GEOMETRIC RESULTS ON LINEAR ACTIONS OF REDUCTIVE LIE GROUPS FOR APPLICATIONS TO HOMOGENEOUS DYNAMICS

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Abstract. Several problems in number theory when reformulated in terms of homogenous dynamics involve study of limiting distributions of translates of algebraically defined measures on orbits of reductive groups. The general non-divergence and linearization techniques, in view of Ratner's measure classification for unipotent flows, reduce such problems to dynamical questions about linear actions of reductive groups on finite dimensional vectors spaces. This article provides general results which resolve these linear dynamical questions in terms of natural group theoretic or geometric conditions.

1. Introduction

Many questions in number theory that involve two interacting groups of symmetries contained in a Lie group can be addressed using methods of homogeneous dynamics. One such class of problems was proposed by Duke, Rudnick and Sarnak [DRS93], where one wants to study the density of integer points on affine varieties defined over $\mathbf{Q}$ which admit transitive action of a semisimple Lie group. In this case one symmetry group is an arithmetic lattice which preserves the integer points and the other being a reductive group, which is the stabilizer of a rational point in the variety. In such a situation, due to a finiteness result of Borel and Harish-Chandra, the question reduces to considering density of points on discrete orbits of lattices in semisimple Lie groups with reductive stabilizers. The approach suggested by [DRS93] involves understanding the limit distributions of translates of closed orbits of reductive subgroups on finite volume (or periodic) homogeneous spaces of semisimple groups by various sequences of elements of the semisimple group. Finding the precise algebraic relation between the translating sequences, the reductive subgroup, and the limiting distributions is a key to this method.

In [DRS93] it was shown that if H is a symmetric subgroup of a semisimple Lie groups G with an irreducible lattice Γ which also intersects H in a lattice, then for any sequence $y_n \to \infty$ in G modulo H, one has that the

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sequence $y_n\mu_H$ gets equidistributed in G/Γ (with respect to a G-invariant probability measure), where μ_H denotes the H-invariant probability measure on $H/H\cap\Gamma \hookrightarrow G/\Gamma$. The result was proved using deep results on spectral theory of $L^2(G/\Gamma)$. Later the same conclusion was obtained by Eskin and McMullen [EM93] using the mixing property of certain subgroup actions on homogeneous spaces.

In the general case, when H is not a symmetric subgroup, the limit distributions of the sequences $y_n\mu_H$ depend on algebraic or geometric relations between H and the sequence y_n. This relation was analyzed in fairly general situation by Eskin, Mozes and Shah [EMS97, EMS96], where a conditional answer to a question in [DRS93] on the density of integer points was obtained. The main steps of this technique are as follows: (1) find the condition under which the sequence $\{y_n\mu_H\}$ is pre-compact in the space of probability measures on G/Γ with respect to the weak* topology; (2) showing that any of the accumulation points, which are colloquially referred to as limiting measures, of this sequence of measures is invariant under a nontrivial unipotent subgroup; (3) apply Ratner's Theorem [Ra91] classifying ergodic invariant measures of unipotent flows to analyze the limiting measures using the linearization technique developed in [DS84, DM90, Sha91, DM93]. Due to Ratner's measure classification theorem, this type of linearization technique reduces all the three steps into geometric questions about linear representations. The purpose of this article is to develop new techniques and obtain results to answer these linear dynamical questions. These results are fundamental for further developments on the above program. For example, Richard and Zamojski [RZ16]) have vastly generalized and enhanced the core theorems of [EMS97, EMS96] using the main results of this article for non-arithmetic situations and also for S-arithmetic situations.

1.1. **Terminological conventions.** All Lie groups are assumed to be *finite dimensional real Lie groups*. By a connected *reductive subgroup H in a semisimple Lie group G*, we mean a closed and connected subgroup whose Lie algebra $\mathfrak{h}$ is a *reductive subalgebra* in the Lie algebra $\mathfrak{g}$ of G, according to [Bou60, §6 N^{0}6 Def. 5]. Namely we ask for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ to be semisimple. We record three equivalent criteria for H to be reductive in G:

(a) the radical of H does not contain non-trivial unipotent elements of G (cf. [Bou60, §6 N^{0}5 Theorem 4]) and hence H, having central radical, is reductive;
(b) every representation of H induced by a finite dimensional linear representation of G is semisimple [Bou60, §6 N^{0}6 Corollary 1 and §6 N^{0}2 Theorem 2];
(c) H is stable under at least one global Cartan involution of G (see [Mos55a] and section 2.1).

1.2. **Statements of the main results.** In this article we prove the following:

Theorem 1.1 (Linearized Non-divergence)**.** *Let G be a connected semisimple real Lie group and let H be a connected* reductive subgroup in G. *Let $Z_G(H)$ denote the centralizer of H inside G and $Z_G(H)^0$ its identity component.*

Then there exists a closed subset Y of G such that

1) *on the one hand we have*
$$G = Y \cdot Z_G(H)^0,$$
2) *on the other hand, given*
   (i) *a subset Ω of H with nonempty interior,*
   (ii) *a finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$,*
   (iii) *and a norm $\|-\|$ on V,*
   there exists a constant $c > 0$ such that

$$\forall y \in Y, \forall v \in V, \sup_{\omega\in\Omega} \|\rho(y \cdot \omega)(v)\| \geq c \cdot \|v\| . \tag{1.1}$$

This result generalizes [EMS97, Proposition 4.4], where H is assumed to be an algebraic torus. The Theorem 1.1 will be formulated for "reductive Lie groups" G and H in §5.

The next result complements the above theorem to provide a more complete picture.

Theorem 1.2 (Linearized Focusing)**.** *We consider the setup of Theorem* 1.1. *Let Y be a subset of G satisfying the conclusion 2) of Theorem* 1.1.

3) *Then given*
   (i) *a sequence $(y_n)_{n\in\mathbf{N}}$ in Y,*
   (ii) *a bounded subset Ω of H with nonempty interior,*
   (iii) *a finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$,*
   (iv) *a vector v of V,*
   we have the equivalence between the following properties
   (A) *the sequence $(y_n\Omega v)_{n\in\mathbf{N}}$ is uniformly bounded in V,*
   (B) *the sequence $(y_n)_{n\in\mathbf{N}}$ is bounded in G modulo the point-wise stabilizer of Ωv.*

1.2.1. *About Theorem* 1.1. Roughly Theorem 1.1 means that one cannot uniformly contract a piece $\Omega \cdot v$ of a H-orbit if one acts with an element y which is "orthogonal"[1] to the centralizer of H. The heuristic is the following: might v itself be contracted by y, the $y\Omega y^{-1}$ part in

$$y \cdot \Omega v = y\Omega y^{-1} \cdot yv \tag{1.2}$$

would be sufficiently expanded in some direction. For y in $Z_G(H)$, (1.2) yields $y{\cdot}\Omega v = \Omega{\cdot}yv$. Assuming that Ω is bounded, the inequality in (1.1) cannot hold with a uniform constant for y in $Z_G(H)$ and v in $V \smallsetminus \{0\}$, provided

[1] Cf. Theorem 4.2 below.

that $\|yv\|/\|v\|$ can take arbitrarily small values (equivalently, $\rho(Z_G(H))$ is not compact). As a result, the first condition of Theorem 1.1 is essentially optimal, for such an Ω.

Our proof uses a different and novel approach as compared to the original proof of [EMS97, Proposition 4.4], and hence it is applicable in greater generality. It also involves a general decomposition theorem of Mostow (Theorem 4.2 below), compatibility of Cartan decompositions under Lie algebra representations (Remark 2.1), and the convexity of the exponential function (Proposition 3.2).

Actually, we will prove Theorem 1.1 under weaker hypotheses on Ω (condition $(*)$ of Corollary 3.5), for an explicitly defined subset Y (see 2.2), and obtain an effective constant c (see equation (4.4)).

1.2.2. *About Theorem* 1.2. Let us spell out the terminology used. In property (A), the sequence of subsets $(y_n\Omega v)_{n\in\mathbf{N}}$ is said to be *uniformly bounded* if there is a compact subset of V containing simultaneously each of the $y_n\Omega v$; equivalently,

$$\text{for any norm on } V, \exists c\in\mathbf{R}, \forall n\in\mathbf{N}, \forall\omega\in\Omega, \|y_n\omega v\|<c. \tag{1.3}$$

Concerning the property (B), the *point-wise stabilizer* of Ωv denotes the subgroup

$$F=\{g\in G \mid \forall\omega\in\Omega, g\omega v=\omega v\} \tag{1.4}$$

of elements of G fixing point-wise each element of Ωv.

The following are notable differences with Theorem 1.1. We are interested in boundedness for the translated piece of orbit $y_n\Omega v$, instead of uniform convergence to 0; at the very end of the proof of Theorem 1.2 we deduces the uniform convergence from Theorem 1.1. The subset Ω is therefore required to be bounded (and here again, one can replace the nonempty interior condition by Zariski density if H is linear, and even weaker conditions, allowing Ω to be finite are possible). More importantly, one fixes a vector v of V. The focusing condition (B) depends greatly on v.

Remark 1.3. The p-adic analogue of Theorem 1.1 has been established by Richard [Ric09, *Exposé V*] by following up and extending some of the ideas of the current article through extensive work. Its S-adic analogue is deduced in [Ric09, *Exposé VI*]. The p-adic analogue of Theorem 1.2 follows from Remark 6.4, and the corresponding S-arithmetic analogue easily follows and is derived and used in [RZ16].

