Irregular sets and conditional variational principles in dynamical systems

by

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Contents

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Declarations

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work, except where otherwise indicated, cited or commonly known. The argument of lemma 2.3.2 was provided by Peter Walters. The idea of §5.3 was provided by Thomas Jordan. The example of §7.4.1 was suggested by Yakov Pesin.

A paper based on the material of chapter 5 has been accepted for publication in the Journal of the London Mathematical Society. I have submitted three more papers for publication, based on the material of chapters 4, 6 and 7 respectively, which are currently under consideration.

Abstract

We derive key results from dimension theory in dynamical systems and thermodynamic formalism at a level of generality suitable for the study of systems which are beyond the scope of the standard uniformly hyperbolic theory. Let (X, d) be a compact metric space, $f : X \mapsto X$ be a continuous map and $\varphi : X \mapsto \mathbb{R}$ be a continuous function.

The subject of chapters 4 and 5 is the multifractal analysis of Birkhoff averages for φ when topological pressure (in the sense of Pesin and Pitskel) is the dimension characteristic and f has the specification property. In chapter 4, we consider the set of points for which the Birkhoff average of φ does not exist (which we call the irregular set for φ) and show that this set is either empty or has full topological pressure. We formulate various equivalent natural conditions on φ that completely describe when the latter situation holds. In chapter 5, we prove a conditional variational principle for topological pressure for non-compact sets of the form

$$\left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \right\},$$

generalising a previously known result for topological entropy. As one application, we prove multifractal analysis results for the entropy spectrum of a suspension flow over a continuous map with specification.

In chapter 6, we assume that $f : X \mapsto X$ is a continuous map satisfying a property we call almost specification (which is weaker than specification). We show that the set of points for which the Birkhoff average of φ does not exist is either empty or has full topological entropy. Every β-shift satisfies almost specification and we show that the irregular set for any β-shift or β-transformation is either empty or has full topological entropy and Hausdorff dimension.

In chapter 7, we introduce an alternative definition of topological pressure for arbitrary (non-compact, non-invariant) Borel subsets of metric spaces. This new quantity is defined via a suitable conditional variational principle, leading to an alternative definition of an equilibrium state. We study the properties of this new quantity and compare it with existing notions of topological pressure. We apply our new definition to some interesting examples, including the level sets of the pointwise Lyapunov exponent for the Manneville-Pomeau family of maps.

Chapter 1

Introduction

The results of this thesis fall within the category of dimension theory in dynamical systems and thermodynamic formalism. We give a summary of the main results here and give detailed introductions at the beginning of each chapter, where we motivate each topic and explain carefully the history of our results. We give full reference to previously known results which arise as special cases of our theorems.

The work is focused on deriving key results from dimension theory and thermodynamic formalism at a level of generality suitable for the study of systems which are beyond the scope of the standard uniformly hyperbolic theory. Our focus is mainly on the development of abstract results rather than on applications. That said, we emphasise that our theory applies to interesting examples (many of which are inaccessible by other methods) and we take care to point these out. In particular, we give a detailed application to the β-transformation.

Much of the work focuses on the class of maps with the specification property. The specification property was introduced by Bowen [Bow2]. He showed that uniformly hyperbolic systems satisfy specification (a stronger version than the one we use) and gave important results about the abundance of periodic orbits in a hyperbolic set. Among the many dynamical properties which can be derived from the specification property, there are results on large deviations [Rue1], dimension theory [TV1], thermodynamic formalism [Bow5] and distributional chaos [OS].

The class of maps with the specification property includes the usual array of uniformly hyperbolic examples as well as interesting non-uniformly and partially hyperbolic examples such as the Manneville-Pomeau map and quasi-hyperbolic toral automorphisms. In chapter 6, we study the class of maps with a property which we call almost specification and prove results which are applicable to every β-transformation.

Key results of the thesis

Topological pressure in multifractal analysis

Topological pressure is a well understood topological invariant of dynamical systems in the compact setting [Wal], [PP1]. It is a tool that is used to prove, for example, results on multifractal analysis, statistical properties of dynamical systems and ergodic optimisation. We study topological pressure for non-compact sets, which is less well understood and was defined by Pesin and Pitskel [PP2] analogously to Hausdorff dimension, an idea that Bowen introduced for entropy [Bow4]. For a compact metric space (X, d), a continuous map $f : X \mapsto X$ and continuous functions $\psi, \varphi : X \mapsto \mathbb{R}$, we undertake a programme to understand the topological pressure of the multifractal decomposition $X = \bigcup_{\alpha \in \mathbb{R}} X(\varphi, \alpha) \cup \widehat{X}(\varphi)$, where $X(\varphi, \alpha)$ denotes the level sets of the Birkhoff average, i.e.

$$X(\varphi, \alpha) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \right\} \tag{1.1}$$

and $\widehat{X}(\varphi)$ denotes the set of points for which the Birkhoff average does not exist. The motivation for proving multifractal analysis results where pressure is the dimension characteristic is twofold. Firstly, topological pressure is a non-trivial and natural generalisation of topological entropy, which is the standard dynamically defined dimension characteristic. Secondly, understanding the topological pressure of the multifractal decomposition allows us to prove results about the topological entropy of systems related to the original system, for example, suspension flows (see §4.3).

In chapters 4 and 5 respectively, we show for maps f with specification that the following holds.

Theorem. *$\widehat{X}(\varphi)$ is either empty or has full topological pressure.*

Theorem. *Let $P_{X(\varphi,\alpha)}(\psi)$ denote the topological pressure of ψ on $X(\varphi, \alpha)$ and h_μ be the measure-theoretic entropy of an f-invariant probability measure μ. Then*

$$P_{X(\varphi,\alpha)}(\psi) = \sup \left\{ h_\mu + \int \psi d\mu : \mu \in \mathcal{M}_f(X) \textit{ and } \int \varphi d\mu = \alpha \right\}. \tag{1.2}$$

Our results generalise and unify various previously known results. We mention some of these here and give a fuller description in the introductions of chapters 4 and 5.

It is an increasingly well known phenomenon that the irregular set can be large from the point of view of dimension theory (despite being a null set with respect to any invariant measure). Symbolic dynamics methods have confirmed this in the uniformly hyperbolic setting [BS5], for certain non-uniformly hyperbolic examples [PW] and for a large class of multimodal maps [Tod]. The irregular set has also been the focus of a great deal of work by Olsen and collaborators [BOS].

Formulae similar to (1.2) have a key role in multifractal analysis and the theorem generalises and unifies results by Takens and Verbitskiy [TV2], Luzia [Luz], Barreira and Schmeling [BS2]. Barreira has used the phrase 'conditional variational principle' to describe formulae similar to (1.2) and we follow suit. We recommend Barreira's book [Bar] as reference for the symbolic dynamics approach to the study of both irregular sets and conditional variational principles.

Our results apply to some interesting examples which are not covered by the standard uniformly hyperbolic theory. For example, the class of maps satisfying the specification property includes the time-1 map of the geodesic flow of compact connected negative curvature manifolds and certain quasi-hyperbolic toral automorphisms as well as any system which can be modelled by a topologically mixing shift of finite type. We discuss these examples and others in §2.4.

Suspension flows

We apply our results to suspension flows, proving in §4.3 that

Theorem. *The irregular set for a suspension flow over a map with specification is either empty or has full topological entropy.*

We only assume continuity of the roof function and along the way we derive some basic properties of suspension flows which, to the best of our knowledge, have previously only been investigated when the roof function is Hölder continuous. We also prove a conditional variational principle for entropy for the suspension flow in §5.2.

The almost specification property

A recent weakening of the specification property provides new tools to study interesting systems beyond the scope of uniformly hyperbolic dynamics such as the β-transformation. This property was introduced by Pfister and Sullivan [PS2] as the g-almost product property. The version we study is a priori slightly weaker and we rename it the almost specification property. The main results of chapter 6 are

Theorem. *When f satisfies the almost specification property, the irregular set is either empty or has full topological entropy.*

Theorem. *The irregular set for an arbitrary β-transformation (or β-shift) is either empty or has full entropy $\log \beta$ and Hausdorff dimension 1.*

The proof relies on a generalisation of the techniques of chapters 4-5. We are required to develop a theory of 'strongly separated' and 'almost spanning' sets and a modified version of the Katok formula for measure-theoretic entropy. These should be of independent interest.

Thermodynamic formalism in non-compact spaces

The non-compact definition of topological pressure of Pesin and Pitskel has an important role in dimension theory. In chapter 7, we contribute an alternative definition of topological entropy and pressure. The definition is made via a suitable 'conditional variational principle' and leads to a new definition of equilibrium state. The advantage of the new definition is that it is more tractable than the Pesin and Pitskel definition and is well adapted to certain problems in thermodynamic formalism. We study the properties of this new quantity and compare it with existing notions of topological pressure, clarifying the literature on this topic [PP1], [HKR], [HNP], [Sar]. We note that the new definition agrees with the old in the classical compact setting. We motivate the naturality of this definition by applying it to some important examples. In particular, we calculate the equilibrium states for the level sets of $\log f'$ (defined as in (1.1)) when f is the Manneville-Pomeau map of the interval (ie. $f(x) = x + x^{1+s}(\text{mod}1)$, where $s \in (0,1)$ is a fixed parameter value). The Manneville-Pomeau map is an important example of a map which displays non-uniform expansion. The result fits in naturally with work of Takens and Verbitskiy [TV2] as well as that of Pollicott, Sharp & Yuri [PSY].

Chapter 2

Preliminaries

We collect the definitions and fix notation for objects which we consider repeatedly through the thesis. Theorems, definitions, lemmas and remarks are numbered in seperate sequences by section. For example, lemma 4.2.3 is the third lemma in §4.2.

2.1 Notation for some standard definitions

Let (X, d) be a compact metric space and $f : X \mapsto X$ a continuous map. We call such a pair (X, f) a (topological) dynamical system. Let $C(X)$ denote the space of continuous functions from X to $\mathbb{R}$, and $\varphi, \psi \in C(X)$. Let

$$S_n\varphi(x) := \sum_{i=0}^{n-1} \varphi(f^i(x)),$$

and for $c > 0$, let

$$\mathrm{Var}(\varphi, c) := \sup\{|\varphi(x) - \varphi(y)| : d(x, y) < c\}.$$

For $Z \subset X$, let $\mathrm{Diam}(Z) = \sup\{d(x, y) : x, y \in Z\}$. For a collection of subsets ξ, let $\mathrm{Diam}(\xi) = \sup\{\mathrm{Diam}(Z) : Z \in \xi\}$. Let $\mathcal{M}_f(X)$ denote the space of f-invariant probability measures and $\mathcal{M}_f^e(X)$ denote those which are ergodic. If $X' \subseteq X$ is an f-invariant subset, let $\mathcal{M}_f(X')$ denote the subset of $\mathcal{M}_f(X)$ for which the measures μ satisfy $\mu(X') = 1$.

Definition 2.1.1. *We define probability measures $\delta_{x,n}$ (sometimes called the empirical measures) as*

$$\delta_{x,n} := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)},$$

where δ_x is the Dirac measure at x.

Definition 2.1.2 (Bowen balls)**.** *Given $\varepsilon > 0, n \in \mathbb{N}$ and a point $x \in X$, define the open (n, ε)-ball at x by*

$$B_n(x, \varepsilon) = \{y \in X : d(f^i(x), f^i(y)) < \varepsilon \text{ for all } i = 0, \dots, n-1\}.$$

Alternatively, let us define a new metric

$$d_n(x,y) = \max\{d(f^i(x), f^i(y)) : i = 0, 1, \ldots, n-1\}.$$

It is clear that $B_n(x,\varepsilon)$ is the open ball of radius ε around x in the d_n metric, and that if $n \leq m$ we have $d_n(x,y) \leq d_m(x,y)$ and $B_m(x,\varepsilon) \subseteq B_n(x,\varepsilon)$.

Definition 2.1.3. *Let $Z \subset X$. We say a set $\mathcal{S} \subset Z$ is an (n,ε) spanning set for Z if for every $z \in Z$, there exists $x \in \mathcal{S}$ with $d_n(x,z) \leq \varepsilon$. We say a set $\mathcal{R} \subset Z$ is an (n,ε) separated set for Z if for every $x, y \in \mathcal{R}$, $d_n(x,y) > \varepsilon$.*

See [Wal] for the basic properties of spanning sets and separated sets.

2.1.1 Definition of the topological pressure

Let $Z \subset X$ be an arbitrary Borel set, not necessarily compact or invariant. We use the definition of topological pressure as a characteristic of dimension type, due to Pesin and Pitskel [PP2]. The definition generalises Bowen's definition of topological entropy for non-compact sets [Bow4]. We consider finite and countable collections of the form $\Gamma = \{B_{n_i}(x_i,\varepsilon)\}_i$. For $s \in \mathbb{R}$, we define the following quantities:

$$Q(Z,s,\Gamma,\psi) = \sum_{B_{n_i}(x_i,\varepsilon)\in\Gamma} \exp\left(-sn_i + \sup_{x\in B_{n_i}(x_i,\varepsilon)} \sum_{k=0}^{n_i-1} \psi(f^k(x))\right),$$

$$M(Z,s,\varepsilon,N,\psi) = \inf_{\Gamma} Q(Z,s,\Gamma,\psi),$$

where the infimum is taken over all finite or countable collections of the form $\Gamma = \{B_{n_i}(x_i,\varepsilon)\}_i$ with $x_i \in X$ such that Γ covers Z and $n_i \geq N$ for all $i = 1, 2, \ldots$. Define

$$m(Z,s,\varepsilon,\psi) = \lim_{N\to\infty} M(Z,s,\varepsilon,N,\psi).$$

The existence of the limit is guaranteed since the function $M(Z,s,\varepsilon,N)$ does not decrease with N. By standard techniques, we can show the existence of

$$P_Z(\psi,\varepsilon) := \inf\{s : m(Z,s,\varepsilon,\psi) = 0\} = \sup\{s : m(Z,s,\varepsilon,\psi) = \infty\}.$$

Definition 2.1.4. *The topological pressure of ψ on Z is given by*

$$P_Z(\psi) = \lim_{\varepsilon\to 0} P_Z(\psi,\varepsilon).$$

See [Pes] for verification that the quantities $P_Z(\psi,\varepsilon)$ and $P_Z(\psi)$ are well defined. If Z is compact and invariant, our definition agrees with the usual topological pressure as defined in [Wal]. We denote the topological pressure of the whole space by $P_X^{classic}(\psi)$, to emphasise that we are dealing with the familiar compact, invariant definition.

Remark 2.1.1. It is sometimes convenient to use an equivalent definition of topological pressure where, in place of covers by Bowen balls, we consider covers by strings of open sets taken from an arbitrary open cover. We use this in §7.3.1, so we save a formal definition until then. We also implicitly use the alternative definition in §4.3.3.

2.1.2 Topological entropy for maps with discontinuities

When $\psi = 0$, we write $h_{top}(Z) := P_Z(0)$. Pesin and Pitskel [PP2] gave a definition of pressure (and hence entropy) which is suitable for maps f which admit discontinuities. We state the topological entropy version of this definition, which we use in chapter 6, where we consider the β-transformation.

Suppose X is a compact metric space, Y is a (generally non-compact) subset of X and $f : Y \mapsto Y$ is continuous. We do not assume that f extends continuously to X. When $f : X \mapsto X$ is continuous, we set $Y = X$. In chapter 6, when we consider the β-transformation f_β, we set

$$Y = X \setminus \{\beta^{-i} : i \in \mathbb{N}\} = X \setminus \bigcup_i f_\beta^{-i}(0).$$

Let $Z \subset Y$ be an arbitrary Borel set, not necessarily compact or invariant. We consider finite and countable collections of the form $\Gamma = \{B_{n_i}(x_i, \varepsilon)\}_i$. For $s \in \mathbb{R}$, we define the following quantities:

$$Q(Z, s, \Gamma) = \sum_{B_{n_i}(x_i,\varepsilon)\in\Gamma} \exp -sn_i,$$

$$M(Z, s, \varepsilon, N) = \inf_\Gamma Q(Z, s, \Gamma),$$

where the infimum is taken over all finite or countable collections of the form $\Gamma = \{B_{n_i}(x_i, \varepsilon)\}_i$ with $x_i \in X$ such that Γ covers Z and $n_i \geq N$ for all $i = 1, 2, \ldots$. Define

$$m(Z, s, \varepsilon) = \lim_{N\to\infty} M(Z, s, \varepsilon, N).$$

The existence of the limit is guaranteed since the function $M(Z, s, \varepsilon, N)$ does not decrease with N. By standard techniques, we can show the existence of

$$h_{top}(Z, \varepsilon) := \inf\{s : m(Z, s, \varepsilon) = 0\} = \sup\{s : m(Z, s, \varepsilon) = \infty\}.$$

Definition 2.1.5. *The topological entropy of Z is given by*

$$h_{top}(Z) = \lim_{\varepsilon\to 0} h_{top}(Z, \varepsilon).$$

When $X = Y$, we denote the topological entropy of the dynamical system (X, f) by $h_{top}(f)$ and we note that $h_{top}(X) = h_{top}(f)$. We sometimes write $h_{top}(Z, f)$ in place of $h_{top}(Z)$ when we wish to emphasise the dependence on f.

2.1.3 Topological entropy for shift spaces

Let Σ be a closed subset of $\prod_{i=1}^{\infty}\{0,\dots,n-1\}$ and σ be the shift map $\sigma((x_i)_{i=1}^{\infty})=(x_i)_{i=2}^{\infty}$. If Σ is σ-invariant, then the pair (Σ,σ) defines a dynamical system. We call such a dynamical system a (one-sided) shift space.

For shift spaces, which we consider in chapter 6, the definition of topological entropy can be simplified and we introduce notation that reflects this. For $x=(x_i)_{i=1}^{\infty}$, let $C_n(x)=\{y\in\Sigma : x_i = y_i \text{ for } i=1,\dots,n\}$.

Let $Z\subset\Sigma$ be an arbitrary Borel set, not necessarily compact or invariant. We consider finite and countable collections of the form $\Gamma=\{C_{n_i}(x_i)\}_i$. For $s\in\mathbb{R}$, we define the following quantities:

$$Q(Z,s,\Gamma)=\sum_{C_{n_i}(x_i)\in\Gamma}\exp -sn_i,$$

$$M(Z,s,N)=\inf_{\Gamma}Q(Z,s,\Gamma),$$

where the infimum is taken over all finite or countable collections of the form $\Gamma=\{C_{n_i}(x_i)\}_i$ with $x_i\in\Sigma$ such that Γ covers Z and $n_i\geq N$ for all $i=1,2,\dots$. Define

$$m(Z,s)=\lim_{N\to\infty}M(Z,s,N).$$

The existence of the limit is guaranteed since the function $M(Z,s,N)$ does not decrease with N.

Lemma 2.1.1. *The topological entropy of $Z\subset\Sigma$ is given by*

$$h_{top}(Z):=\inf\{s:m(Z,s)=0\}=\sup\{s:m(Z,s)=\infty\}.$$

The proof, which we omit, follows from the fact that every open ball $B(x,\varepsilon)$ in Σ is a set of the form $C_n(x)$, where the value of n depends on ε and the metric on Σ.

