The Isocohomological Property

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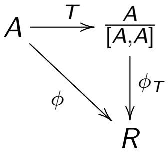
Traces

- $\phi : A \to R$ a trace, $\phi(ab) = \phi(ba)$.
- Universal trace $T : A \to \frac{A}{[A,A]}$.



What happens for $A = \mathbb{Z}G$?

Traces for $\mathbb{Z}G$

- G - finitely generated group.
- $$\frac{\mathbb{Z}G}{[\mathbb{Z}G, \mathbb{Z}G]} \cong \bigoplus_{x \in <G>} \mathbb{Z}$$
- For $x \in <G>$, $\pi_x : \bigoplus_{x \in <G>} \mathbb{Z} \to \mathbb{Z}$.
- Let P a finitely generated projective $\mathbb{Z}G$-module.
- There is Q with $P \oplus Q \cong (\mathbb{Z}G)^n$.
- $\mathrm{End}_{\mathbb{Z}G}(P \oplus Q) \cong M_{n \times n}(\mathbb{Z}G)$
- Id_P extends by zero to an element of $M_{n \times n}(\mathbb{Z}G)$.

Analogue Over $\mathbb{C}$

- Finitely generated projective $\mathbb{C}$-modules are finite dimensional $\mathbb{C}$-vector spaces.
- $P \oplus Q \cong \mathbb{C}^n$
- Id_P extends by zero to an element of $M_{n \times n}(\mathbb{C})$.
- Is the projection onto P.
- The trace of this projection is the rank of P, an integer.

Bass Conjecture

- $Tr : M_{n \times n}(\mathbb{Z}G) \to \frac{\mathbb{Z}G}{[\mathbb{Z}G, \mathbb{Z}G]}$.
- $Tr(Id_P)$ is an element of $\bigoplus_{x \in <G>} \mathbb{Z}$.
- For $g \in G$, the P-rank of g is the number

$$r_P(g) = \pi_{<g>} Tr(Id_P)$$

Conjecture (Classical Bass Conjecture, 1976)

For any finitely generated projective $\mathbb{Z}G$-module P, and any non-identity element $g \in G$, $r_P(g) = 0$.

Theorem (Linell)

The classical Bass conjecture holds for all non-identity torsion elements.

Hochschild Homology

Let A be a unital algebra, and $C_*(A)$ the chain complex

$$\ldots \xrightarrow{b} A^{\otimes 4} \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A \otimes A \xrightarrow{b} A$$

where $b : A^{\otimes n+1} \to A^{\otimes n}$ is given by

$$\begin{aligned} b(a_0, a_1, \ldots, a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\ &\quad + (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}) \end{aligned}$$

- Hochschild Homology: $HH_*(A)$.
- $HH_0(A) \cong \frac{A}{[A,A]}$.

Hattori-Stallings Trace

The map associating, to a finitely generated projective A-module P, $Tr(Id_P) \in HH_0(A)$ induces a homomorphism of abelian groups, the Hattori-Stallings trace.

$$Tr^{HS} : K_0(A) \to HH_0(A)$$

The Bass conjecture for $\mathbb{Z}G$ is giving the range of Tr^{HS}.

Cyclic Bicomplex

$CC_{*,*}(A)$ the first quadrant bicomplex

$$
\begin{array}{ccccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \longleftarrow & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & & b\downarrow & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \longleftarrow & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & -b'\downarrow & & b\downarrow & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \longleftarrow &
\end{array}
$$

Cyclic Homology

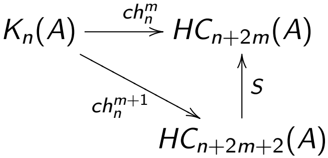
$$
\begin{aligned}
b'(a_0, a_1, \ldots, a_n) &= \sum_{i=0}^{n-1} (-1)^i \, (a_0, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\
b(a_0, a_1, \ldots, a_n) &= \sum_{i=0}^{n-1} (-1)^i \, (a_0, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\
&\quad + (-1)^n \, (a_n a_0, a_1, \ldots, a_{n-1}) \\
t(a_0, a_1, \ldots, a_n) &= (-1)^n \, (a_n, a_0, \ldots, a_{n-1}) \\
N &= 1 + t + t^2 + \ldots + t^n
\end{aligned}
$$

