

Cyclic Cohomology for Discrete Groups and its Applications

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Abstract

We survey the cyclic cohomology associated with various algebras related to discrete groups. We then discuss the motivation and techniques of the cyclic theory approach to various problems in algebra and analysis.

1 Introduction

The invention and initial development of the field of non-commutative differential geometry by Alain Connes [12, 14] may be viewed as an (extremely successful) effort to extend the Index theory of Atiyah-Singer for elliptic differential operators on closed manifolds (representing “commutative” differential geometry) to a more general “non-commutative” setting. Central to Connes’ program was the development of cyclic theory, which allows one to associate to an arbitrary discrete, respectively topological, algebra A its cyclic cohomology groups $HC^*(A)$ (shortly after Connes’ initial work, the dual theory of cyclic homology and its relation to Lie algebra homology, was discovered independently by Loday-Quillen [34] and Tsygen [53]). When $A = C^\infty(M)$ is the algebra of complex-valued C^∞ functions on a compact, closed, smooth manifold M , Connes showed that the continuous cyclic cohomology groups of A (except for one factor) are isomorphic to a direct sum of shifted copies of the complex DeRham homology groups of M , $H_*^{DR}(M; \mathbb{C})$ (computed in terms of currents).

An important feature of cyclic cohomology is the existence of a periodicity operator $S : HC^*(A) \rightarrow HC^{*+2}(A)$. Using this operator, one is able to define the periodic cyclic cohomology groups of A as the inductive limit

$$PHC^*(A) = \lim_{\substack{\rightarrow \\ S}} HC^{*+2k}(A) \quad * = 0, 1$$

Note that, by construction, periodic cyclic cohomology is a two-periodic theory. Again, where A is the Fréchet algebra $C^\infty(M)$, Connes showed that $PHC^*(A) = \bigoplus_{k=0}^{\infty} H_*^{DR}(M; \mathbb{C})$. Moreover, there is a Chern-Connes character ch^* mapping the complex K -homology groups of M to $PHC^*(C^\infty(M))$ [1, 7]

$$ch^* : K_*(M) \rightarrow PHC^*(C^\infty(M))$$

This Chern-Connes character is a natural transformation of two-periodic theories, functorial in M . Let us describe ch^* in more detail. To begin with, a (continuous) cyclic cohomology class of degree n on the (topological) algebra A is represented by a (continuous) $(n+1)$ -linear functional f on A satisfying

$$(bf)(a_0, \dots, a_{n+1}) := \sum_{i=0}^{n-1} (-1)^i f(a_0, \dots, a_i a_{i+1}, \dots, a_n) + (-1)^n f(a_n a_0, a_1, \dots, a_{n-1}) = 0$$

and

$$(t_{n+1}f)(a_0, \dots, a_n) := (-1)^n f(a_n, a_0, \dots, a_{n-1}). \quad (*)$$

Let M be a closed manifold, E and F complex Hermitian vector bundles over M . Let D be a 0th-order elliptic pseudo-differential operator

$$D : C^\infty(M, E) \rightarrow C^\infty(M, F)$$

acting on smooth sections of the bundles. The kernel of D is $\ker D = \{f \in C^\infty(M, E) \mid Df = 0\}$ and cokernel is $\text{coker } D := \ker D^*$, where $D^* : C^\infty(M, F) \rightarrow C^\infty(M, E)$ is the adjoint operator of D on L^2 -sections. Both $\ker D$ and $\text{coker } D$ are finite dimensional and $\text{Index } D$ is thus defined as $\text{Ind } D = \dim_{\mathbb{C}}(\ker D) - \dim_{\mathbb{C}}(\text{coker } D)$, which is an integer. The Atiyah-Singer index theorem states that

$$\text{Ind } D = \langle ch^*[D], [M] \rangle \quad (**)$$

where D defines a K -homology class $[D]$ on M and $[M]$ is the fundamental class of M . The precise way in which D defines a K -homology class is based on the Baum-Douglas formulation of complex K -homology given in [6], and involves the following data: 1) D has a paramatrix $Q : C^\infty(M, F) \rightarrow C^\infty(M, E)$ of order 0 such that $QD - I$ and $DQ - I$ are pseudo-differential operators of order 0 and are Schatten p class operators $\mathcal{C}^p(H_\pm)$ on the Hilbert spaces $H_+ = L^2(M, E)$ and $H_- = L^2(M, F)$ for any $p > \dim M = 2n$. Let $P = \begin{pmatrix} 0 & Q \\ D & 0 \end{pmatrix}$ acting on the Hilbert space $H = H_+ \oplus H_-$. Then the algebra $C^\infty(M)$ acts on H as multiplication diagonal operators. The triple $(H, P, C^\infty(M))$ is called a Fredholm module over $C^\infty(M)$ in the sense that 1) $[P, f] \in \mathcal{C}^p(H)$ and 2)

$P^2 - I \in \mathcal{C}^p(H)$. This Fredholm module defines the K-homology class $[D]$. Without changing the class we may modify the triple so that $P^2 = I$. Connes defined the character $ch^*[D]$ of $[D]$ as a cyclic cohomology class represented by the formula

$$(2\pi i)^n n! Tr(f_0 P^{-1}[P, f_1] P^{-1}[P, f_2] \cdots P^{-1}[P, f_{2k}])$$

for f_i 's in $C^\infty(M)$, where $k > n$. One can show that the above formulas in (*) are satisfied. The pairing on the right side of (**) is the pairing between deRham homology and cohomology. This construction can also be done for operators of higher order.