1.3. **Non-divergence of translated homogeneous measures.** The following main theorem of [EMS97] can be obtained from the general set up of [KM98] and Theorem 1.1 in a straightforward manner as explained below (cf. Corollary 1.6).

Theorem 1.4 (Eskin-Mozes-Shah [EMS97, Theorem 1.1])**.** *Let G be reductive real algebraic group defined over $\mathbf{Q}$ with no nontrivial $\mathbf{Q}$-characters, and*

$\Gamma \subset G(\mathbf{Q})$ be a lattice in G. Let H be a reductive subgroup defined over $\mathbf{Q}$ such that $Z(H)$ is $\mathbf{Q}$-anisotropic. Let μ_H denote the H-invariant probability measure on $H/H \cap \Gamma$. Then the collection of measures $\{g\mu_H : g \in G\}$ is pre-compact in the space of probability measures on G/Γ.

In [EMS97, Proposition 4.4] the Theorem 1.1 was proved for abelian H. As Theorem 1.1 was not available for semisimple groups H, Theorem 1.4 was first proved for semi-simple groups without compact factors using non-divergence of unipotent flows due to Dani and Margulis [DM90], and then for compact simple factors in a very indirect manner using a trick involving its complexification. Then another argument was required for combining the toral and semisimple parts of H. This method is somewhat artificial, and hence does not generalize in a natural way. Therefore Theorem 1.1 is an important missing component from [EMS97] from the point of view of having a natural method of proving non-divergence of translates of pieces orbits of reductive groups on homogeneous spaces.

It might be worthwhile to note that our proof is effective in most aspects. In particular, it makes it possible to quantitatively estimate the divergence of translated measures in the context of [EMS97].

Remark 1.5. We note that the S-arithmetic generalization of Theorem 1.4 has been obtained in [Ric09, Exp VI], by putting together [KT07], Theorem 1.1 and its p-adic analogue in [Ric09, Exp V] in a straightforward manner. This more natural approach allows some improvements on Theorem 1.4 above: the real Lie group H need not be defined over $\mathbf{Q}$, not even algebraic over $\mathbf{R}$, and only assume that $Z(H)$ projects to a pre-compact set in G/Γ.

1.4. **Equidistribution of translates and the focusing condition.** One of our goals is to analyze the limit distribution of $y_n\mu_H$ for a sequence $y_n \to \infty$ in Y as in Theorem 1.2. For this purpose we consider a small pre-compact nonempty open set $\Omega \subset H$ with its boundary having zero Haar measure of H, and consider the Haar measure of H restricted to Ω, normalize it and let μ_Ω be its pushforward on G/Γ. We want to understand the weak* accumulation points of the sequence of translated measures $\{y_n\mu_\Omega\}_{n\in\mathbf{N}}$. In view of non-divergence of unipotent trajectories after Margulis [Mar75] and Dani [Dan79, Dan84], and its generalisations in [DM91, EMS97, Sha96, KM98, KT07] one reduces to showing that the sequence of measures $\{y_n\mu_\Omega\}$ is pre-compact in the space of probability measures on G/Γ if and only if for certain finite dimensional representation V of G and any nonzero vector $p \in V$ with Γp discrete, the set $y_n\Omega\Gamma p$ avoids a fixed neighborhood of 0 for all n. This condition is precisely the conclusion of Theorem 1.1 as explained in 1.2.1. Therefore we pass to a subsequence of $\{y_n\}$ so that the sequence $\{y_n\mu_\Omega\}$ converges to a probability measure μ on G/Γ. Again using Theorem 1.1 and Theorem 1.2 for a specific representation, one can show that μ is invariant under a nontrivial unipotent subgroups of G. More precisely, one obtains

Corollary 1.6. *Let the notation be as in Theorem* 1.2*. Suppose further that* $\{y_n\}$ *has no convergent subsequence and* $\partial\Omega$ *admits null Haar measure of* H*, and consider the restriction of the Haar measure of* H *to* Ω*. Let* μ_Ω *denote its normalized pushforward on the image of* Ω *on* G/Γ*, where* Γ *is a lattice in* G*. Then after passing to a subsequence,* $y_n\mu_\Omega$ *converges to a probability measure* μ *on* G/Γ*, and* μ *is invariant under a nontrivial* Ad*-unipotent one-parameter subgroup of* G*.*

Then we can apply Ratner's classification [Ra91] of ergodic invariant measures for the unipotent group action to μ. Using the linearization techniques as developed in [DS84, DM90, Sha91, DM93, EMS96, Tom00], we can show that if for certain finite dimensional representation V and a discrete orbit of a nonzero $p \in V$ under Γ, the sets $y_n\Omega\Gamma p$ avoid any given large ball in V for all large n, then μ is invariant under G; that is, the sequence $\{y_n\mu_\Omega\}$ is equidistributed in G. Therefore if μ is not G-invariant then there exists a sequence $\{\gamma_n\} \subset \Gamma$ such that $\{y_n\Omega\gamma_n p\}$ is uniformly bounded. Because of Theorem 1.1, $\{\gamma_n p\}$ must be bounded, and since Γp is discrete, by passing to a subsequence, we have that $\gamma_n p = \gamma_1 p := v$ for all n. Now we can apply Theorem 1.2 to deduce that $\{y_n\}$ is bounded modulo the point-wise stabilizer of Ωv. Since Ω is Zariski dense in H, we have that $\{y_n\}$ is bounded in G modulo $\bigcap_{h\in H} hG_v h^{-1}$, where G_v being the stabilizer of v in G. This is a linear algebraic condition relating $\{y_n\}$, H, and $G_{\gamma_1 p}$. This condition is referred to by the *focusing* condition. The focusing condition is further analyzed by Richard and Zamojski [RZ16] to obtain very general results on limiting distributions of translates of measures.

1.4.1. *On applications in Arithmetic Geometry.* Various new arithmetical applications of Theorems 1.1 and 1.2 are obtained in [RZ16] by extending the above strategy to the S-arithmetic setting. This method would also strengthen some of the results proved in [GO11].

One type of application of this work is to counting problems, in a setup analogous to Manin-Peyre conjecture for homogeneous varieties G/H (cf. [GMO08] for the case of symmetric subgroups H). These problems can be translated to a homogeneous dynamical problem about translates of measures through the unfolding argument. The dynamical problem can be divided into sub-issues: non-divergence (see [Ric09, *Exposé VI*]); focusing (see [RZ16]); and volume computations ([EMS96, CLT10, GO11]).

Another type of application of the measure classification, via adelic mixing or equidistribution of Hecke points, was, in the thesis [Ric09] providing a conditional proof of an refinements of a conjecture of Pink: equidistribution of sequences of Galois orbits originating from a given Hodge-generic Hecke orbit. This was conditioned to a form of the Mumford-Tate conjecture.

The preprint [RZ16] is expected to allow an unconditional statement, though restricted to S-arithmetic Hecke orbits. Both the above applications involve Theorems 1.1 and 1.2 and their p-adic analogues.

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2. Preliminaries

We assume all Lie algebras to be real or complex, and finite dimensional.

Let us first recall some more or less well known facts on Cartan involutions and Cartan decompositions. This is for convenience and because no precise references were found for some of these facts; most importantly the criterion (c) of §1.1 and functoriality properties of the Killing form 2.1.10. In order to prove this criterion (c), we use a variant of [Mos55b, Theorem 6] in the general context of fully reducible linear Lie groups as in [Mos55a, Theorem 7.3]. This variant, although not stated, actually follows from proofs of [Mos55a]. Later will provide more precise definitions, as we do not assume our Lie groups to be *linear*; i.e. to admit a finite dimensional faithful representation. Our proof of Theorem 1.1 will only rely on criterion (c) of §1.1 (proved in 2.1.16), on construction 2.1.15, and on facts collected in Remark 2.1.

2.1. **Cartan involutions.** Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and denote its adjoint representation by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Recall that the *Killing form* on $\mathfrak{g}$ denotes the real bilinear form on $\mathfrak{g} \times \mathfrak{g}$ which sends (X, Y) to $B(X, Y) = \mathbf{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$ [Bou60, §3 $\mathrm{N}^0$6 Def. 4]. This form is obviously symmetric, and is nondegenerate if and only if $\mathfrak{g}$ is semisimple [Bou60, §6 $\mathrm{N}^0$1 Theorem 1].

2.1.1. If G is a *linear* real Lie group with Lie algebra $\mathfrak{g}$, then isotropic nonzero vectors of B are exactly the generators of (one dimensional and non trivial) Ad-unipotent subgroups; nonzero non-isotropic vectors of positive (resp. negative) norm are exactly the generators of noncompact one-parameter subgroups of semisimple elements (resp. one-parameter compact subgroup). The Killing form is *completely invariant* [Bou60, §3 $\mathrm{N}^0$6 Proposition 10]; equivalently, the image $\exp(\mathrm{ad}(\mathfrak{g}))$ of $\mathrm{ad}(\mathfrak{g})$ in $GL(\mathfrak{g})$ is contained in the orthogonal group of B. In particular, for any X and Y in $\mathfrak{g}$, $\exp(\mathrm{ad}(X))(Y)$ is isotropic (resp. positive, negative) if and only if Y is.

2.1.2. A *Cartan involution* of a real Lie algebra $\mathfrak{g}$ means an involution θ of the algebra $\mathfrak{g}$ such that the bilinear form $(X, Y) \mapsto B_\theta(X, Y) = -B(X, \theta(Y))$

is symmetric and strictly positive definite [Hel78, III 7]. In particular B is nondegenerate and $\mathfrak{g}$ is semisimple.