Cyclic Homology:

$$HC_*(A) = H_*(Tot(CC_{*,*}(A)))$$

Chern-Connes Characters

- $HC_0(A) \cong HH_0(A)$.
- Chern-Connes characters $ch_n^m : K_n(A) \to HC_{n+2m}(A)$.
- $ch_0^0 = Tr^{HS}$.
- $S \circ ch_n^m = ch_n^{m-1}$



Cyclic Homology of Group Algebras

Theorem (Burghelea)

$$HC_*(\mathbb{C}G) \cong \bigoplus_{x \in <G>} HC_*(\mathbb{C}G)_x$$

For $h \in x$, let G_h be the centralizer of h in G, $N_h = G_h/(h)$.

- For x elliptic (torsion), $HC_*(\mathbb{C}G)_x \cong H_*(N_h) \otimes HC_*(\mathbb{C})$.
- For x non-elliptic (non-torsion), $HC_*(\mathbb{C}G)_x \cong H_*(N_h)$.
- $S = \bigoplus_{x \in <G>} S_x$.
- 'Elliptic Summand' $HC_*(\mathbb{C}G)_{ell} = \bigoplus_{x\ ell} HC_*(\mathbb{C}G)_x$.
- 'Non-Elliptic Summand' $HC_*(\mathbb{C}G)_{non-ell} = \bigoplus_{x\ non-ell} HC_*(\mathbb{C}G)_x$.

Strong Bass Conjecture

Conjecture (Strong Bass Conjecture)

For each non-elliptic conjugacy class x, the image of the composition $\pi_x \circ ch_* : K_*(\mathbb{C}G) \to HC_*(\mathbb{C}G) \to HC_*(\mathbb{C}G)_x$ is zero.

$x \in < G >$ satisfies the 'nilpotency condition' if
$S_x : HC_*(\mathbb{C}G)_x \to HC_{*-2}(\mathbb{C}G)_x$ is nilpotent.

Observation (Eckmann, Ji)

Let x be a non-elliptic conjugacy class satisfying the nilpotency condition. The composition $K_*(\mathbb{C}G) \to HC_*(\mathbb{C}G) \stackrel{\pi_x}{\to} HC_*(\mathbb{C}G)_x$ is zero. In particular the Strong Bass Conjecture holds for G if each non-elliptic conjugacy class satisfies the nilpotency condition.

When is this nilpotency condition satisfied?

- $S_x : HC_*(\mathbb{C}G)_x \to HC_{*-2}(\mathbb{C}G)_x$.
- $S_x : H_*(N_h) \to H_{*-2}(N_h)$.
- The extension below determines a 2-cocycle of N_h.

$$0 \to (h) \to G_h \to N_h \to 0$$

- S_x acts as the cap product with this cocycle.
- If N_h has finite virtual cohomological dimension, G satisfies nilpotency condition.

Group Cohomology

Suppose G is a finitely generated discrete group.

- $C^n(G) = \{\phi : G^n \to \mathbb{C}\}$
- $d : C^n(G) \to C^{n+1}(G)$

$$(d\phi)(g_0, g_1, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i \phi(g_0, g_1, \ldots, \widehat{g_i}, \ldots, g_n)$$

$$0 \to C^0(G) \to C^1(G) \to \ldots$$

A usual cochain complex for calculating group cohomology, $H^*(G)$.

Accounting for Growth

Endow G with a word-length function ℓ_G.