2.1.4 Topological entropy for flows

Let $Z\subset X$ be an arbitrary Borel set, not necessarily compact or invariant. Let $\Psi=\{\psi_t\}_{t\geq 0}$ be a semi-flow on X (i.e. a continuous family of continuous maps $\psi_t : X\mapsto X$ such that $\psi_0=\mathrm{Id}$ and $\psi_s\circ\psi_t=\psi_{s+t}$ for all $s,t\geq 0$). We consider finite and countable collections of the form $\Gamma=\{B_{t_i}(x_i,\varepsilon)\}_i$, where $t_i\in(0,\infty)$, $x_i\in X$ and

$$B_t(x,\varepsilon)=\{y\in X : d(\psi_\tau(x),\psi_\tau(y))<\varepsilon \text{ for all } \tau\in[0,t)\}.$$

For $s\in\mathbb{R}$, we define the following quantities:

$$Q(Z,s,\Gamma)=\sum_{B_{t_i}(x_i,\varepsilon)\in\Gamma}\exp\left(-st_i\right),$$

$$M(Z, s, \varepsilon, T) = \inf_{\Gamma} Q(Z, s, \Gamma),$$

where the infimum is taken over all finite or countable collections of the form $\Gamma = \{B_{t_i}(x_i, \varepsilon)\}_i$ with $x_i \in X$ such that Γ covers Z and $t_i \geq T$ for all $i = 1, 2, \ldots$. Define

$$m(Z, s, \varepsilon) = \lim_{T \to \infty} M(Z, s, \varepsilon, T).$$

The existence of the limit is guaranteed since the function $M(Z, s, \varepsilon, T)$ does not decrease with T. By standard techniques, we can show the existence of

$$h_{top}(Z, \varepsilon) := \inf\{s : m(Z, s, \varepsilon) = 0\} = \sup\{s : m(Z, s, \varepsilon) = \infty\}.$$

Definition 2.1.6. *The topological entropy of Z with respect to Ψ is given by*

$$h_{top}(Z, \Psi) = \lim_{n \to \infty} h_{top}(Z, \varepsilon).$$

2.1.5 Upper and lower capacity pressure

The usual definition of $P_X^{classic}(\psi)$ in terms of spanning sets generalises to non-compact and non-invariant subsets of a compact metric space. Let

$$Q_n(Z, \psi, \varepsilon) = \inf\{\sum_{x \in S} \exp S_n\psi(x) : \text{ S is an } (n, \varepsilon) \text{ spanning set for Z } \}.$$

$\overline{CP}_Z(\psi)$ is defined to be $\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Q_n(Z, \psi, \varepsilon)$ and called in [Pes] the upper capacity topological pressure. The lower capacity topological pressure $\underline{CP}_Z(\psi)$ is given by repacing the $\limsup$ with $\liminf$. In chapter 11 of [Pes], Pesin shows that these quantities can be formulated as characteristics of dimension type and example 11.1 of [Pes] shows that they do not always coincide with $P_Z(\varphi)$, even for compact non-invariant sets. It is proved in [Pes] that $P_Z(\psi) \leq \underline{CP}_Z(\psi)$. For $Z \subset X$, let

$$P_n(Z, \psi, \varepsilon) = \sup \left\{ \sum_{x \in S} \exp \left\{ \sum_{k=0}^{n-1} \psi(f^k x) \right\} : S \text{ is an } (n, \varepsilon) \text{ separated set for } Z \right\}.$$

We have $Q_n(Z, \psi, \varepsilon) \leq P_n(Z, \psi, \varepsilon)$ and $Q_n(Z, \psi, \varepsilon)$ may be replaced with $P_n(Z, \psi, \varepsilon)$ in the definitions of lower and upper capacity pressure. We consider the capacity topological pressure in §5.1.1 and chapter 7.

2.1.6 Measure-theoretic entropy

For $\mu \in \mathcal{M}_f(X)$ and a partition ξ of X into finitely many measurable sets, we define

$$H_\mu(\xi) = -\sum_{A \in \xi} \mu(A) \log \mu(A),$$

$$h_\mu(f,\xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=1}^{n} f^{-i}\xi).$$

In the above, $0\log 0$ is set to be 0 and $\log$ denotes the natural logorithm. We define the measure-theoretic entropy of (X,f) with respect to μ to be

$$h_\mu := \sup\{h_\mu(f,\xi) : \xi \text{ is a finite partition of } X\}.$$

We refer the reader to [Wal] for details. We could write $h_\mu(f)$ in place of h_μ to emphasise the dependence of h_μ on f, but we choose not to.

2.1.7 The variational principle

The variational principle states that

$$P_X^{classic}(\psi) = \sup\{h_\mu + \int \psi d\mu : \mu \in \mathcal{M}_f(X)\}.$$

We sometimes call this formula the classical variational principle to differentiate it from the conditional variational principles which are the subject of chapters 5 and 7.

2.1.8 Hausdorff dimension

For Hausdorff dimension, we fix the notation

$$H(Z,s,\delta) = \inf\{\sum_i \delta_i^s : Z \subseteq \bigcup_i B(x_i,\delta_i), \delta_i \leq \delta\},$$

$H(Z,s) = \lim_{\delta\to 0} H(Z,s,\delta)$ and $\text{Dim}_H(Z) = \inf\{s : H(Z,s) = 0\}$. We sometimes write $\text{Dim}_H(Z,d)$ in place of $\text{Dim}_H(Z)$ when we wish to emphasise the dependence on the metric d. For more information on Hausdorff dimension, we refer the reader to [Pes] or [Fal].

2.2 Specification properties

In chapters 4 and 5, we study transformations f of the following type:

Definition 2.2.1. *A continuous map $f : X \mapsto X$ satisfies the specification property if for all $\varepsilon > 0$, there exists an integer $m = m(\varepsilon)$ such that for any collection $\{I_j = [a_j,b_j] \subset \mathbb{N} : j = 1,\dots,k\}$ of finite intervals with $a_{j+1} - b_j \geq m(\varepsilon)$ for $j = 1,\dots,k-1$ and any $x_1,\dots,x_k$ in X, there exists a point $x \in X$ such that*

$$d(f^{p+a_j}x, f^p x_j) < \varepsilon \text{ for all } p = 0,\dots,b_j - a_j \text{ and every } j = 1,\dots,k. \quad (2.1)$$

The original definition of specification, due to Bowen, was stronger.

Definition 2.2.2. *We say $f : X \mapsto X$ satisfies Bowen specification if under the assumptions of definition 2.2.1 and for every $p \geq b_k - a_1 + m(\varepsilon)$, there exists a periodic point $x \in X$ of period p satisfying (2.1).*

One can describe a map f with specification intuititively as follows. For any set of points $x_1, \ldots, x_k$ in X, there is an $x \in X$ whose orbit follows the orbits of all the points $x_1, \ldots, x_k$. In this way, one can connect together arbitrary pieces of orbit. If f has Bowen specification, x can be chosen to be a periodic point of any sufficiently large period. A good reference for results about the specification property (particularly Bowen specification) is [DGS].

One can verify that a map with the specification property is topologically mixing. The following converse result holds [Blo], a recent proof of which is available in [Buz].

Theorem 2.2.1 (Blokh Theorem)**.** *A continuous topologically mixing map of the interval has Bowen specification.*

A factor of a system with specification has specification. We give a survey of many interesting examples of maps with the specification property in §2.4.

In chapter 4, we study a weakening of the definition of specification as follows. Let $X' \subseteq X$ be an f-invariant (but not necessarily compact) Borel set.

Definition 2.2.3. *A continuous map $f : X \mapsto X$ satisfies specification on X' if for all $\varepsilon > 0$, there exists an integer $m = m(\varepsilon)$ such that for any collection $\{I_j = [a_j, b_j] \subset \mathbb{N} : j = 1, \ldots, k\}$ of finite intervals with $a_{j+1} - b_j \geq m(\varepsilon)$ for $j = 1, \ldots, k-1$ and any $x_1, \ldots, x_k$ in X', there exists a point $x \in X$ such that*

$$d(f^{p+a_j}x, f^p x_j) < \varepsilon \text{ for all } p = 0, \ldots, b_j - a_j \text{ and every } j = 1, \ldots, k.$$

The main theorem of chapter 4 generalises to this setting naturaly with little extra difficulty in the proofs. Although we do not offer an application of this extra generality, we think that there may be examples of non-uniformly hyperbolic systems where definition 2.2.3 holds on an interesting (non-compact) subset but where definition 2.2.1 is not verifiable.

2.2.1 Almost specification

In chapter 6, we consider a weak version of the specification property, which was introduced by Pfister and Sullivan as the g-almost product property, and which we rename as the almost specification property. We define this property and study it in chapter 6. We mention here that the specification property implies the almost specification property. Thus the class of maps with the almost specification property is strictly larger than the class of maps with specification. Motivating examples of maps with almost specification but not specification are provided by a large class of β-shifts (see §6.5.1).

2.3 Cohomology

2.3.1 The multifractal spectrum of Birkhoff averages

For $\alpha \in \mathbb{R}$, we define

$$X(\varphi, \alpha) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \right\}.$$

We define the multifractal spectrum for φ to be

$$\mathcal{L}_\varphi := \{\alpha \in \mathbb{R} : X(\varphi, \alpha) \neq \emptyset\}.$$

Some authors reserve the terminology 'multifractal spectrum' for the pair $(\mathcal{L}_\varphi, \mathcal{F})$, where $\mathcal{F}$ is a dimension characteristic (eg. Hausdorff dimension or topological entropy). Our terminology agrees with Takens and Verbitskiy [TV2]. The following lemma (whose proof is included for completeness) is essentially contained in [TV2].

Lemma 2.3.1. *When f has the specification property, $\mathcal{L}_\varphi$ is a non-empty bounded interval. Furthermore, $\mathcal{L}_\varphi = \{\int \varphi d\mu : \mu \in \mathcal{M}_f(X)\}$.*

Proof. We first show that $\mathcal{L}_\varphi = \mathcal{I}_\varphi$ where $\mathcal{I}_\varphi = \{\int \varphi d\mu : \mu \in \mathcal{M}_f(X)\}$. By Proposition 21.14 of [DGS], when f has the Bowen specification property, every $f-$invariant (not necessarily ergodic) measure has a generic point (i.e. a point x which satisfies $\frac{1}{n} S_n \varphi(x) \to \int \varphi d\mu$ for all continuous functions φ). One can verify that this remains true under the specification property. Thus, given $\mu \in \mathcal{M}_f(X)$, any choice x of generic point for μ lies in $X(\varphi, \int \varphi d\mu)$ and so $\mathcal{I}_\varphi \subseteq \mathcal{L}_\varphi$. Now take $\alpha \in \mathcal{L}_\varphi$ and any $x \in X(\varphi, \alpha)$. Let μ be any weak* limit of the sequence $\delta_{x,n}$. It is a standard result that μ is invariant, and easy to verify that $\int \varphi d\mu = \alpha$. Thus $\mathcal{I}_\varphi = \mathcal{L}_\varphi$.

It is clear that $\mathcal{I}_\varphi \subseteq [\inf_{x \in X} \varphi(x), \sup_{x \in X} \varphi(x)]$ and is non-empty. To show $\mathcal{I}_\varphi$ is an interval we use the convexity of $\mathcal{M}_f(X)$. Assume $\mathcal{I}_\varphi$ is not a single point. Let $\alpha_1, \alpha_2 \in \mathcal{I}_\varphi$. Let $\beta \in (\alpha_1, \alpha_2)$. Let μ_i satisfy $\int \varphi d\mu_i = \alpha_i$ for $i = 1, 2$. Let $t \in (0, 1)$ satisfy $\beta = t\alpha_1 + (1-t)\alpha_2$. One can easily see that $m := t\mu_1 + (1-t)\mu_2$ satisfies $\int \varphi dm = \beta$, and we are done. □

Let $\phi_1, \phi_2 \in C(X)$. We say ϕ_1 is cohomologous to ϕ_2 if they differ by a coboundary, i.e. there exists $h \in C(X)$ such that

$$\phi_1 = \phi_2 + h - h \circ f.$$

If ϕ_1 and ϕ_2 are cohomologous, then $\mathcal{L}_{\varphi_1}$ equals $\mathcal{L}_{\varphi_2}$.

For a constant c, let $\mathrm{Cob}(X, f, c)$ denote the space of functions cohomologous to c and $\overline{\mathrm{Cob}(X, f, c)}$ be the closure of $\mathrm{Cob}(X, f, c)$ in the sup norm.

2.3.2 Cohomology and the irregular set

We recall that $\widehat{X}(\varphi, f)$ is the irregular set for φ, defined as

$$\widehat{X}(\varphi, f) = \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \text{ does not exist} \right\}.$$

By Birkhoff's ergodic theorem, $\mu(\widehat{X}(\varphi, f)) = 0$ for all $\mu \in \mathcal{M}_f(X)$. The following lemma describes conditions equivalent to $\widehat{X}(\varphi, f)$ being non-empty.

Lemma 2.3.2. *When f has specification (or almost specification), the following are equivalent for $\varphi \in C(X)$:*

(a) $\widehat{X}(\varphi, f)$ is non-empty;

(b) $\frac{1}{n} S_n\varphi$ does not converge pointwise to a constant;

(c) $\inf_{\mu\in\mathcal{M}_f(X)} \int \varphi d\mu < \sup_{\mu\in\mathcal{M}_f(X)} \int \varphi d\mu$;

(d) $\inf_{\mu\in\mathcal{M}^e_f(X)} \int \varphi d\mu < \sup_{\mu\in\mathcal{M}^e_f(X)} \int \varphi d\mu$;

(e) $\varphi \notin \bigcup_{c\in\mathbb{R}} \overline{Cob(X, f, c)}$;

(f) $\frac{1}{n} S_n\varphi$ does not converge uniformly to a constant;

(g) $\mathcal{L}_\varphi$ is not equal to a single point.

The argument for (c) $\iff$ (e) $\iff$ (f) was given to the author by Peter Walters and is sketched here. In fact, no assumption on f other than continuity is required except to prove that (a) is implied by the other properties. We note that (c) $\Rightarrow$ (a) is a corollary of theorem 4.1.2 for maps f with specification (and of theorem 6.4.1 for maps f with almost specification), so we omit it for now. For expository reasons, we give a direct proof of (c) $\Rightarrow$ (a) when f has specification as lemma 3.0.2.

Proof. Statement (g) is just a different way of saying (b). We show the contrapositive of (e) $\Rightarrow$ (f). Suppose $\frac{1}{n} S_n\varphi$ converges uniformly to c. Define for $n \in \mathbb{N}$

$$h_n(x) = \frac{1}{n} \sum_{i=1}^{n-1} (n - i)\varphi(f^{i-1}x).$$

We can verify that $\varphi - \frac{1}{n} S_n\varphi = h_n - h_n \circ f$ and it follows that $\varphi \in \overline{Cob(X, f, c)}$. The contrapositive of (c) $\Rightarrow$ (e) is straight forward. Now we prove (f) $\Rightarrow$ (c). Let $\mu_1 \in \mathcal{M}_f(X)$ and let $c := \int \varphi d\mu_1$. From (f), there exists $\varepsilon > 0$ and sequences $n_k \to \infty$ and $x_k \in X$ such that

$$|\frac{1}{n_k} S_{n_k}\varphi(x_k) - c| > \varepsilon.$$

Let $\nu_k = \delta_{x_k,n_k}$ and let μ_2 be a limit point of the sequence ν_k. Then $\mu_2 \in \mathcal{M}_f(X)$ and $\int \varphi d\mu_2 \neq c$, so we are done.

The contrapositive of (a) $\Rightarrow$ (f) is clearly true and (b) $\Rightarrow$ (f) is trivial. We use an ergodic decomposition argument for (c) $\Rightarrow$ (d). For (d) $\Rightarrow$ (b), we take $\mu_1, \mu_2 \in \mathcal{M}_f^e(X)$ such that $\int \varphi d\mu_1 < \int \varphi d\mu_2$. We can find x_i such that $\frac{1}{n} S_n \varphi(x_i) \to \int \varphi d\mu_i$ for $i = 1, 2$ and we are done.

$\square$

2.4 Examples

We now describe some examples of systems with the specification property. The main results of chapters 4 and 5 thus apply to all of these examples.

2.4.1 Standard examples

We recall that any factor of a topologically mixing shift of finite type has the specification property. Bowen's specification theorem tells us that a compact locally maximal hyperbolic set of a topologically mixing diffeomorphism f has the Bowen specification property [Bow3]. In particular, the class of topologically mixing Anosov diffeomorphisms (which includes any Anosov diffeomorphism of a compact connected manifold whose wandering set is empty) has specification.

2.4.2 The Manneville-Pomeau family of maps

Let $I = [0, 1]$. The Manneville-Pomeau family of maps, parametrised by $\alpha \in (0, 1)$ are given by

$$f_\alpha : I \mapsto I, f_\alpha(x) = x + x^{1+\alpha} \mod 1.$$

Considered as a map of S^1, f_α is continuous. Since $f'_\alpha(0) = 1$, the system is not uniformly hyperbolic. However, since the Manneville-Pomeau maps are all topologically conjugate to a full shift on two symbols, they satisfy the specification property.

2.4.3 Beyond symbolic dynamics

As remarked in §2.2, by the Blokh theorem, any continuous topologically mixing interval map satisfies specification. For example, Jakobson [Jak] showed that there exists a set of parameter values $\Lambda \subset [0, 4]$ of positive Lebesgue measure such that if $\lambda \in \Lambda$, then the logistic map $f_\lambda(x) = \lambda x(1 - x)$ is topologically mixing.

Lind [Lin] showed that a quasi-hyperbolic toral automorphism satisfies specification but not Bowen specification if and only if the matrix representation of the automorphism in Jordan normal form admits no 1's off the diagonal in the central direction. Such maps cannot be factors of topologically mixing shifts of finite type or they would inherit the Bowen specification property.

Theorems 17.6.2 and 18.3.6 of [KH] (originally due to Anosov) ensure that the geodesic flow of any compact connected Riemannian manifold of negative sectional curvature is topologically mixing and Anosov. The specification theorem for flows (proved in [Bow3]) ensures that such a flow has the specification property 18.3.13 of [KH]. It is easy to see that the time-t map of a flow with the specification property satisfies our specification property 2.2.1. We conclude that our results apply to the time-t map of the geodesic flow of any compact connected Riemannian manifold of negative sectional curvature.

Chapter 3

Techniques

We introduce some of the techniques which underpin our results. The technique for proving the main results of chapters 4-6 was inspired by the proof of the conditional variational principle of Takens and Verbitskiy [TV2] and we describe it here (although we note that §5.1.1 contains what we believe to be a necessary correction to their proof). The Takens and Verbitskiy proof was in turn inspired by large deviations arguments of Young [You]. In chapter 6, we were inspired by ideas of Pfister and Sullivan [PS1], [PS2]. We have also used ideas from the proof of Pesin and Pitskel's variational principle [PP2] on two occasions (theorem 6.3.1 and theorem 7.3.2).

3.0.4 Constructing points in $\widehat{X}(\varphi, f)$

Proofs which use the specification property are typically constructive, and ours are no exception. The general strategy is to choose sets of points which have a dynamical property that we are interested in, and to use the specification property to construct new points which shadow the orbits of the original points.

We show how to construct a single irregular point for a continuous function φ which satisfies one of the equivalent conditions of lemma 2.3.2. The method for constructing points in $X(\varphi, \alpha)$ is similar.

In the case of topologically mixing shifts of finite type, the specification property is equivalent to the much simpler operation of concatenation of finite words. This example offers insight into our technique. We show how to construct an irregular point for a full one-sided shift as a warm-up, then we show how to construct irregular points for maps with specification.