2.1.3. Note that if a linear subspace $\mathfrak{z}$ of $\mathfrak{g}$ is invariant under θ, then its orthogonal complements with respect to B and B_θ coincide. Consequently, as B_θ is anisotropic, $\mathfrak{z}$ and its orthogonal complement are supplementary. Moreover, θ stable subspace are stable under taking adjoint with respect to B_θ.

2.1.4. Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and a Cartan involution θ on $\mathfrak{g}$. Then for any $X \in \mathfrak{g}$, the negative of the adjoint with respect to B_θ of the endomorphism ad_X of $\mathfrak{g}$ equals $\mathrm{ad}_{\theta(X)}$.

Proof. We need to show that for any Y and Z in $\mathfrak{g}$,

$$B_\theta(\mathrm{ad}_X(Y), Z) = B_\theta(Y, -\mathrm{ad}_{\theta(X)} Z).$$

By definition of ad and B_θ this equality means

$$B(-[X,Y], \theta(Z)) = B(Y, \theta([\theta(X), Z])).$$

Now $B([X,Y],\theta(Z)) + B(Y,[X,\theta(Z)]) = 0$ follows from invariance of the Killing form, whereas the identity $\theta([\theta(X), Z]) = [X, \theta(Z)]$ follows from the fact that θ is an algebra involution. □

2.1.5. A *global Cartan involution* of a connected real Lie group G with Lie algebra $\mathfrak{g}$ is an involution $\Theta : G \to G$ whose differential at the neutral element is a Cartan involution of $\mathfrak{g}$. Every Cartan involution of $\mathfrak{g}$ extends to G.

Proof. We may assume G is semisimple, else there is no Cartan involution. According to [Bou72, §6 N^{0}2 Theorem 1], any Cartan involution θ has an extension $\widetilde{\Theta}$ to the universal cover $\widetilde{G}$ of G. It will be enough to show that $\widetilde{\Theta}$ fixes element-wise the center $Z(\widetilde{G})$ of $\widetilde{G}$. As $Z(\widetilde{G})$ is a characteristic subgroup, it is stable under automorphisms. Hence $\widetilde{\Theta}$ descends to an involution Θ^{ad} of the adjoint group $G^{\mathrm{ad}} = \widetilde{G}/Z(\widetilde{G})$, which is linear. It will be enough to show that the induced action of $\widetilde{\Theta}$ on the fundamental group $\pi_1(G^{\mathrm{ad}})$ is trivial. But, from Cartan decomposition [Mos55a, Theorem 3.2], G^{ad} retracts to any of its maximal compact subgroups, and one such a maximal compact subgroup is the fixed locus of Θ^{ad}. □

2.1.6. We will say that a closed subgroup of a *connected semisimple* Lie group G is *projectively compact* if its image under the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is compact. When G is *linear*, this property is equivalent to compactness. In general this property is stable under direct image by morphisms of semisimple connected Lie groups, and inverse images by isogenies. When G is connected, semisimple and *linear*, the set of fixed point of a given global Cartan involution defines a maximal compact subgroup (recall that G is connected). If G is only assumed to be connected and semisimple, writing G as a central covering of its adjoint group, we deduce that the set of fixed point of a given global Cartan involution defines a maximal projectively compact subgroup.

2.1.7. A semisimple real Lie algebra is said to be *compact* if its Killing form is totally negative. One actually needs only to ask the Killing form to be *anisotropic*.

2.1.8. Every semisimple Lie algebra $\mathfrak{g}$ is a direct product of its simple ideals [Bou60, §6 N^{0}2 Corollary 1], and these simple ideals are pairwise orthogonal for the Killing form [Bou60, §6 N^{0}1 Corollary 1]. For any Cartan involution θ of $\mathfrak{g}$, one has $-B(X, \theta(X)) > 0$ whenever $X \neq 0$. Consequently the image $\theta(X)$ of a non zero element X of $\mathfrak{g}$ cannot be orthogonal to X with respect to the Killing form. *A fortiori* θ cannot send any simple ideal of $\mathfrak{g}$ to an orthogonal ideal. Consequently, a Cartan involution of $\mathfrak{g}$ stabilizes each simple ideal of $\mathfrak{g}$. As ideals of $\mathfrak{g}$ are sum of simple ideals [Bou60, §6 N^{0}2 Corollary 1], a Cartan involution of $\mathfrak{g}$ stabilizes each ideal of $\mathfrak{g}$.

2.1.9. According to [Bou60, §3 N^{0}6 Proposition 9], the restriction of the Killing form of $\mathfrak{g}$ to an ideal $\mathfrak{a}$ is the Killing form of $\mathfrak{a}$. It follows that the restriction of a Cartan involution to an ideal is a Cartan involution, and that an endomorphism of $\mathfrak{g}$ is a Cartan involution if and only if it stabilizes each simple ideal and its restriction to each simple ideal is a Cartan involution. These restrictions being nondegenerate, a simple ideal does not intersect its orthogonal: the orthogonal of a simple ideal, and more generally of any ideal $\mathfrak{a}$, is the sum of simple ideals not contained in $\mathfrak{a}$.

2.1.10. Consider a semisimple Lie algebra $\mathfrak{g}$ and a morphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{g}'$. Its kernel $\ker(\phi)$ is an ideal whose orthogonal $\ker(\phi)^{\perp}$ is a supplementary ideal; ϕ sends $\ker(\phi)^{\perp}$ bijectively onto $\phi(\mathfrak{g})$. In particular $\phi(\mathfrak{g})$ is a semisimple Lie algebra; any Cartan involution of $\mathfrak{g}$ stabilizes $\ker(\phi)$; the induced involution on $\phi(\mathfrak{g})$ is a Cartan involution. We will call the latter the *image Cartan involution*.

2.1.11. Given a Cartan involution θ on a semisimple Lie algebra $\mathfrak{g}$, the associated *Cartan decomposition* denotes the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ as a direct sum of the eigenspaces $\mathfrak{k}$ (resp. $\mathfrak{p}$) of θ associated with the eigenvalue $+1$ (resp.-1). Clearly Cartan involutions and associated Cartan decompositions determine each other. By definition θ is self-adjoint with respect to B_θ, hence the eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal complements of each other. Consequently, the Cartan involution θ is determined by $\mathfrak{k}$ only (knowing B). Note that $\mathfrak{k}$ is a maximal negative anisotropic subspace of B and a subalgebra of $\mathfrak{g}$ (it satisfies Frobenius integrability condition), and that $\mathfrak{p}$ is maximal positive anisotropic linear subspace and is $\mathrm{ad}(\mathfrak{k})$-invariant.

2.1.12. Let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, the complexified Lie algebra obtained from $\mathfrak{g}$ by extending the base field. According to [Hel78, III. Proposition 7.4], every *Cartan decomposition* of $\mathfrak{g}$, as defined in [Hel78, p. 183] actually comes from a Cartan involution, as defined above, and these Cartan decompositions are exactly the restriction to $\mathfrak{g}$ of the Cartan decompositions of the real Lie algebra $\mathfrak{g}_{\mathbf{C}}$ which are invariant (factor-wise) under complex conjugation. The anisotropic subalgebra of $\mathfrak{g}_{\mathbf{C}}$ corresponding to the latter are the *compact* (see 2.1.7) real forms of the complex Lie algebra $\mathfrak{g}_{\mathbf{C}}$ which are *invariant* under the complex conjugation relative to the real structure induced by $\mathfrak{g}$ (see also [Mos55a, section 2]).

2.1.13. For a *reductive* lie algebra $\mathfrak{g}$, *together with* an embedding $\mathfrak{g} \to \mathfrak{gl}(V)$, for some V of finite dimension, a *real form* $\mathfrak{k}$ of $\mathfrak{g}_{\mathbf{C}}$ is said to be *compact* if $\exp(\mathfrak{k})$ is compact in $\mathrm{GL}(V \otimes \mathbf{C})$, and *invariant* if $\mathfrak{k}$ is invariant under the complex conjugation on $\mathfrak{g}_{\mathbf{C}}$ relative to $\mathfrak{g}$. According to [Mos55a, Lemma 6.2], there exists such a compact form if and only if the action of $\mathfrak{g}$ on V is semisimple (the "only if" part is known as "Weyl's unitary trick"). In such a case, we will say that $\mathfrak{g}$ together with its embedding is a *linear fully reducible* subalgebra of $\mathfrak{gl}(V)$.

2.1.14. The preceding points imply that invariant compact forms generalizes to linear fully reducible Lie algebra the Cartan decompositions of semisimple Lie algebras. By using [Mos55a, Theorem 4.1] as in proof of [Mos55a, Theorem 5.1[1]], given a nested sequence of linear fully reducible subalgebras of $\mathfrak{gl}(V)$, one can form a nested sequence of invariant compact real forms of the corresponding complexified linear algebras.

2.1.15. Consider now a finite dimensional linear representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of a semisimple Lie algebra. Then, given a Cartan involution of $\mathfrak{g}$ we get a Cartan involution of $\rho(\mathfrak{g})$, by 2.1.10. This Cartan involution corresponds to an invariant compact real form on $\rho(\mathfrak{g})_{\mathbf{C}}$, by 2.1.12, which we can extend to $\mathfrak{gl}(V)$, by 2.1.14.

