A GENERALIZATION OF THE LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCE FOR POLYNOMIAL COHOMOLOGY

A Dissertation

Submitted to the Faculty

of

Purdue University

by

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In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

May 2008

Purdue University

Indianapolis, Indiana

I dedicate this dissertation to my family. Thank you.

ACKNOWLEDGMENTS

I wish to express my gratitude to my advisor, Ronghui Ji, for his invaluable guidance and constant feedback. I also wish to thank Jerry Kaminker for his guidance through the early portion of my graduate career. Finally I wish to thank the IUPUI Department of Mathematical Sciences as a whole for providing an absolutely wonderful environment in which to work.

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ABSTRACT

Ramsey Jr., Bobby William Ph.D., Purdue University, May, 2008. A Generalization of the Lyndon-Hochschild-Serre Spectral Sequence for Polynomial Cohomology. Major Professor: Ronghui Ji.

We construct an analogue of the Lyndon-Hochschild-Serre spectral sequence in the context of polynomial cohomology with E_2-term $HP^*(Q; HP^*(H;\mathbb{C}))$. For the polynomial extensions of Noskov, with the normal subgroup isocohomological, the spectral sequence converges to $HP^*(G;\mathbb{C})$. In the case that both H and Q are isocohomological G must also be isocohomological. By referring to results of Connes-Moscovici and Noskov if H and Q are both isocohomological and have the Rapid Decay property, then G satisfies the Novikov conjecture.

1. INTRODUCTION

In [1] Connes and Moscovici verified the Novikov Conjecture for word hyperbolic groups by showing that they satisfy two properties. The first is the Rapid Decay property of Jolissaint [2], which ensures the existance of a smooth dense subalgebra of the reduced group C^*-algebra. The second property is that every cohomology class can be represented by a cocycle of polynomial growth, with respect to some (hence any) word-length function on the group. Connes and Moscovici show that any group with these two properties satisfies the Novikov conjecture on the homotopy invariance of higher signatures.

The polynomial cohomology of a group G, denoted $HP^*(G;\mathbb{C})$, obtained by considering only cochains of polynomial growth, has been of interest recently. The inclusion of these polynomially bounded cochains into the full cochain complex yields a homomorphism from the polynomial cohomology to the full cohomology of the group. The second property of Connes-Moscovici above is that this polynomial comparison homomorphism is surjective. We say that a group G is isocohomological for M if $HP^*(G;M)$ is bornologically isomorphic to $H^*(G;M)$. A group which is isocohomological for the trivial coefficients $\mathbb{C}$ is said to be isocohomological. This definition of isocohomological differs from that found in [3]. Meyer's notion is much stronger than that used here.

In [4] Ji defined polynomial cohomology and showed that virtually nilpotent groups, are isocohomological. In [5] Meyer showed that combable groups are isocohomological. By citing a result of Gersten regarding classifying spaces for combable groups [6], Ogle independantly showed that combable groups are isocohomological.

For a group extension, $0 \to H \to G \to Q \to 0$, the Lyndon-Hochschild-Serre (LHS) spectral sequence is a first-quadrant spectral sequence with E_2 term isomorphic to $H^*(Q;H^*(H))$ and which converges to $H^*(G)$. In [7], Noskov generalized the construction of the LHS spectral sequence to obtain a spectral sequence in bounded cohomology for which, under suitable topological circumstances, one could identify the E_2 term as $H_b^*(Q; H_b^*(H))$ and which converges to $H_b^*(G)$. Ogle considered the LHS spectral sequence in the context of P-bounded cohomology in [8]. However he also needs additional technical considerations to ensure the appropriate E_2 term. It is not clear for which class of extensions his condition is satisfied.

In this paper we resolve this issue for polynomial extensions, which were proposed by Noskov in [9]. Let ℓ, ℓ_H, and ℓ_Q be word-length functions on G, H, and Q respectively, and let

$$0 \to H \to G \xrightarrow{\pi} Q \to 0$$

be an extension of Q by H. Let $q \mapsto \overline{q}$ be a cross section of π. To this cross section associate a function $[\cdot, \cdot] : Q \times Q \to H$ by $\overline{q}_1\overline{q}_2 = \overline{q_1q_2}[q_1, q_2]$. This is the factor set of the extension. The factor set has polynomial growth if there exists constants C and r such that $\ell_H([q_1, q_2]) \leq C((1+\ell_Q(q_1))(1+\ell_Q(q_2)))^r$. The cross section also determines an "action" of Q on H given by $h^q = \overline{q}^{-1}h\overline{q}$. For nonabelian groups H this need not be an actual group action but we follow the terminology of Noskov. The action is polynomial if there exists constants C and r such that $\ell_H(h^q) \leq C\ell_H(h)(1 + \ell_Q(q))^r$. Note that if $\mathcal{Q}$ is the finite generating set for Q and $\mathcal{A}$ is the finite generating set for H, then as the generating set for G we will take the set of $h \in \mathcal{A}$ and $\overline{q}$ for $q \in \mathcal{Q}$.

Definition 1.1 *An extension G of a finitely generated group Q by a finitely generated group H is said to be a polynomial extension if there is a cross section yielding a factor set of polynomial growth and inducing a polynomial action of Q on H.*

Our main theorem is as follows.

Theorem 1.2 *If $0 \to H \to G \to Q \to 0$ is a polynomial extension of groups with H be isocohomological, then there exists a bornological spectral sequence with $E_2^{p,q} \cong HP^p(Q; HP^q(H; \mathbb{C}))$ which converges to $HP^*(G; \mathbb{C})$.*

Using the methods of [3] Meyer is able to show that this theorem holds if one uses his notion of isocohomological. This results in a stronger conclusion but requires a stronger hypothesis.

In the case when Q is isocohomological with the appropriate coefficients, we can compare our spectral sequence with the usual LHS spectral sequence, and see that they have isomorphic E_2-terms, thus the same limits.

Corollary 1.3 *Let $0 \to H \to G \to Q \to 0$ be a polynomial extension of groups. If Q is isocohomological with coefficients $HP^*(H;\mathbb{C})$ and H is isocohomological, then G is isocohomological.*

This yields a new class of groups for which the isocohomological property was previously unknown, as these types of polynomial extensions need not be virtually nilpotent or polynomially combable. Applying a result of Noskov, [9], regarding the polynomial extension of groups with the Rapid Decay property, as well as the results of Connes-Moscovici, [1], we obtain:

Corollary 1.4 *Let $0 \to H \to G \to Q \to 0$ be a polynomial group extension, let Q be isocohomological with coefficients $HP^*(H;\mathbb{C})$, and let H be isocohomological. If both Q and H have the Rapid Decay property, G satisfies the Novikov conjecture.*

The class of groups which have the Rapid Decay property and are isocohomological thus contains all virtually nilpotent groups, hyperbolic groups, and is closed under polynomial extensions.

It would be convient to work in the category of Fréchet and DF spaces, however there are many quotients involved in the construction, yielding spaces which are neither Fréchet nor DF. This is the issue Ogle overcomes by use of a technical hypothesis in [8], and it is at the heart of the topological consideration in [7], to identify the E_2-terms in their spectral sequences. Moreover, at this point in our analysis we need to utilize an adjointness relationship of the form $\mathrm{Hom}(A\hat{\otimes}B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$ which is not true in the category of locally convex topological vector spaces. To overcome these obstacles we work mostly in the bornological category. A bornology on a space is an analogue of a topology, in which boundedness replaces openness as the key consideration. In this context, we are also able to bypass many of the issues involved in the topological analysis of algebraic vector spaces. When endowed with the fine bornology, as defined later, an algebraic vector space is a complete bornological vector space. The finest topology yielding a complete topological structure on such a space is cumbersome. This bornology allows us to replace analysis of continuity in this topology, to boundedness in a finite dimensional vector space. This bornological view serves to verify the identification of the 'iterated' cohomology spaces $H^*(Q; H^*(H))$ and $HP^*(Q; HP^*(H))$.

In Chapter 2 we review the cohomology of groups and define the polynomial growth cohomology of discrete groups, in terms of the topological ℓ^1-rapid decay algebra. In Chapter 3 we introduce the bornological framework in which we perform our later analysis. We also show that the bornological approach to polynomial cohomology is equivalent to the topological approach. In Chapter 4 we define bornological spectral sequences and develop the necessary tools to work with them. In Chapter 5 we construct our bornological spectral sequence.

2. COHOMOLOGY OF GROUPS

2.1 A Little Homological Algebra

Let R be a ring. An R-module P is said to be projective if, for any R-modules A and B, any R-module epimorphism $\phi : A \to B$, and any R-module homomorphism $f : P \to B$, there exists an R-module homomorphism $\hat{f} : P \to A$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "\hat{f}"'] \arrow[d, "f"] \\ A \arrow[r, "\phi"'] & B \end{tikzcd}$$

That is to say, that P is a projective object in the category of R-modules. More formally, let $\mathfrak{C}$ be a category. Using the terminology of [10], an object $\mathcal{P}$ of $\mathfrak{C}$ is said to be projective if for any objects $\mathcal{A}$ and $\mathcal{B}$ of $\mathfrak{C}$, any admissible epimorphism $\Phi \in \mathfrak{C}(\mathcal{A}, \mathcal{B})$, and any morphism $\sigma \in \mathfrak{C}(\mathcal{P}, \mathcal{B})$, there exists a morphism $\hat{\sigma} \in \mathfrak{C}(\mathcal{P}, \mathcal{A})$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "\hat{\sigma}"'] \arrow[d, "\sigma"] \\ \mathcal{A} \arrow[r, "\Phi"'] & \mathcal{B} \end{tikzcd}$$

(Recall that a morphism $\Phi \in \mathfrak{C}(\mathcal{A}, \mathcal{B})$ is epimorphic if $\alpha\Phi = \beta\Phi$ implies $\alpha = \beta$, for all morphisms α and β for which the compositions are defined.) In what follows, we shall be mainly interested in the categories of R-modules with R-module homomorphisms, complete locally convex topological vector spaces with continuous linear functions, and complete locally convex bornological vector spaces with bounded linear functions.

Of course in different categories, 'admissible' has different meanings. In the category of algebraic R-modules, any epimorphism is admissible. In the category of topological vector spaces, a continuous linear epimorphism $\mathcal{A} \to \mathcal{B}$ is admissible if there is a continuous $\mathbb{C}$-linear cross-section. In the category of bornological vector spaces, a bounded linear epimorphism $\mathcal{A} \to \mathcal{B}$ is admissible if there is a bounded $\mathbb{C}$-linear cross-section.

Given an R-module A, it is possible to find a projective R-module P_0 and an R-module homomorphism $\partial_0 : P_0 \to A$, which is epimorphic. Denote by K_0 the kernel of ∂_0, and ι_0 the inclusion of K_0 into P_0. We have a short exact sequence

$$0 \to K_0 \xrightarrow{\iota_0} P_0 \xrightarrow{\partial_0} A \to 0$$

Let P_1 be a projective R-module with $\partial_1' : P_1 \to K_0$ an R-module epimorphism. If we denote the kernel of φ_1' by K_1, and let $\partial_1 = \iota_0 \partial_1'$. This yields the following exact sequence

$$0 \to K_1 \xrightarrow{\iota_1} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \to 0$$

Continuing on in this way we build an exact sequence

$$\ldots \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \to 0$$

Definition 2.1 *A projective resolution of an R-module A is a chain complex of R-modules*

$$\ldots \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} A \to 0$$

where each M_i is a projective R-module, admitting $\mathbb{Z}$ or $\mathbb{C}$-linear maps $s_i : M_i \to M_{i+1}$ and $s_{-1} : A \to M_0$ satisfying the following condition.

$$d_{i+1}s_i + s_{i-1}d_i = id$$

The existance of the function s_i guarantees that the chain complex is an exact sequence. It follows from our earlier comments that every R-module admits at least one projective resolution.

Let A and B be R-modules, and let

$$\ldots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0$$

be a projective resolution of A. The deleted resolution is given by

$$\ldots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0$$

Applying the $\mathrm{Hom}_R(\cdot, B)$ functor yields the following cochain complex.

$$\ldots \xleftarrow{d^*_{n+1}} \mathrm{Hom}_R(P_n, B) \xleftarrow{d^*_n} \mathrm{Hom}_R(P_{n-1}, B) \xleftarrow{d^*_{n-1}} \ldots \xleftarrow{d^*_2} \mathrm{Hom}_R(P_1, B) \xleftarrow{d^*_1} \mathrm{Hom}_R(P_0, B)$$

where $d^*_n(f)(x) = f(d_n x)$.

Definition 2.2 $\mathrm{Ext}^n_R(A, B) = \ker d^*_{n+1} / \operatorname{im} d^*_n$ *is the n-th cohomology module of this complex.*

As any two projective resolutions are homotopy equivalent, this is well-defined.

