction hypothesis for $\lambda'$. Hence, we have constructed a morphism $\lambda:(M, [f]) \to (X, [g])$. If $\delta:(M, [f]) \to (X, [g])$ is another such morphism, then a relative version of the above construction can be used to build a homotopy from $\lambda$ to $\delta$ through morphisms. It follows that $(X, [g])$ is universal.

*Proof of necessity.* It remains to prove that if $(X, [g])$ is universal, then each twist invariant is a homotopy equivalence. Fix a normal orbit type $\alpha = [H, V]$ in $\mathfrak{N}_G(X)$ and consider the admissible $G$-manifold

$$M = (G \times_H V) \times_{S_\alpha} S_\pi .$$

Clearly,

$$B_\alpha(M) = S_\pi / S_\alpha ,$$

$$P_\alpha(M) = S_\pi ,$$

and

$$P_\pi(M) = P_\pi(G \times_H V) \times_{S_\alpha} S_\pi .$$

Let $\epsilon : P_\pi(G \times_H V) \to S_\pi$ be a $(S_\pi, S_\alpha)$-bundle trivialization. Denote a typical point in $P_\pi(M)$ by $[x, y]$ where $x \in P_\pi(G \times_H V)$ and $y \in S_\pi$, and define a trivialization $f : P_\pi(M) \to S_\pi$ by $f([x, y]) = \epsilon(x)y$. Choose a base point $* \in U$, where as in 2.2, $U$ denotes the restriction of $P_\pi(G \times_H V)$ to the fixed point set of $S_\alpha$ on $B_\pi(G \times_H V)$. We may assume that $\epsilon(*) = 1$, and that $\tilde{f}_\alpha : P_\alpha(M) \to S_\pi$ is given by the composition

$$P_\alpha(M) \xrightarrow{i} P_\pi(M) \xrightarrow{f} S_\pi$$

where $i(y) = [*, y]$. Thus, $\tilde{f}_\alpha(y) = \epsilon(*)y = y$. By the universal property of $(X, [g])$, there is a morphism $\lambda:(M, [f]) \to (X, [g])$. By Proposition 2.4.1, in the diagram,

$$\begin{tikzcd} S_\pi = P_\alpha(M) \arrow[rr, "P_\alpha(\lambda)"] \arrow[dr, "id = \tilde{f}_\alpha"'] & & P_\alpha(X) \arrow[dl, "\tilde{g}_\alpha"] \\ & S_\pi & \end{tikzcd}$$

the map $\tilde{g}_\alpha \circ P_\alpha(\lambda)$ is $S_\alpha$-homotopic to the identity. It remains to show that $P_\alpha(\lambda)$ is also a left inverse for $\tilde{g}_\alpha$.

As before, let $\bar{X}$ denote the complement of an invariant regular neighborhood of the strata in $X$ of index less than $\alpha$ and let $T_\alpha(\bar{X})$ be a closed tubular neighborhood of $\bar{X}_\alpha$. Since $f_\alpha$ is a homotopy equivalence, the proof of sufficiency shows that there is a morphism $\delta:(T_\alpha(\bar{X}), [g]) \to (M, [f])$. As before, by 2.4.1, $P_\alpha(\delta)$ is $S_\alpha$-homotopic to $\tilde{g}_\alpha$. Since the inclusion $T_\alpha(\bar{X}) \subset X$ and the composition $\lambda \circ \delta: T_\alpha(\bar{X}) \to X$ are both morphisms, we conclude that they are homotopic. Therefore, $P_\alpha(\lambda)$ is also a left $S_\alpha$-homotopy inverse for $\tilde{g}_\alpha$. This completes the proof. □

**4. Multi-axial actions.** In this section $\Lambda$ will denote an associative division algebra over $\mathbf{R}$. Of course, $\Lambda$ is isomorphic to either the real, complex, or quaternionic numbers.

**4.1 The linear models.** The canonical anti-involution on $\Lambda$ is denoted by $c \to \bar{c}$. By a "$\Lambda$-module" we shall always mean a *right* $\Lambda$-module. By a *hermitian inner product* on a $\Lambda$-module $V$, we shall mean a $\mathbf{R}$-bilinear map $\langle\ ,\ \rangle: V \times V \to \Lambda$ such that for all $(v, w) \in V \times V$ and $q \in \Lambda$, the following conditions hold:

(a) $\langle vq, w\rangle = \langle v, w\rangle q$
(b) $\langle v, wq\rangle = \bar{q}\langle v, w\rangle$
(c) $\langle v, w\rangle = \langle w, v\rangle$
(d) $\langle v, v\rangle \in [0, \infty)$ and $\langle v, v\rangle = 0$ if and only if $v = 0$.

The standard hermitian inner product on $\Lambda^n$ is defined by

$$\langle a, b\rangle = \sum_{i=1}^{i=n} \bar{b}_i a_i.$$

If $V$ is a $\Lambda$-module with hermitian inner product, then denote by $G^\Lambda(V)$ the group of $\Lambda$-module automorphisms of $V$ which leave the hermitian inner product invariant. Set $G^\Lambda(n) = G^\Lambda(\Lambda^n)$. Of course, as $\Lambda = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, $G^\Lambda(n)$ is either $O(n)$, $U(n)$ or $Sp(n)$.

Consider the real vector space, $M^\Lambda(n, k) = \mathrm{Hom}_\Lambda(\Lambda^k, \Lambda^n)$. We may identify $M^\Lambda(n, k)$ with the space of $n$ by $k$ matrices with coefficients in $\Lambda$. If $x \in M^\Lambda(n, k)$, then define $x^* \in M^\Lambda(k, n)$ by the condition that $\langle xv, w\rangle = \langle v, x^*w\rangle$ for all $(v, w) \in \Lambda^k \times \Lambda^n$, i.e., $x^*$ is the conjugate transpose of $x$. The $k$ by $k$ matrix $x^*x$ is $\Lambda$-hermitian and positive semi-definite. Define a real inner product on $M^\Lambda(n, k)$ by $x \cdot y = \frac{1}{2}$ trace $(x^*y + y^*x)$.

We shall consider the natural linear $G^\Lambda(n)$-action on $M^\Lambda(n, k)$ defined by matrix multiplication. If $g \in G^\Lambda(n)$ and $x \in M^\Lambda(n, k)$, then $(gx)^*(gx) = x^*g^*gx = x^*x$; whence, $|gx| = |x|$. Thus, $M^\Lambda(n, k)$ is an orthogonal $G^\Lambda(n)$-module. It is isomorphic to the direct sum of $k$-copies of the $G^\Lambda(n)$-module $\Lambda^n$.

A smooth $G^\Lambda(n)$-manifold is said to be *k-axial* if it is stably modeled on $M^\Lambda(n, k)$, i.e., if its normal orbit types occur among those of $M^\Lambda(n, k)$.

**4.2 The universality of the linear models.** To simplify notation we shall now set $G = G^\Lambda(n)$ and $X = M^\Lambda(n, k)$. From now on, we shall assume that $n \geq k$. Under this assumption, we propose to show that $X$ is 1) a trivializable admissible $G$-manifold and 2) universal.

First we calculate the normal $G$-orbit types of $X$. Regard $x \in X$ as a linear map $\Lambda^k \to \Lambda^n$. Denote by $I_x \subset \Lambda^n$, the image of $x$ and by $K_x \subset \Lambda^k$, the kernel of $x$. Let $W_x$ be the orthogonal complement of $I_x$ in $\Lambda^n$. The isotropy group at $x$ is clearly $G^\Lambda(W_x)$. The normal representation at $x$ is $\mathrm{Hom}_\Lambda(K_x, W_x)$ with the natural inner product and $G^\Lambda(W_x)$-action. (See page 18 in [7].) The pair $(G^\Lambda(W_x), \mathrm{Hom}_\Lambda(K_x, W_x))$ is obviously equivalent to $(G^\Lambda(n - i), M^\Lambda(n - i, k - i))$, where $i = \dim_\Lambda(I_x)$, ($i$ is the *rank* of $x$). In summary, we have the following lemma.