- $\phi \in PC^n(G) \subset C^n(G)$ if there is a polynomial P such that

$$|\phi(g_1, \ldots, g_n)| \leq P(\ell_G(g_1) + \ell_G(g_2) + \ldots + \ell_G(g_n))$$

- $PC^*(G)$ forms a subcomplex of $C^*(G)$.
- $HP^n(G)$, the polynomial cohomology of G.
- $PC^*(G) \to C^*(G)$ induces a comparison map $HP^*(G) \to H^*(G)$.
- For many groups this map is an isomorphism.

With Coefficients

For a $\mathbb{C}G$-module V:

- $H^*(G; V) = \text{Ext}^*_{\mathbb{C}G}(\mathbb{C}, V)$.
- $$0 \leftarrow \mathbb{C} \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$$
- $$0 \rightarrow \text{Hom}_{\mathbb{C}G}(P_0, V) \rightarrow \text{Hom}_{\mathbb{C}G}(P_1, V) \rightarrow \text{Hom}_{\mathbb{C}G}(P_2, V) \rightarrow \dots$$
- $\text{Ext}^*_{\mathbb{C}G}(\mathbb{C}, V)$ is cohomology of this complex.

With Coefficients

$$\mathcal{S}G = \left\{ \phi : G \to \mathbb{C} \mid \ \forall_k \sum_{g \in G} |\phi(g)| \left(1 + \ell_G(g)\right)^k < \infty \right\}$$

Suppose V is a bornological $\mathcal{S}G$-module.

- $HP^*(G; V) = \mathrm{bExt}^*_{\mathcal{S}G}(\mathbb{C}, V)$.
- $$0 \leftarrow \mathbb{C} \rightleftarrows P_0 \rightleftarrows P_1 \rightleftarrows P_2 \rightleftarrows \ldots$$
- $$0 \to \mathrm{bHom}_{\mathcal{S}G}(P_0, V) \to \mathrm{bHom}_{\mathcal{S}G}(P_1, V) \to \mathrm{bHom}_{\mathcal{S}G}(P_2, V) \to \ldots$$
- $\mathbb{C}G \hookrightarrow \mathcal{S}G$ induces $HP^*(G; V) \to H^*(G; V)$ for all bornological $\mathcal{S}G$-modules V.

The Isocohomological Property

Definition

G has the (strong) isocohomological property if for all bornological $\mathcal{S}G$-modules V, the comparison map $HP^*(G;V) \to H^*(G;V)$ is an isomorphism. G is V-isocohomological, for a particular $\mathcal{S}G$-module V, if that particular comparison map is an isomorphism.

- Nilpotent groups (Ron Ji, Ralf Meyer)
- Combable groups (Crichton Ogle, Ralf Meyer)

Other Bounding Classes

$\mathcal{B} \subset \{\phi : [0, \infty) \to (0, \infty) | \phi$ is nondecreasing $\}$

- $1 \in \mathcal{B}$.
- If ϕ and $\phi' \in \mathcal{B}$, there is $\psi \in \mathcal{B}$ such that $\lambda\phi + \mu\phi' \leq \psi$, for nonnegative real λ, μ.
- If $\phi \in \mathcal{B}$ and g is a linear function, there is $\psi \in \mathcal{B}$ such that $\phi \circ g \leq \psi$.

Examples: $\mathbb{R}^+$, $\left\{e^f \mid f \text{ is linear } \right\}$.

ℓ^1 Bass Conjecture

- $\mathcal{S}G \hookrightarrow \ell^1 G$ induces $K_*(\mathcal{S}G) \cong K_*(\ell^1 G)$.
- $ch : K_*(\ell^1 G) \to HC_*(\ell^1 G)$ factors through $HC_*(\mathcal{S}G)$.

Conjecture (Strong ℓ^1-Bass Conjecture)

For each non-elliptic conjugacy class x, the image of the composition $\pi_x \circ ch_* : K_*(\mathcal{S}G) \to HC_*(\mathcal{S}G) \to HC_*(\mathcal{S}G)_x$ is zero.