Lemma 3.0.1. *Let (Σ, σ) be a full (one-sided) shift on finitely many symbols. Let $\varphi \in C(\Sigma)$ satisfy $\inf_{\mu \in \mathcal{M}^e_f(\Sigma)} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}^e_f(\Sigma)} \int \varphi d\mu$. Then $\widehat{\Sigma}(\varphi, f) \neq \emptyset$.*

Proof. Let $\mu_1, \mu_2 \in \mathcal{M}_f^e(\Sigma)$ with $\int \varphi d\mu_1 < \int \varphi d\mu_2$. Let $\delta > 0$ be such that

$$\left| \int \varphi d\mu_1 - \int \varphi d\mu_2 \right| > 9\delta.$$

Let $x = (x_i)_{i=1}^{\infty}$ satisfy $\frac{1}{n} S_n\varphi(x) \to \int \varphi d\mu_1$ and $y = (y_i)_{i=1}^{\infty}$ satisfy $\frac{1}{n} S_n\varphi(y) \to \int \varphi d\mu_2$. Let $N_k \to \infty$ sufficiently rapidly that $N_{k+1} > \exp(N_1 + \ldots + N_k)$. Concatenation of a countable sequence of finite words defines a point in Σ. For $i \geq 1$, we define the finite words

$$w_{2i-1} = (x_1, \ldots, x_{N_{2i-1}}),$$

$$w_{2i} = (y_1, \ldots, y_{N_{2i}}),$$

and define $p = w_1 w_2 w_3 \ldots \in \Sigma$. Let $t_k = N_1 + \ldots + N_k$. Let

$$\mathsf{Var}(\varphi, n) := \sup\{|\varphi(w) - \varphi(v)| : w, v \in \Sigma, w_i = v_i \text{ for } i = 1, \ldots, n\},$$

and choose M such that $\mathsf{Var}(\varphi, M) < \delta$. Assume without loss of generality that N_1 was chosen so that $N_1 > M$. For $k \geq 1$, let $p_k = \sigma^{t_{k-1}} p$. For k odd, we have

$$|S_{N_k}\varphi(p_k) - S_{N_k}\varphi(x)| \leq (N_k - M)\mathsf{Var}(\varphi, M) + 2M\|\varphi\|.$$

Thus, for sufficiently large odd k, we have

$$\left| \frac{1}{N_k} S_{N_k}\varphi(p_k) - \int \varphi d\mu_1 \right| < 3\delta.$$

Similarly, for sufficiently large even k, we have

$$\left| \frac{1}{N_k} S_{N_k}\varphi(p_k) - \int \varphi d\mu_2 \right| < 3\delta.$$

Note that $t_{k-1}/t_k \to 0$ and $N_k/t_k \to 1$. We have

$$|S_{t_k}\varphi(p) - S_{N_k}\varphi(p_k)| \leq t_{k-1}\|\varphi\|,$$

and it is thus easily verified that

$$\left| \frac{1}{t_k} S_{t_k}\varphi(p) - \frac{1}{N_k} S_{N_k}\varphi(p_k) \right| \to 0.$$

It follows that for all sufficiently large k

$$\left| \frac{1}{t_k} S_{t_k}\varphi(p) - \int \varphi d\mu_{\rho(k)} \right| < 4\delta,$$

where $\rho(k) = (k+1)(\text{mod}2) + 1$. Hence, $p \in \widehat{\Sigma}(\varphi, f)$. $\square$

Lemma 3.0.2. *Let (X, f) be a dynamical system with the specification property. Let $\varphi \in C(X)$ satisfy $\inf_{\mu \in \mathcal{M}_f^e(X)} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}_f^e(X)} \int \varphi d\mu$. Then $\widehat{X}(\varphi, f) \neq \emptyset$.*

Proof. Let μ_1, μ_2 be ergodic measures with $\int \varphi d\mu_1 < \int \varphi d\mu_2$. Let x_i satisfy $\frac{1}{n} S_n \varphi(x_i) \to \int \varphi d\mu_i$ for $i = 1, 2$. Let $m_k := m(\varepsilon/2^k)$ be as in the definition of specification and $N_k \to \infty$ sufficiently rapidly that $N_{k+1} > \exp\{\sum_{i=1}^k (N_i + m_i)\}$ and $N_k > \exp m_k$. We define $z_i \in X$ inductively using the specification property. Let $t_1 = N_1$, $t_k = t_{k-1} + m_k + N_k$ for $k \geq 2$ and $\rho(k) := (k+1)(\text{mod} 2) + 1$. Let $z_1 = x_1$. Let z_2 satisfy

$$d_{N_1}(z_2, z_1) < \varepsilon/4 \text{ and } d_{N_2}(f^{N_1+m_2} z_2, x_2) < \varepsilon/4.$$

Let z_k satisfy

$$d_{t_{k-1}}(z_{k-1}, z_k) < \varepsilon/2^k \text{ and } d_{N_k}(f^{t_{k-1}+m_k} z_k, x_{\rho(k)}) < \varepsilon/2^k.$$

Note that if $q \in \overline{B}_{t_k}(z_k, \varepsilon/2^{k-1})$, then

$$\begin{aligned} d_{t_{k-1}}(q, z_{k-1}) &\leq d_{t_{k-1}}(q, z_k) + d_{t_{k-1}}(z_k, z_{k-1}) \\ &< \frac{\varepsilon}{2^{k-1}} + \frac{\varepsilon}{2^k} < \frac{\varepsilon}{2^{k-2}}, \end{aligned}$$

and thus $\overline{B}_{t_k}(z_k, \varepsilon/2^{k-1}) \subset \overline{B}_{t_{k-1}}(z_{k-1}, \varepsilon/2^{k-2})$. Hence, we can define a point by

$$p := \bigcap_{k \geq 1} \overline{B}_{t_k}(z_k, \varepsilon/2^{k-1}).$$

For $k \geq 2$, let $p_k := f^{t_{k-1}+m_k} p$. Since

$$d_{N_k}(p_k, f^{t_{k-1}+m_k} z_k) \leq \varepsilon/2^{k-1} \text{ and } d_{N_k}(f^{t_{k-1}+m_k} z_k, x_{\rho(k)}) < \varepsilon/2^k,$$

it follows that $d_{N_k}(p_k, x_{\rho(k)}) < \varepsilon/2^{k-2}$ and hence

$$|S_{N_k}\varphi(p_k) - S_{N_k}\varphi(x_{\rho(k)})| < N_k \text{Var}(\varphi, \varepsilon/2^{k-2}).$$

Since $\lim_{k\to\infty} \text{Var}(\varphi, \varepsilon/2^{k-2}) = 0$, we have

$$\left| \frac{1}{N_k} S_{N_k}\varphi(p_k) - \int \varphi d\mu_{\rho(k)} \right| \to 0.$$

We also have

$$|S_{N_k}\varphi(p_k) - S_{t_k}\varphi(p)| \leq (t_{k-1} + m_k)\|\varphi\|,$$

so we can use the fact that $\frac{N_k}{t_k} \to 1$ and $\frac{t_{k-1}+m_k}{t_k} \to 0$ to prove that

$$\left| \frac{1}{N_k} S_{N_k}\varphi(p_k) - \frac{1}{t_k} S_{t_k}\varphi(p) \right| \to 0.$$

It follows that

$$\left| \frac{1}{t_k} S_{t_k}\varphi(p) - \int \varphi d\mu_{\rho(k)} \right| \to 0,$$

and hence $p \in \widehat{X}(\varphi, f)$. $\square$

3.0.5 Lower bounds on topological entropy and pressure

We have shown how to construct one irregular point using the specification property. Now we describe our strategy to construct sufficiently many irregular points that the irregular set has full topological entropy. The result that the irregular set has full topological entropy (if it is non-empty) for maps with the specification property is due to [EKL]. The author gave an independent proof of this before he was aware of this paper. We sketch the ideas behind the proof of the 'full entropy' result. This will be useful for understanding the more general 'full pressure' result of chapter 4. The same technique is also used in chapter 5 and chapter 6.

We require two key technical ingredients - the Entropy Distribution Principle (proof included for completeness [TV2]) and the Katok formula for measure-theoretic entropy [Kat].

Proposition 3.0.1 (Entropy Distribution Principle). *Let $f : X \mapsto X$ be a continuous transformation. Let $Z \subseteq X$ be an arbitrary Borel set. Suppose there exists a constant $s \geq 0$ such that for sufficiently small $\varepsilon > 0$ one can find a Borel probability measure $\mu = \mu_\varepsilon$ (which is not assumed to be invariant), a constant $C(\varepsilon) > 0$ and an integer $N(\varepsilon)$ satisfying $\mu_\varepsilon(Z) > 0$ and $\mu_\varepsilon(B_n(x,\varepsilon)) \leq C(\varepsilon)e^{-ns}$ for every ball $B_n(x,\varepsilon)$ with $B_n(x,\varepsilon) \cap Z \neq \emptyset$ and $n \geq N(\varepsilon)$. Then $h_{top}(Z) \geq s$.*

Proof. Choose $\varepsilon > 0$ and μ_ε satisfying the conditions of the theorem. Let $\Gamma = \{B_{n_i}(x_i,\varepsilon)\}_i$ cover Z with all $n_i \geq N$ for some $N \geq N(\varepsilon)$. We may assume that $B_{n_i}(x_i,\varepsilon) \cap Z \neq \emptyset$ for every i. Then

$$\begin{aligned} Q(Z,s,\Gamma) &= \sum_i \exp(-sn_i) \\ &\geq C(\varepsilon)^{-1} \sum_i \mu_\varepsilon(B_n(x,\varepsilon)) \\ &\geq C(\varepsilon)^{-1}\mu_\varepsilon(Z) > 0. \end{aligned}$$

So $M(Z,s,\varepsilon,N) \geq C(\varepsilon)^{-1}\mu_\varepsilon(Z) > 0$ for all $N \geq N(\varepsilon)$. Thus $m(Z,s,\varepsilon) > 0$ and $h_{top}(Z,\varepsilon) \geq s$. The result follows. $\square$

Proposition 3.0.2 (Katok's formula for measure-theoretic entropy). *Let (X,d) be a compact metric space, $f : X \mapsto X$ be a continuous map and μ be an ergodic invariant measure. For $\varepsilon > 0$ and $\gamma \in (0,1)$, denote by $N^\mu(\gamma,\varepsilon,n)$ the smallest cardinality of any set which (n,ε)-spans a set with μ-measure greater than $1-\gamma$. We have*

$$h_\mu = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N^\mu(\gamma,\varepsilon,n) = \lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log N^\mu(\gamma,\varepsilon,n).$$

Loosely, our strategy is as follows. Let $\varepsilon > 0$ be arbitrary.

- Take two ergodic measures μ_1, μ_2 with $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$ and $h_{\mu_i} > h_{top}(f) - \varepsilon$ for $i = 1, 2$ (that we can do this is a slightly subtle point).

- Use Katok's formula to find a sequence $\mathcal{S}_k$ of $(n_k, 2\varepsilon)$ separated sets with $n_k \to \infty$ so that $\#\mathcal{S}_k \sim \exp(n_k h_{\mu_{\rho(k)}})$ and if $x \in \mathcal{S}_k$, then $S_{n_k}\varphi(x) \sim n_k \int \varphi d\mu_{\rho(k)}$.

- By the method of lemma 3.0.2, use the specification property to construct points which shadow points taken from $\mathcal{S}_1, \ldots, \mathcal{S}_k, \ldots$ respectively. The set of all such points is a fractal $F \subset \widehat{X}(\varphi, f)$.

- Construct a measure on F suitable for an application of the Entropy Distribution Principle. The idea is as follows. Let $\mu_k = \frac{1}{\#\mathcal{S}_k}\sum_{x\in\mathcal{S}_k}\delta_x$. Since $\mathcal{S}_k$ is (n_k, ε) separated, then

$$\mu_k(B_{n_k}(q, \varepsilon)) \leq \#\mathcal{S}_k^{-1} \sim \exp\{-n_k(h_{top}(f) - \varepsilon)\}.$$

  We define μ to be the weak* limit of measures defined similarly to μ_k.

Chapter 4

The irregular set for maps with the specification property has full topological pressure

For a compact metric space (X, d), a continuous map $f : X \mapsto X$ and a continuous potential $\varphi : X \mapsto \mathbb{R}$, we recall that the irregular set for φ is defined to be

$$\widehat{X}(\varphi, f) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \text{ does not exist } \right\}.$$

The irregular set arises naturally in the context of multifractal analysis, where one decomposes a space X into the disjoint union

$$X = \bigcup_{\alpha \in \mathbb{R}} X(\varphi, \alpha) \cup \widehat{X}(\varphi, f),$$

where $X(\varphi, \alpha)$ is the set of points for which the Birkhoff average of φ is equal to α. In this chapter, we begin a program to understand the topological pressure of the multifractal decomposition by focusing on the irregular set $\widehat{X}(\varphi, f)$. We consider the topological pressure of the sets $X(\varphi, \alpha)$ in chapter 5.

As a consequence of Birkhoff's ergodic theorem, the irregular set is not detectable from the point of view of an invariant measure. However, it is an increasingly well known phenomenon that the irregular set can be large from the point of view of dimension theory [Bar]. Symbolic dynamics methods have confirmed this in the uniformly hyperbolic setting [BS5], for certain non-uniformly hyperbolic examples [PW] and for a large class of multimodal maps [Tod]. The irregular set has also been the focus of a great deal of work by Olsen and collaborators [BOS].

The irregular set could also have a special role in physical applications. Ruelle uses the terminology 'set of points with historic behaviour' to describe the irregular set [Rue2]. The idea is that points for which the Birkhoff average does not exist are capturing the 'history' of the system, whereas points whose Birkhoff average converge only see average behaviour. For example, in the dynamics of the weather, the irregular points are the ones that have observed epochs of climate change. In [Tak], Takens asks for which smooth dynamical systems the irregular set has positive Lebesgue measure. We take a topological point of view and prove that the irregular set is as large as it can be with respect to the topological pressure, which is a family of dimension characteristics parametrised by the continuous functions.

Main result of chapter 4. *When f has the specification property, $\widehat{X}(\varphi, f)$ has full topological pressure or is the empty set. We give conditions on φ which completely describe which of the two cases hold.*

This result is stated formally as theorem 4.1.2. The first to notice the phenomenon of the irregular set carrying full entropy were Pesin and Pitskel [PP2] in the case of the Bernoulli shift on 2 symbols. Barreira and Schmeling [BS5] studied the irregular set for a variety of uniformly hyperbolic systems using symbolic dynamics. They showed that, for example, the irregular set of a generic Hölder continuous function on a conformal repeller has full entropy (and Hausdorff dimension). These arguments can be found in Barreira's book [Bar] in which the result is also proved for subshifts with the specification property. We note that these arguments do not extend to the more general class of maps with the specification property. Furthermore, we consider irregular sets for any continuous functions, whereas Barreira considers only funcions φ for which $t\varphi$ has a unique equilibrium state for every $t \in \mathbb{R}$.

Takens and Verbitskiy have obtained multifractal analysis results for the class of maps with specification, using topological entropy as the dimension characteristic [TV2], [TV1]. However, they do not consider the irregular set. Ercai, Kupper and Lin [EKL] proved that the irregular set is either empty or carries full entropy for maps with the specification property. Our results were derived independently and include the result of [EKL] as a special case. Our methods are largely inspired by those of Takens and Verbitskiy [TV2], and we follow the strategy that we sketched in chapter 3.

We apply our main result to show that the irregular set for a suspension flow over a map with specification has full topological entropy. By considering the 'u-dimension' of the irregular set in the base, Barreira and Saussol [BS1] proved analogous results which apply when the suspension is over a shift of finite type. They assume Hölder continuity of φ and the roof function, whereas we require only continuity.

We expect that an analogue of our main theorem 4.1.2 holds for flows with the specification property, and that our current method of proof can be adapted to this setting (although we do not pursue this here). Such an approach would not cover every suspension flow to which our current results apply. In particular, a special flow (i.e. a suspension flow with constant roof function) over a map with specification never has the specification property itself, but is in the class of flows treated in §4.3.

In §4.1, we state the main results of the chapter and key ideas of the proof. In §4.2, we prove the main theorem of the chapter. In §4.3, we apply our main result to suspension flows.

4.1 Results

We state our results and introduce the key technical tools of the proof.

Theorem 4.1.1. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a continuous map with the specification property. Assume that $\varphi \in C(X)$ satisfies $\inf_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu$. Let $\widehat{X}(\varphi, f)$ be the irregular set for φ, then $P_{\widehat{X}(\varphi,f)}(\psi) = P_X^{classic}(\psi)$ for all $\psi \in C(X)$.*

We remark that lemma 2.3.2 provides us with other natural interpretations of the assumption $\inf_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu$. We state the assumption in this way because it is natural for the method of proof. If our assumption fails, then $\widehat{X}(\varphi, f) = \emptyset$. In fact, we prove a slightly stronger version of the theorem.

Theorem 4.1.2. *Let (X, d) be a compact metric space, $f : X \mapsto X$ be a continuous map and $X' \subseteq X$ be an f-invariant Borel set. Assume f satisfies the specification property on X'. Assume that $\varphi \in C(X)$ satisfies $\inf_{\mu \in \mathcal{M}_f(X')} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}_f(X')} \int \varphi d\mu$. Let $\widehat{X}(\varphi, f)$ be the irregular set for φ, then for all $\psi \in C(X)$,*

$$P_{\widehat{X}(\varphi,f)}(\psi) \geq \sup \left\{ h_\mu + \int \psi d\mu : \mu \in \mathcal{M}_f(X') \right\}.$$

If $\sup \{h_\mu + \int \psi d\mu : \mu \in \mathcal{M}_f(X')\} = P_X^{classic}(\psi)$, then we have $P_{\widehat{X}(\varphi,f)}(\psi) = P_X^{classic}(\psi)$.

If $\mathcal{M}_f(X')$ is dense in $\mathcal{M}_f(X)$, we need only assume $\inf_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu$. As described in chapter 3, we follow the method of Takens and Verbitskiy [TV2]. The key ingredients for the Takens and Verbitskiy proof are an application of the Entropy Distribution Principle [TV2] and Katok's formula for measure-theoretic entropy [Kat]. We are required to generalise both. We offer two generalisations of the Entropy Distribution Principle. While the first offers a more straight forward generalisation, we will use the second as it offers us a short cut in the proof later on. We offer a proof of only the second version, since it is more general than the first.

Proposition 4.1.3 (Pressure Distribution Principle)**.** *Let $f : X \mapsto X$ be a continuous transformation. Let $Z \subseteq X$ be an arbitrary Borel set. Suppose there exists a constant $s \geq 0$ such that for sufficiently small $\varepsilon > 0$ one can find a Borel probability measure μ_ε, an integer $N(\varepsilon)$ and a constant $K(\varepsilon) > 0$*

satisfying $\mu_\varepsilon(Z) > 0$ and $\mu_\varepsilon(B_n(x,\varepsilon)) \leq K(\varepsilon)\exp\{-ns + \sum_{i=0}^{n-1}\psi(f^i x)\}$ for every ball $B_n(x,\varepsilon)$ such that $B_n(x,\varepsilon)\cap Z \neq \emptyset$ and $n \geq N(\varepsilon)$. Then $P_Z(\psi) \geq s$.

Proposition 4.1.4 (Generalised Pressure Distribution Principle)**.** *Let $f : X \mapsto X$ be a continuous transformation. Let $Z \subseteq X$ be an arbitrary Borel set. Suppose there exists $\varepsilon > 0$ and $s \geq 0$ such that one can find a sequence of Borel probability measures μ_k, a constant $K > 0$ and an integer N satisfying*

$$\limsup_{k\to\infty} \mu_k(B_n(x,\varepsilon)) \leq K\exp\{-ns + \sum_{i=0}^{n-1}\psi(f^i x)\}$$

for every ball $B_n(x,\varepsilon)$ such that $B_n(x,\varepsilon)\cap Z \neq \emptyset$ and $n \geq N$. Furthermore, assume that at least one limit measure ν of the sequence μ_k satisfies $\nu(Z) > 0$. Then $P_Z(\psi,\varepsilon) \geq s$.