4.2.1 LEMMA. *Suppose $G = G^\Lambda(n)$ and $X = M^\Lambda(n, k)$. Then $\mathfrak{N}_G(X) = \{[H_i, V_i] \mid 0 \leq i \leq k\}$, where $H_i = G^\Lambda(n - i)$ and $V_i = M^\Lambda(n - i, k - i)$. Obviously, $[H_i, V_i] < [H_j, V_j]$ if and only if $i < j$. Hence, the correspondence $[G_i, V_i] \leftrightarrow i$ sets up an isomorphism of partially ordered sets $\mathfrak{N}_G(X) \leftrightarrow \{0, 1, 2, \ldots, k\}$. Under this identification, the map $X \to \mathfrak{N}_G(X)$ is given by $x \to rk(x)$.*

Henceforth, we shall index the strata of a $k$-axial $G$-manifold by $\{0, 1, 2, \ldots, k\}$. However, in order to be consistent with our previous notation, we shall sometimes use the letter "$\pi$" (rather than "$k$") to denote the principal orbit type.

Next we compute the orbit space of $X$ as in 1.6. Let $H^\Lambda(k)$ be the real vector space of $k$ by $k$ $\Lambda$-hermitian matrices and let $B^\Lambda(k) \subset H^\Lambda(k)$ be the positive semi-definite cone. Consider the polynomial mapping $p: X \to H^\Lambda(k)$ defined by $x \to x^*x$. Since $p(gx) = x^*g^{-1}gx = p(x)$, the mapping is $G$-invariant. The image of $p$ is $B^{\Lambda}(k)$. Choose a linear system of coordinates on $H^{\Lambda}(k)$ and let $p_j$ denote the $j$th component of $p$. According to [20], the $\{p_j\}$ generate $\mathbf{R}[X]^G$. It follows from the remarks in 1.6, that the induced map $\bar{p}: B(X) \to B^{\Lambda}(k)$ is a smooth isomorphism. Let $B_i{}^{\Lambda}(k)$ denote the positive semi-definite matrices of rank $i$. Since $rk(p(x)) = rk(x)$, we see that $p$ maps the $i$-stratum of $X$ onto $B_i{}^{\Lambda}(k)$, i.e., $B_i(X) \cong B_i{}^{\Lambda}(k)$. Henceforth, we shall identify $B(X)$ with $B^{\Lambda}(k)$.

Next we compute the group $S_i$ associated to the pair $(H, V) = (G^{\Lambda}(n - i), M^{\Lambda}(n - i, k - i))$. The group $L = G^{\Lambda}(k - i)$ also acts on $V$ by $a \cdot x = xa^{-1}$. The $L$-action is orthogonal and clearly commutes with the $H$-action. In fact, $L = C_{O(V)}(H)$. Since every automorphism of $H$ is inner, we have by Lemma 1.3.1 that

$$S_i = S_{[H, V]} = C_G(H) \times_{Z(H)} C_{O(V)}(H)$$

Clearly, $C_G(H) = G^{\Lambda}(i) \times Z(H)$. Hence

$$S_i \cong G^{\Lambda}(i) \times G^{\Lambda}(k - i). \tag{4.2.2}$$

Suppose $(u, v) \in G^{\Lambda}(i) \times G^{\Lambda}(k - i)$ and that $(g, x) \in G \times V$. The action of $S_i$ on $G \times_H V$ is given by

$$(u, v) \cdot [g, x] = [gu^{-1}, xv^{-1}]. \tag{4.2.3}$$

According to Section 1.6, we may identify $B(G \times_H V)$ with $B^{\Lambda}(k - i)$ and the orbit map with $[g, x] \to x^*x$. If $z \in B^{\Lambda}(k - i)$, then the action of $S_i$ on $B^{\Lambda}(k - i)$ is defined by $(u, v) \cdot z = vzv^{-1}$. Consider the $S_i$-action on $B_{\pi}(G \times_H V) = B_{k-i}^{\Lambda}(k - i)$. The space $B_{k-i}^{\Lambda}(k - i)$ is $G^{\Lambda}(k - i)$-equivariantly diffeomorphic to $H^{\Lambda}(k - i)$. (The exponential map $\exp: H^{\Lambda}(k - i) \to B_{k-i}^{\Lambda}(k - i)$ is such an equivariant diffeomorphism.) Thus, $B_{\pi}(G \times_H V)$ has the $S_i$-homotopy type of a point.

Finally we must compute the homomorphism $\varphi: S_i \to S_k = S_{\pi}$ defined in Section 2.1. Let

$$e = \begin{pmatrix} I \\ 0 \end{pmatrix} \in V$$

be the matrix which is the $(k - i)$ by $(k - i)$ identity matrix followed by

$(n - k)$ rows of zeroes. Let $\mathcal{O} = \{[g, e]\}$ be the orbit of $[1, e]$ in $G \times_H V$. Then $\mathcal{O}$ is a fixed point of $S_i$ on $B_\pi(G \times_H V)$. Consider the principal bundle (over a point), $P_\pi(\mathcal{O})$. Then $S_\pi$ acts freely and transitively on $P_\pi(\mathcal{O})$. The homomorphism $\varphi: S_i \to S_\pi$ is defined by considering the $S_i$-action on $P_\pi(\mathcal{O})$. A point in $P_\pi(\mathcal{O})$ is an equivariant diffeomorphism from $\mathcal{O}$ to the standard principal orbit $G/K$, where $K = G^\Lambda(n - k)$ is embedded in $G = G^\Lambda(n)$ as the lower right hand block. Embed $S_\pi = G^\Lambda(k)$ in $G^\Lambda(n)$ via

$$a \to \hat{a} = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}$$

and define $R_a$, an equivariant self-diffeomorphism of $G/K$ by $gK \to g\hat{a}^{-1}K$. Let $c: \mathcal{O} \to G/K$ be the equivariant diffeomorphism $[g, e] \to gK$. The canonical action of $S_\pi$ on $P_\pi(\mathcal{O})$ is defined by $a \cdot c = c \circ R_a$. If $(u, v) \in G^\Lambda(i) \times G^\Lambda(k - i) = S_i$, then $\varphi: S_i \to S_\pi$ is defined by the diagram

$$\begin{tikzcd} \mathcal{O} \arrow[r, "c"] \arrow[d, "(u,v)\cdot"'] & G/K \arrow[d, dashed, "R_{\varphi(u,v)}"] \\ \mathcal{O} \arrow[r, "c"] & G/K \end{tikzcd}$$

We have that $(u, v) \cdot [1, e] = [u^{-1}, ev^{-1}] = [\hat{g}, e]$, where

$$\hat{g} = \begin{bmatrix} u^{-1} & 0 & 0 \\ 0 & v^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}$$

Hence, $\varphi: G^\Lambda(i) \times G^\Lambda(k - i) \to G^\Lambda(k)$ is the standard embedding

$$\varphi(u, v) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}. \tag{4.2.4}$$

In particular, $\varphi$ is injective. Thus, every $[H, V] \in \mathfrak{N}_G(X)$ is admissible.

Since $B_\pi(X) = B_k{}^\Lambda(k)$ is contractible, the principal orbit bundle $P_\pi(X)$ is trivial. Let $g: P_\pi(X) \to S_\pi$ be a trivialization and for each $i$,

$0 \leq i \leq k$, let $g_i : B_i(X) \to S_k/S_i$ be the associated twist invariant. (Note that $S_k/S_i = G^\Lambda(k)/\{G^\Lambda(i) \times G^\Lambda(k - i)\}$ is the Grassmann manifold of $i$-planes in $\Lambda^k$.) On page 36 of [7], it is proved that each $g_i$ is a homotopy equivalence. Therefore, as an application of our main result, Theorem 3.2.1, we have the following.