2.2 Definition of Group Cohomology

Let G be a discrete group. The complex group algebra $\mathbb{C}G$ is the set of finite sums of the form $\sum_{g\in G} a_g g$, with $a_g \in \mathbb{C}$. It is a ring under the natural operations. The trivial action of G on $\mathbb{C}$ yields a $\mathbb{C}G$-module structure on $\mathbb{C}$, with module action $\left(\sum_{g\in G} a_g g\right) \cdot z = \sum_{g\in G} a_g z$.

Definition 2.3 *The complex group cohomology of a discrete group G, denoted by $H^*(G; \mathbb{C})$, is given by $\mathrm{Ext}^*_{\mathbb{C}G}(\mathbb{C}, \mathbb{C})$.*

For $k \geq 0$, let $C_k(G) = \mathbb{C}[G^{k+1}]$. $C_k(G)$ is the complex vector space with G^k as a basis. In particular, $C^0(G)$ is just $\mathbb{C}G$. There is a G-action on $C_k(G)$ given on basis vectors by $g \cdot (g_0, g_1, \dots, g_k) = (gg_0, g_1, \dots, g_k)$. This G-action extends to a left $\mathbb{C}G$-module structure on $C_k(G)$. There is a $\mathbb{C}G$-equivariant map $d_k : C_k(G) \to C_{k-1}(G)$ given on basis elements by

$$\begin{aligned} d_k(g_0, g_1, \dots, g_k) &= (g_1, g_2, \dots, g_k) \\ &\quad + \sum_{j=0}^{k} (-1)^{j+1} (g_0, g_1, \dots, g_{j-1}, g_j g_{j+1}, g_{j+2}, \dots, g_k) \\ &\quad + (-1)^k (g_0, g_1, \dots, g_{k-1}) \end{aligned}$$

It is a standard calculation that $d_k d_{k+1} = 0$. We thus have a chain complex

$$\dots \xrightarrow{d_3} C_2(G) \xrightarrow{d_2} C_1(G) \xrightarrow{d_1} \mathbb{C}G \xrightarrow{d_0} \mathbb{C} \to 0$$

where $d_0 : \mathbb{C}G \to \mathbb{C}$ is the augmentation map $d_0 \left(\sum_{g \in G} \lambda_g(g) \right) = \sum_{g \in G} \lambda_g$. For $k > 0$, each $C_k(G)$ is a free $\mathbb{C}G$-module, having as a basis those tuples of $(g_0, g_1, \dots, g_k) \in G^k$ with $g_0 = e$. Thus $C_k(G)$ is a projective left $\mathbb{C}G$-module. Moreover, let $s_k : C_k(G) \to C_{k+1}(G)$ be given on basis tuples by $s_k(g_0, g_1, \dots, g_k) = (e, g_0, g_1, \dots, g_k)$ and extend by $\mathbb{C}$-linearity to all of $C_k(G)$. Define $s_{-1} : \mathbb{C} \to \mathbb{C}G$ by $s_{-1}(\lambda) = \lambda(e)$.

For $k \geq 1$ we have

$$\begin{aligned} (s_{k-1} d_k + d_{k+1} s_k)(g_0, g_1, \dots, g_k) &= s_{k-1} \sum_{j=0}^{k} (-1)^{j+1} (g_0, \dots, g_j g_{j+1}, \dots, g_k) \\ &\quad + s_{k-1}(g_1, \dots, g_k) + (-1)^k s_{k-1}(g_0, \dots, g_{k-1}) \\ &\quad + d_{k+1}(e, g_0, \dots, g_k) \\ &= \sum_{j=0}^{k} (-1)^{j+1} (e, g_0, \dots, g_j g_{j+1}, \dots, g_k) \\ &\quad + (e, g_1, g_2, \dots, g_k) + (-1)^k (e, g_0, \dots, g_{k-1}) \\ &\quad + (g_0, g_1, \dots, g_k) \\ &\quad + \sum_{j=0}^{k} (-1)^{j+2} (e, g_0, g_1, \dots, g_j g_{j+1}, \dots, g_k) \\ &\quad + (-1)^{k+1} (e, g_0, g_1, \dots, g_{k-1}) \\ &= (g_0, g_1, \dots, g_k) \end{aligned}$$

For $k = 0$ we have

$$\begin{aligned}(s_{-1}d_0 + d_1 s_0)(g_0) &= s_{-1}(1) + d_1(e, g_0)\\ &= (e) + (g_0) - (e)\\ &= (g_0)\end{aligned}$$

It follows that s_* is a $\mathbb{C}$-linear contracting homotopy, so that

$$\ldots \xrightarrow{d_3} C_2(G) \xrightarrow{d_2} C_1(G) \xrightarrow{d_1} \mathbb{C}G \xrightarrow{d_0} \mathbb{C} \to 0$$

Is in fact a projective resolution of $\mathbb{C}$ over $\mathbb{C}G$, which is commonly referred to as the Bar Resolution of G. We can use this resolution in the calculation of $\mathrm{Ext}_{\mathbb{C}G}(\mathbb{C}, \mathbb{C})$. To this end, consider $\mathrm{Hom}_{\mathbb{C}G}(C_k(G), \mathbb{C})$. If $\phi \in \mathrm{Hom}_{\mathbb{C}G}(C_k(G), \mathbb{C})$, then ϕ is a $\mathbb{C}G$-equivariant map. In particular ϕ is completely determined by its value on tuples in G^{k+1} of the form $(e, g_1, g_2, \ldots, g_n)$. Let $C^k(G) = \{\psi : G^k \to \mathbb{C}\}$, consisting of all complex valued functions from G^k. To $\phi \in \mathrm{Hom}_{\mathbb{C}G}(C_k(G), \mathbb{C})$ we associate $\psi \in C^k(G)$ defined by $\psi(g_1, g_2, \ldots, g_k) = \phi(e, g_1, g_2, \ldots, g_k)$. Similarly given a $\psi \in C^k(G)$ we can define a $\phi \in \mathrm{Hom}_{\mathbb{C}G}(C_k(G), \mathbb{C})$ given on basis elements by $\phi(g_0, g_1, \ldots, g_k) = \psi(g_0^{-1}g_1, \ldots, g_0^{-1}g_k)$. For $\phi \in \mathrm{Hom}_{\mathbb{C}G}(C_k(G), \mathbb{C})$, we have $d_k^*(\phi)(g_0, g_1, \ldots, g_{k+1}) = \sum_{j=0}^{k+1}(-1)^j\phi(g_0, \ldots, g_{j-1}, g_j g_{j+1}, g_{j+2}, \ldots, g_{k+1})$. We use this boundary map to determine the appropriate boundary map $\partial^k : C^k(G) \to C^{k+1}(G)$. $\partial^k\psi(g_1, g_2, \ldots, g_{k+1}) = \sum_{j=1}^{k+1}(-1)^j\psi(g_1, \ldots, g_{j-1}, g_j g_{j+1}, g_{j+2}, \ldots, g_{k+1})$.

In this way the cochain complex

$$\mathrm{Hom}_{\mathbb{C}G}(C_0(G), \mathbb{C}) \xrightarrow{d_0^*} \mathrm{Hom}_{\mathbb{C}G}(C_1(G), \mathbb{C}) \xrightarrow{d_1^*} \ldots \xrightarrow{d_{k-1}^*} \mathrm{Hom}_{\mathbb{C}G}(C_k(G), \mathbb{C}) \xrightarrow{d_k^*} \ldots$$

is equivalent to the cochain complex

$$C^0(G) \xrightarrow{\partial^0} C^1(G) \xrightarrow{\partial^1} \ldots \xrightarrow{\partial^{k-1}} C^k(G) \xrightarrow{\partial^k} \ldots$$

This is the usual cochain complex utilized in the calculation of complex group cohomology.

2.3 Polynomial Cohomology

A natural question is what happens to the cohomology when we restrict to only cochains which satisfy some growth condition. As a first example, denote by $C_b^n(G)$ those $\phi \in C^n(G)$ with

$$\|\phi\|_\infty = \sup_{(x_1,\ldots,x_n)\in G^n} |\phi(x_1, ..., x_n)| < \infty$$

If $\phi \in C_b^n(G)$, then consider $\partial^n \phi \in C^{n+1}(G)$

$$\begin{aligned}(\partial^n \phi)(x_1, ..., x_{n+1}) &= \phi(x_2, ..., x_{n+1}) - \phi(x_1 x_2, x_3, ..., x_{n+1}) \\ &+ ... + (-1)^i \phi(x_1, ..., x_i x_{i+1}, ..., x_{n+1}) + ... \\ &+ (-1)^{n+1} \phi(x_1, ..., x_n)\end{aligned}$$

So that when considering boundedness:

$$\begin{aligned}\|\partial^n \phi\|_\infty &\leq \|\phi\|_\infty + \|\phi\|_\infty \\ &+ \ldots + \|\phi\|_\infty + \ldots \\ &+ \|\phi\|_\infty \\ &\leq (n+1)\|\phi\|_\infty\end{aligned}$$

Thus the differentials preserve boundedness. That is

$$C_b^0(G) \xrightarrow{\partial^0} C_b^1(G) \xrightarrow{\partial^1} C_b^2(G) \xrightarrow{\partial^2} \ldots$$

is a subcomplex of the usual cochain complex. The cohomology of this complex, denoted by $H_b^*(G; \mathbb{C})$, is the *bounded cohomology* of the group G.

The following result is attributed to Trauber in [11]. We give an elementary proof of the result here.

Theorem 2.4 *Let G be a discrete amenable group. For $n \geq 1$, $H_b^n(G; \mathbb{C}) = 0$.*

Recall that a discrete group G is amenable if there is a left-invariant mean $m : \ell^\infty G \to \mathbb{C}$. This means that m is a positive linear functional on $\ell^\infty G$, with $\|m\| = m(1) = 1$. Moreover, for each $\phi \in \ell^\infty G$ and $g \in G$, $m(\phi) = m(\phi_g)$ where $\phi_g(x) = \phi(g \cdot x)$. See Patterson's monograph [12] for more details on amenable groups.

Proof Let $\phi \in C_b^1(G)$ be a bounded cocycle. That is, $\phi : G \to \mathbb{C}$ is a bounded map satisfying $\partial^0\phi = 0$. That is, for all $a, b \in G$ we have $(\partial^0\phi)(a,b) = \phi(b) - \phi(ab) + \phi(a) = 0$. Thus $\phi(ab) = \phi(a) + \phi(b)$ so ϕ is a group homomorphism from G to the additive group $\mathbb{C}$. Let $g \in G$ such that $\phi(g) \neq 0$. Denote $\phi(g) = z$. For all $n \in \mathbb{N}$, $\phi(g^n) = nz$. As n tends to infinity, the norm of $\phi(g^n)$ itself grows without bound, contradicting that ϕ is bounded. Therefore $C_b^1(G)$ is trivial. Consequently, $H_b^1(G;\mathbb{C})$ itself is trivial.

Denote a left-invariant mean by m. We continue by examining $H_b^2(G;\mathbb{C})$.

Let $\phi \in C_b^2(G)$ with $\partial^2\phi = 0$. For all a, b, and $c \in G$

$$(\partial^2\phi)(a,b,c) = \phi(b,c) - \phi(ab,c) + \phi(a,bc) - \phi(a,b) = 0$$

So $\phi(a,b) = \phi(b,c) - \phi(ab,c) + \phi(a,bc)$. Let $f : G \to \mathbb{C}$ be defined by $f(a) = m(\phi(a,x))$, where $\phi(a,x)$ denotes the bounded function from G to $\mathbb{C}$ obtained by fixing the first argument of ϕ. As $f(a) = m(\phi(a,x)) \leq \|m\|\|\phi\|_\infty < \infty$, $f \in C_b^1(G)$.

$$\begin{aligned}
(\partial^1 f)(a,b) &= f(b) - f(ab) + f(a) \\
&= m(\phi(b,x)) - m(\phi(ab,x)) + m(\phi(a,x)) \\
&= m(\phi(b,x)) - m(\phi(ab,x)) + m(\phi(a,bx)) \\
&= m(\phi(b,x) - \phi(ab,x) + \phi(a,bx)) \\
&= m(\phi(a,b)) \\
&= \phi(a,b) \cdot m(1) \\
&= \phi(a,b)
\end{aligned}$$

Thus ϕ is in the image of ∂^1, so $H_b^2(G;\mathbb{C}) = 0$.