More generally, let Γ be a countable discrete group acting properly and freely on a smooth manifold \tilde{M} , with compact quotient $M = \Gamma \backslash \tilde{M}$. Fix a Riemannian metric on M and lift it to \tilde{M} . Let $\pi : \tilde{M} \rightarrow M$ be the projection. One can pull back bundles E and F on M to π^*E and π^*F on \tilde{M} . The pull back of D , \tilde{D} defines an elliptic operator:

$$\tilde{D} : C_c^\infty(\tilde{M}, \pi^*E) \rightarrow C_c^\infty(\tilde{M}, \pi^*F)$$

The kernel and cokernel of \tilde{D} are not finite dimensional in general, but the closure of $\ker \tilde{D}$ and $\text{coker } \tilde{D}$ in the space of L^2 -sections are of finite rank as projective $C_r^*(\Gamma) \otimes \mathcal{K}$ modules, where $C_r^*(\Gamma)$ is the reduced group C^* -algebra of the group Γ (generated by left translation of Γ on $\ell^2(\Gamma)$) and where \mathcal{K} is the C^* -algebra of compact operators on some separable Hilbert space. The analytic index of \tilde{D} , given as $\text{Ind } \tilde{D} = [\ker \tilde{D}] - [\text{coker } \tilde{D}]$, thus defines an element in $K_0(C_r^*(\Gamma) \otimes \mathcal{K}) \cong K_0(C_r^*(\Gamma))$. Taking Connes-Karoubi character [15], one obtains that $ch_*(\text{Ind } \tilde{D})$ lands in $PHC_*(C_r^*\Gamma)$. Unfortunately, the continuous periodic cyclic groups $PHC_*(C_r^*\Gamma)$ contains little information about the group Γ . If Γ is amenable $PHC_*(C_r^*\Gamma)$ is canonically isomorphic to $PHC_*(\mathbb{C})$, induced by the inclusion $\mathbb{C} \rightarrow C_r^*\Gamma$, [14]. A similar phenomenon occurs for commutative C^* -algebras, $C(M)$. When M is a compact Hausdorff space $PHC_*(C(M))$ is isomorphic to $PHC_*(\mathbb{C})$. The topological information of M is not captured by $PHC_*(C(M))$. To correct this problem, one replaces $C(M)$ by $C^\infty(M)$, when M is a compact and closed manifold. The periodic cyclic homology of $C^\infty(M)$ will then reflect many of the topological properties of M . Thus one needs to consider a smooth version of $C_r^*\Gamma$ to reveal deeper insight into $\text{Ind } \tilde{D}$. Recall that a dense Fréchet subalgebra A^∞ of a C^* -algebra A is called "smooth" if A^∞ is closed in A under holomorphic functional calculus [9]. This smoothness ensures that $K_*(A^\infty)$ is canonically isomorphic to $K_*(A)$. If $C_r^*\Gamma$ has a "good" smooth subalgebra $S(\Gamma)$, then $ch_*(\text{Ind } \tilde{D})$ will land in $PHC_*(S(\Gamma))$. In the case of Γ being a word hyperbolic group [20], Jolissaint [29] and de la Harpe [21] discovered that $C_r^*\Gamma$ does have such

a smooth subalgebra $S(\Gamma)$, which exhibits many properties similar to smooth functions on a manifold. Connes and Moscovici [16] shows that for any cyclic cohomology class $[\varphi]$ on $S(\Gamma)$ induced from a group cocycle φ of Γ , the pairing $\langle ch_*(\text{Ind } \tilde{D}), [\varphi] \rangle$ is a higher signature of the manifold M . This result was used by the authors to verify the Novikov Conjecture for word-hyperbolic groups. A key ingredient in their proof is a result, attributed to Gromov, that for hyperbolic groups every complex group cohomology class above dimension 1 is represented by a bounded cocycle (a detailed proof of this result appears in [40]). Connes-Moscovici's approach to the Novikov conjecture using index theory and cyclic cohomology theory for groups opens the door to many other applications in geometry, topology, algebra and analysis, notably the work done in [35], [55], [22], and more recently [49, 50], [38, 36], [26] and [27].

Let us take another look at Connes-Moscovici's results from a more perspective view point. In early 1980's Baum and Connes [3] introduced a geometric K -theory $K^*(X, \Gamma)$ for a manifold X with an action by a discrete group. This theory reduces to the K -homology of the classifying space of Γ when Γ is torsion free and X is a point. There is a map μ , known as the Baum-Connes map, from $K^*(X, \Gamma)$ to $K_*(C_0(X) \times_r \Gamma)$, where $C_0(X) \times_r \Gamma$ is the reduced crossed product C^* -algebra of $C_0(X)$ by the group Γ [47]. They conjectured that μ is always an isomorphism. This is known as the Baum-Connes conjecture. The validity of this conjecture implies many other important conjectures. For instance, rational injectivity of μ implies the Novikov conjecture and the Gromov-Lawson-Rosenberg conjecture, while the surjectivity implies both the generalized Kadison-Kaplansky conjecture that there is no nontrivial idempotent in $C_r^*\Gamma$ for any torsion-free discrete group. The Baum-Connes conjecture has been verified for many classes of groups and we refer the reader to consult with the recent survey by Alain Valette [54] for a more up to date state of the conjecture. Following [3], the geometric K -groups of Baum-Connes were replaced in [5] with the (complex) K -homology groups of the classifying space for proper actions. However the statement of the Baum-Connes conjecture remains the same. In [4] Baum and Connes proved that there is an analogous equivariant cohomology theory $H^*(X, \Gamma)$ for a group acting on a manifold and a Chern Character

$$\tilde{ch}_* : K^*(X, \Gamma) \rightarrow H^*(X, \Gamma)$$

which is rationally an isomorphism. Thus [23] there is a diagram:

$$\begin{array}{ccc}
K^*(X, \Gamma) & \xrightarrow{\mu} & K_*(C_0(X) \times_r \Gamma) \\
\tilde{ch}_* \downarrow & & \searrow ch_* \\
H^*(X, \Gamma) & \xrightarrow{ch\mu} & PHC_*(C_c^\infty(X) \times_{alg} \Gamma) \xrightarrow{j_*} PHC_*(S(X, \Gamma))
\end{array}$$

where $ch\mu$ is the analogue Baum-Connes map and where $C_c^\infty(X) \times_{alg} \Gamma$ is the algebraic crossed product of $C_c^\infty(X)$ by Γ . Also, $S(X, \Gamma)$ is any smooth subalgebra of $C_0(X) \times_r \Gamma$ that contains $C_c^\infty(X) \times_{alg} \Gamma$ and j_* is induced by the inclusion. In the case that X is a point the diagram simplifies to

$$\begin{array}{ccc}
K^*(\cdot, \Gamma) & \xrightarrow{\mu} & K_*(C_r^*\Gamma) \\
\tilde{ch}_* \downarrow & & \searrow ch_* \\
H^*(\cdot, \Gamma) & \xrightarrow{ch\mu} & PHC_*(\mathbb{C}\Gamma) \xrightarrow{j_*} PHC_*(S(\Gamma))
\end{array}$$