4.2.5 THEOREM. *If $n \geq k$, then $M^\Lambda(n, k)$ is a universal trivializable admissible $G^\Lambda(n)$-manifold.*

4.2.6 *Remarks.* This result was proved in my thesis [5]; the proof is also sketched in [7]. There are many applications. For example, this result is the starting point for the classification, up to concordance, of $k$-axial $G^\Lambda(n)$-actions on homotopy spheres in [9], [11].

**4.3 Actions of $SO(n)$ and $SU(n)$.** Suppose that $\Lambda = \mathbf{R}$ or $\mathbf{C}$ and set $SG^\Lambda(n) = \{g \in G^\Lambda(n) | \det(g) = 1\}$. We now change our notation by setting $G = SG^\Lambda(n)$ and $\hat{G} = G^\Lambda(n)$. As before, $X = M^\Lambda(n, k)$. We shall sometimes write $(X, \hat{G})$ and $(X, G)$ in order to keep track of which group is acting.

We consider the action of $G$ on $X$. It turns out that there are three distinct situations depending on whether $k = n$, $k = n - 1$, or $k \leq n - 2$. If $k \leq n - 2$, then the situation is exactly as before: $(X, G)$ is admissible, trivializable and universal. (In fact, when $k \leq n - 2$ every $k$-axial $G$-action extends uniquely to a $k$-axial $\hat{G}$-action.) If $k = n - 1$, then $(X, G)$ is admissible and trivializable but *not* universal (the twist invariants fail to be homotopy equivalences). If $k = n$, then the action is no longer admissible.

Let us first dispose of the case $k = n$.

$$\mathfrak{N}_G(X) = \{[H_i, V_i] | 0 \leq i \leq k - 1\},$$

where $H_i = SG^\Lambda(n - i)$ and $V_i = M^\Lambda(n - i, k - i)$. The natural projection $q : B(X, G) \to B(X, \hat{G})$ has fiber over the top stratum equal to 2 points and fiber over the lower strata equal to 1 point. Also, the top stratum of $B(X, G)$ is mapped by $q$ onto the top two strata of $B(X, \hat{G})$. (Algebraically, $\mathbf{R}[X]^{\hat{G}} \subsetneqq \mathbf{R}[X]^G$; $\mathbf{R}[X]^G$ has an extra generator and an extra relation.) It follows that $B_\pi(X, G)$ is homotopy equivalent to the suspension of $B^\Lambda_{k-1}(k)$, i.e., it is homotopy equivalent to the suspension of $\Lambda P^{k-1}$. Since this is not contractible, $[G, X]$ is *not* an admissible normal $G$-orbit type.

If $k \leq n-1$, then it is proved in [20] that $\mathbf{R}[X]^G = \mathbf{R}[X]^{\hat{G}}$. This implies that $B(X, G) = B(X, \hat{G})$. Also

$$\mathfrak{N}_G(X) = \{[H_i, V_i] \mid 0 \leq i \leq k\},$$

where $H_i = SG^\Lambda(n-i)$ and $V_i = M^\Lambda(n-i, k-i)$. Thus, the stratifications of $B(X, G)$ and $B(X, \hat{G})$ are identical. The differences between the cases $k \leq n-2$ and $k = n-1$ arise when we try to compute the groups $S_i$ associated to $[H_i, V_i]$.

Set $H = SG^\Lambda(n-i)$, $\hat{H} = G^\Lambda(n-i)$ and $V = M^\Lambda(n-i, k-i)$. Suppose that $k \leq n-2$, or that $k = n-1$ and $i \leq k-2$. Then

1) $H = \hat{H} \cap G$
2) $N_G(H) = S(G^\Lambda(i) \times G^\Lambda(n-i)) = N_{\hat{G}}(\hat{H}) \cap G$.
3) $N_{O(V)}(H) = N_{O(V)}(\hat{H})$

Hence, $N_{G\times O(V)}(H) = N_{\hat{G}\times O(V)}(\hat{H}) \cap \{G \times O(V)\}$. It follows from our earlier computation that if $k \leq n-2$ or if $k = n-1$ and $i \leq k-2$, then

$$\begin{aligned} S_i &= N_{G\times O(V)}(H)/H \\ &= N_{\hat{G}\times O(V)}(\hat{H})/H \\ &= G^\Lambda(i) \times G^\Lambda(k-i). \end{aligned} \tag{4.3.1}$$

Hence, if $k \leq n-2$ and $0 \leq i \leq k$ the situation is exactly as before. We therefore, have the following result.

4.3.2 THEOREM. *If $k \leq n-2$, then $M^\Lambda(n, k)$ is a universal trivial admissible $SG^\Lambda(n)$-manifold.*

**4.4 The case $k = n-1$.** Assume $k = n-1$. We have computed $S_i$ for $i \leq k-2$ in (4.3.1). It also follows as before that $B_\pi(G \times_H V)$ has the $S_i$-homotopy type of a point for $i \leq k-2$. If $i = k$, then $H = \{1\}$. Hence

$$\begin{aligned} S_k &= N_G(H)/H \\ &= G \\ &= SG^\Lambda(k+1). \end{aligned} \tag{4.4.1}$$

For $i \leq k - 2$, we easily check that the homomorphism $\varphi: S_i \to S_k$ sends $(u, v) \in G^\Lambda(i) \times G^\Lambda(k - i)$ to the matrix

$$\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & \gamma \end{pmatrix} \in SG^\Lambda(k + 1)$$

where $\gamma = (\det(u)\det(v))^{-1}$. In particular, $\varphi$ is injective. It follows that $[H, V]$ is admissible for $i \leq k - 2$.

It remains to compute $S_{k-1}$. In this case, $H = SG^\Lambda(2)$ and $V = M^\Lambda(2, 1)$. Thus,

$$N_{O(V)}(H) = \begin{cases} SO(4); & \text{if } \Lambda = \mathbf{C} \\ O(2); & \text{if } \Lambda = \mathbf{R}. \end{cases}$$

If $\Lambda = \mathbf{C}$, then since every automorphism of $SU(2)$ is inner, we can apply Lemma 1.3.1 to conclude that

$$S_{k-1} = C_G(H) \times_{Z(H)} C_{O(V)}(H)$$

where

$$C_G(H) = S(U(k - 1) \times Z(U(2))),$$

$$C_{O(V)}(H) = SU(2),$$

and $Z(H) = \mathbf{Z}/2$. Suppose that $(a, c) \in U(k - 1) \times U(1)$ is such that $\det(a) = c^{-2}$ and that $q \in SU(2)$. Define

$$\psi: C_G(H) \times C_{O(V)}(H) \to S(U(k - 1) \times U(2)) \quad \text{by} \quad (a, c, q) \to (a, [c, q])$$

where $[c, q] \in U(1) \times_{\mathbf{Z}/2} SU(2) \cong U(2)$. The kernel of $\psi$ is clearly $Z(H) = \{\pm 1\}$. Hence, for $\Lambda = \mathbf{C}$, $S_{k-1} = S(U(k - 1) \times U(2))$. We leave it to the reader to produce an argument similar to the proof of Lemma 1.3.1 to show for $\Lambda = \mathbf{R}$ we also have

$$S_{k-1} = S(O(k - 1) \times O(2))$$

and hence, that

$$S_{k-1} = S(G^{\Lambda}(k-1) \times G^{\Lambda}(2)). \tag{4.4.2}$$

Also, it is easily checked that $\varphi : S_{k-1} \to S_k$ is the standard embedding $S(G^{\Lambda}(k-1) \times G^{\Lambda}(2)) \subset SG^{\Lambda}(k+1)$. The orbit space of $G \times_H V$ is isomorphic to $B^{\Lambda}(1)$ and the orbit map is defined by $[g, x] \to x^*x$. The action of $S_{k-1}$ on $G \times_H V$ is defined by $(k, a) \cdot [g, x] = [g\varphi(k, a)^{-1}, ax]$, where $(k, a) \in S(G^{\Lambda}(k-1) \times G^{\Lambda}(2))$. Hence, $(k, a) \cdot (x^*x) = x^*a^{-1}ax = x^*x$. Thus, $S_{k-1}$ acts trivially on $B^{\Lambda}(1) \cong [0, \infty)$ and in particular on the interior of $B^{\Lambda}(1)$, the set of principal orbits. It follows that $[H, V]$ is also admissible when $i = k - 1$, and hence, that $X$ is an admissible $G$-manifold. Since $B_k{}^{\Lambda}(k) = B_{\pi}(X)$ is contractible, $X$ is also trivializable. The $i$-stratum of $B^{\Lambda}(k)$ has the homotopy type of the Grassman $G^{\Lambda}(k)/\{G^{\Lambda}(i) \times G^{\Lambda}(k-i)\}$. On the other hand, the $i$-twist invariant takes values in $S_k/S_i$. From (4.3.1), (4.4.1) and (4.4.2) we see that for $i < k$, the domain and range of the $i$-twist invariant have distinct homotopy types. Thus, $X$ cannot be universal. In summary, we have proved the following result.