Any such extension is the unitary group of a euclidean structure on V, unique up to proportionality. Now $\rho(\mathfrak{g})$ is stable under taking adjoint with respect to the euclidean structure on V [Mos55a, proof of Theorem 5.1], and such that the euclidean adjunction extends the negative of the image Cartan involution 2.1.10 on $\rho(\mathfrak{g})$ (*loc. cit.*, equation (2)).

2.1.16. Criterion (c) of §1.1 clearly follows from the corresponding statement at the level of Lie algebras, which we now prove. Namely *a real subalgebra $\mathfrak{h}$ of a (finite dimensional) real semisimple Lie algebra $\mathfrak{g}$ is invariant under some Cartan involution of $\mathfrak{g}$ if and only if the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is fully reducible.*

Proof. Consider a subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is invariant under θ, its image under $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is stable under taking adjoint with respect to B_θ, according to 2.1.4. As B_θ is *anisotropic*, the orthogonal complement of a $\mathrm{ad}_{\mathfrak{h}}$-stable subspace defines a *supplementary* $\mathrm{ad}_{\mathfrak{h}}$-stable subspace. It implies that $\mathfrak{h}$ acts fully reducibly on $\mathfrak{g}$.

Assume now that $\mathfrak{h}$ acts fully reducibly on $\mathfrak{g}$. We shall denote the extension of scalars by subscripts. Then the linear subalgebra $\mathrm{ad}(\mathfrak{h})_{\mathbf{C}}$ of $\mathfrak{gl}(\mathfrak{g})_{\mathbf{C}}$ has an invariant compact real form 2.1.13. This real form is contained in an invariant compact real form of $\mathrm{ad}(\mathfrak{g})_{\mathbf{C}}$ 2.1.14. The latter is associated with a Cartan involution θ of $\mathfrak{g}$ 2.1.12. Applying to $\mathfrak{h}$ the Lie algebra analogue of decomposition (2) in proof of Theorem 5.1 in [Mos55a], we see that $\mathfrak{h}$ is invariant under θ, each factor being contained in a factor of the corresponding Cartan decomposition of $\mathfrak{g}$. $\square$

2.2. **Notational conventions.** In the next sections we will often consider the following situation. Let us fix, once for all, our notations.

2.2.1. *General notations.* Let G be a connected semisimple Lie group, let H be a connected reductive Lie subgroup in G. According to criterion (c) of §1.1, let Θ be a global Cartan involution (cf. 2.1.2, 2.1.5) of G under which H is invariant. We denote by $Z_G(H)$ the centralizer of H in G, and by K the maximal projectively compact subgroup consisting of fixed points of Θ (cf. 2.1.6).

Denote by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{z}_{\mathfrak{g}}$, $\mathfrak{k}$ the Lie algebra of G, H, $Z_G(H)$, and K respectively, and denote by θ the differential of Θ at the identity element.

We write $\mathfrak{k}^{\perp}$ and ${\mathfrak{z}_{\mathfrak{g}}}^{\perp}$ for the orthogonal complements, with respect to the Killing form (cf. 2.1) on $\mathfrak{g}$, to $\mathfrak{k}$ and $\mathfrak{z}_{\mathfrak{g}}$ respectively. We define $\mathfrak{p} = \mathfrak{k}^{\perp} \cap {\mathfrak{z}_{\mathfrak{g}}}^{\perp}$, $P = \exp_G(\mathfrak{p})$ and $Y = K \cdot P$. Finally, $B_\theta : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ will be the strictly positive definite symmetric bilinear form on $\mathfrak{g}$ associated with θ (cf. 2.1.2).

2.2.2. *Relative notations.* When considering, in situation 2.2.1, a finite dimensional linear representation $\rho : G \to GL(V)$, we shall use the following notations.

We denote by $\mathbf{d}\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ the differential of the representation ρ and by $\mathfrak{z}$ the centralizer of $\rho(H)$ in $\mathfrak{gl}(V)$. The Trace map on $\mathfrak{gl}(V)$ is denoted by $\mathbf{Tr} : \mathfrak{gl}(V) \to \mathbf{R}$, and the *trace form* means the bilinear form $(X, Y) \mapsto \mathbf{Tr}(XY)$ on $\mathfrak{gl}(V)$ (it is the *bilinear form associated with the* $\mathfrak{gl}(V)$-*module* V, according to [Bou60, §3 $\mathrm{N}^0$6 Definition 4]). We write $\mathfrak{z}^{\perp}$ for the orthogonal complement of $\mathfrak{z}$ with respect to the trace form.

2.2.3. *Choice of a euclidean structure.* Using 2.1.15, we can fix a euclidean structure (inner product) on V such that the involution $\theta_V : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ given by $X \mapsto -X^{\mathrm{T}}$, the negative of the euclidean adjoint, stabilizes $\mathbf{d}\rho(\mathfrak{g})$ and extend the image Cartan involution of θ on $\mathbf{d}\rho(\mathfrak{g})$.

Remark 2.1. In such a situation, using our observations in section 2.1 on Cartan involutions we will deduce the following:

1. In $\mathfrak{g}$, the subspaces $\mathfrak{k}$, $\mathfrak{h}$, and hence $\mathfrak{z}_{\mathfrak{g}}$, are invariant under θ.

2. The orthogonal complements of $\mathfrak{k}$ (resp. $\mathfrak{h}$, $\mathfrak{z}_{\mathfrak{g}}$) in $\mathfrak{g}$ with respect to the Killing form or with respect to B_θ are the same. This orthogonal complement is supplementary to $\mathfrak{k}$ (resp. $\mathfrak{h}$, $\mathfrak{z}_{\mathfrak{g}}$) in $\mathfrak{g}$.
3. The subspace ${\mathfrak{z}_{\mathfrak{g}}}^\perp$ of $\mathfrak{g}$ is invariant under the adjoint action of H; it is the unique supplementary H-stable subspace to the isotypic component $\mathfrak{z}_{\mathfrak{g}}$ in the H-module $\mathfrak{g}$.
4. The map $\mathbf{d}\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ commute with the involutions θ on $\mathfrak{g}$ and θ_V on $\mathfrak{gl}(V)$.
5. The subspace $\mathfrak{z}^\perp$ of $\mathfrak{gl}(V)$ is invariant under the adjoint action of H; it is the unique supplementary H-stable subspace to the isotypic component $\mathfrak{z}$ in the H- module $\mathfrak{gl}(V)$.
6. The map $\mathbf{d}\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ sends $\mathfrak{z}_{\mathfrak{g}}$ to $\mathfrak{z}$ and ${\mathfrak{z}_{\mathfrak{g}}}^\perp$ to $\mathfrak{z}^\perp$.
7. In $\mathfrak{gl}(V)$, the subspaces $\mathbf{d}\rho(\mathfrak{k})$, $\mathbf{d}\rho(\mathfrak{z}_{\mathfrak{g}})$, $\mathbf{d}\rho(\mathfrak{h})$ and hence $\mathfrak{z}$ and $\mathfrak{z}^\perp$, are invariant under θ_V.
8. The orthogonal projection $\pi_{\mathfrak{z}} : \mathfrak{gl}(V) \to \mathfrak{z}$ with respect to the trace form commutes with θ_V; it sends self-adjoint endomorphisms to self-adjoint endomorphisms.
9. The map $\mathbf{d}\rho$ sends elements of $\mathfrak{k}^\perp$ to self-adjoint endomorphisms of V.

Proof. **1** follows from the definition of $\mathfrak{k}$, the assumption on H, and the construction of $\mathfrak{z}_{\mathfrak{g}}$ from $\mathfrak{h}$. **2** follows from 1, definition of B_θ and that B_θ is anisotropic (cf. 2.1.3). To prove **3**, note that both $\mathfrak{z}_{\mathfrak{g}}$ and B are invariant under adjoint action of H; and that the isotypic components are uniquely defined. **4** follows from the choice of θ_V in 2.2.3. **5** follows from the same arguments as in 3. To prove **6**, note that as $\mathbf{d}\rho$ commutes with the adjoint action of H, it preserves the isotypic decomposition. **7** follows from 1, from 4, from the construction of $\mathfrak{z}$ from $\mathbf{d}\rho(\mathfrak{h})$, and that Θ_V preserves the trace form. **8** follows from the observations that both the subspace $\mathfrak{z}$ and the orthogonality condition are invariant under θ_V, and that self-adjoint means fixed by $-\theta_V$. **9** follows from 4. □

3. Effective statements

Our effective statements will rely on the Corollary 3.3 of the following result. Note that the condition that p "can only go to infinity in directions orthogonal to $\mathfrak{z}_{\mathfrak{g}}$" is used in the next proof only in order to get a uniform lower bound on the eigenvalues.

In this section we follow the notation and convention of section 2.2.

Theorem 3.1. *For any p in P, the endomorphism $\pi_{\mathfrak{z}}(\rho(p))$ of V is self-adjoint, positive definite, and has no eigenvalue smaller than* 1.

Proof. Fix p in P and write $p = \exp(\wp)$ for some $\wp$ in $\mathfrak{p}$. From $\mathfrak{p} = \mathfrak{k}^\perp \cap {\mathfrak{z}_{\mathfrak{g}}}^\perp$ follows that $\mathbf{d}\rho(\wp)$ belongs to both $\mathbf{d}\rho(\mathfrak{k}^\perp)$ and $\mathbf{d}\rho({\mathfrak{z}_{\mathfrak{g}}}^\perp)$. Consequently $\mathbf{d}\rho(\wp)$ is self-adjoint (Remark 9) and orthogonal to $\mathfrak{z}$ with respect to the trace form (Remark 6). We will write S for $\mathbf{d}\rho(\wp)$. By [Bou72, §4 Corollary 2], we get

$\rho(p) = \exp(S)$, so that $\rho(p)$ is self- adjoint and positive definite. As a result, $\pi_{\mathfrak{z}}(\rho(p))$ is self-adjoint (Remark 8), and belongs to $\mathfrak{z}$, the image of $\pi_{\mathfrak{z}}$.