What Connes-Moscovici's higher index theorem for proper actions [16] implies is that the diagram is in fact commutative [23]. The analogue Baum-Connes map $ch\mu$ is the bivariant cyclic version of the KK-product of Kasparov [30] [9] constructed by using a bivariant Chern character [43]. Further one computes that the range of $ch\mu$ is in the elliptic part $PHC_*(\mathbb{C}\Gamma)_{ell}$ of $PHC_*(\mathbb{C}\Gamma)$. This means that $PHC_*(\mathbb{C}\Gamma)_{ell}$ is the direct summand of $PHC_*(\mathbb{C}\Gamma)$ that corresponds to conjugacy classes of Γ whose elements are of finite orders. Thus, the previous diagram becomes

$$\begin{array}{ccc}
K^*(\cdot, \Gamma) & \xrightarrow{\mu} & K_*(C_r^*\Gamma) \\
\tilde{ch}_* \downarrow & & \searrow ch_* \\
H^*(\cdot, \Gamma) & \xrightarrow{ch\mu} & PHC_*(\mathbb{C}\Gamma)_{ell} \xrightarrow{j_*} PHC_*(S(\Gamma))_{ell} \oplus PHC_*(S(\Gamma))_{non-ell}
\end{array}$$

for some appropriate $S(\Gamma)$. Since $ch\mu$ is an isomorphism as well [23], the rational injectivity of μ is then a consequence of the injectivity of j_* for appropriate choice of $S(\Gamma)$. Note that this injectivity implies that the Novikov conjecture is true for the group Γ . In a future paper [28] we will show that this is, in fact, true for a large class of groups and the result for some of these groups was previously unknown.

At this point we see the importance of computing $PHC_*(\mathbb{C}\Gamma)$ and $PHC_*(S(\Gamma))$. The computation of the cyclic homology of group algebras was done by Burghelea [10]. An algebraic version was given by Nistor in [42]. We will give more details in the following sections.

It turns out that the computation not only gives information about the internal structure of the group but can also provide solutions to several conjectures in algebra and in analysis, e.g. Bass conjecture and ℓ^1 -Bass conjecture, which in turn imply the corresponding idempotent conjectures. It is conceivable that such approach will also shed light on the conjecture that there is no nontrivial projection in the group C^* -algebras of torsion free discrete groups. This is known as the generalized Kadison-Kaplansky conjecture. In fact, Puschnigg [49], using a scheme very similar to above described program, proved that the Kadison-Kaplansky conjecture is true for word hyperbolic groups although the result also follows from the works of Lafforgue [32] and Miniyev-Yu [41] on the Baum-Connes conjecture. We will give a brief account of Puschnigg's result in Section 3.2.

In what follows we will discuss various possible choices of $S(\Gamma)$, give details in computing the cyclic and periodic cyclic homology groups of $\mathbb{C}\Gamma$ and $S(\Gamma)$, and formulate Bass conjectures for various group related algebras. We conclude with a new result on the ℓ^1 -Bass conjecture for relatively hyperbolic groups.

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2 Definitions

2.1 Hochschild Homology

The following definitions are standard and can be found in, for instance [33]. By a first-quadrant bicomplex, we mean a collection $C_{p,q}, p, q \geq 0$, of k -algebras equipped with linear maps $d_{p,q} : C_{p,q} \rightarrow C_{p-1,q}$ and $\partial_{p,q} : C_{p,q} \rightarrow C_{p,q-1}$ referred to as the horizontal and vertical differentials respectively, satisfying $d_{p,q-1}\partial_{p,q} + \partial_{p-1,q}d_{p,q} = d_{p-1,q}d_{p,q} = \partial_{p,q-1}\partial_{p,q} = 0$. Given a general first quadrant bicomplex $C_{p,q}$, one forms the total complex $Tot_*(C_{*,*})$ by $Tot_n(C_{*,*}) = \bigoplus_{q=0}^n C_{n-q,q}$. If the horizontal and vertical differentials of $C_{*,*}$ are denoted by d and ∂ , the total complex differential $Tot_n(C_{*,*}) \rightarrow Tot_{n-1}(C_{*,*})$ is the map $d \oplus \partial$.

Let A be an algebra over a field k of characteristic zero. By $A^{\otimes n}$ we mean the tensor product of n copies of A , over k . We will denote the elementary tensor $a_1 \otimes a_2 \otimes \dots \otimes a_n$ by (a_1, a_2, \dots, a_n) . Denote by $b' : A^{\otimes n} \rightarrow A^{\otimes n-1}$,

$b : A^{\otimes n} \rightarrow A^{\otimes n-1}$, $t : A^{\otimes n} \rightarrow A^{\otimes n}$, and $N : A^{\otimes n} \rightarrow A^{\otimes n}$ the functions

$$\begin{aligned}
b'(a_0, a_1, \dots, a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0, a_1, \dots, a_i a_{i+1}, \dots, a_n) \\
b(a_0, a_1, \dots, a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0, a_1, \dots, a_i a_{i+1}, \dots, a_n) \\
&\quad + (-1)^n (a_n a_0, a_1, \dots, a_{n-1}) \\
t(a_0, a_1, \dots, a_n) &= (-1)^n (a_n, a_0, \dots, a_{n-1}) \\
N &= 1 + t + t^2 + \dots + t^n
\end{aligned}$$

By a routine calculation we are able to establish that the maps t , N , b' and b satisfy the following relations.

$$\begin{aligned}
bb &= 0 \\
b'b' &= 0 \\
(1-t)N &= 0 \\
N(1-t) &= 0 \\
(1-t)b' &= b(1-t) \\
Nb &= b'N
\end{aligned}$$

We can then consider the following bicomplex, which we will denote $C_{*,*}(A)$.

$$\begin{array}{ccc}
& \downarrow & \downarrow \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} \\
b \downarrow & & -b' \downarrow \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} \\
b \downarrow & & -b' \downarrow \\
A & \longleftarrow & A
\end{array}$$

Definition 2.1 *Let A be a not necessarily unital k -algebra. The Hochschild homology of A is the homology of the total complex of this bicomplex, $HH_n(A) = H_n(\text{Tot}(C_{*,*}(A)))$.*

In the case of a unital algebra, simplifications are possible. Denote the second column of this bicomplex by $C_*^a(A)$. Let $u : A^{\otimes n} \rightarrow A^{\otimes n+1}$ be given by $u(a_1, \dots, a_n) = (-1)^n (a_1, \dots, a_n, 1_A)$. It is easily seen that $ub' + b'u = 1$ so that $C_*^a(A)$ is acyclic.