The case for general $n > 1$ is similar in technique to this case. Let $\phi \in C_b^n(G)$ with $\partial^n\phi = 0$. For all $x_1, ..., x_{n+1} \in G$

$$\begin{aligned}
(\partial^n\phi)(x_1, ..., x_{n+1}) &= \phi(x_2, ..., x_{n+1}) - \phi(x_1x_2, x_3, ..., x_{n+1}) \\
&\quad + ... + (-1)^i\phi(x_1, ..., x_ix_{i+1}, ..., x_{n+1}) + ... \\
&\quad + (-1)^{n+1}\phi(x_1, ..., x_n) \\
&= 0
\end{aligned}$$

Consequently

$$\begin{aligned}(-1)^n\phi(x_1,...,x_n) &= \phi(x_2,...,x_{n+1}) - \phi(x_1x_2,x_3,...,x_{n+1}) \\ &\quad +... + (-1)^i\phi(x_1,...,x_ix_{i+1},...,x_{n+1}) + ... \\ &\quad +(-1)^n\phi(x_1,...,x_nx_{n+1})\end{aligned}$$

Let $f : G^{n-1} \to \mathbb{C}$ be defined by

$$f(x_1,...,x_{n-1}) = m(\phi(x_1,...,x_{n-1},x))$$

where, as before, $\phi(x_1,...,x_{n-1},x)$ is treated as a bounded function of one variable obtained by fixing the first $n-1$ arguments of ϕ. By applying ∂^{n-1} to f we find

$$\begin{aligned}(\partial^{n-1}f)(x_1,...,x_n) &= f(x_2,...,x_n) - f(x_1x_2,x_3,...,x_n) \\ &\quad +... + (-1)^if(x_1,...,x_ix_{i+1},...,x_n) + ... \\ &\quad +(-1)^nf(x_1,...,x_{n-1}) \\ &= m(\phi(x_2,...,x_n,x)) - m(\phi(x_1x_2,x_3,...x_n,x)) \\ &\quad +... + (-1)^im(\phi(x_1,...,x_ix_{i+1},...,x_n,x)) + ... \\ &\quad +(-1)^nm(\phi(x_1,...,x_{n-1},x)) \\ &= m(\phi(x_2,...,x_n,x)) - m(\phi(x_1x_2,x_3,...x_n,x)) \\ &\quad +... + (-1)^im(\phi(x_1,...,x_ix_{i+1},...,x_n,x)) + ... \\ &\quad +(-1)^nm(\phi(x_1,...,x_{n-1},x_nx)) \\ &= m(\phi(x_2,...,x_n,x) - \phi(x_1x_2,x_3,...x_n,x) \\ &\quad +... + (-1)^i\phi(x_1,...,x_ix_{i+1},...,x_n,x) + ... \\ &\quad +(-1)^n\phi(x_1,...,x_{n-1},x_nx)) \\ &= m((-1)^n\phi(x_1,...,x_n)) \\ &= (-1)^n\phi(x_1,...,x_n)\cdot m(1) \\ &= (-1)^n\phi(x_1,...,x_n)\end{aligned}$$

Thus $\phi = \partial^{n-1}(-1)^nf$, whence $H_b^n(G;\mathbb{C}) = 0$. ■

As mentioned in the introduction, examining polynomial growth cochains played an integral role in the results of [1]. In other directions, the word hyperbolic groups of Gromov [13] have all cocycles represented by bounded cocycles [14]. The polynomial cohomology of a group G was initially defined by Ji in [4], under the name of Schwartz cohomology. It has also been studied by Meyer in [5], as well as by Ogle in [8]. Our development in this section follows that of [4].

Definition 2.5 *A length function on a group G is a function $\ell : G \to [0, \infty)$ satisfying*

1) *For the identity element e, $\ell(e) = 0$.*

2) *For all $g \in G$, $\ell(g^{-1}) = \ell(g)$.*

3) *For all g and $h \in G$, $\ell(gh) \leq \ell(g) + \ell(h)$.*

Definition 2.6 *Let G be a finitely generated group, and let X be a finite symmetric generating set. The word-length function on G, induced by X, is given by*

$$\ell_X(g) = \min\{n\,;\, g = x_1 x_2 \ldots x_n\,,\, x_i \in X\}$$

It is well known that if X and Y are two finite symmetric generating sets of G, then there is a constant $\lambda > 1$ such that $\frac{1}{\lambda}\ell_Y \leq \ell_X \leq \lambda \ell_Y$. Thus the word-length functions are, in this sense, equivalent. As such, we will typically denote an arbitrary word-length function by ℓ_G without regard to the exact generating set.

A length function, ℓ, on the group G induces a metric on G by the formula $d_\ell(g, g') = \ell(g^{-1}g')$. Note that this metric is left-invariant under the action of G. If ℓ is a word-length function then we will usually denote the induced metric by d_G. This is the word-metric on G.

Let G be a discrete group equipped with length-function ℓ, and let $\mathcal{S}G = \{\phi : G \to \mathbb{C};\, \forall_{k \in \mathbb{Z}} \sum_{g \in G} |\phi(g)|(1 + \ell(g))^k < \infty\}$. The topology on $\mathcal{S}G$ is generated by the countable family of norms $\|\phi\|_k = \sum_{g \in G} |\phi(g)|(1 + \ell(g))^k$.

Lemma 2.7 *$\mathcal{S}G$ is closed under convolution.*

Proof Let ϕ and $\psi \in \mathcal{S}G$.

$$\begin{aligned}
\|\phi * \psi\|_k &= \sum_{g\in G} |(\phi * \psi)(g)|(1+\ell(g))^k \\
&= \sum_{g\in G} |\sum_{h\in G} \phi(h)\psi(h^{-1}g)|(1+\ell(g))^k \\
&\leq \sum_{g\in G}\sum_{h\in G} |\phi(h)\psi(h^{-1}g)|(1+\ell(g))^k \\
&\leq \sum_{g\in G}\sum_{h\in G} |\phi(h)\psi(h^{-1}g)|(1+\ell(h)+\ell(h^{-1}g))^k \\
&\leq \sum_{g\in G}\sum_{h\in G} |\phi(h)\psi(h^{-1}g)|(1+\ell(h))^k(1+\ell(h^{-1}g))^k \\
&\leq \sum_{h'\in G}\sum_{h\in G} |\phi(h)|(1+\ell(h))^k|\psi(h')|(1+\ell(h'))^k \\
&\leq \left(\sum_{h\in G} |\phi(h)|(1+\ell(h))^k\right)\left(\sum_{h'\in G} |\psi(h')|(1+\ell(h'))^k\right) \\
&\leq \|\phi\|_k\|\psi\|_k
\end{aligned}$$

■

$\mathcal{S}G$ is a Fréchet Algebra in the topology given by the family of norms. Let $\mathcal{A}$ be a complete, locally convex topological $\mathbb{C}$-algebra.

Definition 2.8 *A topological $\mathcal{A}$-module is a complete locally convex space, endowed with a jointly continuous $\mathcal{A}$-module structure.*

In the category of topological A-modules with continuous module homomorphisms, a module is projective if and only if it is a direct summand of a topological $\mathcal{A}$-module of the form $\mathcal{A}\hat{\otimes}_\pi E$, for E a complete locally convex space, and where $\hat{\otimes}_\pi$ is the complete projective topological tensor product [15], [4].

Definition 2.9 *The Polynomial Cohomology of G is given by*

$$HP^*(G;\mathbb{C}) = \mathrm{Ext}^*_{\mathcal{S}G}(\mathbb{C},\mathbb{C})$$

where this Ext *is taken over the topological category.*

When dealing with the cohomology of topological algebras, there is a question of whether to quotient out the image of the boundary map, or to quotient out the closure of the image of the boundary map. These correspond to the unreduced and reduced cohomologies respectively. In some sense, the reduced cohomology is 'topologically correct' while the unreduced theory is 'algebraically correct'. We have defined, and will work exclusively with, the unreduced theory.

Consider the following sequence:

$$\ldots \xrightarrow{\partial_3} \mathcal{S}G^{\hat{\otimes}^3_\pi} \xrightarrow{\partial_2} \mathcal{S}G\hat{\otimes}_\pi \mathcal{S}G \xrightarrow{\partial_1} \mathcal{S}G \xrightarrow{\epsilon} \mathbb{C} \to 0$$

where $\epsilon\phi = \sum_{g\in G}\phi(g)$ and

$$\partial_i(\phi_0\hat{\otimes}\ldots\hat{\otimes}\phi_i) = \sum_{j=0}^{i}(-1)^j\epsilon(\phi_j)\phi_0\hat{\otimes}\ldots\hat{\otimes}\phi_{j-1}\hat{\otimes}\phi_{j+1}\hat{\otimes}\ldots\hat{\otimes}\phi_i$$

This is a chain complex of projective $\mathcal{S}G$-modules. Let

$$s_i(\phi_0\hat{\otimes}\ldots\hat{\otimes}\phi_i) = \delta_e\hat{\otimes}\phi_0\hat{\otimes}\ldots\hat{\otimes}\phi_i$$

One readily verifies that this is a continuous homotopy, contracting the above complex. Thus this is a topological projective resolution of $\mathbb{C}$ over $\mathcal{S}G$. We call it the *Standard Resolution*.

The more standard approach to polynomial cohomology is as follows. Consider the usual cochain complex for calculating the group cohomology of a discrete group G.

$$C^0(G) \to C^1(G) \to C^2(G) \to C^3(G) \to \ldots$$

where $C^i(G)$ consists of all functions from G^i to $\mathbb{C}$. Consider $PC^i(G)$ consisting of those cochains in $C^i(G)$ which are polynomially bounded. That is, $PC^i(G)$ consists of those $\phi \in C^i(G)$ for which there exists a polynomial P such that $|\phi(g_1,\ldots,g_i)| \leq P(1+\ell_G(g_1)+\ldots+\ell_G(g_i))$.

$$PC^0(G) \to PC^1(G) \to PC^2(G) \to PC^3(G) \to \ldots$$

is then a subcomplex of the usual complex. The cohomology of this complex is what is typically referred to as the polynomial cohomology of G, with complex coefficients.

Lemma 2.10 *These two definitions of Polynomial Cohomology coincide as complex vector spaces.*

Proof In [5], Meyer proves that the cohomology obtained by using the bornological derived functor $\mathrm{Ext}^k_{\mathcal{S}G}(\mathbb{C}, \mathbb{C})$ is the same as the cohomology of the above $PC^*(G)$ cochain complex. We show in Chapter 6 that $\mathrm{Ext}^k_{\mathcal{S}G}(\mathbb{C}, \mathbb{C})$ over the bornological category is equal to $\mathrm{Ext}^k_{\mathcal{S}G}(\mathbb{C}, \mathbb{C})$ over the topological category, as $\mathcal{S}G$ and $\mathbb{C}$ are Fréchet spaces. ∎

3. BORNOLOGIES

3.1 Definitions

Definition 3.1 *Let A and B be subsets of a vector space V. A is circled if $\lambda A \subset A$ for all $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$. A is a disk if it is both circled and convex. A absorbs B if there is an $\alpha > 0$ such that $B \subset \lambda A$ for all $|\lambda| \geq \alpha$. A is absorbent if it absorbs every singleton in V. The circled (disked, convex) hull of A is the smallest circled (disked, convex) subset of V containing A. For an absorbent set A there is an associated gauge ρ_A on V given by $\rho_A(v) = \inf\{\alpha > 0; v \in \alpha A\}$. For an arbitrary subset A, denote V_A by the subspace of V spanned by A. For a disked set A, ρ_A is a semi-norm on V_A. A is a completant disk if V_A is a Banach space in the topology induced by ρ_A.*

Lemma 3.2 *[16] The circled hull of A is given by $\bigcup_{|\lambda|\leq 1} \lambda A$. The disked hull of A is the convex hull of the circled hull of A.*

Definition 3.3 *Let V be a locally convex topological vector space. A convex vector space bornology on V is a collection $\mathfrak{B}$ of subsets of V such that:*

1) For every $v \in V$, $\{v\} \in \mathfrak{B}$.

2) If $A \subset B$ and $B \in \mathfrak{B}$ then $A \in \mathfrak{B}$.

3) If A, $B \in \mathfrak{B}$ and $\lambda \in \mathbb{C}$, then $A + \lambda B \in \mathfrak{B}$.

4) Let $A \in \mathfrak{B}$, and B is the disked hull of A, then $B \in \mathfrak{B}$.

An $A \in \mathfrak{B}$ is said to be a bounded subset of V.

Definition 3.4 *Let $\mathfrak{B}$ be a convex vector space bornology for V. A base for the bornology is a subset $\mathcal{B} \subset \mathfrak{B}$ such that for all $A \in \mathfrak{B}$ there is $B \in \mathcal{B}$ with $A \subset B$. $\mathfrak{B}$ is complete if it has a base consisting of completant disks.*

If W is a bornological space and $V \subset W$, then there is a bornology on V induced from that on W in the following way. A subset A is bounded in V if and only if A is bounded as a subset of W. There is also a bornology on W/V induced from W. A subset $B \subset W/V$ is bounded if and only if there is a bounded $C \subset W$ which maps to B under the canonical projection $W \to W/V$. In this way there is a natural bornological structure on subspaces and quotients of bornological spaces.