Let λ be an eigenvalue of $\pi_{\mathfrak{z}}(\rho(p))$, and let π_λ be the corresponding spectral projection. We saw that $\pi_{\mathfrak{z}}(\rho(p))$ is self-adjoint and commutes with H, and it is well known that π_λ belongs to the subalgebra generated by $\pi_{\mathfrak{z}}(\rho(p))$. Consequently π_λ is self-adjoint and commutes with H. In particular, π_λ belongs to $\mathfrak{z}$.

The difference $\pi_{\mathfrak{z}}(\rho(p)) - \rho(p)$ belongs to the kernel of the projection $\pi_{\mathfrak{z}}$: it is orthogonal to $\mathfrak{z}$, and, in particular, to π_λ. Consequently,

$$\mathbf{Tr}(\pi_{\mathfrak{z}}(\rho(p))\pi_\lambda) = \mathbf{Tr}(\rho(p)\pi_\lambda) = \mathbf{Tr}(\exp(S)\pi_\lambda).$$

On the other hand, $\mathbf{Tr}(\pi_{\mathfrak{z}}(\rho(p))\pi_\lambda)$ equals $d_\lambda \cdot \lambda$, where d_λ is the rank of π_λ and $d_\lambda > 0$.

From Proposition 3.2 below, we have the inequality

$$\mathbf{Tr}(\exp(S)\pi_\lambda) \geq d_\lambda \cdot \exp(\mathbf{Tr}(S\pi_\lambda)/d_\lambda).$$

Because π_λ is in $\mathfrak{z}$, and S is orthogonal to $\mathfrak{z}$, $\mathbf{Tr}(S\pi_\lambda) = 0$. As a consequence $\lambda \geq 1$. Indeed

$$d_\lambda \cdot \lambda = \mathbf{Tr}(\pi_{\mathfrak{z}}(\rho(p))\pi_\lambda) = \mathbf{Tr}(\exp(S)\pi_\lambda) \geq d_\lambda \cdot \exp(\mathbf{Tr}(S\pi_\lambda)/d_\lambda) = d_\lambda \cdot 1.$$

□

Proposition 3.2. *Let V be a finite dimensional euclidean vector space, let S be a self-adjoint endomorphism of V and let π be a non zero orthogonal projection in V. Then follows*

$$\mathbf{Tr}(\exp(S)\pi) \geq \mathbf{rank}(\pi) \cdot \exp\left(\mathbf{Tr}(S\pi)/\mathbf{rank}(\pi)\right). \tag{3.1}$$

Proof. Let $S = \sum_\lambda \lambda \cdot \pi_\lambda$ be the spectral decomposition of S. Then each of the idempotents π_λ is self-adjoint and $\exp(S) = \sum_\lambda \exp(\lambda) \cdot \pi_\lambda$. One computes

$$\mathbf{Tr}(S\pi) = \sum_\lambda \lambda \cdot \mathbf{Tr}(\pi_\lambda \pi) \text{ and } \mathbf{Tr}(\exp(S)\pi) = \sum_\lambda \exp(\lambda) \cdot \mathbf{Tr}(\pi_\lambda \pi). \tag{3.2}$$

Since π_λ and π are self-adjoints and idempotents,

$$\mathbf{Tr}(\pi_\lambda\pi) = \mathbf{Tr}(\pi_\lambda\pi_\lambda\pi\pi) = \mathbf{Tr}(\pi\pi_\lambda\pi_\lambda\pi) = \mathbf{Tr}((\pi_\lambda\pi)^{\mathrm{T}}\pi_\lambda\pi) \geq 0 \tag{3.3}$$

by idempotence of π_λ and π, by cyclicity of $\mathbf{Tr}$, by self-adjointness of π_λ and π, and by positivity of $X \mapsto \mathbf{Tr}(X^{\mathrm{T}}X)$ respectively.

The sum $\sum_\lambda \mathbf{Tr}(\pi_\lambda\pi)$ has value $\mathbf{Tr}(\mathrm{Id}\pi) = \mathbf{Tr}(\pi) = \mathbf{rank}(\pi)$. The coefficients $\frac{\mathbf{Tr}(\pi_\lambda\pi)}{\mathbf{rank}(\pi)}$ are well defined, because π is assumed to be non zero, they are nonnegative, by (3.3), and they have sum 1, as $\sum_\lambda \mathbf{Tr}(\pi_\lambda\pi) = \mathbf{rank}(\pi)$. From the convexity of the exponential function, one gets

$$\exp\left(\sum_\lambda \lambda \cdot \frac{\mathbf{Tr}(\pi_\lambda\pi)}{\mathbf{rank}(\pi)}\right) \leq \sum_\lambda \exp(\lambda) \cdot \frac{\mathbf{Tr}(\pi_\lambda\pi)}{\mathbf{rank}(\pi)}, \tag{3.4}$$

which, together with (3.2), yields inequality (3.1). □

Corollary 3.3. *In situation of Theorem* 3.1*, for any p in P, $\pi_{\mathfrak{z}}(\rho(p))$ is expanding:*

$$\forall v \in V, \forall p \in P, \|\pi_{\mathfrak{z}}(\rho(p))(v)\| \geq \|v\| \,. \tag{3.5}$$

Proof. Indeed $\pi_{\mathfrak{z}}(\rho(p))$ can be diagonalized in an orthonormal basis with all diagonal coefficients greater than or equal to 1. □

3.1. **Application.** Consider the adjoint representation Ad_ρ of G on $\mathfrak{gl}(V)$ by conjugation. Let $C(\mathrm{Ad}_\rho)$ be the vector space of functions on H generated by the matrix coefficients of Ad_ρ; that is, by functions $\langle \phi, g\rangle : h \mapsto \phi(\rho(h) g \rho(h)^{-1})$ for g in $\mathfrak{gl}(V)$ and ϕ in its algebraic dual $\mathfrak{gl}(V)^\vee$. The function $\langle \phi, g\rangle$ depends linearly on both g and ϕ. Consequently $C(\mathrm{Ad}_\rho)$ is finite dimensional: its dimension is bounded by $\dim\left(\mathfrak{gl}(V) \otimes \mathfrak{gl}(V)^\vee\right)$.

The following identity has two consequences (cf. [Bou60, §7 N^{0}1]):

$$\langle \phi, g\rangle(h \cdot h') = \langle \phi, \rho(h') g \rho(h'^{-1})\rangle(h). \tag{3.6}$$

Firstly the space of matrix coefficients of Ad_ρ is stable under the action of H by translation: as a result we get a linear action of H on $C(\mathrm{Ad}_\rho)$. Secondly (3.6) expresses that, for a fixed ϕ in $\mathfrak{gl}(V)^\vee$ the map $g \mapsto \langle \phi, g\rangle$ is an equivariant morphism from $\mathfrak{gl}(V)$ to $C(\mathrm{Ad}_\rho)$.

Recall that H being *reductive in* G, by criterion (b) of §1.1 the restriction to H of a finite dimensional representation of G is semisimple. Consequently, in both $\mathfrak{gl}(V)$ and $C(\mathrm{Ad}_\rho)$ there is a unique isotypic projection onto the subspaces of invariant elements, namely onto the centralizer $\mathfrak{z}$ of H in $\mathfrak{gl}(V)$, and onto the subset of constant functions (canonically isomorphic to $\{0\}$ or $\mathbf{R}$ according to $\dim(V)$ being zero or not[2]). Moreover these projections, say $\pi_{\mathfrak{z}}$ and $\pi_{\mathbf{R}}$ respectively, commute with any equivariant morphism. In particular, for the morphism $g \mapsto \langle \phi, g\rangle$ from $\mathfrak{gl}(V)$ to $C(\mathrm{Ad}_\rho)$, for any ϕ in $\mathfrak{gl}(V)^\vee$, we get

$$\pi_{\mathbf{R}}\left(\langle \phi, g\rangle\right) = \langle \phi, \pi_{\mathfrak{z}}(g)\rangle; \tag{3.7}$$

note that this is a constant function on H.

Theorem 3.4. *Let $C(\mathrm{Ad}_\rho)$ the vector space of functions on H generated by the matrix coefficients of Ad_ρ, let Ω be a nonempty subset of H, write $\pi_{\mathbf{R}}$ for the equivariant projection of $C(\mathrm{Ad}_\rho)$ onto constant functions. Set*

$$c = \sup\{\pi_{\mathbf{R}}(f) | f \in C(\mathrm{Ad}_\rho), \sup_{\omega\in\Omega} |f(w)| \leq 1\} \tag{3.8}$$

so that, for any f in $C(\mathrm{Ad}_\rho)$, we have:

$$\sup_{\omega\in\Omega} |f(\omega)| \cdot c \geq |\pi_{\mathbf{R}}(f)|. \tag{3.9}$$

Then

$$\forall g \in \mathfrak{gl}(V), \forall v \in V, \sup_{\omega\in\Omega} \left\|\rho\left(\omega^{-1}\right) \cdot g \cdot \rho\left(\omega\right)(v)\right\| \cdot c \geq \|\pi_{\mathfrak{z}}(g)(v)\| \,. \tag{3.10}$$

[2]If $\dim(V) \neq 0$, then $\langle \mathbf{Tr}, \mathrm{Id}\rangle : h \mapsto \dim(V)$ is a nonzero constant matrix coefficient.