$CC_{p,q}(A)[2,0] = CC_{p-2,q}(A)$. The quotient map on the total complex, $s_* = \{s_n : Tot_n(CC_{*,*}(A)) \rightarrow Tot_n(CC_{*,*}(A))[2,0]\}_{n \geq 0}$ induces a map on cyclic homology which is referred to as the periodicity operator $S : HC_n(A) \rightarrow HC_{n-2}(A)$.

The short exact sequence

$$0 \rightarrow C_{*,*}(A) \rightarrow CC_{*,*}(A) \rightarrow CC_{*,*}(A)[2,0] \rightarrow 0$$

induces a long exact sequence in the homology of their respective total complexes.

Theorem 2.5 *Let A be a not necessarily unital k -algebra. There is a long exact sequence relating Hochschild homology to Cyclic homology*

$$\dots \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} \dots$$

This sequence is commonly known as the Connes-Gysin sequence.

The Connes-Gysin sequence is a powerful tool for deriving information about cyclic homology from information about Hochschild homology.

In the case of a unital algebra A , the boundary map B of the above long exact sequence is defined on the bicomplex level $B : A^{\otimes n} \rightarrow A^{\otimes n+1}$ by $B = (1-t)uN$, where u is the homotopy defined above. Interestingly this B operator gives rise to an alternate bicomplex for calculating cyclic homology known as the (b, B) -bicomplex.

$$\begin{array}{ccccccc}
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A^{\otimes 4} & \xleftarrow{B} & A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\
b \downarrow & & b \downarrow & & b \downarrow & & \\
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A & & \\
b \downarrow & & b \downarrow & & & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & & & \\
b \downarrow & & & & & & \\
A & & & & & &
\end{array}$$

In [15], Connes and Karoubi define a family of chern characters

$$ch_n^m : K_n(A) \rightarrow HC_{n+2m}(A)$$

These maps fit naturally into the structure of the cyclic theory, as one has for all $m \geq 1$,

$$S \circ ch_n^m = ch_n^{m-1}$$

We will denote ch_n^0 by ch_n in what follows.

We have seen that $S : HC_n(A) \rightarrow HC_{n-2}(A)$. We can then look at the limits under these maps:

Definition 2.6 *The periodic cyclic homology of A , $PHC_*(A)$, is given by the inverse limits*

$$\begin{aligned} PHC_0(A) &= \varprojlim_S HC_{2k}(A) \\ PHC_1(A) &= \varprojlim_S HC_{1+2k}(A) \end{aligned}$$

Cyclic cohomology has several equivalent definitions, as well. One can dualize the above bicomplex construction to arrive at one of these. Over a field of characteristic 0, a more descriptive construction, due to Connes [12, 14], is as follows. Let A be a k -algebra, for k a subring of \mathbb{C} . Let C_λ^n be the space of $n + 1$ -linear functionals ϕ on A such that

$$\phi(a_1, \dots, a_n, a_0) = (-1)^n \phi(a_0, a_1, \dots, a_n)$$

The Hochschild coboundary map $b^* : C_\lambda^n \rightarrow C_\lambda^{n+1}$ defined as follows. For $\phi \in C_\lambda^n$, $b^*\phi$, is defined

$$\begin{aligned} (b^*\phi)(a_0, \dots, a_{n+1}) &= \sum_{j=0}^n (-1)^j \phi(a_0, a_1, \dots, a_j a_{j+1}, \dots, a_n, a_{n+1}) \\ &\quad + (-1)^{n+1} \phi(a_{n+1} a_0, a_1, \dots, a_n) \end{aligned}$$

The cohomology of the cochain complex (C_λ^n, b^*) is the cyclic cohomology of A , denoted by $HC^*(A)$. As with cyclic homology there is a periodicity operator $S : HC^n(A) \rightarrow HC^{n+2}(A)$. As with the homology theory, we define the periodic cyclic cohomology $PCH^*(A)$ as follows:

Definition 2.7 *The periodic cyclic homology of A , $PHC_*(A)$, is given by*

$$\begin{aligned} PHC^0(A) &= \varprojlim_S HC^{2k}(A) \\ PHC^1(A) &= \varprojlim_S HC^{1+2k}(A) \end{aligned}$$

There are several other variations on cyclic cohomology. Among these are Connes' entire cyclic cohomology [13], Meyer's analytic cyclic cohomology [38], and Puschnigg's local cyclic cohomology [50].

2.3 Incorporating Groups

Let G be a finitely generated discrete group. Among the first questions to be asked is “How does the cyclic homology of the group algebra relate to the usual homology of the group algebra?”. We start with some notation. Let k be a commutative subring of \mathbb{C} . Then $k[G]$ is the usual group algebra with coefficients in k . $\langle G \rangle$ denotes the set of conjugacy classes of G . For an element $h \in G$, let G_h denote the centralizer of h in G , and N_h be the quotient of G_h by the cyclic subgroup generated by h itself. Notice that if h is conjugate to h' then G_h and N_h are naturally isomorphic to $G_{h'}$ and $N_{h'}$, so we will refer to these simply as G_x and N_x , where h and h' are elements of the conjugacy class $x \in \langle G \rangle$. $\langle G \rangle_{ell}$ is the set of conjugacy classes of elliptic elements, that is, elements of finite order, while $\langle G \rangle_{non-ell}$ denotes the set of nonelliptic conjugacy classes. As a first example, let us start with the following well-known result.

Lemma 2.8 *Let G be a finitely generated discrete group and k a subring of \mathbb{C} . Then $HH_0(k[G]) \cong \bigoplus_{x \in \langle G \rangle} k$.*

An important characteristic of this calculation is the splitting as a direct sum over the conjugacy classes. This behavior carries over to the Hochschild and Cyclic homologies at all degrees. The first complete calculation for the cyclic homology of group rings along this line was performed by Burghelea in [10] using topological arguments. Nistor later gave an algebraic proof of the theorem in [42].