4.4.3 PROPOSITION. *$M^{\Lambda}(k+1, k)$ is a trivializable admissible $SG^{\Lambda}(k+1)$-manifold. However, it is not universal.*

In the next section we shall find the desired universal manifold for $k = 2$.

**5. Bi-axial actions on projective planes.** In this section $\Lambda$ denotes a (possibly non-associative) real division algebra of dimension $d$. Thus $\Lambda$ is isomorphic to the algebra of real, complex, quaternion, or Cayley numbers. Denote these algebras by $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, and $\mathbf{O}$, respectively.

**5.1 The linear models.** A real inner product on $\Lambda$ is defined by $x \cdot y = \frac{1}{2}(x\bar{y} + y\bar{x})$, where $x \to \bar{x}$ is the canonical anti-involution. The subalgebra of real multiples of the identity is denoted by $\mathbf{R} \subset \Lambda$ and the perpendicular $(d-1)$-dimensional subspace is denoted by $W^{\Lambda}$.

Let $A^{\Lambda}$ denote the group of $\mathbf{R}$-algebra automorphisms of $\Lambda$. Then

$$A^{\Lambda} \cong \begin{cases} \{1\}; & \text{if } \Lambda \cong \mathbf{R} \\ \mathbf{Z}/2; & \text{if } \Lambda \cong \mathbf{C} \\ SO(3); & \text{if } \Lambda \cong \mathbf{H} \\ G_2; & \text{if } \Lambda \cong \mathbf{O} \end{cases}$$

Every automorphism commutes with the anti-involution. Hence, the inner product is invariant. Also, $A^{\Lambda}$ acts trivially on $\mathbf{R}$. Thus, $W^{\Lambda}$ is an orthogonal $A^{\Lambda}$-module.

The action of $A^{\Lambda}$ on $W^{\Lambda} \oplus W^{\Lambda}$ is called the *linear bi-axial $A^{\Lambda}$-action.* A smooth $A^{\Lambda}$-manifold is *bi-axial* if it is stably modeled on $(W^{\Lambda} \oplus W^{\Lambda}, A^{\Lambda})$.

The action of $A^{\mathbf{H}}$ on $W^{\mathbf{H}}$ is equivalent to the standard action of $SO(3)$ on $\mathbf{R}^3$. It follows from Proposition 4.4.3, that $(W^{\mathbf{H}} \oplus W^{\mathbf{H}}, SO(3))$ is a trivializable admissible $SO(3)$-manifold.

Next, consider the action of $G_2$ on $\mathbf{O}$. If $x$ is a non-real Cayley number, then it generates a 2-dimensional subalgebra isomorphic to the complex numbers. The isotropy group at $x$ is the stabilizer of this subalgebra. The subgroup of $G_2$ which fixes $\mathbf{C}$ may be identified with $SU(3)$ and the action of $SU(3)$ on $\mathbf{C}^{\perp} \cong \mathbf{C}^3$ is standard. In particular, this implies that the restriction of the bi-axial $G_2$-action on $W^{\mathbf{O}} \oplus W^{\mathbf{O}}$ to $SU(3)$ is bi-axial. If $x$ and $y$ are non-commuting Cayley numbers, then they generate a 4-dimensional subalgebra isomorphic to the quaternions. The isotropy group at $(x, y) \in \mathbf{O} \oplus \mathbf{O}$ is equal to the stabilizer of the subalgebra generated by $x$ and $y$. The subgroup of $G_2$ which fixes $\mathbf{H}$ is isomorphic to $SU(2)$. It follows that the normal $G_2$-orbit type of $(x, y) \in W^{\mathbf{O}} \oplus W^{\mathbf{O}}$ is given by

$$\begin{cases} [G_2, W^{\mathbf{O}} \oplus W^{\mathbf{O}}]; & \text{if } (x, y) = (0, 0) \\ [SU(3), \mathbf{C}^3]; & \text{if } x \text{ and } y \text{ commute} \\ [SU(2), \bar{0}]; & \text{if } x \text{ and } y \text{ do not commute.} \end{cases}$$

Thus, $W^{\mathbf{O}} \oplus W^{\mathbf{O}}$ has 3 strata and the principal isotropy type is $SU(2)$.

**5.2 Reduction to a trivial principal isotropy group.** We digress for a moment to discuss a well-known reduction principle in compact transformation groups (see [14]). Suppose that $M$ is a smooth $G$-manifold with principal isotropy group $K$. If $[H, V] \in \mathfrak{N}_G(M)$, then, after replacing $(H, V)$ by an equivalent pair (if necessary), we may assume that $K \subset H$. Set

$$G' = N_G(K)/K, \qquad H' = N_H(K)/K, \qquad V' = V^K.$$

Note that the equivalence class of $(H', V')$ is determined by that of $(H, V)$. $G'$ acts on the fixed point set of $K$ in any $G$-space. In particular, it acts on $(G/H)^K$. The $G'$-orbit containing the identity coset in $(G/H)^K$ is precisely $G'/H'$. The restriction of the vector bundle $(G \times_H V)^K$ to $G'/H'$ is $G' \times_{H'} V'$. Let $V_\pi'$ stand for the points in $V'$ with isotropy group equal to $K$. The intersection of $(G \times_H V)^K$ with the principal orbits of $G \times_H V$ is $G' \times_{H'} V_\pi'$. Hence, $G'/H'$ may be characterized as the union of those components of $(G/H)^K$ which are contained in the closure of the principal orbits in $(G \times_H V)^K$. Denote by $M'$ the union of those components of $M^K$ which intersect $M_\pi \cap M^K$. Note that if $M^K$ is connected (which is true, for example, if $M$ is a linear $G$-space), then $M^K = M'$. It follows from these observations and the results of [17], that the inclusion $M' \hookrightarrow M$ induces a smooth isomorphism $B(M', G') \cong B(M, G)$. Also, the map $\xi: \mathfrak{N}_G(M) \to \mathfrak{N}_{G'}(M')$ defined by $[H, V] \to [H', V']$ commutes with the normal orbit structure maps.

$$\begin{array}{ccc} B(M, G) & \cong & B(M', G') \\ \downarrow & & \downarrow \\ \mathfrak{N}_G(M) & \xrightarrow{\xi} & \mathfrak{N}_{G'}(M') \end{array}$$

This implies that if each stratum of $B(M, G)$ is connected, then $\xi$ is bijective.

Consider what happens to the structure groups of the normal orbit bundles under the correspondence $\xi$. Since $S_{[H, V]}$ acts through $G$-equivariant isomorphisms on $G \times_H V$, both the fixed point set of $K$ and the top stratum are $S_{[H, V]}$-invariant. Hence, their intersection, $G' \times_{H'} V'$, is also invariant. The action of $S_{[H, V]}$ on $G' \times_{H'} V'$ is clearly effective and through orthogonal equivalences. Hence, $S_{[H, V]}$ is a subgroup of the full group of such orthogonal equivalences $S_{[H', V']}$. We say that $M$ has *very fine structure* if $\xi$ is a bijection and if $S_\alpha = S_{\xi(\alpha)}$.