Proof. The remark (3.9) follows from the homogeneity of $\pi_{\mathbf{R}}$.

Fix g in $\mathfrak{gl}(V)$, v in V, and denote by w the vector $\pi_{\mathfrak{z}}(g)(v)$. Applying Cauchy-Schwarz inequality in V, we get, for any ω in H,

$$\left\|\rho\left(\omega^{-1}\right)\cdot g\cdot\rho\left(\omega\right)\left(v\right)\right\|\cdot\|w\|\geq\left(\rho\left(\omega^{-1}\right)\cdot g\cdot\rho\left(\omega\right)\left(v\right)\mid w\right), \tag{3.11}$$

where $(\cdot\mid\cdot)$ denotes the inner product on V. Note that the right-hand side, as a function of ω, is a matrix coefficient belonging to $C(\mathrm{Ad}_\rho)$. Consequently, by definition of c,

$$\sup_{\omega\in\Omega}\left(\rho\left(\omega^{-1}\right)\cdot g\cdot\rho\left(\omega\right)\left(v\right)\mid w\right)\cdot c\geq\pi_{\mathbf{R}}\Big(\left(\rho\left(\omega^{-1}\right)\cdot g\cdot\rho\left(\omega\right)\left(v\right)\mid w\right)\Big). \tag{3.12}$$

Equation (3.7) with $\phi : X\mapsto (X(v)\mid w)$ specializes in

$$\pi_{\mathbf{R}}\Big(\left(\rho\left(\omega^{-1}\right)\cdot g\cdot\rho\left(\omega\right)\left(v\right)\mid w\right)\Big)=\left(\pi_{\mathfrak{z}}(g)(v)\mid w\right)=\|w\|^2. \tag{3.13}$$

Applying $\sup_{\omega\in\Omega}$ to both sides of (3.11), combining with (3.12), substituting (3.13), we finally get

$$\sup_{\omega\in\Omega}\left\|\rho(\omega^{-1})\cdot g\cdot\rho(\omega)(v)\right\|\cdot\|w\|\cdot c\geq\|w\|^2, \tag{3.14}$$

which implies (3.10), as $\|w\|\geq 0$. □

Corollary 3.5. *In the situation of Theorem* 3.4*, assume moreover that*

(∗) *every matrix coefficient in* $C(\mathrm{Ad}_\rho)$ *that vanishes on* Ω *also vanishes on the whole* H*.*

Assuming (∗) *and* $\dim(V)>0$*, we get* $1\leq c\neq\infty$*, and*

$$\forall p\in\exp_G(\mathfrak{p}),\forall v\in V,\sup_{\omega\in\Omega}\left\|\rho\left(\omega^{-1}\cdot p\cdot\omega\right)(v)\right\|\geq\|v\|/c. \tag{3.15}$$

Proof. Assuming $\dim(V)>0$, constant functions are matrix coefficients of Ad_ρ, hence by (3.8), $c\geq\pi_{\mathbf{R}}(1)=1$. Condition (∗) ensure that the map $f\mapsto\sup_{\omega\in\Omega}|f(w)|$ actually defines a *norm*, instead of a mere semi-norm, on the subspace of $C(\mathrm{Ad}_\rho)$ on which it takes finite values. By (3.8), c is the operator norm of the restriction to this subspace of the bounded linear application $\pi_{\mathbf{R}}$. Whence $c<\infty$.

The inequality (3.15) follows from combining 3.4 and 3.1, and then dividing by $c>0$. □

Remark 3.6.
1. Condition (∗) is satisfied for any Zariski dense subset Ω of H, and in particular[3] if Ω has nonempty interior or positive Haar measure.
2. If moreover Ω is bounded, then the map $f\mapsto\sup_{\omega\in\Omega}|f(w)|$ defines a norm on whole of $C(\mathrm{Ad}_\rho)$.
3. Note that condition (∗) means that the evaluation maps $f\mapsto f(\omega)$, with ω in Ω, generate the algebraic dual of $C(\mathrm{Ad}_\rho)$. Choosing a basis from this generating set, one can see that condition (∗) can still be met by replacing Ω by a subset of cardinality at most $\dim(C(\mathrm{Ad}_\rho))$.

[3]Recall H is assumed to be connected, and being smooth, it is irreducible.

4. Note that in terms of such a basis of $C(\mathrm{Ad}_\rho)$ (which can be obtained using a basis of V) one can effectively bound the constant c from the above in Corollary 3.5.

4. Proof of Theorem 1.1

We will show how to derive Theorem 1.1 from Corollary 3.5. Actually we will establish the following more precise statement. The existence of Θ follows from criterion (c) of §1.1.

Proposition 4.1. *In the situation of Theorem* 1.1*, let* Θ *be a Cartan involution of* G *under which* H *is invariant, and let* K *be the maximal projectively compact subgroup of* G *consisting of all the fixed points of* Θ*. Write* $\mathfrak{g}$*,* $\mathfrak{z}_G$*, and* $\mathfrak{k}$ *for the Lie algebras of* G*,* $Z_G(H)$*, and* K *respectively. Let* $\mathfrak{p} = (\mathfrak{k} + \mathfrak{z}_G)^\perp$*, the orthogonal complement of the sum of* $\mathfrak{z}_G$ *and* $\mathfrak{k}$ *with respect to the Killing form on* $\mathfrak{g}$*.*

Then the subset $Y = K \cdot \exp_G(\mathfrak{p})$ *of* G *satisfies both the conditions of Theorem* 1.1*.*

We will prove that Y satisfies each of these conditions in the next two subsections.

4.1. **The first condition.** We recall another theorem due to G.D. Mostow.

Theorem 4.2 (Mostow [Mos55b, Theorem 5])**.** *Let* G *be a connected semi-simple real Lie group and let* K *be a maximal projectively compact subgroup of* G*. Let* $\mathfrak{g}$ *denote the Lie algebra of* G *and* $\mathfrak{k}$ *the Lie algebra of* K*. Let* $\mathfrak{z}$ *be any Lie subalgebra of* $\mathfrak{g}$*. Orthogonality is understood with respect to the Killing form.*

Then the following application is a diffeomorphism.

$$\begin{array}{rcc} K \times \left(\mathfrak{k}^\perp \cap (\mathfrak{k}^\perp \cap \mathfrak{z})^\perp\right) \times \left(\mathfrak{k}^\perp \cap \mathfrak{z}\right) & \to & G \\ (k, Q, Z) & \mapsto & k \cdot \exp_G(Q) \cdot \exp_G(Z) \end{array} \tag{4.1}$$

Mostow states that G "decomposes topologically", meaning that we have a homeomorphism. This is enough to establish the first condition of Theorem 1.1, but we can verify this directly, as below, that map (4.1) is an immersion. As both sides of (4.1) have equal dimension, (4.1) will be a local diffeomorphism, but being bijective, it will be an (analytic) diffeomorphism. Let us prove that at each (k, Q, Z) in $K \times \left(\mathfrak{k}^\perp \cap (\mathfrak{k}^\perp \cap \mathfrak{z})^\perp\right) \times \left(\mathfrak{k}^\perp \cap \mathfrak{z}\right)$ the tangent map is injective.

Proof. Left and right translating one is reduced to the case where $Z = 0$ and $k = \exp_G(0)$. Write $q = \exp_G(Q)$, and let dK, dQ, and dZ be arbitrarily small in $\mathfrak{k}$, $\mathfrak{p}$, and $\mathfrak{k}^\perp \cap \mathfrak{z}$ respectively. At first order,

$$\exp_G(dK)\exp_G(Q + dQ)\exp_G(dZ) \sim \\ q \cdot (q^{-1}\exp_G(dK)q)\exp_G(dQ)\exp_G(dZ).$$

The latter equals $q\cdot\exp_G(\mathrm{Ad}_{q^{-1}}(dK))\exp_G(dQ)\exp_G(dZ)$, or, up to first order,

$$q\cdot\exp_G(\mathrm{Ad}_{q^{-1}}(dK)+dQ+dZ).$$

We will be done showing that $\mathrm{Ad}_{q^{-1}}(dK)+dQ+dZ$ cannot be zero for arbitrarily small and not simultaneously zero dK, dQ and dZ, namely that $\mathrm{Ad}_{q^{-1}}(\mathfrak{k})$, $\left(\mathfrak{k}^{\perp}\cap(\mathfrak{k}^{\perp}\cap\mathfrak{z})^{\perp}\right)$ and $\mathfrak{k}^{\perp}\cap\mathfrak{z}$ are in direct sum. Note that $\mathfrak{k}$ and $\mathfrak{k}^{\perp}$ are anisotropic of opposite sign (negative and positive resp.). By invariance of the Killing form, $\mathrm{Ad}(\exp_G(Q))(\mathfrak{k})$ is negative (cf. 2.1.1), hence has intersection $\{0\}$ with $\mathfrak{k}^{\perp}$. Consequently $\mathrm{Ad}_{q^{-1}}(\mathfrak{k})$ and $\mathfrak{k}^{\perp}$ are in direct sum. As $\mathfrak{k}^{\perp}$ is anisotropic, $\mathfrak{z}\cap\mathfrak{k}^{\perp}$ is supplementary to its orthogonal complement in $\mathfrak{k}^{\perp}$. □

Let K and Y be as in Proposition 4.1. Applying Theorem 4.2 to $\mathfrak{z}=\mathfrak{z}_{\mathfrak{g}}$, it follows that the equality $G=Y\cdot Z_G(H)^0$ is satisfied and that Y defines a closed submanifold of G. In particular Y satisfies condition 1 of Theorem 1.1.