Proof. Choose $\varepsilon > 0$ and ν satisfying the conditions of the theorem. Let μ_{k_j} denote a subsequence of measures which converges to ν. Let $\Gamma = \{B_{n_i}(x_i,\varepsilon)\}_i$ cover Z with all $n_i \geq N'$ for some $N' \geq N$. We may assume that $B_{n_i}(x_i,\varepsilon)\cap Z \neq \emptyset$ for every i. Then

$$\begin{aligned}
Q(Z,s,\Gamma,\psi) &= \sum_i \exp\left\{-sn_i + \sup_{y\in B_{n_i}(x_i,\varepsilon)} \sum_{k=0}^{n_i-1}\psi(f^k(y))\right\} \\
&\geq \sum_i \exp\left\{-sn_i + \sum_{k=0}^{n_i-1}\psi(f^k(x_i))\right\} \\
&\geq K^{-1}\sum_i \limsup_{k\to\infty}\mu_k(B_n(x_i,\varepsilon)) \\
&\geq K^{-1}\sum_i \liminf_{j\to\infty}\mu_{k_j}(B_n(x_i,\varepsilon)) \\
&\geq K^{-1}\sum_i \nu(B_n(x_i,\varepsilon)) \geq K^{-1}\nu(Z) > 0.
\end{aligned}$$

Our arrival at the last line is because for any open set U, if ν_k converges to ν in the weak* topology, then $\liminf_{k\to\infty}\nu_k(U) \geq \nu(U)$ (see [Wal], p.149). We conclude that $M(Z,s,\varepsilon,N',\psi) \geq K^{-1}\nu(Z) > 0$ for all $N' \geq N$. Thus $m(Z,s,\varepsilon,\psi) > 0$ and $P_Z(\psi,\varepsilon) \geq s$. □

The following result generalises Katok's formula for measure-theoretic entropy. In [Men], Mendoza gave a proof based on ideas from the Misiurewicz proof of the variational principle. Although he states the result under the assumption that f is a homeomorphism, his proof works for f continuous.

Proposition 4.1.5. *Let (X,d) be a compact metric space, $f : X \mapsto X$ be a continuous map and μ be an ergodic invariant measure. For $\varepsilon > 0$, $\gamma \in (0,1)$ and $\psi \in C(X)$, define*

$$N^\mu(\psi,\gamma,\varepsilon,n) = \inf\left\{\sum_{x\in S}\exp\left\{\sum_{i=0}^{n-1}\psi(f^i x)\right\}\right\},$$

where the infimum is taken over all sets S which (n,ε) span some set Z with $\mu(Z) \geq 1-\gamma$. We have

$$h_\mu + \int \psi d\mu = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N^\mu(\psi,\gamma,\varepsilon,n).$$

The formula remains true if we replace the $\liminf$ *by* $\limsup$.

We prove a modified version of proposition 4.1.5 (for entropy) as theorem 6.3.1. The technique used there is also suitable for a proof of proposition 4.1.5.

We now begin the proof of theorem 4.1.2. For the sake of clarity, it will be convenient to give the proof under a certain additional hypothesis, which we will later explain how to remove.

Theorem 4.1.6. *Let us assume the hypotheses of theorem 4.1.2 and fix $\psi \in C(X)$. Let*

$$C := \sup\left\{h_\mu + \int \psi d\mu : \mu \in \mathcal{M}_f(X')\right\}.$$

Let us assume further that $P_X^{classic}(\psi)$ is finite and for all $\gamma > 0$, there exist ergodic measures $\mu_1, \mu_2 \in \mathcal{M}_f(X')$ which satisfy

(1) $h_{\mu_i} + \int \psi d\mu_i > C - \gamma$ for $i = 1,2$,

(2) $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$.

Then $P_{\widehat{X}(\varphi,f)}(\psi) \geq C$. If $C = P_X^{classic}(\psi)$, for example when $X' = X$, then $P_{\widehat{X}(\varphi,f)}(\psi) = P_X^{classic}(\psi)$.

The assumption that $P_X^{classic}(\psi)$ is finite is trivial to remove and is included only for notational convenience. Given a result from [PS1], we give a short proof that the hypotheses of theorem 4.1.1 imply those of theorem 4.1.6. We explain how to modify the proof of theorem 4.1.6 to obtain a self contained proof of theorem 4.1.2 in §4.2.2.

Proof of theorem 4.1.1. Let μ_1 be ergodic and satisfy $h_{\mu_1} + \int \psi d\mu_1 > C - \gamma/3$, Let $\nu \in \mathcal{M}_f(X)$ satisfy $\int \varphi d\mu_1 \neq \int \varphi d\nu$. Let $\nu' = t\mu_1 + (1-t)\nu$ where $t \in (0,1)$ is chosen sufficiently close to 1 so that $h_{\nu'} + \int \psi d\nu' > C - 2\gamma/3$. By [PS1], when f has a property called the g-almost product property (see chapter 6), which is weaker than specification, we can find a sequence of ergodic measures $\nu_n \in \mathcal{M}_f(X)$ such that $h_{\nu_n} \to h_{\nu'}$ and $\nu_n \to \nu'$ in the weak-$*$ topology (this also follows from theorem B of [EKW] when f has specification and the map $\mu \to h_\mu$ is upper semi-continuous). Therefore, we can choose a measure belonging to this sequence which we call μ_2 which satisfies $h_{\mu_2} + \int \psi d\mu_2 > C - \gamma$ and $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$. $\square$

4.2 Proof of the main theorem 4.1.6

Let us fix a small $\gamma > 0$, and take the measures μ_1 and μ_2 provided by our hypothesis. Choose $\delta > 0$ sufficiently small so

$$\left| \int \varphi d\mu_1 - \int \varphi d\mu_2 \right| > 4\delta.$$

Let $\rho : \mathbb{N} \mapsto \{1, 2\}$ be given by $\rho(k) = (k+1)(\text{mod}2) + 1$. Choose a strictly decreasing sequence $\delta_k \to 0$ with $\delta_1 < \delta$ and a strictly increasing sequence $l_k \to \infty$ so the set

$$Y_k := \left\{ x \in X' : \left| \frac{1}{n} S_n \varphi(x) - \int \varphi d\mu_{\rho(k)} \right| < \delta_k \text{ for all } n \geq l_k \right\} \tag{4.1}$$

satisfies $\mu_{\rho(k)}(Y_k) > 1 - \gamma$ for every k. This is possible by Birkhoff's ergodic theorem.

The following lemma follows readily from proposition 4.1.5.

Lemma 4.2.1. *For any sufficiently small $\varepsilon > 0$, we can find a sequence $n_k \to \infty$ and a countable collection of finite sets $\mathcal{S}_k$ so that each $\mathcal{S}_k$ is an $(n_k, 4\varepsilon)$ separated set for Y_k and $M_k := \sum_{x \in \mathcal{S}_k} \exp\left\{ \sum_{i=0}^{n_k - 1} \psi(f^i x) \right\}$ satisfies*

$$M_k \geq \exp(n_k(C - 4\gamma)).$$

Furthermore, the sequence n_k can be chosen so that $n_k \geq l_k$ and $n_k \geq 2^{m_k}$, where $m_k = m(\varepsilon/2^k)$ is as in definition 2.2.3 of the specification property.

Proof. By proposition 4.1.5, let us choose ε sufficiently small so

$$\liminf_{n \to \infty} \frac{1}{n} \log N^{\mu_i}(\psi, \gamma, 4\varepsilon, n) \geq h_{\mu_i} + \int \psi d\mu_i - \gamma \geq C - 2\gamma \text{ for } i = 1, 2.$$

For $A \subset X$, recall that

$$Q_n(A, \psi, \varepsilon) = \inf \left\{ \sum_{x \in S} \exp\left\{ \sum_{k=0}^{n-1} \psi(f^k x) \right\} : S \text{ is } (n, \varepsilon) \text{ spanning set for } A \right\},$$

$$P_n(A, \psi, \varepsilon) = \sup \left\{ \sum_{x \in S} \exp\left\{ \sum_{k=0}^{n-1} \psi(f^k x) \right\} : S \text{ is } (n, \varepsilon) \text{ separated set for } A \right\}.$$

We have $Q_n(A, \psi, \varepsilon) \leq P_n(A, \psi, \varepsilon)$ and since $\mu_{\rho(k)}(Y_k) > 1 - \gamma$ for every k, it is immediate that

$$Q_n(Y_k, \psi, 4\varepsilon) \geq N^{\mu_{\rho(k)}}(\psi, \gamma, \varepsilon, n).$$

Let $M(k, n) = P_n(Y_k, \psi, 4\varepsilon)$. For each k, we obtain

$$\liminf_{n \to \infty} \frac{1}{n} \log M(k, n) \geq \liminf_{n \to \infty} \frac{1}{n} \log N^{\mu_{\rho(k)}}(\psi, \gamma, 4\varepsilon, n) \geq C - 2\gamma.$$

We may now choose a sequence $n_k \to \infty$ satisfying the hypotheses of the lemma so

$$\frac{1}{n_k} \log M(k, n_k) \geq C - 3\gamma.$$

Now for each k, let $\mathcal{S}_k$ be a choice of $(n_k, 4\varepsilon)$ separated set for Y_k which satisfies

$$\frac{1}{n_k}\log\left\{\sum_{x\in\mathcal{S}_k}\exp\left\{\sum_{i=0}^{n-1}\psi(f^ix)\right\}\right\} \geq \frac{1}{n_k}\log M(k,n_k) - \gamma.$$

Let $M_k := \sum_{x\in\mathcal{S}_k}\exp\left\{\sum_{i=0}^{n-1}\psi(f^ix)\right\}$, then

$$\frac{1}{n_k}\log M_k \geq \frac{1}{n_k}\log M(k,n_k) - \gamma \geq C - 4\gamma.$$

We rearrange to obtain the desired result. □

We choose ε sufficiently small so that $\mathrm{Var}(\psi, 2\varepsilon) < \gamma$ and $\mathrm{Var}(\varphi, 2\varepsilon) < \delta$, and fix all the ingredients provided by lemma 4.2.1.

Our strategy is to construct a certain fractal $F \subset \widehat{X}(\varphi, f)$, on which we can define a sequence of measures suitable for an application of the generalised Pressure Distribution Principle.

4.2.1 Construction of the fractal F

We begin by constructing two intermediate families of finite sets. The first such family we denote by $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$ and consists of points which shadow a very large number N_k of points from $\mathcal{S}_k$. The second family we denote by $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$ and consist of points which shadow points (taken in order) from $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$. We choose N_k to grow to infinity very quickly, so the ergodic average of a point in $\mathcal{T}_k$ is close to the corresponding point in $\mathcal{C}_k$.

Construction of the intermediate sets $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$

Let us choose a sequence N_k which increases to ∞ sufficiently quickly so that

$$\lim_{k\to\infty}\frac{n_{k+1}+m_{k+1}}{N_k} = 0, \lim_{k\to\infty}\frac{N_1(n_1+m_1)+\ldots+N_k(n_k+m_k)}{N_{k+1}} = 0. \tag{4.2}$$

We enumerate the points in the sets $\mathcal{S}_k$ provided by lemma 4.2.1 and write them as follows

$$\mathcal{S}_k = \{x_i^k : i = 1, 2, \ldots, \#\mathcal{S}_k\}.$$

We make a choice of k and consider the set of words of length N_k with entries in $\{1, 2, \ldots, \#\mathcal{S}_k\}$. Each such word $\underline{i} = (i_1, \ldots, i_{N_k})$ represents a point in $\mathcal{S}_k^{N_k}$. Using the specification property, we can choose a point $y := y(i_1, \ldots, i_{N_k})$ which satisfies

$$d_{n_k}(x_{i_j}^k, f^{a_j}y) < \frac{\varepsilon}{2^k}$$

for all $j \in \{1, \ldots, N_k\}$, where $a_j = (j-1)(n_k+m_k)$. In other words, y shadows each of the points $x_{i_j}^k$ in order for length n_k and gap m_k. We define

$$\mathcal{C}_k = \left\{y(i_1, \ldots, i_{N_k}) \in X : (i_1, \ldots, i_{N_k}) \in \{1, \ldots, \#\mathcal{S}_k\}^{N_k}\right\}.$$

Let $c_k = N_k n_k + (N_k - 1)m_k$. Then c_k is the amount of time for which the orbit of points in $\mathcal{C}_k$ has been prescribed. It is a corollary of the following lemma that distinct sequences $(i_1, \ldots, i_{N_k})$ give rise to distinct points in $\mathcal{C}_k$. Thus the cardinality of $\mathcal{C}_k$, which we shall denote by $\#\mathcal{C}_k$, is $\#S_k^{N_k}$.

Lemma 4.2.2. *Let $\underline{i}$ and $\underline{j}$ be distinct words in $\{1, 2, \ldots, \#S_k\}^{N_k}$. Then $y_1 := y(\underline{i})$ and $y_2 := y(\underline{j})$ are $(c_k, 3\varepsilon)$ separated points (i.e. $d_{c_k}(y_1, y_2) > 3\varepsilon$).*

Proof. Since $\underline{i} \neq \underline{j}$, there exists l so $i_l \neq j_l$. We have

$$d_{n_k}(x^k_{i_l}, f^{a_l} y_1) < \frac{\varepsilon}{2^k}, d_{n_k}(x^k_{j_l}, f^{a_l} y_2) < \frac{\varepsilon}{2^k} \text{ and } d_{n_k}(x^k_{i_l}, x^k_{j_l}) > 4\varepsilon.$$

Combining these inequalities, we have

$$\begin{aligned} d_{c_k}(y_1, y_2) &\geq d_{n_k}(f^{a_l} y_1, f^{a_l} y_2) \\ &\geq d_{n_k}(x^k_{i_l}, x^k_{j_l}) - d_{n_k}(x^k_{i_l}, f^{a_l} y_1) - d_{n_k}(x^k_{j_l}, f^{a_l} y_2) \\ &> 4\varepsilon - \varepsilon/2 - \varepsilon/2 = 3\varepsilon. \end{aligned}$$

□

Construction of the intermediate sets $\{\mathcal{T}_k\}_{k \in \mathbb{N}}$

We use the specification property to construct points whose orbits shadow points (taken in order) from $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$. Formally, we define $\mathcal{T}_k$ inductively. Let $\mathcal{T}_1 = \mathcal{C}_1$. We construct $\mathcal{T}_{k+1}$ from $\mathcal{T}_k$ as follows. Let $x \in \mathcal{T}_k$ and $y \in \mathcal{C}_{k+1}$. Let $t_1 = c_1$ and $t_{k+1} = t_k + m_{k+1} + c_{k+1}$. Using specification, we can find a point $z := z(x, y)$ which satisfies

$$d_{t_k}(x, z) < \frac{\varepsilon}{2^{k+1}} \text{ and } d_{c_{k+1}}(y, f^{t_k + m_{k+1}} z) < \frac{\varepsilon}{2^{k+1}}.$$

Define $\mathcal{T}_{k+1} = \{z(x, y) : x \in \mathcal{T}_k, y \in \mathcal{C}_{k+1}\}$. Note that t_k is the amount of time for which the orbit of points in $\mathcal{T}_k$ has been prescribed. Once again, points constructed in this way are distinct. So we have

$$\#\mathcal{T}_k = \#\mathcal{C}_1 \ldots \#\mathcal{C}_k = \#S_1^{N_1} \ldots \#S_k^{N_k}.$$

This fact is a corollary of the following straightforward lemma:

Lemma 4.2.3. *For every $x \in \mathcal{T}_k$ and distinct $y_1, y_2 \in \mathcal{C}_{k+1}$*

$$d_{t_k}(z(x, y_1), z(x, y_2)) < \frac{\varepsilon}{2^k} \textit{ and } d_{t_{k+1}}(z(x, y_1), z(x, y_2)) > 2\varepsilon.$$

Thus $\mathcal{T}_k$ is a $(t_k, 2\varepsilon)$ separated set. In particular, if $z, z' \in \mathcal{T}_k$, then

$$\overline{B}_{t_k}(z, \frac{\varepsilon}{2^k}) \cap \overline{B}_{t_k}(z', \frac{\varepsilon}{2^k}) = \emptyset.$$

Proof. Let $p := z(x, y_1)$ and $q := z(x, y_2)$. The first inequality is trivial since by construction, $d_{t_k}(x, z_i) < \varepsilon/2^{k+1}$ for $i = 1, 2$.

Using lemma 4.2.2, we obtain the second inequality as follows:

$$\begin{aligned}
d_{t_{k+1}}(p,q) &\geq d_{c_{k+1}}(f^{t_k+m_{k+1}}p, f^{t_k+m_{k+1}}q) \\
&\geq d_{c_{k+1}}(y_1,y_2) - d_{c_{k+1}}(y_1, f^{t_k+m_{k+1}}p) - d_{c_{k+1}}(y_2, f^{t_k+m_{k+1}}q) \\
&> 3\varepsilon - \varepsilon/2 - \varepsilon/2 = 2\varepsilon.
\end{aligned}$$

The third statement is a straightforward consequence of the second. □

Following the terminology of Takens and Verbitskiy, we say $z \in \mathcal{T}_{k+1}$ descends from $x \in \mathcal{T}_k$ if $z = z(x, y)$ for some $y \in \mathcal{C}_{k+1}$.

Lemma 4.2.4. *If $z \in \mathcal{T}_{k+1}$ descends from $x \in \mathcal{T}_k$ then*

$$\overline{B}_{t_{k+1}}(z, \frac{\varepsilon}{2^k}) \subset \overline{B}_{t_k}(x, \frac{\varepsilon}{2^{k-1}}).$$

Proof. Let $z' \in \overline{B}_{t_{k+1}}(z, \frac{\varepsilon}{2^k})$. Then

$$\begin{aligned}
d_{t_k}(z',x) &\leq d_{t_{k+1}}(z',z) + d_{t_k}(z,x) \\
&\leq \varepsilon/2^k + \varepsilon/2^{k+1} \leq \varepsilon/2^{k-1}.
\end{aligned}$$

□

Construction of the fractal F and a special sequence of measures μ_k

Let $F_k = \bigcup_{x \in \mathcal{T}_k} \overline{B}_{t_k}(x, \frac{\varepsilon}{2^{k-1}})$. By lemma 4.2.4, $F_{k+1} \subset F_k$. Since we have a decreasing sequence of compact sets, the intersection $F = \bigcap_k F_k$ is non-empty. Further, every point $p \in F$ can be uniquely represented by a sequence $\underline{p} = (\underline{p}_1, \underline{p}_2, \underline{p}_3, \ldots)$ where each $\underline{p}_i = (p_1^i, \ldots, p_{N_i}^i) \in \{1, 2, \ldots, \#S_i\}^{N_i}$. Each point in $\mathcal{T}_k$ can be uniquely represented by a finite word $(\underline{p}_1, \ldots \underline{p}_k)$. We introduce some useful notation to help us see this. Let $y(\underline{p}_i) \in \mathcal{C}_i$ be defined as in 4.2.1. Let $z_1(\underline{p}) = y(\underline{p}_1)$ and proceeding inductively, let $z_{i+1}(\underline{p}) = z(z_i(\underline{p}), y(\underline{p}_{i+1})) \in \mathcal{T}_{i+1}$ be defined as in 4.2.1. We can also write $z_i(\underline{p})$ as $z(\underline{p}_1, \ldots, \underline{p}_i)$. Then define $p := \pi \underline{p}$ by

$$p = \bigcap_{i \in \mathbb{N}} \overline{B}_{t_i}(z_i(\underline{p}), \frac{\varepsilon}{2^{i-1}}).$$

It is clear from our construction that we can uniquely represent every point in F in this way.