Example 3.5 *Let V be a locally convex topological vector space. The fine bornology on V is defined as follows. A subset $A \subset V$ is bounded in the fine bornology if and only if it is contained in a compact subset of some finite dimensional subspace of V. If $\mathcal{B}$ is a base for this bornology, then let $\mathcal{B}'$ consist of the disked hulls of elements of $\mathcal{B}$. Then $\mathcal{B}'$ is also a base for this bornology, consisting of disks. The span of such a disk is a finite dimensional subspace of V. As any finite dimensional subspace of V is isomorphic to $\mathbb{C}^n$, for some $n \geq 0$, it is a Banach space. The fine bornology is then a complete convex vector space bornology.*

Example 3.6 *Let V be a locally convex topological vector space. A subset A of V is von Neumann bounded if A is absorbed by every neighborhood of $0 \in V$. The set of all von Neumann bounded set forms the von Neumann Bornology. We denote by $vN(V)$ the topological vector space endowed with this bornology.*

Example 3.7 *Let V be a locally convex topological vector space. A subset A of V is precompact if for all neighborhoods, N, of 0 there exists finitely many points $v_1, \ldots,$ $v_k \in V$ such that $A \subset \bigcup_{i=1}^{n}(v_i + N)$. The set of all precompact set forms a bornology called the Precompact Bornology. We denote by $Pt(V)$ the topological vector space endowed with the Precompact Bornology.*

Definition 3.8 *Let V and W be two bornological spaces. A map $V \to W$ is said to be bounded if the image of every bounded set in V is bounded in W. A bornological isomorphism is a bounded bijection with bounded inverse.*

Definition 3.9 *The collection of all bounded linear maps $V \to W$ is denoted by* $\mathrm{Hom}(V, W)$. *There is a canonical complete convex bornology on* $\mathrm{Hom}(V, W)$, *given by the families of equi-bounded functions. A family $\mathcal{U}$ is equi-bounded if for every bounded $A \subset V$, $\mathcal{U}[A]$ is bounded in W.*

Definition 3.10 *Let V and W be bornological vector spaces. The complete projective bornological tensor product $V\hat{\otimes}W$ is given by the following universal property: For any complete bornological vector space X, a jointly bounded linear map $V \times W \to X$ extends uniquely to a bounded map $V\hat{\otimes}W \to X$.*

Unlike the complete topological projective tensor product on the category of locally compact vector spaces, this bornological tensor product admits an adjoint.

Lemma 3.11 *[3]* $\mathrm{Hom}(V\hat{\otimes}W, M) \cong \mathrm{Hom}(V, \mathrm{Hom}(W, M))$

The following theorems show a useful interaction between the topological structure and the bornological structure on a nice category of spaces which will be useful in the following section.

Theorem 3.12 *[17] Let V and W be two Fréchet spaces. There is a bornological isomorphism $Pt(V\hat{\otimes}_\pi W) \cong Pt(V)\hat{\otimes}Pt(W)$.*

Theorem 3.13 *[17] Let V and W be two Fréchet spaces. Then there is an isomorphism between the continuous* $\mathrm{Hom}(V, W)$ *and the bounded* $\mathrm{Hom}(Pt(V), vN(W))$.

Definition 3.14 *Let V and W be two bornological vector spaces. The tensor product bornology on the algebraic tensor product $V \otimes W$ has, as a basis, the sets $A \otimes B$ with A bounded in V and B bounded in W.*

There is a notion, due to Meyer in [18], of the completion of a convex bornological vector space. As one would expect, the completed projective tensor product is related to the algebraic tensor product through this completion.

Lemma 3.15 *Let V and W be two bornological vector spaces. The completed projective bornological tensor product $V \hat{\otimes} W$ is the bornological completion of $V \otimes W$.*

Definition 3.16 *A sequence $\{v_i\}$ in a bornological vector space V is said to converge bornologically to v if there is a bounded subset A and a sequence $\{r_i\}$ of real numbers converging to 0, such that $v_i - v \in r_i A$ for all i.*

In a Fréchet space equipped with either the von Neumann bornology or the Precompact bornology, this is equivalent to the usual notion of topological convergence.

Definition 3.17 *Let V_i be a directed system of bornological vector spaces. As a set, the bornological direct limit V is given by the vector space direct limit. A set U is bounded in V if there is some i for which U is contained in and bounded in V_i.*

In what follows, we will be interested in Fréchet spaces. For a Fréchet space F, there is a countable directed family of seminorms, $\| \cdot \|_n$ yielding the topology. In this case, a set U is von Neumann bounded if $\|U\|_n < \infty$ for all n. This will be our bornology of choice on Fréchet spaces, due to the relation with topological constructs.

Definition 3.18 *A net in a bornological vector space F is a family $\mathcal{R}$ of disks of F, $e_{i_1,i_2,\ldots,i_k}$, with $k \in \mathbb{N}$, and $i_j \in \mathcal{I}$, for some countable index set $\mathcal{I}$, which satisfy the following conditions:*

1) *$F = \bigcup_{i \in \mathcal{I}} e_i$, and $e_{i_1,\ldots,i_k} = \bigcup_{i_{k+1} \in \mathcal{I}} e_{i_1,\ldots,i_k,i_{k+1}}$ for $k > 1$.*

2) *For every sequence (i_k) in $\mathcal{I}$, there is a sequence (ν_k) of positive reals such that for each $f_k \in e_{i_1,\ldots,i_k}$ and each $\mu_k \in [0, \nu_k]$, the series $\sum_{k=1}^{\infty} \mu_k f_k$ converges bornologically in F, and for each $k_0 \in \mathbb{N}$ $\sum_{k=k_0}^{\infty} \mu_k f_k \in e_{i_1,\ldots,i_{k_0}}$.*

3) *For every sequence (i_k) of elements of $\mathcal{I}$ and every sequence (λ_k) of positive real numbers, $\bigcup_{k=1}^{\infty} \lambda_k e_{i_1,\ldots,i_k}$ is bounded in F.*

As an example, Hogbe-Nlend shows that every bornological space with a countable base has a net.

Lemma 3.19 *Let F be a bornological vector space with a net, and let V be a subspace of F. Then V has a net.*

Proof Let $e_{i_1,\ldots,i_k}$, $i_j \in \mathcal{I}$, denote the net on F. Then $e'_{i_1,\ldots,i_k} = V \cap e_{i_1,\ldots,i_k}$ is a disk in V, satisfying the conditions. ■

Lemma 3.20 *Let F be a bornological vector space with a net, and let V be a subspace of F. Then F/V has a net.*

Proof Let $e_{i_1,\ldots,i_k}$, $i_j \in \mathcal{I}$, denote the net on F, and let $\pi : F \to F/V$ be the projection. Let $e'_{i_1,\ldots,i_k} = \pi e_{i_1,\ldots,i_k}$. That $e'_{i_1,\ldots,i_k}$ is a disk follows from the linearity of π. The properties of the net $e_{i_1,\ldots,i_k}$ on F, yield that $e'_{i_1,\ldots,i_k}$ is a net on F/V. ■

Lemma 3.21 *Let U be a Frèchet space. Then* $\mathrm{Hom}(U, \mathbb{C}^N)$ *has a net.*

Proof For each finite sequence of ordered triples of positive integers (n_1, M_1, K_1), $\ldots$, (n_k, M_k, K_k), define $b_{(n_1,M_1,K_1),\ldots,(n_k,M_k,K_k)}$ to be the set of all $f \in \mathrm{Hom}(U, \mathbb{C}^N)$ such that for all i between 1 and k, $|f(u)| < M_i$ for all $u \in U$ with $\|u\|_{U,n_i} < K_i$. In this case, Hogbe-Nlend shows that bounded homomorphisms are exactly the continuous homomorphisms, and equi-bounded families are precisely the equi-continuous families [16]. If W is an equi-bounded family, for all neighborhoods V of zero in $\mathbb{C}^N$, $W^{-1}[V] = \cap_{f \in W} f^{-1}(V)$ is a neighborhood of zero in U. That is, there exist $n_1, \ldots, n_k$ and $K_1, \ldots, K_k$ such that $\{u \in U \mid \text{ for all } 1 \le i \le k \; \|u\|_{U,n_i} < K_i\}$ is contained in $W^{-1}[U]$. For each $f \in W$ and each u in this set, $f(u) \in V$. Let V be the open ball of radius 1 in $\mathbb{C}^N$, and let $n_1, \ldots, n_k$ and $K_1, \ldots, K_k$ be the positive integers associated to V as above. Then $W \subset b_{(n_1,1,K_1),\ldots,(n_k,1,K_k)}$. Let $B = b_{(n_1,M_1,K_1),\ldots,(n_k,M_k,K_k)}$, $f \in B$, $M = \max M_i$, and let V be the open ball of radius R in $\mathbb{C}$. Then $V = \frac{R}{M} V'$ where V' is the open ball of radius M in $\mathbb{C}^N$. $f^{-1}(V) = \frac{R}{M} f^{-1}(V')$, so if $B^{-1}[V']$ is a neighborhood of zero in U, so is $B^{-1}[V]$. Let $u \in U$ be such that for all $1 \le i \le k$ we have $\|u\|_{U,n_i} < K_i$. $|f(u)| < M$, so $f(u) \in V'$. Let S be the set of all such $u \in U$. It is a neighborhood of zero. Moreover $f(S) \subset V'$ so that $S \subset f^{-1}(V')$, whence $S \subset B^{-1}[V']$. This implies that B is an equi-bounded family, so the bornology on $\mathrm{Hom}(U, \mathbb{C}^N)$ has a countable base. The result follows from our earlier remark. ■

Our main use for nets will be the following theorem from [16].

Theorem 3.22 *Let E and F be convex bornological spaces such that E is complete and F has a net. Every bounded linear bijection $v : F \to E$ is a bornological isomorphism.*

3.2 The Bornological Approach to Cohomology

Definition 3.23 *A bornological $\mathcal{A}$-module is a complete locally convex bornological space, along with a jointly bounded $\mathcal{A}$-module structure.*

In the category of bornological $\mathcal{A}$-modules and bounded module homomorphisms, an A-module is said to be bornologically projective if and only if it is a direct summand of a bornological $\mathcal{A}$-module of the form $\mathcal{A}\hat{\otimes}E$, for E a convex bornological vector space. With this in mind, this resolution is in fact a resolution of bornologically projective $\mathcal{S}G$-modules.

When calculating the usual complex group cohomology, the modules involved are algebraic vector spaces. As such we endow them with the fine bornology. The bar resolution of $\mathbb{C}$ over $\mathbb{C}G$ thus consists of free $\mathbb{C}G$-modules, in the fine bornology. Their algebraic freeness yields that they are bornologically free bornological $\mathbb{C}G$-modules. For V a vector space equipped with the fine bornology and any bornological vector space W, any homomorphisms $\phi : V \to W$ is a bounded morphism. This shows that $\mathrm{Ext}^k_{\mathbb{C}G}(\mathbb{C}, \mathbb{C})$ is the same, whether we are referring to the algebraic or the bornological functor.

Recall the standard resolution for calculating topological polynomial cohomology.

$$\ldots \xrightarrow{\partial_3} \mathcal{S}G^{\hat{\otimes}^3_\pi} \xrightarrow{\partial_2} \mathcal{S}G\hat{\otimes}_\pi\mathcal{S}G \xrightarrow{\partial_1} \mathcal{S}G \xrightarrow{\epsilon} \mathbb{C} \to 0$$

$\mathcal{S}G$ is a Fréchet space. Consider the following resolution of bornological vector spaces.

$$\ldots \xrightarrow{\partial_3} Pt(\mathcal{S}G^{\hat{\otimes}^3_\pi}) \xrightarrow{\partial_2} Pt(\mathcal{S}G\hat{\otimes}_\pi\mathcal{S}G) \xrightarrow{\partial_1} Pt(\mathcal{S}G) \xrightarrow{\epsilon} \mathbb{C} \to 0$$

By Theorem 3.12 this is the same as the resolution

$$\ldots \xrightarrow{\partial_3} Pt(\mathcal{S}G)^{\hat{\otimes}^3} \xrightarrow{\partial_2} Pt(\mathcal{S}G)\hat{\otimes}Pt(\mathcal{S}G) \xrightarrow{\partial_1} Pt(\mathcal{S}G) \xrightarrow{\epsilon} \mathbb{C} \to 0$$

As $vN(\mathbb{C}) = Pt(\mathbb{C})$, we see from the above bornological projective resolution the following.