Observation

Let x be a non-elliptic conjugacy class for which S_x^t is nilpotent. The composition $K_*(\mathcal{S}G) \to HC_*(\mathcal{S}G) \xrightarrow{\pi_x} HC_*(\mathcal{S}G)_x$ is zero.

ℓ^1 Bass Conjecture

Question

If S_x is nilpotent, when is S_x^t nilpotent?'

Definition

G satisfies a polynomial conjugacy problem if for each non-elliptic $x \in < G >$ there is P_x such that: $u, v \in x$ then there is $g \in G$ with $g^{-1}ug = v$ such that $\ell_G(g) \leq P_x(\ell_G(u) + \ell_G(v))$.

- Hyperbolic groups
- Mapping class groups

If G satisfies a polynomial conjugacy bound for a non-elliptic class x, $HC_*(\mathcal{S}G)_x \cong HP_*^{\ell_G}(N_h)$.
If in addition N_h isocohomological $S_x^t : HC_*(\mathcal{S}G)_x \to HC_{*-2}(\mathcal{S}G)_x$ is nilpotent, too.

Connes-Moscovici

Theorem (Connes-Moscovici, 1990)

Suppose G is a finitely generated discrete group endowed with word-length function ℓ_G. If G has the Rapid Decay property, and has cohomology of polynomial growth, then G satisfies the Strong Novikov Conjecture.

- $$\sum_{g \in G} |f(g)|^2 (1 + \ell_G(g))^{2k}$$
- $HP^*(G) \to H^*(G)$ surjective. 'epicohomological'

Theorem (Ogle preprint, 2010)

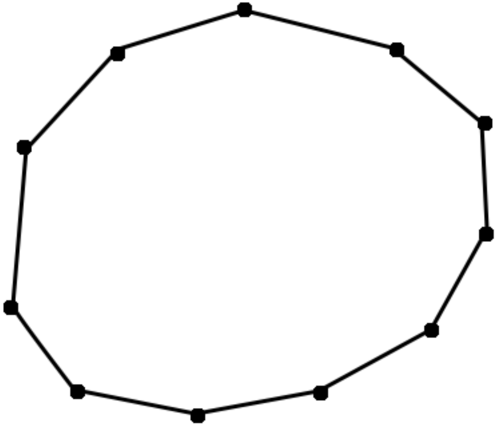
Suppose G is a finitely generated discrete group endowed with word-length function ℓ_G. If G is $\mathbb{C}$-'epicohomological', then G satisfies the Strong Novikov Conjecture.

HF^∞ Groups

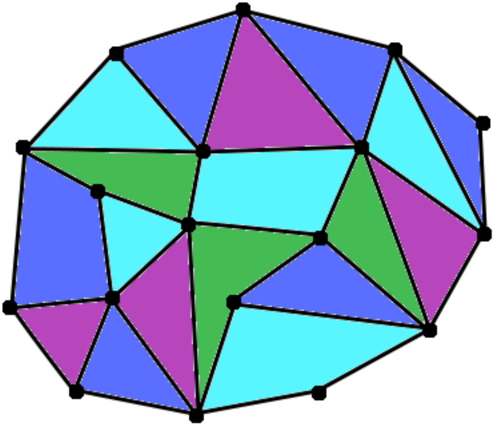
Definition

A group is of type HF^∞ if it has a classifying space the type of a polyhedral complex with finitely many cells in each dimension.

Dehn Functions



Dehn Functions



Weighted Dehn Functions

Suppose that X is a weakly contractible complex with fixed basepoint x_0.

- Define the weight of a vertex v to be $\ell_X(v) = d_{X^{(1)}}(v, x_0)$.
- Define the weight of a higher dimensional simplex to be the sum of the weights of its vertices.
- The weighted volume of an n-dimensional subcomplex is the sum of the weights of its n-dimensional cells.
- Get 'Weighted Dehn Functions' rather than just 'Dehn Functions'.