Lemma 4.2.5. *Given $z = z(\underline{p}_1, \ldots, \underline{p}_k) \in \mathcal{T}_k$, we have for all $i \in \{1, \ldots, k\}$ and all $l \in \{1, \ldots, N_i\}$,*

$$d_{n_i}(x^i_{p^i_l}, f^{t_{i-1}+m_{i-1}+(l-1)(m_i+n_i)}z) < 2\varepsilon.$$

Proof. We fix $i \in \{1, \ldots, k\}$ and $l \in \{1, \ldots, N_i\}$. For $m \in \{1, \ldots, k-1\}$, let $z_m = z(\underline{p}_1, \ldots, \underline{p}_m) \in \mathcal{T}_m$. Let $a = t_{i-1} + m_{i-1}$ and $b = (l-1)(m_i + n_i)$. Then

$$d_{n_i}(x^i_{p^i_l}, f^{a+b}z) < d_{n_i}(x^i_{p^i_l}, f^b y(\underline{p}_i)) + d_{n_i}(f^b y(\underline{p}_i), f^{a+b}z_i) + d_{n_i}(f^{a+b}z_i, f^{a+b}z).$$

We have, by construction,

$$d_{n_i}(x^i_{p^i_l}, f^b y(\underline{p}_i)) < \frac{\varepsilon}{2^i}.$$

We have, by construction,

$$d_{n_i}(f^b y(\underline{p}_i), f^{a+b}z_i) \leq d_{c_i}(y(\underline{p}_i), f^a z) < \frac{\varepsilon}{2^{i+1}}.$$

We have

$$\begin{aligned} d_{n_i}(f^{a+b}z_i, f^{a+b}z) < d_{t_i}(z_i, z) &< d_{t_i}(z_i, z_{i+1}) + \ldots + d_{t_i}(z_{k-1}, z) \\ &< \frac{\varepsilon}{2^{i+1}} + \frac{\varepsilon}{2^{i+2}} + \ldots + \frac{\varepsilon}{2^k}. \end{aligned}$$

Combining the inequalities, we obtain $d_{n_i}(f^{a+b}z, x^i_{p^i_l}) < \sum_{m=i}^{k} \frac{\varepsilon}{2^m} + \frac{\varepsilon}{2^{i+1}} < 2\varepsilon$, as required. $\square$

We now define the measures on F which yield the required estimates for the Pressure Distribution Principle. For each $z \in \mathcal{T}_k$, we associate a number $\mathcal{L}(z) \in (0, \infty)$. Using these numbers as weights, we define, for each k, an atomic measure centred on $\mathcal{T}_k$. Precisely, if $z = z(\underline{p}_1, \ldots \underline{p}_k)$, we define

$$\mathcal{L}(z) := \mathcal{L}(\underline{p}_1) \ldots \mathcal{L}(\underline{p}_k),$$

where if $\underline{p}_i = (p^i_1, \ldots, p^i_{N_i}) \in \{1, \ldots, \#\mathcal{S}_i\}^{N_i}$, then

$$\mathcal{L}(\underline{p}_i) := \prod_{l=1}^{N_i} \exp S_{n_i}\psi(x^i_{p^i_l}).$$

We define

$$\nu_k := \sum_{z \in \mathcal{T}_k} \delta_z \mathcal{L}(z).$$

We normalise ν_k to obtain a sequence of probability measures μ_k. More precisely, we let $\mu_k := \frac{1}{\kappa_k}\nu_k$, where κ_k is the normalising constant

$$\kappa_k := \sum_{z \in \mathcal{T}_k} \mathcal{L}_k(z).$$

Lemma 4.2.6. $\kappa_k = M_1^{N_1} \ldots M_k^{N_k}$.

Proof. We note that

$$\begin{aligned} \sum_{\underline{p}_i \in \{1, \ldots, \#\mathcal{S}_i\}^{N_i}} \mathcal{L}(\underline{p}_i) &= \sum_{p^i_1=1}^{\#\mathcal{S}_i} \exp S_{n_i}\psi(x^i_{p^1_l}) \ldots \sum_{p^i_{N_i}=1}^{\#\mathcal{S}_i} \exp S_{n_i}\psi(x^i_{p^i_{N_i}}) \\ &= M_i^{N_i} \end{aligned}$$

By the definition and since each $z \in \mathcal{T}_k$ corresponds uniquely to a sequence $(\underline{p}_1, \ldots, \underline{p}_k)$, we have

$$\sum_{z \in \mathcal{T}_k} \mathcal{L}_k(z) = \sum_{\underline{p}_1 \in \{1,\ldots,\#\mathcal{S}_1\}^{N_1}} \cdots \sum_{\underline{p}_k \in \{1,\ldots,\#\mathcal{S}_k\}^{N_k}} \mathcal{L}(\underline{p}_1) \ldots \mathcal{L}(\underline{p}_k).$$

The result follows. □

Lemma 4.2.7. *Suppose ν is a limit measure of the sequence of probability measures μ_k. Then $\nu(F) = 1$.*

Proof. Suppose ν is a limit measure of the sequence of probability measures μ_k. Then $\nu = \lim_{k\to\infty} \mu_{l_k}$ for some $l_k \to \infty$. For any fixed l and all $p \geq 0$, $\mu_{l+p}(F_l) = 1$ since $\mu_{l+p}(F_{l+p}) = 1$ and $F_{l+p} \subseteq F_l$. Therefore, $\nu(F_l) \geq \limsup_{k\to\infty} \mu_{l_k}(F_l) = 1$. It follows that $\nu(F) = \lim_{l\to\infty} \nu(F_l) = 1$. □

In fact, the measures μ_k converge. However, by using the generalised Pressure Distribution Principle, we do not need to use this fact and so we omit the proof (which goes like lemma 5.4 of [TV2]).

We verify that $F \subset \widehat{X}(\varphi, f)$.

Lemma 4.2.8. *For any $p \in F$, the sequence $\frac{1}{t_k}\sum_{i=0}^{t_k-1} \varphi(f^i(p))$ diverges.*

Proof. Let us choose a point $p \in F$. Using the notation of 4.2.1, let $y_k := y(\underline{p}_k)$ and $z_k = z_k(\underline{p})$. We first show that

$$\left|\frac{1}{c_k} S_{c_k}\varphi(y_k) - \int \varphi d\mu_{\rho(k)}\right| \to 0. \tag{4.3}$$

We rely on the fact that Var$(\varphi, c) \to 0$ as $c \to 0$ and that

$$\lim_{k\to\infty} \frac{n_k N_k}{c_k} = 1, \lim_{k\to\infty} \frac{m_k(N_k-1)}{c_k} = 0 \text{ and } \lim_{k\to\infty} \delta_k = 0. \tag{4.4}$$

The first two limits follow from the assumption that $n_k \geq 2^{m_k}$. Let $a_j = (j-1)(n_k + m_k)$. We have

$$\begin{aligned}\left|S_{c_k}\varphi(y_k) - c_k \int \varphi d\mu_{\rho(k)}\right| &\leq \left|\sum_{j=1}^{N_k} S_{n_k}\varphi(f^{a_j} y_k) - c_k \int \varphi d\mu_{\rho(k)}\right| \\ &+ m_k(N_k-1)\|\varphi\|\end{aligned}$$

$$\begin{aligned}\leq \left|\sum_{j=1}^{N_k} S_{n_k}\varphi(f^{a_j} y_k) - \sum_{j=1}^{N_k} S_{n_k}\varphi(x_{i_j}^k)\right| &+ \left|\sum_{j=1}^{N_k} S_{n_k}\varphi(x_{i_j}^k) - c_k \int \varphi d\mu_{\rho(k)}\right| \\ &+ m_k(N_k-1)\|\varphi\|\end{aligned}$$

$$\begin{aligned}\leq \sum_{j=1}^{N_k} \left|S_{n_k}\varphi(f^{a_j} y_k) - S_{n_k}\varphi(x_{i_j}^k)\right| &+ \sum_{j=1}^{N_k} \left|S_{n_k}\varphi(x_{i_j}^k) - n_k \int \varphi d\mu_{\rho(k)}\right| \\ &+ m_k(N_k-1)\{\|\varphi\| + \int \varphi d\mu_{\rho(k)}\}\end{aligned}$$

$$\leq n_k N_k\{\mathsf{Var}(\varphi, \varepsilon/2^k) + \delta_k\} + m_k(N_k - 1)\{\|\varphi\| + \int \varphi d\mu_{\rho(k)}\}.$$

We have used the fact $d_{n_k}(x^k_{i_j}, f^{a_j} y_k) < \varepsilon/2^k$ in the last line. The statement of (4.3) follows from this and (4.4).

Let $p' = f^{t_k - c_k} p$ and $z'_k = f^{t_k - c_k} z_k$. Using $d_{t_k}(p, z_k) \leq \varepsilon/2^{k-1}$, we have

$$\begin{aligned} d_{c_k}(p', y_k) &\leq d_{c_k}(p', z'_k) + d_{c_k}(z'_k, y_k) \\ &\leq \varepsilon/2^{k-1} + \varepsilon/2^k \leq \varepsilon/2^{k-2}. \end{aligned}$$

Using this and (4.3), we obtain

$$\left| \frac{1}{c_k} S_{c_k} \varphi(p') - \int \varphi d\mu_{\rho(k)} \right| \leq \mathsf{Var}(\varphi, \varepsilon/2^{k-2}). \tag{4.5}$$

The final ingredient we require is to show that

$$\left| \frac{1}{t_k} S_{t_k} \varphi(p) - \frac{1}{c_k} S_{c_k} \varphi(p') \right| \to 0. \tag{4.6}$$

From the assumptions of (4.2), we can verify that $c_k/t_k \to 1$. Thus for arbitrary $\gamma > 0$ and sufficiently large k, we have $|c_k/t_k - 1| < \gamma$. We have

$$\begin{aligned} \left| \frac{1}{t_k} S_{t_k} \varphi(p) - \frac{1}{c_k} S_{c_k} \varphi(p') \right| &= \left| \frac{1}{t_k} S_{t_k - c_k} \varphi(p) + \frac{1}{c_k} S_{c_k} \varphi(p') \left(\frac{c_k}{t_k} - 1 \right) \right| \\ &\leq \left| \frac{t_k - c_k}{t_k} \|\varphi\| + \gamma \frac{1}{c_k} S_{c_k} \varphi(p') \right| \\ &\leq 2\gamma \|\varphi\| \end{aligned}$$

Since γ was arbitrary, we have verified (4.6). Using (4.5) and (4.6), it follows that

$$\left| \frac{1}{t_k} S_{t_k} \varphi(p) - \int \varphi d\mu_{\rho(k)} \right| \to 0.$$

□

In order to prove theorem 4.1.6, we give a sequence of lemmas which will allow us to apply the generalised Pressure Distribution Principle. Let $\mathcal{B} := B_n(q, \varepsilon/2)$ be an arbitrary ball which intersects F. Let k be the unique number which satisfies $t_k \leq n < t_{k+1}$. Let $j \in \{0, \dots, N_{k+1} - 1\}$ be the unique number so

$$t_k + (n_{k+1} + m_{k+1})j \leq n < t_k + (n_{k+1} + m_{k+1})(j+1).$$

We assume that $j \geq 1$ and leave the details of the simpler case $j = 0$ to the reader.

Lemma 4.2.9. *Suppose $\mu_{k+1}(\mathcal{B}) > 0$, then there exists (a unique choice of) $x \in \mathcal{T}_k$ and $i_1, \dots, i_j \in \{1, \dots, \#\mathcal{S}_{k+1}\}$ satisfying*

$$\nu_{k+1}(\mathcal{B}) \leq \mathcal{L}(x) \prod_{l=1}^{j} \exp S_{n_{k+1}} \psi(x^{k+1}_{i_l}) M_{k+1}^{N_{k+1} - j}.$$

Proof. If $\mu_{k+1}(\mathcal{B}) > 0$, then $\mathcal{T}_{k+1} \cap \mathcal{B} \neq \emptyset$. Let $z = z(x,y) \in \mathcal{T}_{k+1} \cap \mathcal{B}$ where $x \in \mathcal{T}_k$ and $y = y(i_1, \ldots, i_{N_{k+1}}) \in \mathcal{C}_{k+1}$. Let

$$\mathcal{A}_{x;i_1,\ldots,i_j} = \{z(x, y(l_1, \ldots, l_{N_{k+1}})) \in \mathcal{T}_{k+1} : l_1 = i_1, \ldots, l_j = i_j\}.$$

Suppose that $z(x', y(\underline{l})) \in \mathcal{B}$. Since $\mathcal{T}_k$ is $(t_k, 2\varepsilon)$ separated and $n \geq t_k$, $x = x'$. For $l \in \{1, 2, \ldots, j\}$, we have

$$d_{n_{k+1}}(f^{t_k+(l-1)(n_{k+1}+m_{k+1})}q, x_{i_l}^{k+1}) < 2\varepsilon.$$

Since $x_{i_l}^{k+1} \in \mathcal{S}_{k+1}$ and $\mathcal{S}_{k+1}$ is $(n_{k+1}, 4\varepsilon)$ separated, it follows that $l_1 = i_1, \ldots, l_j = i_j$. Thus, if $z \in \mathcal{T}_{k+1} \cap \mathcal{B}$, then $z \in \mathcal{A}_{x;i_1,\ldots,i_j}$. Hence,

$$\begin{aligned}
\nu_{k+1}(\mathcal{B}) &\leq \sum_{z \in \mathcal{A}_{x;i_1,\ldots,i_j}} \mathcal{L}(z) = \mathcal{L}(x) \sum_{\underline{p}_{k+1}:p_1^{k+1}=i_1,\ldots,p_j^{k+1}=i_j} \mathcal{L}(\underline{p}_{k+1}) \\
&= \mathcal{L}(x) \prod_{l=1}^{j} \exp S_{n_{k+1}}\psi(x_{i_l}^{k+1}) \prod_{p=j+1}^{N_{k+1}} \sum_{l_p=1}^{\#\mathcal{S}_{k+1}} \exp S_{n_{k+1}}\psi(x_{l_p}^{k+1}),
\end{aligned}$$

whence the required result. □

Lemma 4.2.10. *Let $x \in \mathcal{T}_k$ and $i_1, \ldots, i_j$ be as before. Then*

$$\begin{aligned}
\mathcal{L}(x) \prod_{l=1}^{j} \exp S_{n_{k+1}}\psi(x_{i_l}^{k+1}) \leq \exp\{S_n\psi(q) &+ 2n\mathit{Var}(\psi, 2\varepsilon) \\
&+ \|\psi\|(\sum_{i=1}^{k} N_i m_i + j m_{k+1})\}.
\end{aligned}$$

Proof. We write $x = x(\underline{p}_1, \ldots \underline{p}_k)$. Lemma 4.2.5 tells us that

$$d_{n_i}(f^{t_{i-1}+m_{i-1}+(l-1)(m_i+n_i)}x, x_{p_l^i}^i) < 2\varepsilon$$

for all $i \in \{1, \ldots, k\}$ and all $l \in \{1, \ldots, N_i\}$ and it follows that

$$\mathcal{L}(x) \leq \exp\{S_{t_k}\psi(x) + t_k Var(\psi, 2\varepsilon) + \sum_{i=1}^{k} \|\psi\| N_i m_i\}.$$

Similarly,

$$\prod_{l=1}^{j} \exp S_{n_{k+1}}\psi(x_{i_l}^{k+1}) \leq \exp\{S_{n-t_k}\psi(z) + (n - t_k)Var(\psi, \frac{\varepsilon}{2^{k+1}}) + \|\psi\| j m_{k+1}\}.$$

We obtain the result from these two inequalities and that $d_n(z,q) < 2\varepsilon$ and $d_{t_k}(x,q) < 2\varepsilon$. □

The proof of the following lemma is similar to that of lemma 4.2.9.

Lemma 4.2.11. *For any $p \geq 1$, suppose $\mu_{k+p}(\mathcal{B}) > 0$. Let $x \in \mathcal{T}_k$ and $i_1, \ldots, i_j$ be as before. Then every $z \in \mathcal{T}_{k+p} \cap \mathcal{B}$ descends from some point in $\mathcal{A}_{x;i_1,\ldots,i_j}$. We have*

$$\nu_{k+p}(\mathcal{B}) \leq \mathcal{L}(x) \prod_{l=1}^{j} \exp S_{n_{k+1}}\psi(x_{i_l}^{k+1}) M_{k+1}^{N_{k+1}-j} M_{k+2}^{N_{k+2}} \ldots M_{k+p}^{N_{k+p}}.$$

Lemma 4.2.12.

$$\mu_{k+p}(\mathcal{B}) \leq \frac{1}{\kappa_k M_{k+1}^j} \exp\left\{S_n\psi(q) + 2nVar(\psi, 2\varepsilon) + \|\psi\|(\sum_{i=1}^{k} N_i m_i + j m_{k+1})\right\}.$$

Proof. Using lemma 4.2.10, it follows from lemma 4.2.11 that

$$\begin{aligned}\nu_{k+p}(\mathcal{B}) \leq M_{k+1}^{N_{k+1}-j} \dots M_{k+p}^{N_{k+p}} \exp\{S_n\psi(q) &+ 2nVar(\psi, 2\varepsilon) \\ &+ \|\psi\|(\sum_{i=1}^{k} N_i m_i + j m_{k+1})\}.\end{aligned}$$

Since $\mu_{k+p} = \frac{1}{\kappa_{k+p}}\nu_{k+p}$ and $\kappa_{k+p} = \kappa_k M_{k+1}^{N_{k+1}} \dots M_{k+p}^{N_{k+p}}$, the result follows. □

Lemma 4.2.13. *For sufficiently large n, $\kappa_k M_{k+1}^j \geq \exp((C-5\gamma)n)$*

Proof. Recall that by construction $M_k \geq \exp((C-4\gamma)n_k)$. We have

$$\begin{aligned}\kappa_k M_{k+1}^j &= M_1^{N_1} \dots M_k^{N_k} M_{k+1}^j \\ &\geq \exp\{(C-4\gamma)(N_1 n_1 + N_2 n_2 + \dots + N_k n_k + j n_{k+1})\} \\ &\geq \exp\{(C-5\gamma)(N_1(n_1+m_1) + N_2(n_2+m_2) + \dots \\ &\quad +N_k(n_k+m_k) + j(n_{k+1}+m_{k+1})\} \\ &= \exp\{(C-5\gamma)(t_k + m_1 + j(n_{k+1}+m_{k+1})\} \geq \exp\{(C-5\gamma)n\}.\end{aligned}$$

Our arrival at the third line may require some explanation. Morally, we are able to add in the extra terms with an arbitrarily small change to the constant s because n_k is much larger than m_k. The reader may wish to verify this. □

Lemma 4.2.14. *For sufficiently large n,*

$$\limsup_{k\to\infty} \mu_k(B_n(q, \frac{\varepsilon}{2})) \leq \exp\{-n(C - 2Var(\psi, 2\varepsilon) - 6\gamma) + \sum_{i=0}^{n-1} \psi(f^i q)\}.$$

Proof. By lemmas 4.2.12 and 4.2.13, for sufficiently large n and any $p \geq 1$,

$$\begin{aligned}\mu_{k+p}(\mathcal{B}) &\leq \frac{1}{\kappa_k M_{k+1}^j} \exp\left\{S_n\psi(q) + 2nV + \|\psi\|(\sum_{i=1}^{k} N_i m_i + j m_{k+1})\right\} \\ &\leq \frac{1}{\kappa_k M_{k+1}^j} \exp\left\{S_n\psi(q) + n\left(2V + \gamma\right)\right)\} \\ &\leq \exp\{-n(C - 6\gamma - 2V)) + S_n\psi(q)\},\end{aligned}$$

where $V = \mathrm{Var}(\psi, 2\varepsilon)$. Our arrival at the second line is because n_k is much larger than m_k. □

Applying the Generalised Pressure Distribution Principle, we have

$$P_F(\psi, \varepsilon) \geq C - 2\mathrm{Var}(\psi, 2\varepsilon) - 6\gamma.$$

Recall that ε was chosen sufficiently small so $\mathrm{Var}(\psi, 2\varepsilon) < \gamma$. It follows that

$$P_{\widehat{X}(\varphi,f)}(\psi, \varepsilon) \geq P_F(\psi, \varepsilon) \geq C - 8\gamma.$$

Since γ and ε were arbitrary, the proof of theorem 4.1.6 is complete.