Lemma 3.24 *The bornological and topological* $\mathrm{Ext}^k_{\mathcal{S}G}(\mathbb{C}, \mathbb{C})$ *groups coincide.*

This equivalence is the reason that we can approach polynomial cohomology, as defined topologically, in this bornological framework. This is the approach utilized by Meyer in [5].

Definition 3.25 *A group G is isocohomological for an $\mathcal{S}G$-module M if there is a bornological isomorphism $HP^*(G; M) \cong H^*(G; M)$. A groups is isocohomological if it is isocohomological for $\mathbb{C}$ with the trivial $\mathcal{S}G$ action.*

We note here that Meyer's notion of isocohomological is more subtle than what is expressed in our notion. His notion requires that a certain chain complex is contractible, which yields an isomorphism between cohomology and polynomial cohomology, but it also implies other things [3]. It is a stronger notion that what we mean by the term.

4. SPECTRAL SEQUENCES

A spectral sequence is, philosophically, a method for approximating cohomology. In general the cohomology of a group is difficult to calculate, but it may be possible to approximate it in terms of a normal subgroup and the corresponding quotient. This information can then help further refine the approximation. This is the point of a spectral sequence. We have a sequence of spaces which "converge" to the cohomology about which, we are trying to gain information.

4.1 Bornological Spectral Sequences

This section contains several results from McCleary's book, [19], translated into the bornological framework. The proofs given are almost entirely from McCleary, making changes only to verify that the algebraic isomorphisms involved are in fact isomorphisms of bornological spaces.

Let (A, d) be a differential graded bornological module. That is, $A = \bigoplus_{n=0}^{\infty} A^n$ is a graded bornological module, $d : A^n \to A^{n+1}$ is a degree 1 bounded linear map, with $d^2 = 0$. Let F be a filtration of A which is preserved by the differential, so that for all p, q we have $d(F^pA^q) \subset F^pA^{q+1}$. We also assume that the filtration is decreasing, in that $\ldots \subset F^{p+1}A^q \subset F^pA^q \subset F^{p-1}A^q \subset \ldots$. Such an (A, F, d) will be referred to as a filtered differential graded bornological module. The filtration F is said to be bounded if for each n, there is $s = s(n)$ and $t = t(n)$ such that

$$0 = F^sA^n \subset F^{s-1}A^n \subset \ldots \subset F^{t+1}A^n \subset F^tA^n = A^n$$

Lemma 4.1 *To every filtered differential graded bornological module (A, F, d), there is an associated spectral sequence of bornological modules $\{E_r^{*,*}, d_r\}$, $r = 1, 2, \ldots$, with d_r of bidegree $(r, 1-r)$ and $E_1^{p,q} \cong H^{p+q}(F^PA/F^{p+1}A)$. If the filtration is bounded, the spectral sequence converges to $H(A, d)$, $E_\infty^{p,q} \cong F^pH^{p+q}(A, d)/F^{p+1}H^{p+q}(A, d)$.*

Proof We begin by fixing some notation. Let

$$\begin{aligned}
Z_r^{p,q} &= F^pA^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1}) \\
B_r^{p,q} &= F^pA^{p+q} \cap d(F^{p-r}A^{p+q-1}) \\
Z_\infty^{p,q} &= F^pA^{p+q} \cap \ker d \\
B_\infty^{p,q} &= F^pA^{p+q} \cap \operatorname{im} d
\end{aligned}$$

where each of these subspaces are given the subspace bornology. Let d^n be the restriction of $d : A \to A$ to $d^n : A^n \to A^{n+1}$.

These definitions yield the following 'tower' of submodules:

$$B_0^{p,q} \subset B_1^{p,q} \subset \ldots \subset B_\infty^{p,q} \subset Z_\infty^{p,q} \subset \ldots \subset Z_1^{p,q} \subset Z_0^{p,q}$$

Moreover,

$$\begin{aligned}
d(Z_r^{p-r,q+r-1}) &= d(F^{p-r}A^{p+q+1} \cap d^{-1}(F^pA^{p+q}) \\
&= F^pA^{p+q+1} \cap d(F^{p-r}A^{p+q-1}) \\
&= B_r^{p,q}
\end{aligned}$$

From the boundedness condition on the filtration, if $r \geq \max\{s(p+q+1)-p, p-t(p+q-1)\}$ then $Z_r^{p,q} = Z_\infty^{p,q}$ and $B_r^{p,q} = B_\infty^{p,q}$. Thus the sequence will converge.

For $0 \leq r \leq \infty$, let $E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}}$ endowed with the quotient bornology. Let $\eta_r^{p,q} : Z_r^{p,q} \to E_r^{p,q}$ be the projection with kernel $Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$.

$d(Z_r^{p,q}) = B_r^{p+r,q-r+1} \subset Z_r^{p+r,q-r+1}$ and

$$\begin{aligned}
d(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) &= d(Z_{r-1}^{p+1,q-1}) + d(B_{r-1}^{p,q}) \\
&= B_{r-1}^{p+r,q-r+1} \\
&\subset Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q-r+1}
\end{aligned}$$

It follows that $d : Z_r^{p,q} \to Z_r^{p+r,q-r+1}$ induces a differential map $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that the following diagram commutes.

$$\begin{tikzcd}
Z_r^{p,q} \arrow[r, "d"] \arrow[d, "\eta_r^{p,q}"'] & Z_r^{p+r,q-r+1} \arrow[d, "\eta_r^{p+r,q-r+1}"] \\
E_r^{p,q} \arrow[r, "d_r"'] & E_r^{p+r,q-r+1}
\end{tikzcd}$$

It is clear that $d_r^2 = 0$. Let U be a bounded subset of $E_r^{p,q}$. Then there is a U' bounded in $Z_r^{p,q}$ with $U = \eta_r^{p,q}(U')$. As d is a bounded map, $d(U')$ is a bounded subset of $Z_r^{p+r,q-r+1}$, so $d_r(U) = \eta_r^{p+r,q-r+1}(d(U'))$ is a bounded subset of $E_r^{p+r,q-r+1}$. For all p and q, $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is a bounded map. $E_r = \bigoplus_{p,q} E_r^{p,q}$ so $d_r = \bigoplus_{p,q} d_r^{p,q}$. Let U be bounded in E_r. There exist $\mathcal{B}^{p,q}$, bounded in $E_r^{p,q}$, such that $U \subset \bigoplus_{p,q} \mathcal{B}^{p,q}$, with $\mathcal{B}^{p,q} = 0$ for all but finitely many pairs (p,q). Then $d_r(U) \subset \bigoplus_{p,q} d_r^{p,q}(\mathcal{B}^{p,q})$. Each of these $d_r^{p,q}$ are bounded maps so d_r is a bounded map.

$\ker d_r^{p,q} \subset E_r^{p,q}$, so $(\eta_r^{p,q})^{-1}(\ker d_r^{p,q})$ makes sense. $d_r^{p,q}\eta_r^{p,q} = \eta_r^{p+r,q-r+1}d$ implies $d_r^{p,q}(\eta_r^{p,q}z) = 0$ if and only if $dz \in Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q-r+1}$. This is true if and only if $z \in Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}$. Thus $(\eta_r^{p,q})^{-1}(\ker d_r^{p,q}) = Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}$. As $Z_{r-1}^{p+1,q-1} \subset \ker \eta_r^{p,q}$, $\ker d_r^{p,q} = \eta_r^{p,q}(Z_{r+1}^{p,q})$.

$\operatorname{im} d_r^{p-r,q+r-1} = \eta_r^{p,q}(d(Z_r^{p-r,q+r-1})) = \eta_r^{p,q}(B_r^{p,q})$ so that

$$\begin{aligned}
(\eta_r^{p,q})^{-1}(\operatorname{im} d_r^{p-r,q+r-1}) &= B_r^{p,q} + \ker \eta_r^{p,q} \\
&= B_r^{p,q} + B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1} \\
&= B_r^{p,q} + Z_{r-1}^{p+1,q-1}
\end{aligned}$$

as $B_{r-1}^{p,q} \subset B_r^{p,q}$. Moreover,

$$\begin{aligned}
Z_{r-1}^{p+1,q-1} \cap Z_{r+1}^{p,q} &= F^{p+1}A^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1}) \\
&\quad \cap F^pA^{p+q} \cap d^{-1}(F^{p+r+1}A^{p+q+1}) \\
&= F^{p+1}A^{p+q} \cap d^{-1}(F^{p+r+1}A^{p+q+1}) \\
&= Z_r^{p+1,q-1}
\end{aligned}$$

So

$$Z_{r+1}^{p,q} \cap (\eta_r^{p,q})^{-1}(\operatorname{im} d_r^{p-r,q+r-1}) = \left(B_r^{p,q} + Z_{r-1}^{p+1,q-1}\right) \cap Z_{r+1}^{p,q}$$

$$
\begin{aligned}
&= B_r^{p,q} + Z_{r-1}^{p+1,q-1} \cap Z_{r+1}^{p,q} \\
&= B_r^{p,q} + Z_r^{p+1,q-1}
\end{aligned}
$$

Let $\gamma : Z_{r+1}^{p,q} \to H^{p,q}(E_r^{*,*}, d_r)$ be the composite

$$Z_{r+1}^{p,q} \xrightarrow{\eta_r^{p,q}} \ker d_r^{p,q} \xrightarrow{\pi} H^{p,q}(E_r^{*,*}, d_r)$$

where π is the usual projection onto $H^{p,q}(E_r^{*,*}, d_r) = \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}$ As γ is the composition of two bounded maps, γ is itself a bounded map.

$\ker \gamma = Z_{r+1}^{p,q} \cap (\eta_r^{p,q})^{-1}(\operatorname{im} d_r^{p-r,q+r-1}) = B_r^{p,q} + Z_r^{p+1,q-1}$ so, at least algebraically, there is an isomorphism

$$\frac{Z_{r+1}^{p,q}}{B_r^{p,q} + Z_r^{p+1,q-1}} = E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$$

This isomorphism is given by $\gamma' : z + (B_r^{p,q} + Z_r^{p+1,q-1}) \mapsto \gamma(z) + (\operatorname{im} d_r^{p-r,q+r-1})$. To verify that this map is a bornological isomorphism we must check that it and its inverse are both bounded maps.

Let U be a bounded subset of $\frac{Z_{r+1}^{p,q}}{B_r^{p,q}+Z_r^{p+1,q-1}} = E_{r+1}^{p,q}$. There is a bounded subset U' of $Z_{r+1}^{p,q}$ such that $\eta_{r+1}^{p,q}(U') = U$. $\gamma'(U) = \eta_r^{p,q}(U') + (\operatorname{im} d_r^{p-r,q+r-1})$. $\eta_r^{p,q}$ is a bounded map so $\eta_r^{p,q}(U')$ is a bounded set in $\ker d_r^{p,q}$, and $\eta_r^{p,q}(U') + (\operatorname{im} d_r^{p-r,q+r-1})$ is bounded in $H^{p,q}(E_r^{*,*}, d_r)$. It follows that γ' is a bounded map.

Let $\phi : \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}} \to \frac{Z_{r+1}^{p,q}}{B_r^{p,q}+Z_r^{p+1,q-1}}$ be given by $z + (\operatorname{im} d_r^{p-r,q+r-1}) \mapsto (\eta_r^{p,q})^{-1}(z) \cap Z_{r+1}^{p,q} + (B_r^{p,q} + Z_r^{p+1,q-1})$. This is the algebraic inverse of γ'. Let U be a bounded subset of $\frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}$. There exists a bounded subset U' of $\ker d_r^{p,q}$ such that $U' + (\operatorname{im} d_r^{p-r,q+r-1})$ contains U in $\frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}$. U' is bounded in $E_r^{p,q}$, as $\ker d_r^{p,q} \subset E_r^{p,q}$, so there exists a bounded subset U'' of $Z_r^{p,q}$ with $U' = \eta_r^{p,q}(U'')$. Thus $U'' + B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1}$ is the full preimage of U' under $\eta_r^{p,q}$.

$$
\begin{aligned}
(\eta_r^{p,q})^{-1}(U') \cap Z_{r+1}^{p,q} &= U'' \cap Z_{r+1}^{p,q} + B_{r-1}^{p,q} \cap Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1} \cap Z_{r+1}^{p,q} \\
&= U'' \cap Z_{r+1}^{p,q} + B_{r-1}^{p,q} + Z_r^{p+1,q-1} \\
&\subset U'' \cap Z_{r+1}^{p,q} + B_r^{p,q} + Z_r^{p+1,q-1}
\end{aligned}
$$

Thus

$$\phi(U) \subset U'' \cap Z_{r+1}^{p,q} + (B_r^{p,q} + Z_r^{p+1,q-1})$$

in $\frac{Z^{p,q}_{r+1}}{B^{p,q}_r+Z^{p+1,q-1}_r}$. As $U'' \cap Z^{p,q}_{r+1}$ is bounded in $Z^{p,q}_{r+1}$, $\phi(U)$ is bounded in $\frac{Z^{p,q}_{r+1}}{B^{p,q}_r+Z^{p+1,q-1}_r}$, whence ϕ is a bounded map. Therefore we have a bornological isomorphism

$$E^{p,q}_{r+1} \cong H^{p,q}(E^{*,*}_r, d_r)$$

Thus this is indeed a bornological spectral sequence. It remains to identify the $E^{p,q}_1$ and $E^{p,q}_\infty$ terms.