Theorem 2.9 *Let G be a finitely generated discrete group, and k a commutative subring of \mathbb{C} . Then $HH_*(k[G]) \cong \bigoplus_{x \in \langle G \rangle} HH_*(k[G])_x$ and $HC_*(k[G]) \cong \bigoplus_{x \in \langle G \rangle} HC_*(k[G])_x$ with*

- 1) $HH_*(k[G])_x \cong H_*(G_x; k)$
- 2) $HC_*(k[G])_x \cong H_*(N_x; k) \otimes HC_*(k)$ if $x \in \langle G \rangle_{ell}$
- 3) $HC_*(k[G])_x \cong H_*(N_x; k)$ if $x \in \langle G \rangle_{non-ell}$

This result is obtained by considering a subcomplex of the cyclic bicomplex. For $x \in \langle G \rangle$, let $\mathcal{L}_n(G, x)$ be the k -span of the tuples (g_1, \dots, g_n) in $k[G]^{\otimes n}$ for which $g_1 g_2 \dots g_n \in x$. The vertical and horizontal differentials of the cyclic bicomplex preserve this subspace, so it is in fact an actual subcomplex. The homology of the total complex of this bicomplex gives $HC_*(k[G])_x$, the cyclic homology over the x conjugacy class, and the cyclic homology of $k[G]$ splits

as a direct sum over conjugacy classes. The rest of the calculation rests in the identification of the results after a reduction process.

An algebra A is said to be G -graded if $A = \bigoplus_{g \in G} A_g$ with the multiplicative structure satisfying $A_g \cdot A_{g'} \subset A_{gg'}$. This is a natural generalization of the concept of the group algebra. For instance, $\ell^1 G$ is a G -graded algebra, as are the rapid decay algebras defined below. In [23], the first author showed that the cyclic homology of A splits into a direct sum over the conjugacy classes, in much the same way as in Burghelca's result. We refer to the sum over just the elliptic conjugacy classes as the elliptic summand, ${}^{ell}HC_*(A) = \bigoplus_{x \in \langle G \rangle_{ell}} HC_*(A)_x$. More interesting is that this calculation shows that for each conjugacy class $x \in \langle G \rangle$, $HC_*(A)_x$ has a natural $H_*(N_x; k)$ module structure given by an operation similar to Connes' sharp product construction [12], [14]. This action extends to an $HC_*(k[G])$ module structure on $HC_*(A)$. Moreover Connes' periodicity operator $S : HC_*(A) \rightarrow HC_{*-2}(A)$ is given by this module structure.

Theorem 2.10 *Let G be a finitely generated group, k a field of characteristic zero, and A a G -graded k -algebra. The periodicity operator $S : HC_*(A)_x \rightarrow HC_{*-2}(A)_x$ is given by the module action of the group 2-cocycle associated to the extension*

$$0 \rightarrow \mathbb{Z}x \rightarrow G_x \rightarrow N_x \rightarrow 0$$

2.4 Rapid Decay Algebras

Let G be a finitely generated group as above, and let ℓ_G be a word-length function on G . That is, we fix some finite symmetric generating set S of G and define

$$\ell_G(g) = \min\{n \mid g = s_1 s_2 \dots s_n, s_i \in S\}$$

Of course ℓ_G depends on the particular generating set chosen, however any two finite generating sets yield equivalent length functions. Note that ℓ_G induces a metric on the group, which is left-invariant under the group action, by $d_G(g_1, g_2) = \ell_G(g_1^{-1}g_2)$. We refer to d_G as a word-metric. For $i = 1, 2$, we define a family of norms on $\mathbb{C}G$ by

$$\|\phi\|_{i,k} = \left[\sum_{g \in G} |\phi(g)|^i (1 + \ell_G(g))^{ik} \right]^{1/i}$$

We denote by $\mathcal{S}_{\ell_G}^i G$ the completion of $\mathbb{C}G$ in the family of norms $\|\cdot\|_{i,k}$. $\mathcal{S}_{\ell_G}^1 G$ is always a Fréchet algebra, sitting inside $\ell^1 G$. Moreover the inclusion $\mathcal{S}_{\ell_G}^1 G \rightarrow \ell^1 G$ induces an isomorphism $K_*(\mathcal{S}_{\ell_G}^1 G) \rightarrow K_*(\ell^1 G)$. The case of $\mathcal{S}_{\ell_G}^2 G$ is more complicated.

The left-regular representation λ of a finitely generated discrete group G is the representation on $\ell^2(G)$ by $(\lambda(g)f)(x) = f(g^{-1}x)$. Extending this action by linearity to a representation of $\mathbb{C}G$ on $\ell^2(G)$ gives an embedding of $\mathbb{C}G$ into the space of bounded operators on the hilbert space $\ell^2(G)$. Denote the operator norm of a $\phi \in \mathbb{C}G$ by $\|\phi\|_*$. The reduced group C^* -algebra, C_r^*G , is the closure of $\mathbb{C}G$ in this operator topology.

Definition 2.11 G has the RD property if $\mathcal{S}_{\ell_G}^2 G$ lies in C_r^*G .

This is equivalent to several alternate conditions, as discussed by Jolissaint in [29].

Lemma 2.12 Let G be a finitely generated discrete group and let $|\cdot|_G$ be a word-length function on G . Then the following are equivalent.

1. G has the RD property
2. $\mathcal{S}_{\ell_G}^2 G$ is a Fréchet algebra.
3. There exists constants $C, s > 0$ such that for all $\phi \in \mathbb{C}G$ one has $\|\phi\|_* \leq C\|\phi\|_s$.