Let $\mathcal{C}_G(M)$ be the category of smooth $G$-manifolds which are stably modeled on $M$. The morphisms are $G$-equivariant stratified maps. The correspondence $X \to X'$ clearly defines a functor $\mathfrak{F}: \mathcal{C}_G(M) \to \mathcal{C}_{G'}(M')$. Suppose now that $M$ has very fine structure. Then the normal orbit bundles and "attaching data" of $X$ are identical with those of $X'$. It follows from [6] that $\mathfrak{F}$ induces a bijection on isomorphism classes. (In the language of [6] the category of normal systems associated to $\mathcal{C}_G(M)$ is identical with the category of normal systems associated to $\mathcal{C}_{G'}(M')$.) If $\psi': X' \to Y'$ is any morphism, then we may alter it by a stratified homotopy so that it is a bundle map on a prescribed tubular neighborhood of each stratum. The new $\psi'$ can clearly be lifted (uniquely) to $\psi: X \to Y$ (since the structure groups of the tubular neighborhoods in $X'$

are identical with those of $X$). In fact, by using the Covering Homotopy Theorem of [19], we see that we could have lifted the original $\psi'$. This proves the following result.

5.2.1 Proposition. *If $M$ has very fine structure, then*

$$\mathfrak{F} : \mathcal{C}_G(M) \to \mathcal{C}_{G'}(M')$$

*is an equivalence of categories.*

Applying these general remarks to the case,

$$(M, G) = (W^{\mathbf{O}} \oplus W^{\mathbf{O}}, G_2)$$

$$(H, V) = (SU(3), \mathbf{C}^3)$$

we have,

$$(M', G') = (W^{\mathbf{H}} \oplus W^{\mathbf{H}}, SO(3))$$

$$(H', V') = (SO(2), \mathbf{C})$$

It follows that the orbit space of $W^{\mathbf{O}} \oplus W^{\mathbf{O}}$ under $G_2$ coincides with the orbit space of $W^{\mathbf{H}} \oplus W^{\mathbf{H}}$ under $SO(3)$. As we saw in Section 4.3, the latter orbit space is $B^{\mathbf{R}}(2)$, which is a solid 3-dimensional cone. Since each stratum of $B^{\mathbf{R}}(2)$ is connected,

$$\xi : \mathfrak{N}_{G_2}(W^{\mathbf{O}} \oplus W^{\mathbf{O}}) \to \mathfrak{N}_{SO(3)}(W^{\mathbf{H}} \oplus W^{\mathbf{H}})$$

is bijective. As we have seen in Section 4.4, the latter set is identified with $\{0, 1, 2\}$ via $0 = [SO(3), W^{\mathbf{H}} \oplus W^{\mathbf{H}}]$, $1 = [SO(2), \mathbf{C}]$, $2 = [\{1\}, \bar{0}]$. We also identify $\mathfrak{N}_{G_2}(W^{\mathbf{O}} \oplus W^{\mathbf{O}})$ with $\{0, 1, 2\}$ via $\xi^{-1}$, i.e., $0 = [G_2, W^{\mathbf{O}} + W^{\mathbf{O}}]$, $1 = [SU(3), \mathbf{C}^3]$, $2 = [SU(2), \bar{0}]$. Denote by $S_i$ the structure groups for $W^{\mathbf{O}} + W^{\mathbf{O}}$ and by $S_i'$ those for $W^{\mathbf{H}} \oplus W^{\mathbf{H}}$. Using the methods of Section 4, we check immediately that

$$S_0 = S_0' \cong O(2) \tag{5.2.2}$$

$$S_1 = S_1' \cong O(2)$$

$$S_2 = S_2' = SO(3).$$

Hence, $W^0 \oplus W^0$ has very fine structure. Therefore, we have the following corollary to 5.2.1.

5.2.3 COROLLARY. *The functor $X \to X^{SU(2)} = X'$ is an equivalence from the category of bi-axial $G_2$-manifolds to the category of bi-axial $SO(3)$-manifolds.*

In particular this implies that $W^0 \oplus W^0$ is a trivial admissible $G_2$-manifold. Also, $(X, G_2)$ is a universal bi-axial $G_2$-manifold if and only if $(X^{SU(2)}, SO(3))$ is a universal bi-axial $SO(3)$-manifold.

**5.3 Jordan algebras and projective spaces.** Let $H^\Lambda(k)$ be the real vector space of $k$ by $k$ $\Lambda$-hermitian matrices. Define a multiplication on $H^\Lambda(k)$ by $A \cdot B = \frac{1}{2}(AB + BA)$.[3] A matrix $A \in H^\Lambda(k)$ is called *positive semi-definite* if it is a square and *idempotent* if $A^2 = A$. The trace of $A$ is a real number. Define $B^\Lambda(k)$ to be the set of positive semi-definite elements in $H^\Lambda(k)$ and define $T^\Lambda(k)$ to be the elements in $B^\Lambda(k)$ of trace 1. Define $\Lambda P^{k-1}$ to be the set of trace 1 idempotents in $H^\Lambda(k)$. Obviously, $\Lambda P^{k-1} \subset T^\Lambda(k)$.

If $\Lambda$ is associative then the idempotents of trace 1 may be identified with the projections onto lines in $\Lambda^k$. Hence, for $\Lambda$ associative, our definition of projective space agrees with the usual one. Also, if $\Lambda$ is associative, then projection onto the line $[1, x_1, x_2, \ldots, x_{k-1}]$ corresponds to the idempotent

$$E = \rho^{-1} \begin{pmatrix} 1 & \bar{x}_1 & \cdots & \bar{x}_{k-1} \\ x_1 & |x_1|^2 & \cdots & x_1\bar{x}_{k-1} \\ x_2 & x_2\bar{x}_1 & \cdots & x_2\bar{x}_{k-1} \\ \vdots & \vdots & & \vdots \\ x_{k-1} & x_{k-1}\bar{x}_1 & \cdots & |x_{k-1}|^2 \end{pmatrix}$$

where $\rho = 1 + |x_1|^2 + \cdots + |x_{k-1}|^2$.

Set $U_i^\Lambda = \{A \in H^\Lambda(k) | A_{ii} \neq 0\} \cap \Lambda P^{k-1}$. Let $\psi_i : \Lambda^{k-1} \to U_1^\Lambda$ be the map which takes $(x_1, \ldots, x_{k-1})$ to $E$. (This definition makes sense even for the Cayley numbers.) One checks immediately that $\psi_1(x_1, \ldots, x_{k-1})$ is an idempotent of trace 1 and that $\psi_1$ is a smooth

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[3] If $\Lambda$ is associative or if $k \geq 3$, then $H^\Lambda(k)$ is a formally real Jordan algebra. This means that $A \cdot B = B \cdot A$, $(A^2 \cdot B) \cdot A = A^2 \cdot (B \cdot A)$, and that $A^2 = 0$ implies $A = 0$. See Chapter II in [1].