From the above definitions $Z^{p+1,q-1}_{-1} = F^{p+1}A^{p+q}$, $B^{p,q}_{-1} = d(F^{p+1}A^{p+q-1})$, and $Z^{p,q}_0 = F^pA^{p+q} \cap d^{-1}(F^pA^{p+q+1})$.

$$\begin{aligned} E^{p,q}_0 &= \frac{Z^{p,q}_0}{Z^{p+1,q-1}_{-1} + B^{p,q}_{-1}} \\ &= \frac{F^pA^{p+q} \cap d^{-1}(F^pA^{p+q+1})}{F^{p+1}A^{p+q} + d(F^{p+1}A^{p+q-1})} \\ &= \frac{F^pA^{p+q}}{F^{p+1}A^{p+q}} \end{aligned}$$

$d^{p,q}_0 : E^{p,q}_0 \to E^{p,q+1}_0$ is induced by $d : F^pA^{p+q} \to F^pA^{p+q+1}$, fitting into the commutative diagram

$$\begin{tikzcd} F^pA^{p+q} \arrow[r, "d"] \arrow[d, "\pi"'] & F^pA^{p+q+1} \arrow[d, "\pi"] \\ E^{p,q}_0 = \dfrac{F^pA^{p+q}}{F^{p+1}A^{p+q}} \arrow[r, "d_0"'] & \dfrac{F^pA^{p+q+1}}{F^{p+1}A^{p+q+1}} = E^{p,q+1}_0 \end{tikzcd}$$

where π are the usual projections. It follows that $H^{p,q}(E^{*,*}_0, d_0)$ is the homology of the complex $(F^pA^*/F^{p+1}A^*, d_0)$, thus $H^{p,q}(E^{*,*}_0, d_0) = H^{p+q}(F^pA/F^{p+1}A)$. The $E^{p,q}_1$ term has been identified as $H^{p,q}(E^{*,*}_0, d_0)$ and we have a bornological isomorphism

$$E^{p,q}_1 \cong H^{p+q}(F^pA/F^{p+1}A)$$

Recall that if (A, F, d) is a filtered differential graded bornological module, then F induces a filtering on $H(A, d)$, given by $F^pH(A, d) = \mathrm{im}\{H(\mathit{inclusion}) : H(F^pA) \to$

$H(A)\}$. Let $\eta_\infty^{p,q} : Z_\infty^{p,q} \to E_\infty^{p,q}$ and $\pi : \ker d \to H(A,d)$ denote the canonical projections.

$$\begin{aligned} F^pH^{p+q}(A,d) &= H^{p+q}(\mathrm{im}(F^pA \to A), d) \\ &= \pi(F^pA^{p+q} \cap \ker d) \\ &= \pi(Z_\infty^{p,q}) \end{aligned}$$

$$\begin{aligned} \pi(\ker \eta_\infty^{p,q}) &= \pi(Z_\infty^{p+1,q-1} + B_\infty^{p,q}) \\ &= \pi(Z_\infty^{p+1,q-1}) \\ &= F^{p+1}H^{p+q}(A,d) \end{aligned}$$

π induces a map

$$d_\infty : E_\infty^{p,q} \to \frac{F^pH^{p+q}(A,d)}{F^{p+1}H^{p+q}(A,d)}$$

$$\begin{aligned} \ker d_\infty &= \eta_\infty^{p,q}(\pi^{-1}(F^{p+1}H^{p+q}(A,d) \cap Z_\infty^{p,q}) \\ &= \eta_\infty^{p,q}(\pi^{-1}((Z_\infty^{p+1,q-1} + d(A)) \cap Z_\infty^{p,q}) \\ &= \eta_\infty^{p,q}(\pi^{-1}(Z_\infty^{p+1,q-1} + B_\infty^{p,q}) \\ &= 0 \end{aligned}$$

Therefore d_∞ is injective, hence an algebraic isomorphism. We must show that d_∞ is a bornological isomorphism. As $\pi : \ker d \to H(A,d)$ is bounded and $\pi(Z_\infty^{p,q}) = F^pH^{p+q}(A,d)$, the restriction $\pi : Z_\infty^{p,q} \to F^pH^{p+q}(A,d)$ is a bounded surjection. Let U be a bounded subset of $E_\infty^{p,q}$. There is a bounded subset U' of $Z_\infty^{p,q}$ such that $\eta_\infty^{p,q}(U') = U$. As π is a bounded map, $\pi(U')$ is a bounded subset of $F^pH^{p+q}(A,d)$, thus $d_\infty(U) = \pi(U') + F^{p+1}H^{p+q}(A,d)$ is a bounded subset of $\frac{F^pH^{p+q}(A,d)}{F^{p+1}H^{p+q}(A,d)}$, whence d_∞ is a bounded map.

Consider the inverse map. Let

$$\phi : \frac{F^pH^{p+q}(A,d)}{F^{p+1}H^{p+q}(A,d)} \to \frac{Z_\infty^{p,q}}{Z_\infty^{p+1,q-1} + B_\infty^{p,q}}$$

be defined by $\phi : z + (F^{p+1}H^{p+q}(A,d)) \mapsto \pi^{-1}(z) \cap Z_\infty^{p,q} + (Z_\infty^{p+1,q-1} + B_\infty^{p,q})$. This is the inverse of d_∞. Let U be a bounded subset of $\frac{F^pH^{p+q}(A,d)}{F^{p+1}H^{p+q}(A,d)}$. There is a bounded U' subset of $F^pH^{p+q}(A,d)$ which maps to U under the canonical projection. As $F^pH^{p+q}(A,d)$ is contained in $H^{p+q}(A,d)$, U' is a bounded subset of $H^{p+q}(A,d)$. There exists a bounded subset U'' of $\ker d^{p+q}$ with $U' = U'' + (\operatorname{im} d^{p+q-1})$. As U' is a subset of $F^pH^{p+q}(A,d)$ and not just of $H^{p+q}(A,d)$, we can assume $U' \subset \ker d^{p+q} \cap F^pA^{p+q} + (\operatorname{im} d^{p+q-1})$.

$$\begin{aligned} U'' &\subset \ker d^{p+q} \cap F^pA^{p+q} + \operatorname{im} d^{p+q-1} \\ &= Z_\infty^{p,q} + B_\infty^{p,q} \end{aligned}$$

Therefore U'' is bounded in the subspace $Z_\infty^{p,q} + B_\infty^{p,q}$, and $\pi(U'') = U'' + (\operatorname{im} d^{p+q-1}) \supset U' = U + F^{p+1}H^{p+q}(A,d)$. Thus $\pi^{-1}(U) \subset U'' + \operatorname{im} d^{p+q-1}$.

$$\begin{aligned} \pi^{-1}(U) \cap Z_\infty^{p,q} &\subset U'' \cap Z_\infty^{p,q} + \operatorname{im} d^{p+q-1} \cap Z_\infty^{p,q} \\ &= U'' \cap Z_\infty^{p,q} + B_\infty^{p,q} \end{aligned}$$

So

$$\begin{aligned} \phi(U) &= \pi^{-1}(U) \cap Z_\infty^{p,q} + (Z_\infty^{p+1,q-1} + B_\infty^{p,q}) \\ &\subset U'' \cap Z_\infty^{p,q} + (Z_\infty^{p+1,q-1} + B_\infty^{p,q}) \end{aligned}$$

As U'' is bounded in $Z_\infty^{p,q} + B_\infty^{p,q}$, $U'' \cap Z_\infty^{p,q}$ is bounded in $Z_\infty^{p,q}$. Thus ϕ is a bounded map. ■

A double complex of bornological modules is a bigraded module $M = \bigoplus_{p,q} M^{p,q}$, where each $M^{p,q}$ is a bornological module, along with two bounded linear maps d' and d'' of bidegree $(1,0)$ and $(0,1)$ respectively. The total complex of the double complex $\{M^{*,*}, d', d''\}$ is the differential graded bornological module with $total(M)^n = \bigoplus_{p+q=n} M^{p,q}$ and bounded differential map $d = d' + d''$.

Lemma 4.2 *To a double complex $\{M^{*,*}, d', d''\}$ of bornological modules and bounded maps, there are two associated spectral sequences of bornological modules, $\{{}_IE_r^{*,*}, {}_Id_r\}$*

and $\{{}_{II}E_r^{*,*}, {}_{II}d_r\}$ *with* ${}_IE_2^{p,q} \cong H_I^{*,*}H_{II}(M)$ *and* ${}_{II}E_2^{p,q} \cong H_{II}^{*,*}H_I(M)$. *If* $M^{*,*}$ *is a first-quadrant double complex, then both spectral sequences converge, and their limit is* $H^*(total(M), d)$.

Proof Consider the following filtrations on $(total(M), d)$.

$$\begin{aligned} F_I^p(total(M))^t &= \bigoplus_{r\geq p} M^{r,t-r} \\ F_{II}^p(total(M))^t &= \bigoplus_{r\geq p} M^{t-r,r} \end{aligned}$$

F_I^* will be referred to as the column-wise filtration, and F_{II}^* will be referred to as the row-wise filtration. Both are decreasing filtrations, respected by the differential. As $M^{*,*}$ is first-quadrant, each of these filtrations are bounded, and by Lemma 4.1 these yield two spectral sequences of bornological modules converging to $H(total(M), d)$. In the case of F_I^p

$${}_IE_r^{p,q} = H^{p+q}\left(\frac{F_I^p(total(M))}{F_I^{p+1}(total(M))}, d\right)$$

The differential on $total(M)$ is given by $d = d' + d''$ so that $d'(F_I^p(total(M))) \subset F_I^{p+1}(total(M))$. There is a bornological isomorphism

$$\left(\frac{F_I^p(total(M))}{F_I^{p+1}(total(M))}\right)^{p+q} \cong M^{p,q}$$

with the induced differential d''. Thus ${}_IE_1^{p,q} \cong H_{II}^{p,q}(M)$.

Consider the following maps

$$\begin{aligned} &i : H^n(F^p) \to H^n(F^{p-1}) \\ &j : H^n(F^p) \to H^n(F^p/F^{p+1}) \\ &k : H^n(F^p/F^{p+1}) \to H^{n+1}(F^{p+1}) \\ &d_1 : H_{II}^{p,q}(M) \to H_{II}^{p+1,q}(M) \end{aligned}$$

where i is induced by the inclusion $F^{p-1} \to F^p$, j is induced by the quotient map $F^p \to F^p/F^{p+1}$, and k is the connecting homomorphism. It is clear that i and j are bounded. The k map sends $[x + F^{p+1}] \in H^n(F^p/F^{p+1})$ to $[dx] \in H^{n+1}(F^{p+1})$. If U

is a bounded subset of $H^n(F^p/F^{p+1})$ there is a bounded subset U' in the kernel of the boundary map $\partial : F^p/F^{p+1} \to F^{p+1}/F^{p+2}$ with $U' + (\operatorname{im}\partial) = U \in H^n(F^p/F^{p+1})$. There is U'' a bounded subset of F^p with $U' = U'' + F^{p+1} \in F^p/F^{p+1}$. As d is a bounded map, $d(U'')$ is a bounded subset of F^{p+1}. It follows that $[d(U'')]$ is bounded in $H^{n+1}(F^{p+1})$, and k is a bounded map.