A Geometric Characterization

Theorem (Ji-R, 2009)

For an HF^∞ group G, the following are equivalent.

(1) *All higher Dehn functions of G are polynomially bounded.*

(2) *G is strongly isocohomological.*

(3) *$HP^*(G; V) \to H^*(G; V)$ is surjective for all $\mathcal{S}G$-modules V.*

(1) implies (2)

- Denote by X is the universal cover of the HF^∞ classifying space.
- $C_*(X)$ is a projective resolution of $\mathbb{C}$ over $\mathbb{C}G$.
- Length function on the vertices of X: $\ell_X(x) = d_X(x, *)$.
- Length function on $X^{(n)}$: $\ell_X(\sigma) = \sum_{v\in\sigma} \ell_X(v)$.
- $S_n(X)$ the completion of $C_n(X)$ under the family of norms given by

$$\|\phi\|_k = \sum_{\sigma \in X^{(n)}} |\phi(\sigma)| \left(1 + \ell_X(\sigma)\right)^k$$

- $S_*(X)$ a projective resolution of $\mathbb{C}$ over $\mathcal{S}G$. (The Dehn function bounds ensure a bounded contracting homotopy of $S_*(X)$)

(1) implies (2)

For each n there is finite dimensional W_n with

$$\begin{aligned} S_n(X) &\cong \mathcal{S}G\hat{\otimes}W_n \\ C_n(X) &\cong \mathbb{C}G\otimes W_n \end{aligned}$$

$$\begin{aligned} \mathrm{bHom}_{\mathcal{S}G}(S_n(X), V) &\cong \mathrm{bHom}_{\mathcal{S}G}(\mathcal{S}G\hat{\otimes}W_n, V) \\ &\cong \mathrm{Hom}(W_n, V) \\ &\cong \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G\otimes W_n, V) \\ &\cong \mathrm{Hom}_{\mathbb{C}G}(C_n(X), V) \end{aligned}$$

After applying $\mathrm{bHom}_{\mathcal{S}G}(\cdot, V)$ to $S_*(X)$ and $\mathrm{Hom}_{\mathbb{C}G}(\cdot, V)$ to $C_*(X)$ we obtain isomorphic cochain complexes.

the rest

- (2) implies (3) is obvious.
- (3) implies (1): This implication is similar to Mineyev's corresponding result on hyperbolic group and bounded cohomology.

Relative Group Cohomology

Let G be a finitely generated group and $\mathcal{H}$ a finite family of finitely generated subgroups.

- $\mathbb{C}G/\mathcal{H} = \bigoplus_{H \in \mathcal{H}} \mathbb{C}G/H$.
- $\varepsilon : \mathbb{C}G/\mathcal{H} \to \mathbb{C}$ defined by $\varepsilon(gH) = 1$.
- $\Delta = \ker \varepsilon$.

For a $\mathbb{C}G$-module V, the relative group cohomology is given by:

$$H^*(G, \mathcal{H}; V) = \mathrm{Ext}^{*-1}_{\mathbb{C}G}(\Delta, V)$$

Relative Classifying Spaces

Definition

G has relative type HF^∞ with respect to H if there is a polyhedral complex X which is a model of BG which contains BH and only finitely many cells not in BH, in each dimension.