embedding (look at the first column vector of $E$). In a similar fashion define maps $\psi_i : \Lambda^{k-1} \to U_i^\Lambda$. If $\Lambda$ is associative, then the $\psi_i$ are just the usual charts on $\Lambda$-projective space (in particular, if $\Lambda$ is associative $\psi_i$ is a diffeomorphism). In fact, $\psi_i : \mathbf{O}^{k-1} \to U_i^{\mathbf{O}}$ is also a diffeomorphism *provided that* $k \leq 3$. (This is the reason it only makes good sense to talk of the Cayley projective line and the Cayley projective plane.) We shall now prove that $\psi_1 : \mathbf{O}^2 \to U_1^{\mathbf{O}}$ is surjective (and similarly for $\psi_2$ and $\psi_3$). So suppose that $k = 3$ and $A \in U_1^{\mathbf{O}}$. This means that trace $A = 1$, $A^2 = A$, and $A_{11} \neq 0$. We may write $A$ as

$$\rho^{-1} \begin{pmatrix} 1 & \bar{x} & \bar{y} \\ x & a & \bar{\alpha} \\ y & \alpha & b \end{pmatrix}$$

where $\rho = A_{11}$ and where $a$ and $b$ are both real. Since $(A^2)_{11} = A_{11}$, $\rho = 1 + |x|^2 + |y|^2$, and since trace $A = 1$, $\rho = 1 + a + b$. If $x$ and $y$ are both zero, then let $\Lambda' \subset \mathbf{O}$ be the subalgebra generated by $\alpha$. Otherwise, let $\Lambda' \subset \mathbf{O}$ be the subalgebra generated by $x$ and $y$. If $(x, y) = (0, 0)$, then, obviously, $A \in U_1^{\Lambda'}$. Otherwise we may suppose (by symmetry) that $y \neq 0$. Then the equation $(A^2)_{12} = A_{12}$ becomes $\bar{x} + \bar{x}a + \bar{y}\alpha = \rho\bar{x}$. Hence, $\alpha \in \Lambda'$ and $A \in U_1^{\Lambda'}$. Since $\Lambda'$ is associative, it follows from our earlier discussion that $A$ is in the image of $\psi_1 | (\Lambda')^2$. Therefore, the embedding $\psi_1 : \mathbf{O}^2 \to U_1^{\mathbf{O}}$ is surjective and hence, a diffeomorphism. Similarly, for $\psi_i$, $i = 2, 3$.

**5.4 Orbit spaces of bi-axial actions on projective planes.** We now assume that $\Lambda$ is associative or that $k \leq 3$. The group $A^\Lambda$ of automorphisms of $\Lambda$ is a subgroup of the full group of automorphisms of the algebra $H^\Lambda(k)$. Any automorphism preserves the idempotents and $A^\Lambda$ clearly preserves the trace. Hence, $A^\Lambda$ leaves $\Lambda P^{k-1}$ invariant. The charts $\psi_i : \Lambda^{k-1} \to U_i^\Lambda$ are obviously $A^\Lambda$-equivariant. In particular, the action on $\Lambda P^{k-1}$ is $(k - 1)$-axial. Thus $SO(3)$ acts bi-axially on $\mathbf{H}P^2$ and $G_2$ acts bi-axially on $\mathbf{O}P^2$. Restricting the $G_2$-action to $SU(3)$ we get a bi-axial $SU(3)$-action on $\mathbf{O}P^2$. In view of 5.2.2, discussion of $G_2$ on $\mathbf{O}P^2$ is superfluous, since it reduces to $SO(3)$ on $\mathbf{H}P^2$.

If $\Lambda'$ is a subalgebra of $\Lambda$ then let $\pi^{\Lambda,\Lambda'} : T^\Lambda(3) \to T^{\Lambda'}(3)$ be the map induced by the orthogonal projection of $\Lambda$ onto $\Lambda'$. Let $\pi : \mathbf{H}P^2 \to T^{\mathbf{R}}(3)$ denote the restriction of $\pi^{\mathbf{H},\mathbf{R}}$ to $\mathbf{H}P^2$ and let $\theta : \mathbf{O}P^2 \to T^{\mathbf{C}}(3)$ be the restriction of $\pi^{\mathbf{O},\mathbf{C}}$ to $\mathbf{O}P^2$.

5.4.1 PROPOSITION. *The map* $\pi : \mathbf{H}P^2 \to T^{\mathbf{R}}(3)$ *is* $SO(3)$*-invariant and induces a smooth isomorphism* $\bar{\pi} : B(\mathbf{H}P^2) \cong T^{\mathbf{R}}(3)$.

5.4.2 PROPOSITION. *The map* $\theta : \mathbf{O}P^2 \to T^{\mathbf{C}}(3)$ *is* $SU(3)$*-invariant and induces a smooth isomorphism* $\bar{\theta} : B(\mathbf{O}P^2) \cong T^{\mathbf{C}}(3)$.

5.4.3 *Remark.* The space $T^{\Lambda}(3)$ of 3 by 3 positive semi-definite $\Lambda$-hermitian matrices of trace 1 can be described quite easily (see pages 23 and 36 in [7].) It is homeomorphic to a closed disk of dimension $3d + 2$, $d = \dim_{\mathbf{R}} \Lambda$. It has 3 strata. The non-singular stratum is the open disk. The most singular stratum is $\Lambda P^2$. The intermediate stratum is the complement of $\Lambda P^2$ in the boundary sphere. This complement is diffeomorphic to a $(d + 1)$-plane bundle over $\Lambda P^2$.

5.4.4 COROLLARY. $\mathbf{H}P^2$ *is a trivializable bi-axial* $SO(3)$*-manifold.* $\mathbf{O}P^2$ *is a trivializable bi-axial* $SU(3)$*-manifold. Moreover, in both cases there is a unique homotopy class of trivialization.*

*Proof.* Let $(X, G)$ stand for $(\mathbf{H}P^2, SO(3))$ or $(\mathbf{O}P^2, SU(3))$. Since $B_\pi(X)$ is an open disk, the bundle $P_\pi(X) \to B_\pi(X)$ is trivial. The structure group $S_\pi$ is $SO(3)$ or $SU(3)$. Since $S_\pi$ is therefore connected, such a trivialization is unique up to an $S_\pi$-homotopy. □

We shall prove only 5.4.2, the proof of 5.4.1 being similar and easier.

*Proof of Proposition* 5.4.2. Since $SU(3)$ fixes $\mathbf{C} \subset \mathbf{O}$, the map $\theta : \mathbf{O}P^2 \to T^{\mathbf{C}}(3)$ is obviously $SU(3)$-invariant and smooth. For $i = 1, 2, 3$, let $V_i^{\Lambda} = \{A \in T^{\Lambda}(3) \mid A_{ii} \neq 0\}$, and let $U_i^{\Lambda} = V_i^{\Lambda} \cap \Lambda P^2$. Let $\psi_i : \mathbf{O}^2 \to U_i^{\mathbf{O}}$, $i = 1, 2, 3$ be the charts defined above. The image of $\theta \circ \psi_i$ is contained in $V_i^{\mathbf{C}}$. To prove that $\bar{\theta} : B(\mathbf{O}P^2) \to T^{\mathbf{C}}(3)$ is a smooth isomorphism it clearly suffices to show that each of the maps $\bar{\theta} \circ \bar{\psi}_i : B(U_i^{\mathbf{O}}) \to V_i^{\mathbf{C}}$ is an isomorphism, or by symmetry, just that $\bar{\theta} \cdot \bar{\psi}_1$ is an isomorphism.