A class in $H^{p+q}(F^p/F^{p+1})$ can be written as $[x + F^{p+1}]$, where $x \in F^p$ and $dx \in F^{p+1}$, or it can be written as a class, $[z] \in H^{p,q}_{II}(M)$, $z \in M^{p,q}$. k sends $[x + F^{p+1}]$ to $[dx] \in H^{p+q+1}(F^{p+1})$. Taking z as a representative, this determines $[d'z] \in H^{p+q+1}(F^{p+1})$, since $d''(z) = 0$. Thus $d'z$ can be considered as an element of $M^{p+1,q}$. j assigns a class in $H^{p+q+1}(F^{p+1})$ to its representative in $H^{p+q+1}(F^{p+1}/F^{p+2})$. This gives $d_1 = j \circ k$ as the induced mapping of d' on $H^{p,q}_{II}(M)$, so $d_1 = \bar{d}'$. Thus ${}_I E_2^{p,q} \cong H_I^{p,q} H_{II}^{*,*}(M)$. Symmetry gives ${}_{II} E_2^{p,q} \cong H_{II}^{p,q} H_I^{*,*}(M)$. ∎

5. THE LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCE

Let H and Q be finitely generated discrete groups with word-length functions ℓ_H and ℓ_Q respectively, and let

$$0 \to H \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 0$$

be an extension of Q by H, with word-length function ℓ. (We will consider H as a subgroup of G and in so doing, we will omit the ι when considering an $h \in H$ as an element of G.) Let $q \mapsto \overline{q}$ be a cross section of π. Note that if $\mathcal{Q}$ is the finite generating set for Q and $\mathcal{A}$ is the finite generating set for H, then as the generating set for G we will take the set of $h \in \mathcal{A}$ and $\overline{q}$ for $q \in \mathcal{Q}$. To this cross section assocaite a function $[\cdot,\cdot] : Q \times Q \to H$ via $\overline{q}_1\overline{q}_2 = \overline{q_1q_2}[q_1,q_2]$. This function is the factor set of the extension. The factor set has polynomial growth if there exists constants C and r such that $\ell_H([q_1,q_2]) \leq C((1+\ell_Q(q_1))(1+\ell_Q(q_2)))^r$. The cross section also determines an 'action' of Q on H given by $h^q = \overline{q}^{-1}h\overline{q}$. For nonabelian groups H, this need not be an actual group action, but we are following the terminology of Noskov. The action is polynomial if there exists constants C and r such that $\ell_H(h^q) \leq C\ell_H(h)(1+\ell_Q(q))^r$.

Definition 5.1 *An extension G of a finitely generated group Q by a finitely generated group H is said to be a polynomial extension if there is some cross section yielding a factor set of polynomial growth and inducing a polynomial action of Q on H.*

An important consequence of this definition is that the word-length function on H is polynomially equivalent to the word-length function on G restricted to H.

Lemma 5.2 *Let G be a polynomial extension of the finitely generated group Q by the finitely generated group H. Then there exists constants C and r such that for all $h \in H$ we have $\ell(h) \leq \ell_H(h) \leq C(1+\ell(h))^r$.*

Proof The right hand side of this inequality is Lemma 1.4 of [9]. The left hand side is obvious as the generators of G contain a subset generating H. ■

It follows that $\mathcal{S}_{\ell_H} H = \mathcal{S}_{\ell|_H} H$.

Lemma 5.3 *$\mathcal{S}G \cong \mathcal{S}H \hat{\otimes} \mathcal{S}G/H$ as bornological left $\mathcal{S}H$-modules, where H is endowed with the restricted length function and G/H is given the minimal length function, $\ell^*(gH) = \min_{h \in H} \ell(gh)$, where ℓ is the length function on G.*

Proof Let R be a set of minimal length representatives for right cosets. Let $r : G \to R$ be the map assigning to g, the representative of Hg. Each $g \in G$ has a unique representation as $g = h_g r(g)$, for $h_g \in H$ and $r(g) \in R$. There is an obvious equivalence between $\mathcal{S}G/H$ and $\mathcal{S}R$. Consider the map $\phi : \mathcal{S}G \to \mathcal{S}H \hat{\otimes} \mathcal{S}R$ given by $\phi(g) = (h_g) \hat{\otimes} (r(g))$. This is the desired bornological isomorphism. ■

Corollary 5.4 *A bornologically projective $\mathcal{S}G$-module is a bornologically projective $\mathcal{S}H$-module by restriction of the $\mathcal{S}G$-action to an $\mathcal{S}H$-action.*

Corollary 5.5 *Let M be an $\mathcal{S}G$-module. Then a bornologically projective $\mathcal{S}G$-module resolution of M is a bornologically projective $\mathcal{S}H$-module resolution of M, by restriction.*

Consider the following:

$$\ldots \xrightarrow{\delta} \mathcal{S}G^{\hat{\otimes} n} \xrightarrow{\delta} \mathcal{S}G^{\hat{\otimes} n-1} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathcal{S}G \hat{\otimes} \mathcal{S}G \xrightarrow{\delta} \mathcal{S}G \xrightarrow{\epsilon} \mathbb{C} \to 0$$

where $\delta : \mathcal{S}G^{\hat{\otimes} n} \to \mathcal{S}G^{\hat{\otimes} n-1}$ is the usual boundary map given by

$$\delta(g_1, \ldots, g_n) = \sum_{i=1}^{n} (-1)^i (x_1, \ldots, \widehat{x_i}, \ldots, x_n)$$

and extended by linearity, where the tuple $(g_1, \ldots, g_n)$ represents the elementary tensor $g_1 \hat{\otimes} \ldots \hat{\otimes} g_n$. δ is a bounded map. Moreover, it is readily verified that the map $s : \mathcal{S}G^{\hat{\otimes} n} \to \mathcal{S}G^{\hat{\otimes} n+1}$ defined by

$$s(g_1, \ldots, g_n) = (1_G, g_1, \ldots, g_n)$$

and extended by linearity, is a bounded contracting homotopy for this complex. It follows that this is a bornologically projective resolution of $\mathbb{C}$ over $\mathcal{S}G$. We call this the standard bornological resolution for the group G. By the above corollary, it is also a bornologically projective resolution of $\mathbb{C}$ over $\mathcal{S}H$.

Theorem 5.6 *Let $0 \to H \to G \to Q \to 0$ be a polynomial extension of groups, and let H be an isocohomological group. There is a bornological spectral sequence with $E_2^{p,q}$ term bornologically isomorphic to $HP^p(Q; HP^q(H;\mathbb{C}))$ which converges to $HP^*(G;\mathbb{C})$.*

Before we prove Theorem 5.6 we first present some corollaries.

Corollary 5.7 *Let $0 \to H \to G \to Q \to 0$ be a polynomial extension of groups, let Q be isocohomological with coefficients $HP^*(H;\mathbb{C})$, and let H be isocohomological. Then G is isocohomological.*

Proof Compare the polynomial cohomology spectral sequence with the usual spectral sequence for the group extension. The usual spectral sequence has E_2 term $H^p(Q; H^q(H;\mathbb{C}))$. Since Q and H are isocohomological we have $HP^p(Q; HP^q(H;\mathbb{C}))$ is bornologically isomorphic to $H^p(Q; H^q(H;\mathbb{C}))$, so the two spectral sequences have isomorphic E_2 terms, hence they have isomorphic limits. ∎

A group G acts on $\ell^2(G)$ via $(g \cdot f)(x) = f(g^{-1}x)$. This action extends by linearity to yield an action by $\mathbb{C}G$ on $\ell^2(G)$ by bounded operators. The completion of $\mathbb{C}G$ in $\mathcal{B}(\ell^2(G))$, the space of all bounded operators on $\ell^2(G)$ endowed with the operator norm, is defined to be the reduced group C^*-algebra, C_r^*G. Let $\mathcal{S}^2G$ be the set of all functions $f : G \to \mathbb{C}$ such that for all k, $\|f\|_{2,k} < \infty$. The group G is said to have the Rapid Decay property if $\mathcal{S}^2G \subset C_r^*G$, [2].

Theorem 5.8 *[9] Let G be a polynomial extension of the finitely generated group Q by the finitely generated group H. If H and Q have the Rapid Decay property, so does G.*

Corollary 5.9 *Let $0 \to H \to G \to Q \to 0$ be a polynomial group extension, let Q be isocohomological with coefficients $HP^*(H;\mathbb{C})$, and let H be isocohomological, both with the Rapid Decay property. Then G satisfies the Novikov conjecture.*

Proof By Corollary 5.7, G has cohomology of polynomial growth. By Theorem 5.8, G has the Rapid Decay property. The result follows by appealing to Connes-Moscovici. ■

5.1 Proof of Theorem 5.6

Throughout this section, we assume the hypotheses of Theorem 5.6. Let (P_*, d_P) be the standard bornological resolution for G, and let (R_*, d_R) be the bornological standard resolution for Q. As the P_q are bornological $\mathcal{S}G$-modules, they are by restriction, bornological $\mathcal{S}H$-modules. Q acts on $\mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C})$ via $(q\phi)(x) = \overline{q} \cdot \phi(\overline{q}^{-1}x)$, where $\bar{\cdot} : Q \to G$ is a cross-section satisfying the polynomial extension properties. This extends to a bornological $\mathcal{S}Q$-module structure on $\mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C})$.

Let $C^{p,q} = \mathrm{Hom}_{\mathcal{S}Q}(R_p, \mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C}))$. d_P and d_R induce maps $\delta_P : C^{p,q} \to C^{p,q+1}$ and $\delta_R : C^{p,q} \to C^{p+1,q}$ respectively, given by the following:

$$
\begin{aligned}
(\delta_P f)(r)(x) &= (-1)^p f(r)(d_P x) \\
(\delta_R f)(r)(x) &= f(d_R r)(x)
\end{aligned}
$$

Lemma 5.10 *With notation as above, δ_R and δ_P are bounded maps.*

Proof Let $f \in \mathrm{Hom}_{\mathcal{S}Q}(R_p, \mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C}))$. For every $r \in R_p$, we have by definition $(\delta_R f)(r) = f(d_R r)$. Let $S \subset C^{p,q}$ be an equibounded family, and let $U \subset R_p$ a bounded set, and consider $(\delta_R S)[U] = S[d_R U]$.

As d_R is a bounded map and U is a bounded set, $d_R U$ is a bounded subset of R_{p-1} and by equiboundedness $S[d_r U]$ is an equibounded family in $\mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C})$. Thus δ_R is a bounded map.

Similarly, $(\delta_P f)(r)(x) = (-1)^p f(r)(d_P x)$ for all $x \in P_q$. Let $S \subset C^{p,q}$ be an equibounded family. For all bounded $U \subset R_p$, $S[U]$ is an equibounded family in $\mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C})$. That is, for all bounded sets $V \subset P_q$, $S[U][V]$ is bounded in $\mathbb{C}$.

$(\delta_P S)[U][V] = S[U][d_P V]$. As d_P is a bounded map, we have that $(\delta_P S)[U][V]$ is bounded in $\mathbb{C}$ for all bounded U and V. Thus $\delta_P S$ is an equibounded family, so δ_P is a bounded map. ■

It is clear that $\delta_R^2 = \delta_P^2 = 0$. Moreover, $\delta_R\delta_P + \delta_P\delta_R = 0$ so that $C^{p,q}$ is a double complex of bornological vector spaces. Our goal is to apply the machinery of Chapter 4 to analyze the spectral sequence to which this double complex gives rise.

Consider the row-wise filtration on this complex. For a fixed q we have the complex

$$\ldots \xrightarrow{\delta_R} C^{*-1,q} \xrightarrow{\delta_R} C^{*,q} \xrightarrow{\delta_R} C^{*+1,q} \xrightarrow{\delta_R} \ldots$$

As R_p is a free $\mathcal{S}Q$-module, $R_p \cong \mathcal{S}Q \hat{\otimes} R'_p$ for some R'_p. Using the adjointness properties of the bornological projective tensor product

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C}) &\cong \mathrm{Hom}_{\mathcal{S}H}(\mathcal{S}G \hat{\otimes} A_q, \mathbb{C}) \\ &\cong \mathrm{Hom}(\mathcal{S}Q \hat{\otimes} A_q, \mathbb{C}) \end{aligned}$$

As $P_q \cong \mathcal{S}H \hat{\otimes} P'_q \cong \mathcal{S}G \hat{\otimes} A_q$,

$$\begin{aligned} C^{p,q} &\cong \mathrm{Hom}_{\mathcal{S}Q}(R_p, \mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C})) \\ &\cong \mathrm{Hom}_{\mathcal{S}Q}(R_p, \mathrm{Hom}(\mathcal{S}Q \hat{\otimes} A_q, \mathbb{C})) \\ &\cong \mathrm{Hom}(R_p \hat{\otimes} A_q, \mathbb{C}) \\ &\cong \mathrm{Hom}(R_p, \mathrm{Hom}(A_q, \mathbb{C})) \end{aligned}$$

The bounded homotopy for the complex R_* induces a contraction on $C^{*,q}$, so $E_1^{p,q} = 0$ for $p \geq 1$, while $E_1^{0,q} = \mathrm{Hom}_{\mathcal{S}Q}(\mathbb{C}, \mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C}))$.

Lemma 5.11 *Let F be a bornological $\mathcal{S}G$-module. There is a bornological isomorphism* $\mathrm{Hom}_{\mathcal{S}Q}(\mathbb{C}, \mathrm{Hom}_{\mathcal{S}H}(F, \mathbb{C})) \cong \mathrm{Hom}_{\mathcal{S}G}(F, \mathbb{C})$.