4.2. **The second condition.** What is left, in order to prove Proposition 4.1, is to show that Y satisfies the condition (1.1) of the Theorem 1.1. Fix ρ as in Theorem 1.1. We will prove (1.1) under a weaker hypothesis on Ω, namely the condition $(*)$ stated in Corollary 3.5.

Proof. If $\dim(V)=0$, then (1.1) is immediate. So assume $\dim(V)>0$.

Note that it is enough to prove the inequality (1.1) for any subset Ω_b of Ω instead of Ω. Moreover, according to Remark 3.6 (3), we can assume this subset to be finite and still satisfy condition $(*)$. In particular such an Ω_b will be bounded.

Because V is finite dimensional, all norms on V are equivalent. Consequently, the validity of the inequality (1.1) doesn't depend on the chosen norm on V, if one allows to change the constant. In particular one can assume that this norm is associated to a euclidean structure on V as in 2.2.3. Then the corresponding inner product is K- invariant.

Recall (proposition 4.1) that $Y=K\cdot\exp_G\mathfrak{p}$. As the euclidean norm on V is K invariant, the inequality (1.1) for y in Y will follow from inequality (1.1) for $y\in\exp_G(\mathfrak{p})$.

We only need to prove that there exists a constant $c>0$ such that

$$\forall y\in\exp_G(\mathfrak{p}),\ \forall v\in V,\quad \sup_{\omega\in\Omega}\|\rho(y\cdot\omega)(v)\|\geq c\cdot\|v\|\,. \tag{4.2}$$

As Ω is bounded, $C=\sup_{\omega\in\Omega_b}\left|\!\left|\!\left|\rho(\omega^{-1})\right|\!\right|\!\right|$ is finite, where $|\!|\!|\cdot|\!|\!|$ denotes the operator norm, and because $\dim(V)>0$, $C>0$. Because of the inequalities

$$\left\|\rho\left(\omega^{-1}\cdot y\cdot\omega\right)v\right\|\leq\left|\!\left|\!\left|\rho\left(\omega^{-1}\right)\right|\!\right|\!\right|\cdot\|\rho\left(y\cdot\omega\right)v\|\leq C\cdot\|\rho(y\cdot\omega)v\|\,,$$

equation (4.2) follows from

$$\forall y\in\exp_G(\mathfrak{p}),\ \forall v\in V,\quad \sup_{\omega\in\Omega}\left\|\rho(\omega^{-1}\cdot y\cdot\omega)(v)\right\|\geq Cc\cdot\|v\|\,. \tag{4.3}$$

Let c' be the constant given by (3.8). According Corollary 3.5, equations (4.3), and hence (4.2) hold for $c = \frac{1}{Cc'}$. $\square$

Remark 4.3. Proposition 4.1 and Theorem 1.1 are proved, with Y given by Proposition 4.1 (or 2.2.1), assuming only that Ω satisfies condition $(*)$ of Corollary 3.5, and, whenever the norm on V is given by 2.2.3, with

$$(4.4)\quad c = \left(\sup_{\omega \in \Omega_b} |||\rho(\omega^{-1})||| \cdot \sup\{\pi_{\mathbf{R}}(f) | f \in C(\mathrm{Ad}_\rho), \sup_{\omega \in \Omega_b} |f(w)| \leq 1\} \right)^{-1}.$$

5. Subsequent enhancements: case of reductive G

Actually, Theorem 1.1 can be generalized in different ways. First of all, if G is a semisimple linear Lie group, one can consider the algebraic structure (given by the algebra of matrix coefficients). In this case the Theorem 1.1 and its proof remain true if one only assume that the Zariski closure of H is Zariski connected, instead of H being connected as a Lie group.

More importantly, in a different direction,

Proposition 5.1. *The conclusion of Theorem* 1.1 *holds under the following relaxed conditions on* G *and* H*: Let* G *be a* reductive Lie group*; that is,* G *has finitely many components and the adjoint action of* $\mathfrak{g}$ *on itself is completely reducible. And Let* H *to be a connected* reductive subgroup of G*; that is, the adjoint action of* H *on* $\mathfrak{g}$ *is completely reducible.*

Proof. Note that, for any compact subset C of G, if we replace Y by CY, the conclusion of Theorem 1.1 still hold, up to a change in the constant c. This remark shows that without loss of generality we may assume that G is connected.

Let Z denote the center of G and $[G, G]$ be the derived subgroup of G. Set $H' = (HZ) \cap [G, G]$. Note that H' is reductive in $[G, G]$, because it has the same action as H on $\mathfrak{g}$ and because $[\mathfrak{g}, \mathfrak{g}]$ is invariant subspace of $\mathfrak{g}$. We can apply Theorem 1.1 to H' in $[G, G]$, in order to get a subset Y' of $[G, G]$. As $Z_G(H) = Z_G(HZ) = Z_{[G,G]}(H')Z$, we have $Y'Z_G(H) = Y'Z_{[G,G]}(H')Z = [G, G]Z = G$. Thus Y as a subset of G satisfies the first condition of Theorem 1.1, with respect to G and H.

Let us check that Y also satisfies the second condition of Theorem 1.1, namely formula (1.1). First note we can replace Ω by a bounded subset, then that, for any bounded subset C of Z, formula (1.1) still holds, up to a change in constant, if we replace Ω by ΩC, and conversely. Consequently we can replace H by HZ and assume Ω to have nonempty interior in HZ. Taking a smaller subset we can assume Ω, which we assumed to be bounded, is a product in G of subsets of $[G, G]$ and Z. Using the converse above, we can replace HZ by H' and assume Ω, to be contained in H' and have nonempty interior in H'. $\square$

6. Proof of Theorem 1.2

We now turn to the proof of Theorem 1.2. We will in fact derive it from Theorem 1.1. This was inspired by an argument of Kempf [Kem78] for reducing S-instability to instability; Kempf credits Mumford for the argument.

We consider $(y_n)_{n\in\mathbf{N}}$, Ω, ρ and v as in the statement and prove the equivalence.

Remark 6.1.
1. The veracity of each of property (A) and property (B) is independent of the choice of the subset Ω of H, provided it is bounded and has nonempty interior in H.
2. Concerning property (B), we first remark that F depends only on the subspace $\langle\Omega v\rangle$ generated by Ωv. This space is contained in $\langle Hv\rangle$, and not in any proper subspace. Indeed, Ω can not be contained in the inverse image by $h\to hv$ of a proper subspace of $\langle Hv\rangle$: this inverse image is a proper differential subvariety, and has empty interior. We proved property (B) depends only on $\langle Hv\rangle$ and not on a specific Ω.
3. Concerning property (A), we first choose a basis of $\langle\Omega v\rangle$ from the generating subset Ωv. Consider any bounded subset Ω' of H. Then $\Omega' v$ is bounded in $\langle Hv\rangle$. Consequently, the coefficients of $\omega' v$ in chosen basis will remain bounded as ω' ranges over Ω'. Write $(e_i)_{0\leq i\leq N}$ for the basis. The vector $y_n\omega' v$ will be written with the same bounded coefficients in the basis $(y_n e_i)_{0\leq i\leq N}$ as $\omega' v$ in the basis $(e_i)_{0\leq i\leq N}$. Let us now assume property (A) for Ω. As each e_i belongs to Ωv, the sequences $(y_n e_i)_{n\in\mathbf{N}}$ will be bounded. Consequently, the sequences $(y_n\omega' v)_{n\in\mathbf{N}}$, which are finite linear combinations of the formers, with uniformly bounded coefficients, are uniformly bounded, as ω' ranges over Ω'. This proves property (A) for Ω'.
4. From the previous argument, we deduce that property (A) is equivalent to each of the following two variants:

(A') $\qquad$ For each ω in Ω, the sequence $(y_n\omega v)_{n\in\mathbf{N}}$ is bounded in V,

(A") $\qquad$ For each w in $\langle Hv\rangle$, the sequence $(y_n w)_{n\in\mathbf{N}}$ is bounded in V.

Proof of (B) ⇒ (A). This implication is the easiest one to prove and does not need the knowledge of $(y_n)_{n\in\mathbf{N}}$ being in Y, or G being semisimple.

Let F denote the point-wise stabilizer Ωv. Assuming (B), we know there is some compact set C in G such that each y_n belongs to CF. The image subset $\rho(C)$ in $\mathrm{End}(V)$ is compact because ρ is continuous. On the other hand Ω is bounded, hence contained in a compact, for instance $\overline{\Omega}$. Again, $\rho(\overline{\Omega})$ is compact in $\mathrm{End}(V)$.

For every n in $\mathbf{N}$, and every ω in Ω, one has

$$y_n\omega v\in\rho(C)\rho(F)\rho(\overline{\Omega})v. \tag{6.1}$$

But F acts trivially on $\rho(\Omega)v$, hence on $\overline{\rho(\Omega)v}$ because the fixed point subspace of F is closed. But, by continuity of ρ, the closure $\overline{\rho(\Omega)v}$ contains

$\overline{\rho(\Omega)}v$. In equation (6.1) above, one can then forget about the action of F, which acts trivially on $\rho(\overline{\Omega})v$. It remains:

$$\forall n \in \mathbf{N}, \forall \omega \in \Omega, y_n \omega v \in \rho(C)\rho(\overline{\Omega})v.$$

But $\rho(C)$ and $\rho(\overline{\Omega})$ are compact, and so is $\rho(C)\rho(\overline{\Omega})v$. This proves the sought for uniform boundedness of property (A). $\square$

The second implication is more involved. To summarize our approach a few words, we first convert boundedness in (A) into convergence in some auxiliary representation space (after passing to a subsequence). We then employ the clever idea due to Kempf for passing from S- instability to $\{0\}$-instability using the following easily provable Lemma 1.1(b) of [Kem78], to convert the situation of convergence toward any vector into convergence to 0 in another auxiliary finite dimensional representation space. As the property property (A) carries over to the new representation, we will be able to reduce everything to Theorem 1.1.