Let $c : \mathbf{O} \to \mathbf{C}$ be the orthogonal projection. Then

$$\theta \circ \psi_1(x_1, x_2) = \rho^{-1} \begin{bmatrix} 1 & \overline{c(x_1)} & \overline{c(x_2)} \\ c(x_1) & |x_1|^2 & c(x_1\bar{x}_2) \\ c(x_2) & c(x_2\bar{x}_1) & |x_2|^2 \end{bmatrix} \qquad (1)$$

where $\rho = 1 + |x_1|^2 + |x_2|^2$. Define a complex hermitian inner product on $\mathbf{O}$ by $\langle x, y\rangle = c(x\bar{y})$. Let $d = 1 - c:\mathbf{O} \to \mathbf{C}^{\perp} \cong \mathbf{C}^3$ be the orthogonal projection onto the orthogonal complement of $\mathbf{C}$. Note that

$$\langle x, y\rangle = \langle c(x), c(y)\rangle + \langle d(x), d(y)\rangle. \tag{2}$$

The map $(c, d):\mathbf{O} \to \mathbf{C} \oplus \mathbf{C}^3$ is an isomorphism of $SU(3)$-spaces, where the action is trivial on $\mathbf{C}$ and standard on $\mathbf{C}^3$. It follows from the remarks in 4.2, that the orbit space of $\mathbf{O}^2$ under $SU(3)$ is $\mathbf{C}^2 \times B^{\mathbf{C}}(2)$ and that the orbit map $p:\mathbf{O}^2 \to \mathbf{C}^2 \times B^{\mathbf{C}}(2)$ is defined by

$$(x_1, x_2) \to \left(u,\ v,\ \begin{pmatrix} e & \bar{z} \\ z & f \end{pmatrix}\right) \tag{3}$$

where $u = c(x_1)$, $v = c(x_2)$ and where $\left[\begin{smallmatrix} e & \bar{z} \\ z & f \end{smallmatrix}\right]$ is the 2 by 2 complex hermitian matrix $(\langle d(x_i), d(x_j)\rangle)$. From (1) it follows that the map $\theta \circ \psi_1:\mathbf{C}^2 \times B^{\mathbf{C}}(2) \to V_1{}^{\mathbf{C}}$ is defined by

$$\left[u,\ v,\ \begin{bmatrix} e & \bar{z} \\ z & f \end{bmatrix}\right] \to \rho^{-1}\begin{bmatrix} 1 & \bar{u} & \bar{v} \\ u & |u|^2 + e & u\bar{v} + \bar{z} \\ v & v\bar{u} + z & |v|^2 + f \end{bmatrix} \tag{4}$$

where $\rho = (1 + |u|^2 + |v|^2 + e + f)$. Here we have also used (2) to deduce that

$$c(x_2\bar{x}_1) = \langle x_2, x_1\rangle = \langle c(x_2), c(x_1)\rangle + \langle d(x_2), d(x_1)\rangle = v\bar{u} + z.$$

Any matrix $A \in V_1{}^{\mathbf{C}}$ can be written in the form

$$A = a\begin{bmatrix} 1 & \bar{\alpha} & \bar{\beta} \\ \alpha & b & \bar{\gamma} \\ \beta & \gamma & c \end{bmatrix}$$

where $a$, $b$, and $c$ are real, $\alpha$, $\beta$, and $\gamma$ are complex, $a \neq 0$, and $a(1 + b + c) = 1$. The inverse map of $\bar{\theta} \circ \bar{\psi}_1$ is given by the equations:

$$u = \alpha, \qquad v = \beta, \qquad z = \gamma - \beta\bar{\alpha}$$

$$e = b - |\alpha|^2, \qquad f = c - |\beta|^2$$

as one can check by verifying that the image of this inverse mapping is contained in $\mathbf{C}^2 \times B^{\mathbf{C}}(2)$. Therefore, $\bar{\theta} \circ \bar{\psi}_1$ (and hence, $\bar{\theta}$) is a smooth isomorphism. □

**5.5. The twist invariants of $SU(3)$ on the Cayley projective plane.** In this section we consider only the bi-axial action of $SU(3)$ on $\mathbf{O}P^2$. The strata are indexed by $\{0, 1, 2\}$ where $0 \leftrightarrow [SU(3); \mathbf{C}^3 \oplus \mathbf{C}^3]$, $1 \leftrightarrow [SU(2), \mathbf{C}^2]$ and 2 corresponds to the principal orbit type. Let $B$ denote the orbit space. As we have just seen $B \cong T^{\mathbf{C}}(3)$ which is a closed 8-disk. According to 5.4.3, $B_2$ is the open 8-disk, $B_1$ is a 3-plane bundle over $\mathbf{C}P^2$, and $B_0$ is $\mathbf{C}P^2$. According to (4.3.1), (4.4.1) and (4.4.2), the structure groups are given by $S_0 = U(2)$, $S_1 = U(2)$, $S_2 = SU(3)$ and for $i = 0, 1$ the homomorphism $S_i \hookrightarrow S_2$ is the standard inclusion. Thus, for $i = 0, 1$, the homogeneous space $S_2/S_i$ is $\mathbf{C}P^2$.

Let $P_i \to B_i$ denote the $i$-normal orbit bundle of $SU(3)$ on $\mathbf{O}P^2$. Choose a trivialization $f: P_2 \to S_2$ and for $i = 0, 1, 2$ define twist invariants $f_i: B_i \to S_2/S_i$ as in Section 2. We wish to verify that each $f_i$ is a homotopy equivalence. For $i = 2$, this is automatic since both the domain and range are contractible. For $i = 0, 1$, both the domain and range have the homotopy type of $\mathbf{C}P^2$. Hence, it suffices to show that $f_i$ induces an isomorphism on cohomology. Since $f_i$ pulls back the bundle $S_2 \to S_2/S_i$ to the bundle $P_i \to B_i$ we can verify this by calculating characteristic classes.

5.5.1 Lemma. *The $U(2)$-bundle $SU(3) \to SU(3)/U(2) = \mathbf{C}P^2$ has total Chern class $1 - \alpha + \alpha^2$ where $\alpha$ is the canonical generator of $H^2(\mathbf{C}P^2)$.*

*Proof.* The bundle in question is the inverse of the canonical line bundle. □

5.5.2 Lemma. *The complement of $\mathbf{C}P^2$ in $\mathbf{H}P^2$ is diffeomorphic to a 3-plane bundle over $S^5$.*

*Proof.* The group $Sp(3)$ acts transitively on $\mathbf{H}P^2$. Consider the restriction of the action to the subgroup $U(3)$. The principal orbit type is $U(3)/U(1) \times U(1)$. Since each principal orbit has codimension one,

$\mathbf{H}P^2/U(3)$ is a compact connected 1-manifold. Hence, either it is a circle and all orbits are principal or it is an interval and there are exactly two singular orbits. The second possibility holds, since

$$\mathbf{C}P^2 = U(3)/U(1) \times U(2)$$

is singular. The isotropy group at $[1, j, 0]$ is $SU(2) \times U(1)$. Hence, the other singular orbit is $U(3)/SU(2) \times U(1) \cong S^5$. In general, if $M$ is a smooth $G$-manifold with $M/G$ an interval, then the complement of either singular orbit is diffeomorphic to a vector bundle over the other. (See page 206 in [2].) □

5.5.3 Corollary. *The inclusion* $\mathbf{C}P^2 \hookrightarrow \mathbf{H}P^2$ *induces an isomorphism* $H^4(\mathbf{H}P^2) \xrightarrow{\cong} H^4(\mathbf{C}P^2)$.

*Proof.*

$$H^4(\mathbf{H}P^2, \mathbf{C}P^2) \cong H_4(\mathbf{H}P^2 - \mathbf{C}P^2) = 0$$

and

$$H^5(\mathbf{H}P^2, \mathbf{C}P^2) \cong H_3(\mathbf{H}P^2 - \mathbf{C}P^2) = 0.$$ □

5.5.4 Corollary. *The Euler class of the normal bundle of* $\mathbf{C}P^2$ *in* $\mathbf{H}P^2$ *is a generator of* $H^4(\mathbf{C}P^2)$.

*Proof.* By the previous result $\mathbf{C}P^2$ has self-intersection number 1 in $\mathbf{H}P^2$. □

Note that $(\mathbf{O}P^2)^{SU(3)} = \mathbf{C}P^2 = B_0$ and that $(\mathbf{O}P^2)^{SU(2)} = \mathbf{H}P^2$. It follows that $P_0 \to B_0$ is the principal $U(2)$-bundle associated to the normal bundle of $\mathbf{C}P^2$ in $\mathbf{H}P^2$. Therefore, its Euler class is the generator of $H^4(\mathbf{C}P^2)$. By 5.5.1, the Euler class of $S_2 \to S_2/S_0$ is also a generator. Since $P_0$ is the pullback of this bundle via $f_0$, we conclude that $f_0 : B_0 \to S_2/S_0$ is a homotopy equivalence.