- $EG = \widetilde{BG}$ with $p : EG \to BG$.
- $E\mathcal{H}$ the disjoint union of the connected components of $p^{-1}(BH)$.
- $EG//E\mathcal{H}$ obtained by collapsing each component in $E\mathcal{H}$ to a point.

$$\ldots \to C_3(EG//E\mathcal{H}) \to C_2(EG//E\mathcal{H}) \to C_1(EG//E\mathcal{H}) \twoheadrightarrow \Delta \to 0$$

Relative Group Cohomology

Applying $\mathrm{Ext}^*_{\mathbb{C}G}(\cdot, V)$ to the short-exact sequence

$$\Delta \hookrightarrow \mathbb{C}G/\mathcal{H} \twoheadrightarrow \mathbb{C}$$

gives a long-exact sequence

$$\ldots \to H^k(G; V) \to H^k(\mathcal{H}; V) \to H^{k+1}(G; \mathcal{H}; V) \to H^{k+1}(G; V) \to \ldots$$

Relative Polynomial Cohomology

$$w : \coprod_{H \in \mathcal{H}} G/H \to \mathbb{N}$$

- $\mathcal{S}G/H = \{f : G/H \to \mathbb{C} \mid \sum_{x \in G/H} |f(x)|\,(1 + w(x))^k < \infty\}$.
- $\mathcal{S}G/\mathcal{H} = \bigoplus_{H \in \mathcal{H}} \mathcal{S}G/H$.
- $\varepsilon : \mathcal{S}G/\mathcal{H} \to \mathbb{C}$ defined by $\varepsilon(gH) = 1$.
- $\mathcal{S}\Delta = \ker \varepsilon$.

For an $\mathcal{S}G$-module V, the relative polynomial cohomology is given by:

$$HP^*(G, \mathcal{H}; V) = \mathrm{bExt}^{*-1}_{\mathcal{S}G}(\mathcal{S}\Delta, V)$$

Relative Isocohomologicality

Definition

G has the (strong) relative isocohomological property with respect to $\mathcal{H}$ if for all bornological $\mathcal{S}G$-modules V, the relative comparison map $HP^*(G, \mathcal{H}; V) \to H^*(G, \mathcal{H}; V)$ is an isomorphism.

- Relatively hyperbolic groups.
- Groups acting cocompactly without inversion and with finite edge stabilizers on a contractible simplicial complex with polynomially bounded Dehn functions.
- Groups acting cocompactly without inversion on trees.

Relative Dehn Functions

Definition

The relative Dehn functions of G with respect to H are the Dehn functions of $EG//E\mathcal{H}$.

Theorem

Suppose the finitely generated group G is HF^∞ relative to the finite family of finitely generated subgroups $\mathcal{H}$, with BG of type DF relative to $\mathcal{H}$.

1. *The relative Dehn functions of G relative to $\mathcal{H}$ are each polynomially bounded.*
2. *G is strongly relatively isocohomological with respect to $\mathcal{H}$.*
3. *The comparison map $HP^*(G, \mathcal{H}; A) \to HP^*(G, \mathcal{H}; A)$ is surjective for all bornological $\mathcal{S}G$-modules A.*

A Long-Exact Sequence

Lemma

Let G be a group with length function L, H a subgroup of G equipped with the restricted length function. For any bornological $\mathcal{S}G$-module A, there is an isomorphism:

$$\mathrm{bExt}^*_{\mathcal{S}G}(\mathcal{S}G/H, A) \cong \mathrm{bExt}^*_{\mathcal{S}H}(\mathbb{C}, A)$$

Follows from an identification of $\mathcal{S}G \cong \mathcal{S}G/H \hat{\otimes} \mathcal{S}H$ as right $\mathcal{S}H$-modules.

A Long-Exact Sequence

Theorem

Let G and $\mathcal{H}$ be as above and let V be a bornological $\mathcal{S}G$-module. Equip each $H \in \mathcal{H}$ with the length restricted from G. There is a long exact sequence:

$$\ldots \to HP^k(G;V) \to HP^k(\mathcal{H};V) \to HP^{k+1}(G,\mathcal{H};V) \to HP^{k+1}(G;V) \to \ldots$$

Corollary

Let each $H \in \mathcal{H}$ be strongly isocohomological, and G strongly relatively isocohomological with respect to $\mathcal{H}$. Then G is strongly isocohomological.