4.2.2 Modification of the construction to obtain theorem 4.1.2

Let us fix a small $\gamma > 0$. Let μ_1 be ergodic and satisfy $h_{\mu_1} + \int \psi d\mu_1 > C - \gamma/2$. Let $\nu \in \mathcal{M}_f^e(X')$ satisfy $\int \varphi d\mu_1 \neq \int \varphi d\nu$. Let $\mu_2 = t_1\mu_1 + t_2\nu$ where $t_1 + t_2 = 1$ and $t_1 \in (0, 1)$ is chosen sufficiently close to 1 so that $h_{\mu_2} + \int \psi d\mu_2 > C - \gamma$. Choose $\delta > 0$ sufficiently small so

$$\left| \int \varphi d\mu_1 - \int \varphi d\mu_2 \right| > 8\delta.$$

Choose a strictly decreasing sequence $\delta_k \to 0$ with $\delta_1 < \delta$. For k odd, we proceed as before, choosing a strictly increasing sequence $l_k \to \infty$ so the set

$$Y_k := \left\{ x \in X' : \left| \frac{1}{n} S_n\varphi(x) - \int \varphi d\mu_1 \right| < \delta_k \text{ for all } n \geq l_k \right\}$$

satisfies $\mu_1(Y_k) > 1 - \gamma$ for every k. For k even, we define $Y_{k,1} := Y_{k-1}$ and find $l_k > l_{k-1}$ so that each of the sets

$$Y_{k,2} := \left\{ x \in X' : \left| \frac{1}{n} S_n\varphi(x) - \int \varphi d\nu \right| < \delta_k \text{ for all } n \geq l_k \right\}$$

satisfies $\nu(Y_{k,2}) > 1 - \gamma$. The proof of the following lemma is similar to that of lemma 4.2.1.

Lemma 4.2.15. *For any sufficiently small $\varepsilon > 0$ and k even, we can find a sequence $\hat{n}_k \to \infty$ so $[t_i\hat{n}_k] \geq l_k$ for $i = 1, 2$ and sets $\mathcal{S}_k^i$ so that $\mathcal{S}_k^i$ is a $([t_i\hat{n}_k], 4\varepsilon)$ separated set for $Y_{k,i}$ with $M_k^i := \sum_{x \in \mathcal{S}_k^i} \exp\left\{\sum_{j=0}^{n_k-1} \psi(f^jx)\right\}$ satisfying*

$$M_k^1 \geq \exp([t_1\hat{n}_k](h_{\mu_1} + \int \psi d\mu_1 - 4\gamma)),$$

$$M_k^2 \geq \exp([t_2\hat{n}_k](h_\nu + \int \psi d\nu - 4\gamma)).$$

Furthermore, the sequence $\hat{n}_k$ can be chosen so that $\hat{n}_k \geq 2^{m_k}$ where $m_k = m(\varepsilon/2^k)$ is as in the definition of specification.

We now use the specification property to define the set $\mathcal{S}_k$ as follows. For $i = 1, 2$, let $y_i \in \mathcal{S}_k^i$ and define $x = x(y_1, y_2)$ to be a choice of point which satisfies

$$d_{[t_1\hat{n}_k]}(y_1, x) < \frac{\varepsilon}{2^k} \text{ and } d_{[t_2\hat{n}_k]}(y_2, f^{[t_1\hat{n}_k]+m_k}x) < \frac{\varepsilon}{2^k}.$$

Let $\mathcal{S}_k$ be the set of all points constructed in this way. Let $n_k = [t_1\hat{n}_k] + [t_2\hat{n}_k] + m_k$. Then n_k is the amount of time for which the orbit of points in $\mathcal{S}_k$ has been prescribed and we have $n_k/\hat{n}_k \to 1$. We note that $\mathcal{S}_k$ is $(n_k, 4\varepsilon)$ separated and so $\#\mathcal{S}_k = \#\mathcal{S}_k^1 \# S_k^2$. Let $M_k = M_k^1 M_k^2$. Given our new construction of $\mathcal{S}_k$, the rest of our constuction goes through unchanged.

4.2.3 Modification to the proof

For every $x \in \mathcal{S}_k$,

$$\begin{aligned} |S_{n_k}\varphi(x) - n_k \int \varphi d\mu_2| &\leq |S_{[t_1\hat{n}_k]}\varphi(x) - [t_1\hat{n}_k] \int \varphi d\mu_1| + m_k\|\varphi\| \\ &+ |S_{[t_2\hat{n}_k]}\varphi(f^{[t_1 n_k]+m_k}x) - [t_2\hat{n}_k] \int \varphi d\nu| \end{aligned}$$

It follows that $|\frac{1}{n_k} S_{n_k}\varphi(x) - \int \varphi d\mu_2| \to 0$. This observation allows us to modify the proof of lemma 4.2.8 and ensures that our construction still gives rise to points in $\widehat{X}(\varphi, f)$. We have for sufficiently large n_k,

$$\begin{aligned} M_k &\geq \exp\{[t_1\hat{n}_k](h_{\mu_1} + \int \psi d\mu_1 - 4\gamma) + [t_2\hat{n}_k](h_\nu + \int \psi d\nu - 4\gamma)\} \\ &\geq \exp\{(1-\gamma)\hat{n}_k(t_1(h_{\mu_1} + \int \psi d\mu_1) + t_2(h_\nu + \int \psi d\nu) - 4\gamma)\} \\ &\geq \exp(1-\gamma)^2 n_k(h_{\mu_2} + \int \psi d\mu_2 - 4\gamma) \geq \exp(1-\gamma)^2 n_k(C - 5\gamma). \end{aligned}$$

Since γ was arbitrary, this observation allows us to modify the estimates in lemma 4.2.13 to cover this more general construction.

4.3 Application to suspension flows

We apply theorem 4.1.2 to suspension flows. Let $f : X \mapsto X$ be a homeomorphism of a compact metric space (X, d). We consider a continuous roof function $\rho : X \mapsto (0, \infty)$. We define the suspension space to be

$$X_\rho = \{(x, s) \in X \times \mathbb{R} : 0 \leq s \leq \rho(x)\},$$

where $(x, \rho(x))$ is identified with $(f(x), 0)$ for all x. Alternatively, we can define X_ρ to be $X \times [0, \infty)$, quotiented by the equivalence relation $(x, t) \sim (y, s)$ iff $(x, t) = (y, s)$ or there exists $n \in \mathbb{N}$ so $(f^n x, t - \sum_{i=0}^{n-1} \rho(f^i x)) = (y, s)$ or $(f^{-n}x, t + \sum_{i=1}^{n} \rho(f^{-i}x)) = (y, s)$. Let π denote the quotient map from $X \times [0, \infty)$ to X_ρ. We extend the domain of definition of π to $X \times (-\inf \rho, \infty)$ by identifying points of the form $(y, -t)$ with $(f^{-1}y, \rho(y) - t)$ for $t \in (0, \inf \rho)$. We write (x, s) in place of $\pi(x, s)$ when $\inf \rho < s < \rho(x)$.We define the flow $\Psi = \{g_t\}$ on X_ρ by

$$g_t(x, s) = \pi(x, s + t).$$

To a function $\Phi : X_\rho \mapsto \mathbb{R}$, we associate the function $\varphi : X \mapsto \mathbb{R}$ by $\varphi(x) = \int_0^{\rho(x)} \Phi(x, t) dt$. Since the roof function is continuous, when Φ is continuous, so is φ. For $\mu \in \mathcal{M}_f(X)$, we define the measure μ_ρ by

$$\int_{X_\rho} \Phi d\mu_\rho = \int_X \varphi d\mu / \int \rho d\mu$$

for all $\Phi \in C(X_\rho)$, where φ is defined as above. We have Ψ-invariance of μ_ρ (ie. $\mu(g_t^{-1}A) = \mu(A)$ for all $t \geq 0$ and measurable sets A). The map $\mathcal{R} : \mathcal{M}_f(X) \mapsto \mathcal{M}_\Psi(X_\rho)$ given by $\mu \mapsto \mu_\rho$ is a bijection. Abramov's theorem [Abr], [PP1] states that $h_{\mu_\rho} = h_\mu / \int \rho d\mu$ and hence,

$$h_{top}(\Psi) = \sup\{h_\mu : \mu \in \mathcal{M}_\Psi(X_\rho)\} = \sup\left\{\frac{h_\mu}{\int \rho d\mu} : \mu \in \mathcal{M}_f(X)\right\},$$

where $h_{top}(\Psi)$ is the topological entropy of the flow. Proposition 6.1 of [PP1] states that $h_{top}(\Psi)$ is the unique solution to the equation $P_X^{classic}(-s\rho) = 0$. We use the notation $h_{top}(Z, \Psi)$ for topological entropy of a non-compact subset $Z \subset X_\rho$ with respect to Ψ (defined below). We define

$$\widehat{X}_\rho = \{(x,s) \in X_\rho : \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi(g_t(x,s))dt \text{ does not exist } \}.$$

By the ergodic theorem for flows, $\mu(\widehat{X}_\rho) = 0$ for any $\mu \in \mathcal{M}_\Psi(X_\rho)$. Our main result on suspension flows is the following (the proof is at the end of the section).

Theorem 4.3.1. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a homeomorphism with the specification property. Let $\rho : X \mapsto (0, \infty)$ be continuous. Let (X_ρ, Ψ) be the corresponding suspension flow over X. Assume that $\Phi : X_\rho \mapsto \mathbb{R}$ is continuous and satisfies $\inf_{\mu \in \mathcal{M}_\Psi(X_\rho)} \int \Phi d\mu < \sup_{\mu\in\mathcal{M}_\Psi(X_\rho)} \int \Phi d\mu$. Then $h_{top}(\widehat{X_\rho}, \Psi) = h_{top}(\Psi)$.*

We remark that the flow Φ may not satisfy specification itself. For example, when ρ is a constant function, Φ is not even topologically mixing.

4.3.1 Properties of suspension flows

The following lemma is similar to one given in [BS4].

Lemma 4.3.1. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a homeomorphism. Let $\rho : X \mapsto (0, \infty)$ be continuous. Let (X_ρ, Ψ) be the corresponding suspension flow over X. Let $\Phi : X_\rho \mapsto \mathbb{R}$ be continuous and $\varphi : X \mapsto \mathbb{R}$ be given by $\varphi(x) = \int_0^{\rho(x)} \Phi(x,t)dt$. We have*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \Phi(g_t(x,s))dt = \liminf_{n\to\infty} \frac{S_n\varphi(x)}{S_n\rho(x)},$$

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^t \Phi(g_t(x,s))dt = \limsup_{n\to\infty} \frac{S_n\varphi(x)}{S_n\rho(x)},$$

$$\widehat{X_\rho} = \{(x,s) : \lim_{n\to\infty} \frac{S_n\varphi(x)}{S_n\rho(x)} \text{ does not exist}, 0 \leq s < \rho(x)\}.$$

Proof. Fix $\gamma > 0$. Given $T > 0$, let n satisfy $S_n\rho(x) \leq T < S_{n+1}\rho(x)$. It follows that $1 - \frac{\|\rho\|}{T} \leq \frac{S_n\rho(x)}{T} \leq 1$. Assume T is sufficiently large that $2T^{-1}\|\rho\|\|\Phi\| < \gamma$. We note that

$$\begin{aligned}\int_0^T \Phi(g_t(x,s))dt &\leq \sum_{i=0}^{n-1}\int_0^{\rho(f^ix)} \Phi(f^ix,t)dt + 2\|\rho\|\|\Phi\| \\ &= S_n\varphi(x) + 2\|\rho\|\|\Phi\|,\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{T}\int_0^T \Phi(g_t(x,s))dt &\leq \frac{S_n\rho(x)}{T}\frac{S_n\varphi(x)}{S_n\rho(x)} + \frac{2}{T}\|\rho\|\|\Phi\| \\ &\leq \frac{S_n\varphi(x)}{S_n\rho(x)} + \gamma.\end{aligned}$$

The result follows from this and a similar calculation for the opposite inequality. □

As the lemma suggests, our result on $\widehat{X}_\rho$ will follow from a corresponding result about the set

$$\widehat{X}(\varphi,\rho) := \left\{x \in X : \lim_{n\to\infty} \frac{S_n\varphi(x)}{S_n\rho(x)} \text{ does not exist}\right\}. \tag{4.7}$$

Lemma 4.3.2. *Under our assumptions, the following are equivalent:*

(a) $\widehat{X}_\rho \neq \emptyset$; *(b)* $\widehat{X}(\varphi,\rho) \neq \emptyset$;

(c) $\inf_{\mu\in\mathcal{M}_\Psi(X_\rho)} \int \Phi d\mu < \sup_{\mu\in\mathcal{M}_\Psi(X_\rho)} \int \Phi d\mu$;

(d) $\inf_{\mu\in\mathcal{M}_f(X)} \int \varphi d\mu / \int \rho d\mu < \sup_{\mu\in\mathcal{M}_f(X)} \int \varphi d\mu / \int \rho d\mu$;

(e) $\inf_{\mu\in\mathcal{M}_f^e(X)} \int \varphi d\mu / \int \rho d\mu < \sup_{\mu\in\mathcal{M}_f^e(X)} \int \varphi d\mu / \int \rho d\mu$;

(f) $S_n\varphi/S_n\rho$ *does not converge (uniformly or pointwise) to a constant;*

(g) $\frac{1}{T}\int_0^T \Phi(g_t)dt$ *does not converge (uniformly or pointwise) to a constant;*

Let $\varphi_T(x) := \int_0^T \Phi(g_t x)dt$.

(h) There exists $T > 0$ *such that* $\varphi_T \notin \bigcup_{c\in\mathbb{R}} \overline{Cob(X_\rho, g_T, c)}$*, i.e* φ_T *is not in the closure of the coboundaries for the time-*T *map of the flow;*

(i) For all $T > 0$, $\varphi_T \notin \bigcup_{c\in\mathbb{R}} \overline{Cob(X_\rho, g_T, c)}$.

Proof. First we note that (d) $\iff$ (e) $\iff$ (f) is similar to the proof of the analogous statements in lemma 2.3.2. For (c) ⇒ (d), let $\mu_1, \mu_2 \in \mathcal{M}_\Psi(X_\rho)$ satisfy $\int \Phi d\mu_1 < \int \Phi d\mu_2$. Let $v_i = \mathcal{R}^{-1}\mu_i$ for $i = 1, 2$. By definition, $\int \varphi dv_i / \int \rho dv_i = \int \Phi d\mu_i$ for $i = 1, 2$ and so $\int \varphi dv_1 / \int \rho dv_1 < \int \varphi dv_2 / \int \rho dv_2$. (d) ⇒ (c) is similar. (f) $\iff$ (g) follows from lemma 4.3.1.

We show (g) $\iff$ (h) $\iff$ (i). It is clear that $\frac{1}{T}\int_0^T \Phi(g_t)dt$ does not converge to a constant iff $\frac{1}{n}S_n\varphi_\tau$ does not converge to a constant for any fixed $\tau > 0$. An appliction of lemma 2.3.2 gives the desired results.

(a) ⇒ (g), (b) ⇒ (f), (b) ⇒ (a) are trivial. (d) ⇒ (b) is a consequence of theorem 4.3.2, so we omit the proof. □

We remark that if $\varphi \in \overline{Cob(X, f, 0)}$ or $\varphi - \rho \in \overline{Cob(X, f, 0)}$, then $S_n\varphi/S_n\rho$ converges uniformly to a constant and so $\widehat{X}_\rho = \emptyset$.

4.3.2 A generalisation of the main theorem

To prove theorem 4.3.1, we require the following generalisation of theorem 4.1.1.

Theorem 4.3.2. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a continuous map with specification. Let $\varphi, \psi \in C(X)$ and $\rho : X \mapsto (0, \infty)$ be continuous with*

$$\inf_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu / \int \rho d\mu < \sup_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu / \int \rho d\mu.$$

Let $\widehat{X}(\varphi, \rho)$ be defined as in (4.7). We have $P_{\widehat{X}(\varphi,\rho)}(\psi) = P_X^{classic}(\psi)$.

Proof. We require only a small modification to the proof of theorem 4.1.2. We replace the family of sets defined at (4.1) by the following:

$$Y_k := \left\{ x \in X : \left| \frac{S_n \varphi(x)}{S_n \rho(x)} - \frac{\int \varphi d\mu_{\rho(k)}}{\int \rho d\mu_{\rho(k)}} \right| < \delta_k \text{ for all } n \geq l_k \right\}$$

chosen to satisfy $\mu_{\rho(k)}(Y_k) > 1 - \gamma$ for every k. This is possible by the ratio ergodic theorem. The rest of the proof requires only superficial modifications. □

4.3.3 The relationship between entropy of a suspension flow and pressure in the base

The natural metric on X_ρ is the Bowen-Walters metric [BW], [BS4]. The appendix of [BS4] contains a study of dynamical balls taken with respect to this metric when the roof function is Hölder. We assume only continuity of ρ. When ρ is non-constant, computations involving this metric are rather unwieldy, particularly when no regularity of the roof function is assumed. We sidestep this problem by making the following definitions. Let $(x, s) \in X_\rho$ with $0 \leq s < \rho(x)$. We define the horizontal segment of (x, s) to be $\{(y, t) : y \in X, 0 \leq t < \rho(y), t = \rho(y) s \rho(x)^{-1}\}$ and the horizontal ball of radius ε at (x, s) to be

$$B^H((x, s), \varepsilon) := \{(y, \frac{s}{\rho(x)} \rho(y)) : (1 - \frac{s}{\rho(x)}) d(x, y) + \frac{s}{\rho(x)} d(fx, fy) < \varepsilon\}.$$

We define

$$B((x, s), \varepsilon) = \bigcup_{t : |s-t| < \varepsilon} B^H((x, t), \varepsilon),$$

$$B_T((x, s), \varepsilon) = \bigcap_{t=0}^{T} g_{-t} B(g_t(x, s), \varepsilon).$$

We are abusing notation, since $B((x, s), \varepsilon)$ is not a ball in the Bowen-Walters metric. We can consider covers by sets of the form $B_T((x, s), \varepsilon)$ in the definition of topological pressure in place of covers consisting of dynamical balls (see remark 2.1.1 and §7.3.1). This is because one can verify that there exists constants $C_1, C_2 > 0$ such that the metric ball of radius $C_1\varepsilon$ at (x,s) is a subset of $B((x,s),\varepsilon)$, that a set of diameter ε is contained in some set $B((x,s),C_2\varepsilon)$ for sufficiently small ε, that $B((x,s),\varepsilon)$ is open and as $\varepsilon \to 0$, $\mathrm{Diam}(\{B((x,s),\varepsilon) : (x,s) \in X_\rho\}) \to 0$. Diameter and topology are taken with respect to the Bowen-Walters metric.

Lemma 4.3.3. *Let $(y,s) \in X \times (-\inf \rho, \infty)$ and suppose $\pi(y,s) \in B((x,\delta),\varepsilon)$, where $|\delta| \le \varepsilon < \inf \rho/4$. Then for ε sufficiently small there exists $n \in \mathbb{N}$ such that*

$$(y,s) \sim (f^n y, s - S_n\rho(y)), |s - S_n\varphi(y)| < K\varepsilon \text{ and } d(x, f^n y) < K\varepsilon,$$

where $K = 4\|\rho\|/\inf \rho$ and $K\varepsilon < \inf \rho$.