The RD property is not closed under taking all extensions, however an interesting extension theorem is due to Noskov in [44]. Let $0 \rightarrow A \xrightarrow{\iota} G \xrightarrow{\pi} Q \rightarrow 0$ be a group extension, and Let $q \mapsto \bar{q}$ be a cross section of π . That is, a map from $Q \rightarrow G$ such that $\pi\bar{q} = q$. To this cross section we associate a function $[\cdot, \cdot] : Q \times Q \rightarrow A$ by the formula $\bar{q}_1\bar{q}_2 = \overline{q_1q_2}[q_1, q_2]$. We call this function the factor set of the extension. We say that the factor set has polynomial growth if there exists constants C and r such that $\ell_A([q_1, q_2]) \leq C((1+\ell_Q(q_1))(1+\ell_Q(q_2)))^r$. The cross section also determines a map $\cdot A \times Q \rightarrow A$ given by $a^q = \bar{q}^{-1}a\bar{q}$. In the case that A is abelian, this is a right group action of Q on A . With this example in mind, we will refer to this map as an action of Q on A , even though it may not satisfy the usual axiom for a group action on a set, $(a_1^q)_2^q = a_1^{q_1q_2}$. We say that the action is polynomial if there exists constants C and r such that $\ell_A(a^q) \leq C\ell_A(a)(1 + \ell_Q(q))^r$.

The following is due to Noskov.

Theorem 2.13 Let $0 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 0$ be an extension with factor set of polynomial growth and polynomial action of Q on A . If A and Q have the RD property, then so does G .

Thus the RD property is closed under ‘‘polynomial extensions’’.

3 Conjectures

3.1 Novikov Conjecture

There are many different approaches to the Novikov conjecture, but we are concerned here only with the cyclic cohomological approach. Readers interested in other approaches are referred to [19]. Of course the Baum-Connes conjecture implies the Novikov conjecture.

Let M be a closed, oriented n -dimensional manifold with fundamental group G , and let $f : M \rightarrow BG$ be a continuous map from M to BG , an Eilenberg-MacLane space for G . Let $x \in H^{n-4i}(G, \mathbb{Q})$ be a rational cohomology class. The cohomology-homology pairing

$$\langle f^*(x) \cup L_i(M), [M] \rangle \in \mathbb{Q}$$

where $L_i(M)$ is the i -th Hirzebruch polynomial in Pontryagin classes of M , and $[M]$ is the fundamental class of the manifold is called a “higher signature” of the manifold. The Novikov conjecture, introduced by S.P. Novikov, is usually stated as follows.

Conjecture 3.1 *The higher signatures of a manifold M are homotopy invariants.*

As stated in the introduction, the Novikov conjecture is implied by the rational injectivity of the Baum-Connes map. As such, most of the work on the conjecture in this direction has been made through the machinery of Kasparov’s bivariant KK-theory. The notable exception to this, however, is the paper by Connes-Moscivici [16]. The main calculation of their paper is the following

Theorem 3.2 *Let G be a finitely generated discrete group. The Novikov conjecture is true for G if the following two conditions hold:*

1. G has the RD property.
2. The comparison map $HP^*(G; \mathbb{C}) \rightarrow H^*(G; \mathbb{C})$ is surjective.

Here $HP^*(G; \mathbb{C})$ is the polynomial cohomology of G with complex coefficients. That is, instead of examining the usual cochain complex to calculate group cohomology, we restrict to those cochains ϕ for which there is a polynomial P such that $|\phi(g_0, \dots, g_n)| \leq P(1 + \ell_G(g_0) + \dots + \ell_G(g_n))$. These polynomially bounded cochains form a subcomplex of the usual cochain complex, and the cohomology of this subcomplex is defined as $HP^*(G; \mathbb{C})$. The inclusion of these polynomially bounded cochains into the complex of all cochains induces the comparison map $HP^*(G; \mathbb{C}) \rightarrow H^*(G; \mathbb{C})$.

This polynomial cohomology was defined by in [22], under the terminology Schwartz cohomology. It is shown that for all groups of polynomial growth this comparison map is in fact an isomorphism. Combining this with a result of Jolissaint [29] stating that all groups of polynomial growth have the RD property, we see that all groups of polynomial growth satisfy the Novikov conjecture.

These results were expanded upon by Meyer in the context of combable groups. The notion of combability was introduced by Thurston, [11].

Definition 3.3 *Let G be a finitely generated group with a word-length metric d_G . A combing on G is a function $p : G \times \mathbb{N} \rightarrow G$ satisfying:*

1. *For all $g \in G$, $p(g, 0) = e$ the identity element of the group.*
2. *For all $g \in G$ there is an $N_g \in \mathbb{N}$ such that for all $n \geq N_g$, $p(g, n) = g$.*
3. *There is a constant S such that for all $g \in G$ and $n \in \mathbb{N}$, $d_G(p(g, n), p(g, n+1)) \leq S$.*
4. *There is a constant C such that for all g and $g' \in G$ and all $n \in \mathbb{N}$, $d_G(p(g, n), p(g', n)) \leq C(d_G(g, g') + 1)$.*

Let $J(g) = \#\{n \in \mathbb{N}; p(g, n) \neq p(g, n+1)\}$. G is polynomially combable if there is a polynomial P such that for all $g \in G$, $J(g) \leq P(\ell_G(g))$. We think of p as assigning to each element $g \in G$, a discrete path σ , in the group, connecting e to g given by $\sigma(n) = p(g, n)$. The original definition of combability in [11] required that these paths be quasi-geodesics. With this restriction, however, the only polynomial growth groups which are combable are the virtually abelian ones. By allowing polynomial combings we allow the possibility for more groups to be included. Notice that the existence of a combing, p , is equivalent to a family of maps $(f_n : G \rightarrow G)$ given by $f_n(g) = p(g, n)$ satisfying the obvious properties.

We say that G is *isocohomologous* if the comparison map $HP^*(G; \mathbb{C}) \rightarrow H^*(G; \mathbb{C})$ is an isomorphism. The following theorem is due to Meyer [37].

Theorem 3.4 *Let G be a polynomially combable group. Then G is isocohomologous.*

Thus any combable group with the RD property satisfies the Novikov conjecture. This includes, for example, all hyperbolic groups and all virtually abelian groups.

By constructing an analogue of the Lyndon-Hochschild-Serre spectral sequence for polynomial cohomology and improving a result in [46] we have the following [51]:

Theorem 3.5 *Let A and Q be two isocohomological groups, and let $0 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 0$ be a polynomial extension. Then G is isocohomological.*

It is not at all clear that a polynomial extension of two polynomially combable groups should itself be polynomially combable, so this is a nontrivial extension of Meyer's result. Combining this with the work of Connes-Moscovici we obtain the following.