If $M$ is any bi-axial $SU(3)$-manifold, then the orbit bundle $M_1 \to B_1$ has associated principal $U(1)$-bundle $M^{SU(3)} - M^{SU(2)} \to B_1(M)$. This bundle may also be identified with $P_1(M)/SU(2) \to B_1(M)$. In our case this yields,

$$P_1/SU(2) \cong \mathbf{H}P^2 - \mathbf{C}P^2.$$

Since $\mathbf{H}P^2 - \mathbf{C}P^2$ is homotopy equivalent to $S^5$ and since $B_1$ is homotopy equivalent to $\mathbf{C}P^2$, we see that the $U(1)$-bundle $P_1/SU(2) \to B_1$ has first Chern class a generator of $H^2(B_1)$. The first Chern class of the bundle $SU(3)/SU(2) \to SU(3)/U(2)$ is obviously also a generator of $H^2(\mathbf{C}P^2)$. We deduce, as before, that $f_1 : B_1 \to S_2/S_1$ is a homotopy equivalence. Applying our main theorem we get the following result.

5.5.5 THEOREM. *The bi-axial $SU(3)$-action on the Cayley projective plane is universal.*

**5.6 The twist invariants of $SO(3)$ on the quaternionic projective plane.** In this section $B$ stands for $T^{\mathbf{R}}(3)$, the orbit space of $SO(3)$ on $\mathbf{H}P^2$. According to 5.4.3, $B$ is a closed 5-disk, $B_2$ is the open 5-disk, $B_1$ is a (non-orientable) 2-plane bundle over $\mathbf{R}P^2$, and $B_0$ is $\mathbf{R}P^2$. According to (4.3.1), (4.4.1) and (4.4.2) the structure groups are $S_0 = O(2)$, $S_1 = O(2)$ and $S_2 = SO(3)$, and for $i = 0, 1$, the homomorphism $S_i \hookrightarrow S_2$ is the standard inclusion. Choose a trivialization $f : P_2 \to S_2$ and define twist invariants $f_i : B_2 \to S_2/S_i$. As before, $f_2$ is a homotopy equivalence since both spaces are contractible and for $i = 0, 1$ both the domain and range have the homotopy type of $\mathbf{R}P^2$. Hence to prove the twist invariants are homotopy equivalences it suffices to show, for i = 0, 1, that $f_i$ induces an isomorphism on $H^1( \ ; \mathbf{Z}/2)$ and on $H^2( \ ; \mathbf{Z}^-)$. Here $\mathbf{Z}^-$ denotes twisted integer coefficients.

5.6.1 LEMMA. *The $O(2)$-bundle $SO(3) \to SO(3)/O(2) = \mathbf{R}P^2$ has non-zero first Stiefel-Whitney class and has as twisted Euler class a generator of $H^2(\mathbf{R}P^2; \mathbf{Z}^-) \cong \mathbf{Z}$.*

*Proof.* This bundle is associated to the tangent bundle of $\mathbf{R}P^2$. □

5.6.2 LEMMA. *$\mathbf{C}P^2 - \mathbf{R}P^2$ is diffeomorphic to a 2-plane bundle over $S^2$.*

*Proof.* Consider the natural action of $O(3)$ on $\mathbf{C}P^2$ and proceed as in 5.5.1. □

5.6.3 LEMMA. *The twisted Euler class of the normal bundle of $\mathbf{R}P^2$ in $\mathbf{C}P^2$ is a generator of $H^2(\mathbf{R}P^2; \mathbf{Z}^-)$.*

*Proof.* Apply the $G$-signature Theorem to conjugation on $\mathbf{C}P^2$. □

Since the $O(2)$-bundle $P_0 \to B_0$ is associated to the normal bundle of $\mathbf{R}P^2$ in $\mathbf{C}P^2$, it follows from 5.6.1 and 5.6.3 that $f_0 : B_0 \to S_2/S_0$ is a homotopy equivalence.

Next, consider $\tilde{B}_1 = P_1/SO(2)$. Then $\tilde{B}_1 \to B_1$ is a double cover. Let $Y$ be the 1-stratum of $\mathbf{H}P^2$. Since $\tilde{B}_1 = Y^{SO(2)} = \mathbf{C}P^2 - \mathbf{R}P^2$, we see that $\tilde{B}_1$ is homotopy equivalent to $S^2$ and therefore, that $\tilde{B}_1 \to B_1$ is a non-trivial double covering. This means that the first Stiefel-Whitney class of $P_1$ is non-zero.

It remains to show that the non-orientable circle bundle $P_1/O(1) \to B_1$ has twisted Euler class a generator of $H^2(B_1; \mathbf{Z}^-) \cong \mathbf{Z}$. We shall prove this by showing that the pullback of this bundle to $\tilde{B}_1$ has (untwisted) Euler class equal to twice a generator of $H^2(\tilde{B}_1; \mathbf{Z}) \cong H^2(S^2; \mathbf{Z})$. The total space of this pullback is $P_1$ and the base space is $\tilde{B}_1$. Let $E$ be the normal 2-plane bundle of $Y$ in $\mathbf{H}P^2$. Then $P_1 \to \tilde{B}_1$ is the principal bundle associated to $E \mid \tilde{B}_1$. Let $Q$ be the normal bundle of $\mathbf{C}P^2$ in $\mathbf{H}P^2$ and let $R$ be the normal bundle of $\tilde{B}_1$ in $Y$. Then

$$Q \mid \tilde{B}_1 \cong E \mid \tilde{B}_1 \oplus R. \tag{1}$$

The $SO(2)$-action on $\mathbf{H}P^2$ induces an action on $Q$ which is trivial on zero-section and free off the zero-section. The bundles $E \mid \tilde{B}_1$ and $R$ are $SO(2)$-invariant sub-bundles of $Q \mid \tilde{B}_1$. Hence, (1) can be interpreted as giving an isomorphism of a complex 2-plane bundle with the Whitney sum of two complex line bundles.

If $M$ is any $G$-manifold with only one orbit type, say $G/H$, then $M \to G/N_G(H)$ is a fiber bundle with fiber $M^H$. This implies, in particular, that the normal bundle of $M^H$ in $M$ is trivial. Applying this observation to the case $M = Y$, we see that $R$ is a trivial bundle. Hence,

$$\chi(E \mid \tilde{B}_1) = c_1(E \mid \tilde{B}_1) = c_1(Q \mid \tilde{B}_1). \tag{2}$$

In the previous section we showed that $Q$ had associated principal $U(2)$-bundle $SU(3) \to SU(3)/U(2)$ and that this bundle had total Chern class $1 - \alpha + \alpha^2 \in H^*(\mathbf{C}P^2)$. Let $i : \tilde{B}_1 = \mathbf{C}P^2 - \mathbf{R}P^2 \hookrightarrow \mathbf{C}P^2$ be the inclusion. Since $H^3(\mathbf{C}P^2, \mathbf{C}P^2 - \mathbf{R}P^2; \mathbf{Z}) \cong H^1(\mathbf{R}P^2; \mathbf{Z}^-) \cong \mathbf{Z}/2$, we see that $i_* : H^2(\mathbf{C}P^2; \mathbf{Z}) \to H^2(\mathbf{C}P^2 - \mathbf{R}P^2; \mathbf{Z})$ is multiplication by 2. Hence, $c_1(Q \mid \tilde{B}_1) = \chi(E \mid \tilde{B}_1)$ is twice a generator of $H^2(\tilde{B}_1; \mathbf{Z})$. It follows from 6.5.1, that $f_1 : B_1 \to S_2/S_1$ is a homotopy equivalence. Applying our main theorem, we therefore, get the following result.

5.6.4 THEOREM. *The bi-axial $SO(3)$-action on the quaternionic projective plane is universal.*

Combining this result with 5.2.3 yields the following.

5.6.5 THEOREM. *The bi-axial $G_2$-action on the Cayley projective plane is universal.*

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