Proof. Suppose $(y,s) \in B^H((x,\gamma),\varepsilon)$ for some γ with $0 \le |\gamma| < 2\varepsilon$. Then $s = \gamma\rho(y)\rho(x)^{-1}$. Therefore, $s < 2\varepsilon\|\rho\|/\inf\rho$. We have

$$(1 - \frac{\gamma}{\rho(x)})d(x,y) + \frac{\gamma}{\rho(x)}d(fx,fy) < \varepsilon.$$

Thus $(1 - \frac{\gamma}{\rho(x)})d(x,y) < \varepsilon$. Rearranging, we have $d(x,y) < \varepsilon\rho(x)(\rho(x)-\gamma)^{-1} < K\varepsilon$. For $-\varepsilon < \gamma < 0$, we apply a similar argument. Now assume $\pi(y,s) \in B((x,\delta),\varepsilon)$. Then $\pi(y,s)$ has a unique representation (y',s') with $|s'| < 2\varepsilon$ and $y' = f^n y$. We apply the previous argument to (y',s'). □

Lemma 4.3.4. *Suppose $|s| < \varepsilon$ and $S_n\rho(x) \le T < S_{n+1}\rho(x)$, then*

$$B_T((x,s),\varepsilon) \subset B_n(x,K\varepsilon) \times (-K\varepsilon, K\varepsilon).$$

Proof. Let $(y,t) \in B_T((x,s),\varepsilon)$, with $|t| < K\varepsilon$. Then $d(x,y) < K\varepsilon$. Let t_i satisfy $s + t_i = S_i\rho(x)$ for $i = 1,\dots n$. Then $g_{t_i}(y,t) \in B((f^{i-1}x,0),\varepsilon)$. Applying the previous lemma, we have $d(f^n y, f^{i-1}x) < K\varepsilon$ for some $n \in \mathbb{N}$. Furthermore, we must have $n = i-1$. Suppose not, then for some time $\tau \in [0, S_i\rho(x))$, $g_\tau(y,t) \notin B(g_\tau(x,s),\varepsilon)$, which is a contradiction. This implies that $y \in B_n(x,K\varepsilon)$. □

Theorem 4.3.3. *Let (X,d) be a compact metric space and $f : X \mapsto X$ be a homeomorphism. Let $\rho : X \mapsto (0,\infty)$ be continuous. Let (X_ρ, Ψ) be the corresponding suspension flow over X. For an arbitrary Borel set $Z \subset X$, define $Z_\rho := \{(z,s) : z \in Z, 0 \le s < \rho(s)\}$. Let β be the unique solution to the equation $P_Z(-t\rho) = 0$. Then $h_{top}(Z_\rho,\Psi) \ge \beta$.*

Proof. The function $t \to P_Z(-t\rho)$ is continuous and decreasing. Since $P_Z(0) \ge 0$, it follows that there exists a unique solution to the equation $P_Z(-t\rho) = 0$. We assume $P_Z(-\beta\varphi) > 0$ and show $h_{top}(Z_\rho,\Psi) \ge \beta$. Let $\varepsilon > 0$ be arbitrary and sufficiently small so lemma 4.3.4 applies and

$P_Z(-\beta\varphi, \varepsilon) > 0$. Choose $\Gamma = \{B_{t_i}((x_i, s_i), \varepsilon)\}$ covering Z_ρ with $t_i \geq T$. Take the subcover Γ' of Γ which covers $Z \times \{0\}$, and assume without loss of generality that $|s_i| < \varepsilon$. Let m_i be the unique number so $S_{m_i}\rho(x) \leq t_i < S_{m_i+1}\rho(x)$. Let $m(\Gamma') = \inf m_i$ obtained in this way. Then $m(\Gamma') \geq \|\rho\|^{-1}(T - \|\rho\|)$ and thus as T tends to infinity so does $m(\Gamma')$. Let $\Gamma'' = \{B_{m_i}(x_i, K\varepsilon)\}$: $B_{t_i}((x_i, s_i), \varepsilon) \in \Gamma'\}$. By lemma 4.3.4, $B_{m_i}(x_i, K\varepsilon) \times (-K\varepsilon, K\varepsilon)$ covers $Z \times \{0\}$ and if we assume ε was chosen sufficiently small, then Γ'' is a cover for Z.

$$\begin{aligned}
Q(Z \times \{0\}, \beta, \Gamma') &\geq \sum_{B_i \in \Gamma'} \exp -\beta(S_{m_i}\rho(x_i) + \|\rho\|) \\
&\geq \sum_{B_i \in \Gamma''} \exp -\beta(\sup_{y \in B_i} S_{m_i}\rho(y) + \|\rho\| + \mathsf{Var}(\rho, K\varepsilon)) \\
&= \exp\{-\beta(\mathsf{Var}(\rho, K\varepsilon) + \|\rho\|)\} Q(Z, 0, \Gamma'', -\beta\rho) \\
&\geq \exp\{-\beta(\mathsf{Var}(\rho, K\varepsilon) + \|\rho\|)\} M(Z, 0, m(\Gamma'), -\beta\rho) \\
&\geq 1,
\end{aligned}$$

if T and hence $m(\Gamma')$ are chosen to be sufficiently large. We have

$$Q(Z_\rho, \beta, \Gamma) \geq Q(Z \times \{0\}, \beta, \Gamma')$$

and since Γ was arbitrary, we have $M(Z_\rho, \beta, T - \|\rho\|, \varepsilon) \geq 1$ and hence $h_{top}(Z_\rho, \Psi, \varepsilon) \geq \beta$. $\square$

4.3.4 Proof of theorem 4.3.1

Given the results we have proved so far, theorem 4.3.1 follows easily. By lemma 4.3.1, $\widehat{X}_\rho = Z_\rho$, where $Z = \widehat{X}(\varphi, \rho)$. We recall that $h_{top}(\Psi)$ is the unique number satisfing $P_X^{classic}(-t\rho) = 0$. By theorem 4.3.2, $P_Z(-t\rho) = P_X^{classic}(-t\rho)$ for all $t \in \mathbb{R}$, and so $h_{top}(\Psi)$ is the unique number such that $P_Z(-t\rho) = 0$. Applying theorem 4.3.3, our result follows.

Chapter 5

A conditional variational principle for topological pressure

We continue the programme started in chapter 4 to understand the topological pressure of the multifractal decomposition

$$X = \bigcup_{\alpha \in \mathbb{R}} X(\varphi, \alpha) \cup \widehat{X}(\varphi, f).$$

In chapter 4, we showed that $\widehat{X}(\varphi)$ is either empty or has full topological pressure. In this chapter, we turn our attention to the sets $X(\varphi, \alpha)$, which we recall are defined as

$$X(\varphi, \alpha) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \right\}.$$

Main result of chapter 5. *Suppose f has specification. For any continuous functions $\varphi, \psi : X \mapsto \mathbb{R}$,*

$$P_{X(\varphi,\alpha)}(\psi) = \sup \left\{ h_\mu + \int \psi d\mu : \mu \in \mathcal{M}_f(X) \text{ and } \int \varphi d\mu = \alpha \right\}. \tag{5.1}$$

Formulae similar to (5.1) have a key role in multifractal analysis (see [Bar], [Pes] for a broad and unified introduction). Following Barreira, we use the terminology 'conditional variational principle' to describe formulae such as (5.1). For hyperbolic maps and Hölder continous φ, Barriera and Saussol established our main result for the case $\psi = 0$, i.e. for the topological entropy of $X(\varphi, \alpha)$ and used it to give a new proof of the multifractal analysis in this setting [BS2]. The study of multifractal analysis for arbitrary (ie. non-Hölder) continuous functions was initiated in the symbolic dynamics setting by Fan and Feng [FF] and Olivier [Oli]. Takens and Verbitskiy proved (5.1) in the case of topological entropy for maps with the specification property [TV2].

Luzia proved our main result for topological pressure when the system is a topologically mixing subshift of finite type and φ, ψ are Hölder, and used it to analyse fibred systems [Luz]. Our current result generalises and unifies the above mentioned results.

Pfister and Sullivan generalised the result of Takens and Verbitskiy still further to the class of maps with the almost specification property [PS2]. We strongly expect that a synthesis of the method in this chapter and the method in chapter 6 can be used to prove (5.1) when f has almost specification. Fan et al. [FLP] proved a version of (1.2) for $\psi = 0$ which holds when φ takes values in a Banach space.

Barreira and Saussol proved an analogue of (5.1) for hyperbolic flows when $\psi = 0$ and φ is Hölder [BS3]. While we expect (5.1) can be established for flows with specification using our current methods, we consider here the class of suspension flows over maps with specification, and show that (5.1) holds true in this setting.

A large part of our argument is the same as that used in chapter 4, which was inspired by Takens and Verbitskiy [TV2]. We do not give a self-contained proof of this part of the argument but state the key ideas and refer the reader to chapter 4 for the details. We remark that we believe the argument in §5.1.1 (also inspired by [TV2]) to be a necessary correction to the corresponding argument of Takens and Verbitskiy.

An interesting application of our main result is a 'Bowen formula' for the Hausdorff dimension of the level sets of the Birkhoff average for a class of non-uniformly expanding maps of the interval, which includes the Manneville-Pomeau family of maps.

In §5.1, we state and prove our main results. In §5.2, we apply our main result to suspension flows. In §5.3, we use our main result to derive a certain Bowen formula for interval maps.

5.1 Results

Theorem 5.1.1. *Suppose f has specification, $\varphi, \psi \in C(X, \mathbb{R})$ and $\alpha \in \mathcal{L}_\varphi$, then*

$$P_{X(\varphi,\alpha)}(\psi) = \sup\left\{h_\mu + \int \psi d\mu : \mu \in \mathcal{M}_f(X) \text{ and } \int \varphi d\mu = \alpha\right\}.$$

As a simple corollary, we note that if $\alpha = \int \varphi dm_\psi$, where m_ψ is an equilibrium measure for ψ (in the usual sense), then $P_{X(\varphi,\alpha)}(\psi) = P_X(\psi)$.

5.1.1 Upper bound on $P_{X(\varphi,\alpha)}(\psi)$

We clarify the method of Takens and Verbitskiy. Our proof relies on analysis of the lower capacity pressure of $X(\varphi, \alpha)$. We recall the notation we set up in §2.1.5. For $Z \subset X$, let

$$Q_n(Z, \psi, \varepsilon) = \inf\left\{\sum_{x \in S} \exp\left\{\sum_{k=0}^{n-1} \psi(f^k x)\right\} : S \text{ is } (n, \varepsilon) \text{ spanning set for } Z\right\},$$

$$P_n(Z, \psi, \varepsilon) = \sup\left\{\sum_{x \in S} \exp\left\{\sum_{k=0}^{n-1} \psi(f^k x)\right\} : S \text{ is } (n, \varepsilon) \text{ separated set for } Z\right\}.$$

We have $Q_n(Z,\psi,\varepsilon) \le P_n(Z,\psi,\varepsilon)$. The lower capacity pressure is

$$\underline{CP}_Z(\psi,\varepsilon) = \liminf_{n\to\infty} \frac{1}{n} \log Q_n(Z,\psi,\varepsilon),$$

$$\underline{CP}_Z(\psi) = \lim_{\varepsilon\to 0} \underline{CP}_Z(\psi,\varepsilon).$$

It is proved in [Pes] that $P_Z(\psi) \le \underline{CP}_Z(\psi)$. We use the specification property to construct a set $Z \subset X(\varphi,\alpha)$ which is almost as large as $X(\varphi,\alpha)$ (from the point of view of lower capacity pressure) and satisfies a certain uniform convergence condition.

Lemma 5.1.1. *When f has the specification property, given $\gamma > 0$, there exists $Z \subset X(\varphi,\alpha)$, $t_k \to \infty$ and $\varepsilon_k \to 0$ such that if $p \in Z$ then*

$$\left|\frac{1}{m} S_m\varphi(p) - \alpha\right| \le \varepsilon_k \text{ for all } m \ge t_k \tag{5.2}$$

and $\underline{CP}_Z(\psi) \ge \underline{CP}_{X(\varphi,\alpha)}(\psi) - 4\gamma$.

Proof. Choose $\varepsilon > 0$ such that $\underline{CP}_{X(\varphi,\alpha)}(\psi, 2\varepsilon) \ge \underline{CP}_{X(\varphi,\alpha)}(\psi) - \gamma$. For $\delta > 0$, let

$$X(\alpha,n,\delta) = \{x \in X(\varphi,\alpha) : \left|\frac{1}{m} S_m\varphi(x) - \alpha\right| \le \delta \text{ for all } m \ge n\}.$$

We have $X(\varphi,\alpha) = \bigcup_n X(\alpha,n,\delta)$ and $X(\alpha,n,\delta) \subset X(\alpha,n+1,\delta)$, thus $\underline{CP}_{X(\varphi,\alpha)}(\psi,2\varepsilon) = \lim_{n\to\infty} \underline{CP}_{X(\alpha,n,\delta)}(\psi,2\varepsilon)$. Fix an arbitrary sequence $\delta_k \to 0$ and for each δ_k pick $M_k \in \mathbb{N}$ so that

$$\underline{CP}_{X(\alpha,M_k,\delta_k)}(\psi,2\varepsilon) \ge \underline{CP}_{X(\varphi,\alpha)}(\psi,2\varepsilon) - \gamma.$$

Write $X_k := X(\alpha, M_k, \delta_k)$. Let $m_k = m(\varepsilon/2^k)$ be as in the definition of specification. Now pick a sequence of natural numbers $N_k \to \infty$ increasing sufficiently rapidly so that

$$N_{k+1} > \max\left\{\exp\sum_{i=1}^{k}(N_i + m_i), \exp M_{k+1}, \exp m_{k+1}\right\}, \tag{5.3}$$

$$Q_{N_k}(X_k,\psi,2\varepsilon) > \exp N_k(\underline{CP}_{X(\varphi,\alpha)}(\psi) - 3\gamma). \tag{5.4}$$

Let $t_1 = N_1$ and $t_k = t_{k-1} + m_k + N_k$ for $k \ge 2$. By (5.3), we have $t_k/N_k \to 1$ and $t_{k-1}/t_k \to 0$.

Fix $x_1 \in X_1, x_2 \in X_2, \ldots, x_k \in X_k, \ldots$. We use the specification property to choose points $z_1, z_2, \ldots, z_k, \ldots$ as follows. Let $z_1 = x_1$ and choose z_2 to satisfy

$$d_{N_1}(z_2,z_1) < \varepsilon/4 \text{ and } d_{N_2}(f^{N_1+m_2}z_2, x_2) < \varepsilon/4$$

and z_k to satisfy

$$d_{t_{k-1}}(z_{k-1},z_k) < \varepsilon/2^k \text{ and } d_{N_k}(f^{t_{k-1}+m_k}z_k, x_k) < \varepsilon/2^k.$$

We can verify that $\overline{B}_{t_{k+1}}(z_{k+1}, \varepsilon/2^k) \subset \overline{B}_{t_k}(z_k, \varepsilon/2^{k-1})$ and so the point

$$p := \bigcap_{k=1}^{\infty} \overline{B}_{t_k}(z_k, \varepsilon/2^{k-1})$$

is well defined. We define Z to be the set of all points p constructed in this way.

Let $p \in Z$. There exists $x_k \in X_k$ such that $d_{N_k}(f^{t_{k-1}+m_{k-1}}p, x_k) < \varepsilon/2^{k-2}$. We have

$$S_{t_k}\varphi(p) \leq S_{N_k}\varphi(x_k) + N_k \mathsf{Var}(\varphi, \varepsilon/2^{k-2}) + t_{k-1} + m_{k-1}\|\varphi\|.$$

Therefore, we can find a sequence $\varepsilon'_k \to 0$ such that for any $p \in Z$,

$$|\frac{1}{t_k} S_{t_k}\varphi(p) - \alpha| < \varepsilon'_k.$$

Now let $t_k < n < t_{k+1}$. There are two cases to consider. First, suppose that $n - t_k + m_k \geq M_{k+1}$. There exists $x \in X_{k+1}$ such that $d_{N_{k+1}}(f^{t_k+m_k}p, x) < \varepsilon/2^{k-1}$ and thus

$$S_n\varphi(p) \quad \leq \quad t_k(\alpha + \varepsilon'_k) + (n - t_k)(\alpha + \delta_{k+1} + \mathsf{Var}(\varphi, \varepsilon/2^{k-1})) + m_{k+1}\|\varphi\|.$$

Now suppose $n - t_k \leq M_{k+1}$. Then

$$\frac{1}{n} S_n\varphi(p) \leq \frac{t_k}{n}(\alpha + \varepsilon'_k) + \frac{n - t_k}{n}\|\varphi\| \leq \alpha + \varepsilon'_k + \frac{M_{k+1}}{N_k}\|\varphi\|.$$

Let $\varepsilon_k = \max\{\varepsilon'_k, \delta_{k+1} + \mathsf{Var}(\varphi, \varepsilon/2^{k+1})\} + \max\{M_{k+1}/N_k, m_{k+1}/N_k\}\|\varphi\|$ and we have shown that (5.2) holds.

Take a (t_k, ε) spanning set S_k satisfying $\sum_{x \in S_k} \exp S_{t_k}\psi(x) = Q_{t_k}(Z, \psi, \varepsilon)$. It follows that $f^{t_{k-1}+m_k}S_k$ is a (N_k, ε) spanning set for $f^{t_{k-1}+m_k}Z$. Since $\sup\{d_{N_k}(x, z) : x \in X_k, z \in f^{t_{k-1}+m_k}Z\} < \varepsilon/2^k$, then $f^{t_{k-1}+m_k}S_k$ is a $(N_k, 2\varepsilon)$ spanning set for X_k. Thus

$$\sum_{x \in S_k} \exp S_{N_k}\psi(f^{t_{k-1}+m_k}x) \geq Q_{N_k}(X_k, \psi, 2\varepsilon) > \exp N_k(\underline{CP}_{X(\varphi,\alpha)}(\psi) - 3\gamma),$$

and for sufficiently large k,

$$\begin{aligned}\sum_{x \in S_k} \exp S_{t_k}\psi(x) \quad &\geq \quad \exp\{N_k(\underline{CP}_{X(\varphi,\alpha)}(\psi) - 3\gamma) + (t_{k-1} + m_k)\inf\psi\} \\ &\geq \quad \exp\{t_k(\underline{CP}_{X(\varphi,\alpha)}(\psi) - 4\gamma)\}.\end{aligned}$$

Taking the $\liminf$ of the sequence $t_k^{-1}\log Q_{t_k}(Z, \psi, \varepsilon)$, it follows that

$$\underline{CP}_Z(\psi, \varepsilon) > \underline{CP}_{X(\varphi,\alpha)}(\psi) - 4\gamma.$$

Since ε was arbitrary, we are done. $\square$

We follow the second half of the proof of the variational principle (Theorem 9.10 of [Wal]). We construct a measure out of (n, ε) separated sets for Z (with a suitable fixed choice of ε). In contrast, Takens and Verbitskiy construct a measure from (n, ε_n) separated sets with $\varepsilon_n \to 0$. We believe it is not clear in this case how to use the proof of the variational principle to give the desired result. The uniform convergence provided by lemma 5.1.1 is designed to avoid this. We fix $\gamma > 0$ and find $\varepsilon > 0$ such that $\underline{CP}_Z(\psi, \varepsilon) > \underline{CP}_Z(\psi) - \gamma$.

Let S_n be a (n, ε) separated set for Z with

$$\sum_{x \in S_n} \exp S_n\psi(x) = P_n(Z, \psi, \varepsilon),$$

and write $P_n := P_n(Z, \psi, \varepsilon)$. Let $\sigma_n \in \mathcal{M}(X)$ be given by

$$\sigma_n = \frac{1}{P_n} \sum_{x \in S_k} \exp S_n\psi(x)\delta_x$$

and let

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \sigma_n \circ f^{-i}.$$

Let n_j be a sequence of numbers so that μ_{n_j} converges, and let μ be the limit measure. We have $\mu \in \mathcal{M}_f(X)$ and we verify that $\int \varphi d\mu = \alpha$. Let $n \in \mathbb{N}$ and k be the unique number so $t_k \leq n < t_{k+1}$. Using lemma 5.1.1, we have

$$\begin{aligned} \int \varphi d\mu_n &= \frac{1}{P_n}\frac{1}{n} \sum_{x \in S_k} S_n\varphi(x) e^{S_n\psi(x)} \\ &\leq \frac{1}{P_n}\frac{1}{n} \sum_{x \in S_k} n(\alpha + \varepsilon_k) e^{S_n\psi(x)} \\ &= \alpha + \varepsilon_k, \end{aligned}$$

and it follows that $\int \varphi d\mu = \alpha$.