Theorem 3.6 *Let A and Q be rapid decay isocohomological groups, and $0 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 0$ be a polynomial extension. Then G satisfies the Novikov conjecture.*

3.2 The Generalized Kadison-Kaplansky Conjecture

In early 1980's Pimsner and Voiculescu [48] proved that there is no non-trivial projection in the reduced group C^* -algebra of the free group of two generators. Their result settled a question raised by Kadison. Their method of solution involved calculation of the K -theory groups of the reduced group C^* -algebra. By introducing a Fredholm module associated to a construction in [48] Connes in [12] proposed another solution based on calculating the pairing of a projection with the Chern-Connes character of the Fredholm module in terms of the canonical trace of the projection in the C^* -algebra. A new and innovative approach to this conjecture, at least for word-hyperbolic groups, was given by Puschnigg in [49]. As mentioned in the introduction there is a commutative diagram

$$\begin{array}{ccc} K^*(\cdot, \Gamma) & \xrightarrow{\mu} & K_*(C_r^*\Gamma) \\ \tilde{ch}_* \downarrow & & \downarrow ch_* \\ H^*(\cdot, \Gamma) & \xrightarrow{ch\mu} & PHC_*(S(\Gamma)) \end{array}$$

Puschnigg showed that if one replaces the periodic cyclic homology groups in the lower-right corner of this diagram, with the localized cyclic homology groups $HC_*^{loc}(A(\Gamma))$, as defined in [49], the diagram still commutes (here Γ represents a word-hyperbolic group and $A(\Gamma) = S_{\ell^2}^2(\Gamma)$ denotes the ℓ^2 rapid-decay algebra of Γ). He then proved that the image

$$H^*(\cdot, \Gamma) \xrightarrow{ch\mu} HC_*^{loc}(A(\Gamma))$$

is concentrated in the homogeneous part $HC_*^{loc}(A(\Gamma))_{\langle hom \rangle}$ in the decomposition of $HC_*^{loc}(A(\Gamma))$ as a direct sum of homogeneous (corresponding to the conjugacy class of the identity element) and inhomogeneous summands.

On the other hand Kasparov and Skandalis [31] proved that for torsion-free hyperbolic groups the equivariant Kasparov KK -group $KK^\Gamma(\mathbb{C}, \mathbb{C})$ has a γ -element, which acts as an idempotent on $K_*(C_r^*(\Gamma))$. Puschnigg was able to show that the bivariant Chern character $ch_*(\gamma) \in HC_*^{loc}(A(\Gamma), A(\Gamma))$, a bivariant version of his local cyclic groups, acts on $HC_*^{loc}(A(\Gamma))$ as the canonical projection onto $HC_*^{loc}(A(\Gamma))_{\langle hom \rangle}$. Moreover, $ch_* : K_*(C_r^*(\Gamma)) \rightarrow HC_*^{loc}(A(\Gamma))$ takes $\gamma K_*(C_r^*(\Gamma))$ to $HC_*^{loc}(A(\Gamma))_{\langle hom \rangle}$ while taking $(1 - \gamma)K_*(C_r^*(\Gamma))$ to $HC_*^{loc}(A(\Gamma))_{\langle inhom \rangle}$. From here the idempotent conjecture can be verified for $A(\Gamma)$ by the L^2 -index theorem of Atiyah and Singer. Since $A(\Gamma)$ is smooth in $C_r^*(\Gamma)$, the generalized Kadison-Kaplansky conjecture is also true. This settles the conjecture for word hyperbolic groups. It is conceivable that due to Puschnigg's results described above and the work of [25], the local cyclic homology of $A(\Gamma)$ can be replaced by the regular continuous cyclic homology described in the previous sections. We will discuss this development in the future paper [28].

3.3 ℓ^1 -Bass Conjecture

Let R and A be two rings. By an A -valued trace on R , we mean a function $\tau : R \rightarrow A$ such that for all a and $b \in R$ we have $\tau(ab) = \tau(ba)$. Notice that the projection $\pi : R \rightarrow HH_0(R) = \frac{R}{[R, R]}$ is itself a trace function. It is universal in the following sense. For any A , and any A -valued trace τ on R , there is a map $\phi_\tau : \frac{R}{[R, R]} \rightarrow A$ such that $\tau = \phi_\tau \pi$. This π map extends to an abelian group homomorphism $\tau_{HS} : K_0(R) \rightarrow HH_0(R)$.

Specifically, let P a finitely generated projective R -module, there is an n such that P is a direct summand of R^n . Any endomorphism of P can be extended by zero to an endomorphism of R^n . The endomorphism ring of R^n can be identified with $M_n(R)$, the $n \times n$ matrices with entries in R . Denote this extension by $\iota : \text{End}_R(P) \rightarrow \text{End}_R(R^n)$. For a matrix $M \in M_n(R)$, let $\text{Trace}(M)$ denote the usual trace of M , i.e. the sum of the diagonal entries. For a finitely generated projective R -module P , let $\text{Tr}^{\text{HS}}(P) = \pi \text{Trace} \iota(\text{Id}_P)$. The image of P depends only on the isomorphism class of P . This map extends to an abelian group homomorphism $\text{Tr}^{\text{HS}} : K_0(R) \rightarrow HH_0(R)$, called the Hattori-Stallings trace.

Let G be a discrete group, and k a commutative subring of \mathbb{C} . We have seen in Lemma 2.8 that $HH_0(k[G]) \cong \bigoplus_{x \in \langle G \rangle} k$. Denote by $\pi_x : HH_0(k[G]) \rightarrow k$ be the projection onto the summand indexed by the class x , sending the other summands to zero. Let P be a finitely generated projective $k[G]$ -module. For an element $g \in G$, the P -rank of g is given by $r_P(g) = \pi_{\langle g \rangle} \text{Tr}^{\text{HS}}(P)$.

Conjecture 3.7 *Let P be a finitely generated projective $\mathbb{Z}G$ -module. Then*

for any nonidentity $g \in G$ we have $r_P(g) = 0$.

This is the classical Bass conjecture as studied in [2]. This conjecture is implied by the following stronger version.