Proof of (A) $\Rightarrow$ (B). *Assume by contradiction that* (A) *holds, but not* (B). Consider the space of functions from Ω to V, and more specifically the G-invariant subspace W generated by the function $f_v : \omega \mapsto \omega v$. We can realize W as a G-subspace of the finite dimensional space $\mathrm{Hom}(\langle f_v(\Omega)\rangle, V)$, where G-acts on the image space. Note that f_v corresponds to the identity homomorphism of $\langle f_v(\Omega)\rangle$ in V.

The G action on f_v is via point-wise translation on the values of f_v. These values span Ωv. The stabilizer in G of the function f_v is then the point-wise stabilizer of Ωv. We denote this stabilizer by F.

Since we assumed that (B) fails to hold, the sequence $(y_n)_{n\in\mathbf{N}}$ is not bounded in G modulo F on the right. By passing to a subsequence, one may assume no subsequence of the sequence $(y_n)_{n\in\mathbf{N}}$ is bounded in G modulo F on the right.

Let us assume that property (A) holds (it then holds for any subsequence). In other words property (A) tells that the vector $y_n f_v$ of W can be bounded independently of n. Replacing by a subsequence, one may assume that the sequence $(y_n f_v)_{n\in\mathbf{N}}$ is convergent in the finite dimensional vector space W. Let f_∞ be its limit.

We claim that the limit f_∞ can not belong to the orbit Gf_v. By contradiction, if $(y_n f_v)_{n\in\mathbf{N}}$ were converging inside the orbit Gv, then its inverse image under the bijective map

$$G/F \to Gf_v,\ gF \longmapsto gf_v \tag{6.2}$$

would be convergent in G/F, hence would be bounded in G/F, contradicting the failure of property (B). For this argument to work, we have to know that the inverse map of (6.2) is continuous. By [PR94, Corollary 2 of Lemma 3.2], Gf_v is open in its closure, and in particular it is locally compact. Therefore as a consequence of the Baire's category theorem for locally compact second countable spaces, the orbit map (6.2) is open, and hence a homeomorphism.

This limit f_∞ then belongs to $\overline{Gf_v} \smallsetminus Gf_v$, which we denote by $\partial(Gv)$.

Let h be in H. Then by Remark 6.1-(4), property (A) holds also for $h\Omega$ instead of Ω. The veracity of Property (B) is clearly untouched by substituting $(y_n)_{n\in\mathbf{N}}$ with $(y_n h)_{n\in\mathbf{N}}$. Arguing with function $hf_v : \omega \mapsto h\omega v$ instead of f_v, we conclude the sequence $(y_n h f_v)_{n\in\mathbf{N}}$ has a limit, say f^h_∞ in W, and that this limit belongs to $\overline{Ghf_v} \smallsetminus Ghf_v$, which equals $\overline{Gf_v} \smallsetminus Gf_v$.

Let $\mathrm{Z}\partial(Gf_v) = \mathrm{Zcl}(Gf_v) \smallsetminus \mathrm{Zcl}(G)f_v$ denote the boundary of Gf_v with respect to the Zariski topology; see Lemma 6.2 stated below. From the closed orbit lemma [PR94, Proposition 2.23], one knows that $S := \mathrm{Z}\partial(Gf_v)$ is a Zariski closed (G-invariant) subset in W. By [Kem78, Lemma 1.1(b)], over $k = \mathbf{R}$, there exists a finite dimensional linear G-space W' and a G-equivariant polynomial map

$$\Phi : W \to W'$$

such that $\Phi^{-1}(0) = S$. Clearly, $f_v \notin S$, therefore $\Phi(f_v) \neq 0$. On the other hand, for each $h \in H$, in view of the above observation and by Lemma 6.2, $f^h_\infty \in S$, and hence $\Phi(f^h_\infty) = 0$.

Let us recall the situation. We have a sequence $(y_n)_{n\in\mathbf{N}}$ in Y such that, for each h in H, the sequence $(y_n h\Phi(f_v))_{n\in\mathbf{N}}$ converges to 0 whereas $h\Phi(f_v)$ is never zero. We now consider a compact subset Ω' of H with nonempty interior. This can be found because H is a connected Lie group. Then the sequence of (continuous) functions $(h \mapsto y_n h\Phi(f_v))_{n\in\mathbf{N}}$ is converging pointwise to 0, hence, on Ω', is uniformly converging to 0 by the argument as in Remark 6.1-(3).

In particular, the subsets $y_n\Omega'\Phi(f_v)$ of W' are uniformly converging to 0 as n goes to ∞. But this contradicts Theorem 1.1 applied to

(i) the bounded subset Ω' of H with nonempty interior;
(ii) the representation of G on W';
(iii) any norm on W'.

This theorem says $y_n\Omega'\Phi(f_v)$ can not be bounded above, in the chosen norm, by $c\,\|\Phi(f_v)\|$.

This contradiction completes the proof of Theorem 1.2, modulo the following lemma. □

Lemma 6.2. *Consider a* $\mathbf{R}$*-vector space* V*, and let a* G *be a connected real Lie subgroup of* $\mathrm{GL}(V)$*. pick any* v *in* V*. We denote*

- *by* $\mathrm{Zcl}(G)$ *the Zariski closure of* G *in* $GL(V)$*,*
- *by* $\mathrm{Zcl}(Gv)$ *the Zariski closure of the orbit* Gv *in* V*,*
- *by* $\partial(Gv) = \overline{Gv} \smallsetminus Gv$*, the boundary for the metric topology, and*
- *by* $\mathrm{Z}\partial(Gv) = \mathrm{Zcl}(Gv) \smallsetminus \mathrm{Zcl}(G)v$*, the boundary for the Zariski topology.*

Assume now that G *is open in* $\mathrm{Zcl}(G)$*, (which means that* G *is a real algebraic Lie group; see* [PR94, Theorem 3.6 and Corollary 1]*,) for instance if* G *is semisimple.*

Then $\mathrm{Z}\partial(Gv) \cap \overline{Gv} = \partial(Gv)$*: one has* $\partial(Gv) \subseteq \mathrm{Z}\partial(Gv)$ *and* $\mathrm{Z}\partial(Gv) \cap G.v = \emptyset$.

Proof. Firstly, as Gv is contained in $\mathrm{Zcl}(Gv)$, which does not meet $\mathrm{Z}\partial(Gv)$, one gets easily: $\mathrm{Z}\partial(Gv) \cap Gv = \emptyset$.

Since G is open in $\mathrm{Zcl}(G)$, for any point $w \in \mathrm{Zcl}(G)v$ we have that Gw is open in $\mathrm{Zcl}(G)w$ (the orbit map $\mathrm{Zcl}(G) \to \mathrm{Zcl}(G)v$ is an open map, by [PR94, Corollary 2 of Lemma 3.2]). Using the Closed orbit lemma ([Bor91, §I.1.8], or [PR94, Proposition 2.23]), we know that $\mathrm{Zcl}(G)v$ is Zariski open in $\mathrm{Zcl}(Gv)$, hence open. It follows that Gw, for w in $\mathrm{Zcl}(G)v$, is also open in $\mathrm{Zcl}(Gv)$. We also know that $\mathrm{Zcl}(Gv)$ is Zariski closed, hence closed, and contains Gv; it hence contains $\overline{Gv}$. Therefore $Gw \cap \overline{Gv}$ is an open subset of $\overline{Gv}$.

Consider now a point x of $\partial(Gv)$. Then the G-orbit Gx of x is distinct from Gv. But x belongs to $\overline{Gv}$, and can be approached along Gv. Hence any neighborhood of x meets at least two G orbits: Gv and Gx. It follows that Gx is not open in $\overline{Gv}$ at x, hence not open in $\overline{Gv}$.

Therefore x can not be of the form w, with $w \in \mathrm{Zcl}(G)v$. Now $\partial(Gv)$ is contained in $\overline{Gv}$, hence in $\mathrm{Zcl}(Gv)$, while it does not meet $\mathrm{Zcl}(G)v$. in other words, $\partial(Gv) \subseteq \mathrm{Z}\partial(Gv)$. ☐

Remark 6.3. The proof of Lemma 6.2 uses only that the metric topology is finer than the Zariski topology, and that the orbit map $\mathrm{Zcl}(G) \to \mathrm{Zcl}(G)v$ is an open map ([PR94, Corollary 2 of Lemma 3.2]). The lemma and the proof then hold for any non-discrete locally compact field k of characteristic zero (hypotheses from [PR94, Sec. 3.1]), for any group G which is open in $\mathrm{Zcl}(G)(k)$.

Remark 6.4. Our proof of Theorem 1.2 is essentially algebraic in nature. It can be transposed mutatis mutandis to other locally compact fields k, provided: (1) That one has the analog of Lemma 6.2, see Remark 6.3; (2) one has the analog of Theorem 1.1 (as, for instance, in [Ric09]).

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