Proof Define a map $\Psi : \mathrm{Hom}_{\mathcal{S}Q}(\mathbb{C}, \mathrm{Hom}_{\mathcal{S}H}(F, \mathbb{C})) \to \mathrm{Hom}_{\mathcal{S}G}(F, \mathbb{C})$ by $\Psi(\xi)(f) = \xi(1)(f)$. Assume that g has the decomposition $h\overline{q}$. For $\xi \in \mathrm{Hom}_{\mathcal{S}Q}(\mathbb{C}, \mathrm{Hom}_{\mathcal{S}H}(F, \mathbb{C}))$

$$\begin{aligned}
\Psi(\xi)(gf) &= \xi(1)(h\overline{q}f) \\
&= h \cdot \xi(1)(\overline{q}f) \\
&= h\overline{q}\overline{q}^{-1} \cdot \xi(1)(\overline{q}f) \\
&= g \cdot \xi(\overline{q}^{-1} \cdot 1)(f) \\
&= g \cdot \xi(1)(f) \\
&= g \cdot \Psi(\xi)(f)
\end{aligned}$$

$\Psi(\xi)$ is $\mathcal{S}G$-equivariant. Let U be an equibounded family in $\mathrm{Hom}_{\mathcal{S}Q}(\mathbb{C}, \mathrm{Hom}_{\mathcal{S}H}(F, \mathbb{C}))$, so $U[1]$ is an equibounded family in $\mathrm{Hom}_{\mathcal{S}H}(F, \mathbb{C}) \subset \mathrm{Hom}(F, \mathbb{C})$. $\Psi(U) = U[1] \subset \mathrm{Hom}_{\mathcal{S}G}(F, \mathbb{C}) \subset \mathrm{Hom}(F, \mathbb{C})$ so Ψ is a bounded map.

Consider the map $\Phi : \mathrm{Hom}_{\mathcal{S}G}(F, \mathbb{C}) \to \mathrm{Hom}_{\mathcal{S}Q}(\mathbb{C}, \mathrm{Hom}_{\mathcal{S}H}(F, \mathbb{C}))$ given by the formula $\Phi(\varphi)(z)(f) = \varphi(zf)$, for all $\varphi \in \mathrm{Hom}_{\mathcal{S}G}(F, \mathbb{C})$, $z \in \mathbb{C}$, $q \in Q$, and $f \in F$.

$$\begin{aligned}
\Phi(\varphi)(qz)(f) &= \Phi(\varphi)(z)(f) \\
&= \varphi(zf) \\
&= \overline{q} \cdot \varphi(\overline{q}^{-1}zf) \\
&= \overline{q} \cdot \varphi(z\overline{q}^{-1}f) \\
&= \overline{q} \cdot \Phi(\varphi)(z)(\overline{q}^{-1}f)
\end{aligned}$$

It follows that $\Phi(\varphi)$ is $\mathcal{S}Q$-equivariant. Moreover for $h \in H$

$$\begin{aligned}
\Phi(\varphi)(z)(hf) &= \varphi(zhf) \\
&= \varphi(hzf) \\
&= h \cdot \varphi(zf) \\
&= h \cdot \Phi(\varphi)(z)(f)
\end{aligned}$$

So $\Phi(\varphi)(z)$ is $\mathcal{S}H$-equivariant. Let U be an equibounded family in $\mathrm{Hom}_{\mathcal{S}G}(F, \mathbb{C}) \subset \mathrm{Hom}(F, \mathbb{C})$, so for any bounded $V \subset F$, $U[V]$ is bounded in $\mathbb{C}$. Let B be a bounded subset of $\mathbb{C}$. Then $\Phi(U)[B][V] = U[BV]$. Since B is bounded in $\mathbb{C}$, and V is bounded in F, BV is bounded in F, so $U[BV]$ is bounded in M. It follows that Φ is a bounded map.

$$\begin{aligned}
\Phi\left(\Psi(\xi)\right)(z)(f) &= \Psi(\xi)(zf) \\
&= \xi(1)(zf) \\
&= z\xi(1)(f) \\
&= \xi(z)(f)
\end{aligned}$$

$$\begin{aligned}
\Psi\left(\Phi(\varphi)\right)(f) &= \Phi(\varphi)(1)(f) \\
&= \varphi(f)
\end{aligned}$$

So $\Phi\left(\Psi(\xi)\right) = \xi$ and $\Psi\left(\Phi(\varphi)\right) = \varphi$. Ψ and Φ are the desired bornological isomorphisms. ■

Applying this lemma, $E_1^{0,q} \cong \mathrm{Hom}_{\mathcal{S}G}(P_q, \mathbb{C})$. As P_* was a projective $\mathcal{S}G$-resolution of $\mathbb{C}$, the E_2 term is precisely $HP^*(G;\mathbb{C})$ and the sequence collapses here.

Consider the column-wise filtration on the double complex. For a fixed p we have the complex

$$\ldots \xrightarrow{\delta_P} C^{p,*-1} \xrightarrow{\delta_P} C^{p,*} \xrightarrow{\delta_P} C^{p,*+1} \xrightarrow{\delta_P} \ldots$$

Note that d_P induces a map $d_P^* : \mathrm{Hom}_{\mathcal{S}H}(P_q, \mathbb{C}) \to \mathrm{Hom}_{\mathcal{S}H}(P_{q+1}, \mathbb{C})$ which is given by $d_P^*(\varphi)(x) = \varphi(d_P x)$.

Lemma 5.12

$$\begin{aligned}
\ker \delta_P &= \mathrm{Hom}_{\mathcal{S}Q}(R_p, \ker d_P^*) \\
\mathrm{im}\, \delta_P &= \mathrm{Hom}_{\mathcal{S}Q}(R_p, \mathrm{im}\, d_P^*)
\end{aligned}$$

Proof If $\varphi \in \ker \delta_P$, for all $r \in R_p$, $(\delta_P \varphi)(r)(x) = 0$ for all $x \in P_*$. $(\delta_P\varphi)(r)(x) = (-1)^* \varphi(r)(d_P x)$, implies $\varphi(r) \in \ker d_P^*$ for all r, so $\ker \delta_P \subset \mathrm{Hom}_{\mathcal{S}Q}(R_p, \ker d_P^*)$. If $\xi \in \mathrm{Hom}_{\mathcal{S}Q}(R_p, \ker d_P^*)$, $d_P^* \xi(r) = 0$ for all $r \in R_p$. That is $\xi(r)(d_P x) = 0$ for all $x \in P_*$, and $\delta_P \xi$ is the zero map, establishing $\mathrm{Hom}_{\mathcal{S}Q}(R_p, \ker d_P^*) \subset \ker \delta_P$.

If $\varphi \in \operatorname{im} \delta_P$ there is a ϕ such that $\delta_P \phi = \varphi$. For all r and x, $\varphi(r)(x) = (-1)^p \phi(r)(d_P x) = (-1)^p d_P^*(\phi(r))(x)$. It follows that $\varphi(r) \in \operatorname{im} d_P^*$, so $\operatorname{im} \delta_P \subset \operatorname{Hom}_{\mathcal{S}Q}(R_p, \operatorname{im} d_P^*)$. If $\xi \in \operatorname{Hom}_{\mathcal{S}Q}(R_p, \operatorname{im} d_P^*)$, for all r there is a $\psi(r) \in \operatorname{Hom}_{\mathcal{S}H}(P_q, \mathbb{C})$ such that $\xi(r) = d_P^* \psi(r)$. That is, for all x, $\xi(r)(x) = \psi(r)(d_P x)$.

$$
\begin{aligned}
\psi(qr)(d_P x) &= \xi(qr)(x) \\
&= \overline{q} \cdot (\xi(r)(\overline{q}^{-1} x)) \\
&= \overline{q} \cdot (\psi(r)(d_P \overline{q}^{-1} x)) \\
&= \overline{q} \cdot (\psi(r)(\overline{q}^{-1} d_P x))
\end{aligned}
$$

It follows that ψ is an SQ-module map, so $\operatorname{Hom}_{\mathcal{S}Q}(R_p, \operatorname{im} d_P^*) \subset \operatorname{im} \delta_P$. ■

Lemma 5.13 $\operatorname{Hom}_{\mathcal{S}Q}(R_p, \frac{\ker d_P^*}{\operatorname{im} d_P^*}) \cong \frac{\operatorname{Hom}_{\mathcal{S}Q}(R_p, \ker d_P^*)}{\operatorname{Hom}_{\mathcal{S}Q}(R_p, \operatorname{im} d_P^*)}$ *as bornological vector spaces.*

Proof Consider the map $\Phi' : \operatorname{Hom}_{\mathcal{S}Q}(R_p, \ker d_P^*) \to \operatorname{Hom}_{\mathcal{S}Q}(R_p, \frac{\ker d_P^*}{\operatorname{im} d_P^*})$. The kernel of this map consists of exactly those maps whose image lies in the image of d_P^*, so $\ker \Phi' = \operatorname{Hom}_{\mathcal{S}Q}(R_p, \operatorname{im} d_P^*)$. Consider $\Phi' : \operatorname{Hom}(R_p', \ker d_P^*) \to \operatorname{Hom}(R_p', \frac{\ker d_P^*}{\operatorname{im} d_P^*})$, where R_p' is as above. $\frac{\ker d_P^*}{\operatorname{im} d_P^*} = HP^*(H; \mathbb{C})$ which, since H is isocohomological, is bornologically isomorphic to $H^*(H; \mathbb{C})$, an algebraic vector space endowed with the fine bornology. Thus the projection $\gamma : \ker d_P^* \to \frac{\ker d_P^*}{\operatorname{im} d_P^*}$ gives a bornological quotient $\gamma' : \ker d_P^* \to H^*(H; \mathbb{C})$. Let $\mathcal{V}$ be a basis for the algebraic vector space $H^*(H; \mathbb{C})$. For each $v \in \mathcal{V}$, let $T'(v)$ be an element of $\ker d_P^*$ such that $\gamma'(T'(v)) = v$, and extend by linearity. This gives a linear $T' : H^*(H; \mathbb{C}) \to \ker d_P^*$, which serves as a cross-section of γ'. As $H^*(H; \mathbb{C})$ is endowed with the fine bornology, T is a bounded map. This induces a bounded $T : \frac{\ker d_P^*}{\operatorname{im} d_P^*} \to \ker d_P^*$ For each $\xi \in \operatorname{Hom}(R_p', \frac{\ker d_P^*}{\operatorname{im} d_P^*})$, there is $T^*(\xi) \in \operatorname{Hom}(R_p', \ker d_P^*)$ given by $T^*(\xi)(r) = T(\xi(r))$. $\Phi'(T^*(\xi))(r) = \xi(r)$, so Φ' is surjective. Φ' then induces a bounded bijection $\Phi : \frac{\operatorname{Hom}(R_p', \ker d_P^*)}{\operatorname{Hom}(R_p', \operatorname{im} d_P^*)} \to \operatorname{Hom}(R_p', \frac{\ker d_P^*}{\operatorname{im} d_P^*})$.

$$
\begin{aligned}
\operatorname{Hom}(R_p', \ker d_P^*) &\cong \operatorname{Hom}_{\mathcal{S}Q}(R_p, \ker d_P^*) \\
&\subset \operatorname{Hom}_{\mathcal{S}Q}(R_p, \operatorname{Hom}_{\mathcal{S}H}(P_q, \mathbb{C})) \\
&\cong \operatorname{Hom}(R_p', \operatorname{Hom}(P_q', \mathbb{C}))
\end{aligned}
$$

$$\cong \quad \mathrm{Hom}(R'_p \hat{\otimes} P'_q, \mathbb{C})$$

As R'_p and P'_q are tensor products of Fréchet spaces, $R'_p \hat{\otimes} P'_q$ itself is a Fréchet space in its canonical bornology. By Lemma 3.21, Lemma 3.19, and Lemma 3.20 we have that $\frac{\mathrm{Hom}(R'_p, \ker d^*_P)}{\mathrm{Hom}(R'_p, \mathrm{im}\, d^*_P)}$ has a net. Moreover $\mathrm{Hom}(R'_p, \ker d^*_P)$ is a complete bornological space. Applying Theorem 3.22, the result follows. ■

Using the column-wise filtration on our double complex, we have that the E_1 term is bornologically isomorphic to $\mathrm{Hom}_{\mathcal{S}Q}(R_p, HP^q(H; \mathbb{C}))$. It follows that E_2 is isomorphic to $HP^p(Q; HP^q(H; \mathbb{C}))$. As this spectral sequence converges to the same sequence as that obtained from the row-wise filtration, it must convergence to $HP^*(G; \mathbb{C})$.

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