Conjecture 3.8 *The image of $ch_* : K_*(\mathbb{C}G) \rightarrow HC_*(\mathbb{C}G)$ consists of exactly the elliptic summand.*

From arguments contained in [45], it follows that the image consists of at least the elliptic summand. The work is showing that it consists of no more than this. Eckmann notes in [17] that if G has finite homological dimension, and one of several other technical conditions, then $HC_*(\mathbb{C}G)_x$ vanishes for non-elliptic conjugacy classes x . Thus such a group satisfies the Bass conjecture. This strategy was exploited more fully in [24]. In particular, let \mathcal{C} denote the class of groups G with the following properties.

(B1) The rational cohomological dimension of G is finite.

(B2) If $h \in G$ is non-elliptic, then the rational cohomological dimension of N_h is finite.

These conditions assure that for any non-elliptic class $x \in \langle G \rangle$, Connes' periodicity operator S_x is nilpotent. Thus, as $(ch_*)_x = (ch_*^k)_x \circ S_x^k$, we have that for non-elliptic conjugacy classes, $(ch_*)_x$ vanishes. Therefore the Bass conjecture holds for these groups. [24] goes on to show that \mathcal{C} is closed under several constructions.

Theorem 3.9 *The class \mathcal{C} satisfies the following properties:*

1. \mathcal{C} contains all finite groups.
2. \mathcal{C} contains all finitely generated abelian groups.
3. Subgroups of a group in \mathcal{C} are again in this class.
4. \mathcal{C} is closed under extensions.
5. \mathcal{C} is closed under the operation defined by groups acting on a tree (with a finite quotient graph) with vertex and edge groups in \mathcal{C} . In particular, it is closed under amalgamated free products and HNN extensions [52].
6. \mathcal{C} contains all the word-hyperbolic groups of Gromov [20].
7. \mathcal{C} contains all arithmetic groups.

This class was later extended by Emmanouil in [18].

Let $\ell^1 G$ denote the ℓ^1 algebra of G . As in the case of the group algebra, $HC_*(\ell^1 G)$ decomposes as a direct sum indexed by conjugacy classes in $\langle G \rangle$. As before we refer to the elliptical summand of $HC_*(\ell^1 G)$ by ${}^{ell}HC_*(\ell^1 G) = \bigoplus_{x \text{ elliptic}} HC_*(\ell^1 G)_x$. The ℓ^1 Bass conjecture has the following form.

Conjecture 3.10 *The image of $ch_* : K_*(\ell^1 G) \rightarrow HC_*(\ell^1 G)$ lies in the elliptic summand.*

The related ℓ^1 idempotent conjecture claims that for any torsion-free discrete group G , the only idempotents in $\ell^1 G$ are the trivial ones. This conjecture is implied by the ℓ^1 Bass conjecture. There is an analogue of the Baum-Connes assembly map $\tilde{\mu}$ which takes values in the K-theory of $\ell^1 G$, $K_*(\ell^1 G)$. (as opposed to the actual Baum-Connes assembly map, which takes values in $K_*(C_r^* G)$, the K-theory of the reduced group C^* algebra) Berrick, Chatterji, and Mislin show in [8] that if this Bost assembly map is rationally surjective in degree zero, then the ℓ^1 Bass conjecture holds for the group. This is then used to show that amenable groups satisfy the ℓ^1 Bass conjecture.

Let $\langle x \rangle$ be a conjugacy class of the group G . $\langle x \rangle$ is said to have *polynomially bounded conjugacy problem* if there is a polynomial P such that if $u, v \in \langle x \rangle$ there is a $g \in G$ such that $\ell_G(g) \leq P(1 + \ell_G(v) + \ell_G(u))$. The group G is said to have polynomially bounded conjugacy problem if each non-elliptic conjugacy class does. This property appears when considering the decomposition into conjugacy classes. As in the case for the usual group algebra, one decomposes the cyclic bicomplex associated to $S_\ell^1(G)$ into a direct sum of sub-bicomplexes, indexed by the conjugacy classes of G . The cohomology of these subcomplexes gives $HC_*(S_\ell^1(G))_x$. In order to reduce to the cohomology of $N_x = G_x/\mathbb{Z}x$, as in the result of Burghelea and Nistor discussed above, we obtain not the usual cohomology of N_x with complex coefficients, but rather the polynomial cohomology of N_x [27]. As these calculations now involve topological algebras, the group needs to have polynomially bounded conjugacy problem in order to ensure the continuity of the morphisms involved in relating the cyclic cohomology groups $HC^*(S_\ell^1(G))_x$ with the polynomial cohomology groups $HP^*(N_x; \mathbb{C})$.

In [27] it is shown that, using a setup similar to that of the class \mathcal{C} utilized above, several classes of groups satisfy the ℓ^1 Bass conjecture. In particular it is shown that if G is a finitely generated nilpotent group or a word-hyperbolic group, then G lies in this new class. More specifically, let $\mathcal{C}_{\mathcal{P}}$ be the class of groups G for which the periodicity operator $S_x^* : HC^*(S_\ell^1(G))_x \rightarrow HC^{*+2}(S_\ell^1(G))_x$ is nilpotent for all non-elliptic conjugacy classes $\langle x \rangle$, where $S_\ell^1(G)$ is the ℓ^1 rapid decay algebra of G . As in the above case, if a group G is in $\mathcal{C}_{\mathcal{P}}$, then the ℓ^1 -Bass conjecture holds for G .

The class $\mathcal{C}_{\mathcal{P}}$ is related to the class \mathcal{C} by the following [27].

Theorem 3.11 *Let G be a discrete group with word-length function ℓ . Suppose that each non-elliptic conjugacy class $\langle x \rangle$ has polynomially bounded conjugacy problem, and that the corresponding comparison maps $HP^*(G_x) \rightarrow H^*(G_x)$ are isomorphisms. If G is in \mathcal{C} then G is in $\mathcal{C}_{\mathcal{P}}$.*

For many classes of groups, checking whether or not the group lies in $\mathcal{C}_{\mathcal{P}}$ rests on the verification that the group satisfies a polynomially bounded conjugacy problem, and appealing to this theorem.

The following interesting result is shown in [27].

Theorem 3.12 *Let G be a group which is relatively hyperbolic to a finite family of subgroups H_1, H_2, \dots, H_m with each $H_i \in \mathcal{C}_{\mathcal{P}}$. Then G is in $\mathcal{C}_{\mathcal{P}}$.*

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