To show that $h_\mu + \int \psi d\mu \geq \liminf_{j \to \infty} \frac{1}{n_j} \log P_{n_j}$, we recall some key ingredients of the proof of the variational principle. Notation for the measure-theoretic entropy is given in §2.1.6 (following [Wal]). See [Wal] for additional details of the proof. Let ξ be a partition of X with diameter less than ε and $\mu(\partial\xi) = 0$.

$$H_{\sigma_n}(\bigvee_{i=1}^{n} f^{-i}\xi) + \int S_n\psi d\sigma_n = \log P_n.$$

Since $\mu(\partial\xi) = 0$, we have for any $k, q \in \mathbb{N}$,

$$\lim_{j \to \infty} H_{\mu_{n_j}}(\bigvee_{i=0}^{q-1} f^{-i}\xi) = H_\mu(\bigvee_{i=0}^{q-1} f^{-i}\xi).$$

For a fixed n and $1 < q < n$ and $0 \leq j \leq q - 1$, we have

$$\frac{q}{n} \log P_n \leq H_{\mu_n}(\bigvee_{i=0}^{q-1} f^{-i}\xi) + q \int \psi d\mu_n + 2\frac{q^2}{n} \log \#\xi.$$

Replacing n by n_j and taking $j \to \infty$, we obtain

$$q \liminf_{j\to\infty} \frac{1}{n_j} \log P_{n_j} \le H_\mu(\bigvee_{i=0}^{q-1} f^{-i}\xi) + q\int \psi d\mu.$$

Dividing by q and letting $q \to \infty$, we obtain

$$\underline{CP}_Z(\psi,\varepsilon) \le \liminf_{n\to\infty} \frac{1}{n}\log P_n \le h_\mu(f,\xi) + \int \psi d\mu \le h_\mu + \int \psi d\mu.$$

It follows that

$$P_{X(\varphi,\alpha)}(\psi) - 5\gamma \le \underline{CP}_{X(\varphi,\alpha)}(\psi) - 5\gamma \le \underline{CP}_Z(\psi) - \gamma \le \underline{CP}_Z(\psi,\varepsilon) \le h_\mu + \int \psi d\mu.$$

Since γ was arbitrary, we are done.

5.1.2 Lower bound on $P_{X(\varphi,\alpha)}(\psi)$

This inequality is harder and the proof is similar to the main theorem of chapter 4, which we follow closely. As in chapter 4, the key ingredients are the Pressure Distribution Principle (proposition 4.1.4) and a Katok formula for measure-theoretic pressure (proposition 4.1.5).

Our strategy is to define a specially chosen family of finite sets $\mathcal{S}_k$ using the Katok formula for mearure-theoretic pressure, which will form the building blocks for the construction of a certain fractal $F \subset X(\varphi,\alpha)$, on which we can define a sequence of measures suitable for an application of the Pressure Distribution Principle.

The first stage of the construction is where our current argument differs from chapter 4. After this modification, the rest of the construction goes through largely verbatim.

5.1.3 Construction of the special sets $\mathcal{S}_k$

Choose a strictly decreasing sequence $\delta_k \to 0$ and fix an arbitrary $\gamma > 0$. Let us fix μ satisfying $\int \varphi d\mu = \alpha$ and

$$h_\mu + \int \psi d\mu \ge \sup\left\{h_\nu + \int \psi d\nu : \nu \in \mathcal{M}_f(X) \text{ and } \int \varphi d\nu = \alpha\right\} - \gamma.$$

We cannot assume that μ is ergodic, so we use the following lemma [You], p.535, to approximate μ arbitrarily well by convex combinations of ergodic measures.

Lemma 5.1.2. *For each $\delta_k > 0$, there exists $\eta_k \in \mathcal{M}_f(X)$ such that $\eta_k = \sum_{i=1}^{j(k)} \lambda_i \eta_i^k$, where $\sum_{i=1}^{j(k)} \lambda_i = 1$ and $\eta_i^k \in \mathcal{M}_f^e(X)$, satisfying $|\int \varphi d\mu - \int \varphi d\eta_k| < \delta_k$ and $h_{\eta_k} > h_\mu - \delta_k$.*

Choose a strictly increasing sequence $l_k \to \infty$ so that each of the sets

$$Y_{k,i} := \left\{x \in X : \left|\frac{1}{n} S_n\varphi(x) - \int \varphi d\eta_i^k\right| < \delta_k \text{ for all } n \ge l_k\right\} \tag{5.5}$$

satisfies $\eta_i^k(Y_{k,i}) > 1-\gamma$ for every $k \in \mathbb{N}, i \in \{1,\dots,j(k)\}$. This is possible by Birkhoff's ergodic theorem. Using proposition 4.1.5, we can establish the following lemma (see the corresponding lemma in chapter 4 for details of the proof). Let $\gamma' > 0$.

Lemma 5.1.3. *For any sufficiently small $\varepsilon > 0$, we can find a sequence $\hat{n}_k \to \infty$ with $[\lambda_i \hat{n}_k] \geq l_k$ and finite sets $\mathcal{S}_{k,i}$ so that each $\mathcal{S}_{k,i}$ is a $([\lambda_i \hat{n}_k], 5\varepsilon)$ separated set for $Y_{k,i}$ and*

$$M_{k,i} := \sum_{x \in \mathcal{S}_{k,i}} \exp\left\{\sum_{i=0}^{n_k-1} \psi(f^i x)\right\}$$

satisfies

$$M_{k,i} \geq \exp\left\{[\lambda_i \hat{n}_k]\left(h_{\eta_i^k} + \int \psi d\eta_i^k - \frac{4}{j(k)}\gamma'\right)\right\}.$$

Furthermore, the sequence $\hat{n}_k$ can be chosen so that $\hat{n}_k \geq 2^{m_k}$ where $m_k = m(\varepsilon/2^k)$ is as in the definition of specification.

We choose ε sufficiently small so that the lemma applies and $\mathrm{Var}(\psi, 2\varepsilon) < \gamma$. We fix all the ingredients provided by the lemma. We now use the specification property to define the set $\mathcal{S}_k$ as follows. Let $y_i \in \mathcal{S}_{k,i}$ and define $x = x(y_1, \dots, y_{j(k)})$ to be a choice of point which satisfies

$$d_{[\lambda_i \hat{n}_k]}(y_l, f^{a_l} x) < \frac{\varepsilon}{2^k}$$

for all $l \in \{1, \dots, j(k)\}$ where $a_1 = 0$ and $a_l = \sum_{i=1}^{l-1} [\lambda_i \hat{n}_k] + (l-1)m_k$ for $l \in \{2, \dots, j(k)\}$. Let $\mathcal{S}_k$ be the set of all points constructed in this way. Let $n_k = \sum_{i=1}^{j(k)} [\lambda_i \hat{n}_k] + (j(k)-1)m_k$. Then n_k is the amount of time for which the orbit of points in $\mathcal{S}_k$ has been prescribed and we have $n_k/\hat{n}_k \to 1$. We can verify that $\mathcal{S}_k$ is $(n_k, 4\varepsilon)$ separated and so $\#\mathcal{S}_k = \#\mathcal{S}_{k,1} \dots \#S_{k,j(k)}$. Let $M_k := M_{k,1} \dots M_{k,j(k)}$.

We assume that γ' was chosen to be sufficiently small so the following lemma holds.

Lemma 5.1.4. *We have*

(1) for sufficiently large k, $M_k \geq \exp n_k(h_\mu + \int \psi d\mu - \gamma)$;

(2) if $x \in \mathcal{S}_k$, $|\frac{1}{n_k} S_{n_k} \varphi(x) - \alpha| < \delta_k +$ Var$(\varphi, \varepsilon/2^k) + 1/k$.

Proof. We have for sufficiently large k,

$$\begin{aligned}
M_k &\geq \exp \sum_{i=1}^{j(k)} \{[\lambda_i \hat{n}_k](h_{\eta_i^k} + \int \psi d\eta_i^k - 4j(k)^{-1}\gamma')\} \\
&\geq \exp\{(1-\gamma')\hat{n}_k \sum_{i=1}^{j(k)} \lambda_i (h_{\eta_i^k} + \int \psi d\eta_i^k) - 4\gamma'\} \\
&\geq \exp(1-\gamma')^2 n_k (h_{\eta_k} + \int \psi d\eta_k - 4\gamma') \\
&\geq \exp(1-\gamma')^2 n_k (h_\mu + \int \psi d\mu - 4\gamma' - 2\delta_k).
\end{aligned}$$

Thus if γ' is sufficiently small, we have (1).

Suppose $x = x(y_1, \ldots, y_{j(k)}) \in \mathcal{S}_k$, then

$$\begin{aligned}
|S_{n_k}\varphi(x) - n_k\alpha| &\leq |S_{n_k}\varphi(x) - n_k(\int \varphi d\eta_k - \delta_k)| \\
&\leq \sum_{i=1}^{j(k)} |S_{[\lambda_i \hat{n}_k]}\varphi(f^{a_i}x) - n_k\lambda_i \int \varphi d\eta_i^k| \\
&\quad + n_k\delta_k + m_k(j(k)-1)\|\varphi\| \\
&\leq \sum_{i=1}^{j(k)} |S_{[\lambda_i \hat{n}_k]}\varphi(y_i) - [\lambda_i \hat{n}_k] \int \varphi d\eta_i^k| + m_k j(k)\|\varphi\| \\
&\quad + n_k \mathsf{Var}(\varphi, \varepsilon/2_k) + n_k\delta_k \\
&< \delta_k \sum_{i=1}^{j(k)} [\lambda_i \hat{n}_k] + m_k j(k)\|\varphi\| + n_k \mathsf{Var}(\varphi, \varepsilon/2_k) + n_k\delta_k
\end{aligned}$$

The result follows on dividing through by n_k. □

We now construct two intermediate families of finite sets. We follow chapter 4, to which we refer the reader for the full details. The first such family we denote by $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$ and consists of points which shadow a very large number N_k of points from $\mathcal{S}_k$. The second family we denote by $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$ and consist of points which shadow points (taken in order) from $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$. We choose N_k to grow to infinity very quickly, so the ergodic average of a point in $\mathcal{T}_k$ is close to the corresponding point in $\mathcal{C}_k$.

5.1.4 Construction of the intermediate sets $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$

Let us choose a sequence N_k which increases to ∞ sufficiently quickly so that

$$\lim_{k\to\infty} \frac{n_{k+1} + m_{k+1}}{N_k} = 0, \lim_{k\to\infty} \frac{N_1(n_1 + m_1) + \ldots + N_k(n_k + m_k)}{N_{k+1}} = 0. \tag{5.6}$$

We enumerate the points in the sets $\mathcal{S}_k$ provided by lemma 5.1.3 and write them as follows

$$\mathcal{S}_k = \{x_i^k : i = 1, 2, \ldots, \#\mathcal{S}_k\}.$$

We make a choice of k and consider the set of words of length N_k with entries in $\{1, 2, \ldots, \#\mathcal{S}_k\}$. Each such word $\underline{i} = (i_1, \ldots, i_{N_k})$ represents a point in $\mathcal{S}_k^{N_k}$. Using the specification property, we can choose a point $y := y(i_1, \ldots, i_{N_k})$ which satisfies

$$d_{n_k}(x_{i_j}^k, f^{a_j}y) < \frac{\varepsilon}{2^k}$$

for all $j \in \{1, \ldots, N_k\}$, where $a_j = (j-1)(n_k + m_k)$. In other words, y shadows each of the points $x_{i_j}^k$ in order for length n_k and gap m_k. We define

$$\mathcal{C}_k = \left\{ y(i_1, \ldots, i_{N_k}) \in X : (i_1, \ldots, i_{N_k}) \in \{1, \ldots, \#\mathcal{S}_k\}^{N_k} \right\}.$$

Let $c_k = N_k n_k + (N_k - 1)m_k$. Then c_k is the amount of time for which the orbit of points in $\mathcal{C}_k$ has been prescribed. It is a corollary of the following lemma that distinct sequences $(i_1, \ldots, i_{N_k})$ give rise to distinct points in $\mathcal{C}_k$. Thus the cardinality of $\mathcal{C}_k$, which we shall denote by $\#\mathcal{C}_k$, is $\#S_k^{N_k}$.

Lemma 5.1.5. *Let $\underline{i}$ and $\underline{j}$ be distinct words in $\{1, 2, \ldots, \#S_k\}^{N_k}$. Then $y_1 := y(\underline{i})$ and $y_2 := y(\underline{j})$ are $(c_k, 3\varepsilon)$ separated points (i.e. $d_{c_k}(y_1, y_2) > 3\varepsilon$).*

Construction of the intermediate sets $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$

We define $\mathcal{T}_k$ inductively. Let $\mathcal{T}_1 = \mathcal{C}_1$. We construct $\mathcal{T}_{k+1}$ from $\mathcal{T}_k$ as follows. Let $x \in \mathcal{T}_k$ and $y \in \mathcal{C}_{k+1}$. Let $t_1 = c_1$ and $t_{k+1} = t_k + m_{k+1} + c_{k+1}$. Using specification, we can find a point $z := z(x, y)$ which satisfies

$$d_{t_k}(x, z) < \frac{\varepsilon}{2^{k+1}} \text{ and } d_{c_{k+1}}(y, f^{t_k + m_{k+1}} z) < \frac{\varepsilon}{2^{k+1}}.$$

Define $\mathcal{T}_{k+1} = \{z(x, y) : x \in \mathcal{T}_k, y \in \mathcal{C}_{k+1}\}$. Note that t_k is the amount of time for which the orbit of points in $\mathcal{T}_k$ has been prescribed. Once again, points constructed in this way are distinct. So we have

$$\#\mathcal{T}_k = \#\mathcal{C}_1 \ldots \#\mathcal{C}_k = \#S_1^{N_1} \ldots \#S_k^{N_k}.$$

This fact is a corollary of the following straightforward lemma:

Lemma 5.1.6. *For every $x \in \mathcal{T}_k$ and distinct $y_1, y_2 \in \mathcal{C}_{k+1}$*

$$d_{t_k}(z(x, y_1), z(x, y_2)) < \frac{\varepsilon}{2^k} \text{ and } d_{t_{k+1}}(z(x, y_1), z(x, y_2)) > 2\varepsilon.$$

Thus $\mathcal{T}_k$ is a $(t_k, 2\varepsilon)$ separated set. In particular, if $z, z' \in \mathcal{T}_k$, then

$$\overline{B}_{t_k}(z, \frac{\varepsilon}{2^k}) \cap \overline{B}_{t_k}(z', \frac{\varepsilon}{2^k}) = \emptyset.$$

Lemma 5.1.7. *Let $z = z(x, y) \in \mathcal{T}_{k+1}$, then*

$$\overline{B}_{t_{k+1}}(z, \frac{\varepsilon}{2^k}) \subset \overline{B}_{t_k}(x, \frac{\varepsilon}{2^{k-1}}).$$

Construction of the fractal F and a special sequence of measures μ_k

Let $F_k = \bigcup_{x \in \mathcal{T}_k} \overline{B}_{t_k}(x, \frac{\varepsilon}{2^{k-1}})$. By lemma 5.1.7, $F_{k+1} \subset F_k$. Since we have a decreasing sequence of compact sets, the intersection $F = \bigcap_k F_k$ is non-empty. Further, every point $p \in F$ can be uniquely represented by a sequence $\underline{p} = (\underline{p}_1, \underline{p}_2, \underline{p}_3, \ldots)$ where each $\underline{p}_i = (p_1^i, \ldots, p_{N_i}^i) \in \{1, 2, \ldots M_i\}^{N_i}$. Each point in $\mathcal{T}_k$ can be uniquely represented by a finite word $(\underline{p}_1, \ldots \underline{p}_k)$. We introduce some useful notation to help us see this. Let $y(\underline{p}_i) \in \mathcal{C}_i$ be defined as in 5.1.4. Let $z_1(\underline{p}) = y(\underline{p}_1)$ and proceeding inductively, let $z_{i+1}(\underline{p}) = z(z_i(\underline{p}), y(\underline{p}_{i+1})) \in \mathcal{T}_{i+1}$ be defined as in 5.1.4. We can also write $z_i(\underline{p})$ as $z(\underline{p}_1, \ldots, \underline{p}_i)$. Then define $p := \pi \underline{p}$ by

$$p = \bigcap_{i \in \mathbb{N}} \overline{B}_{t_i}(z_i(\underline{p}), \frac{\varepsilon}{2^{i-1}}).$$

It is clear from our construction that we can uniquely represent every point in F in this way.

Lemma 5.1.8. *Given $z = z(\underline{p}_1, \ldots, \underline{p}_k) \in \mathcal{T}_k$, we have for all $i \in \{1, \ldots, k\}$ and all $l \in \{1, \ldots, N_i\}$,*

$$d_{n_i}(x^i_{p^i_l}, f^{t_{i-1}+m_{i-1}+(l-1)(m_i+n_i)}z) < 2\varepsilon.$$

We now define the measures on F which yield the required estimates for the Pressure Distribution Principle. For each $z \in \mathcal{T}_k$, we associate a number $\mathcal{L}(z) \in (0, \infty)$. Using these numbers as weights, we define, for each k, an atomic measure centred on $\mathcal{T}_k$. Precisely, if $z = z(\underline{p}_1, \ldots \underline{p}_k)$, we define

$$\mathcal{L}(z) := \mathcal{L}(\underline{p}_1) \ldots \mathcal{L}(\underline{p}_k),$$

where if $\underline{p}_i = (p^i_1, \ldots, p^i_{N_i}) \in \{1, \ldots, \#\mathcal{S}_i\}^{N_i}$, then

$$\mathcal{L}(\underline{p}_i) := \prod_{l=1}^{N_i} \exp S_{n_i}\psi(x^i_{p^i_l}).$$

We define

$$\nu_k := \sum_{z \in \mathcal{T}_k} \delta_z \mathcal{L}(z).$$

We normalise ν_k to obtain a sequence of probability measures μ_k. More precisely, we let $\mu_k := \frac{1}{\kappa_k}\nu_k$, where κ_k is the normalising constant

$$\kappa_k := \sum_{z \in \mathcal{T}_k} \mathcal{L}_k(z).$$

Lemma 5.1.9. $\kappa_k = M_1^{N_1} \ldots M_k^{N_k}$.

Lemma 5.1.10. *Suppose ν is a limit measure of the sequence of probability measures μ_k. Then $\nu(F) = 1$.*

In fact, the measures μ_k converge. However, by using the generalised Pressure Distribution Principle, we do not need to use this fact and so we omit the proof (which goes like lemma 5.4 of [TV2]). The proof of the following lemma is similar to lemma 5.3 of [TV2] or lemma 4.2.8, and relies on (2) of lemma 5.1.4.

Lemma 5.1.11. *For any $p \in F$, the sequence $\lim_{k\to\infty} \frac{1}{t_k} \sum_{i=0}^{t_k-1} \varphi(f^i(p)) = \alpha$. Thus $F \subset X(\varphi, \alpha)$.*

In order to prove theorem 5.1.1, we give a sequence of lemmas which will allow us to apply the generalised Pressure Distribution Principle. The proofs are the same as the corresponding lemmas from chapter 4, with minor modifications coming from the changed definition of $\mathcal{S}_k$ and lemma 5.1.4.

Let $\mathcal{B} := B_n(q, \varepsilon/2)$ be an arbitrary ball which intersects F. Let k be the unique number which satisfies $t_k \leq n < t_{k+1}$. Let $j \in \{0, \ldots, N_{k+1} - 1\}$ be the unique number so

$$t_k + (n_{k+1} + m_{k+1})j \leq n < t_k + (n_{k+1} + m_{k+1})(j + 1).$$

We assume that $j \geq 1$ and leave the details of the simpler case $j = 0$ to the reader. The following lemma reflects the fact that the number of points in $\mathcal{B} \cap \mathcal{T}_{k+1}$ is restricted since $\mathcal{T}_k$ is $(t_k, 2\varepsilon)$ separated and $\mathcal{S}_{k+1}$ is $(n_{k+1}, 4\varepsilon)$ separated.

Lemma 5.1.12. *Suppose $\mu_{k+1}(\mathcal{B}) > 0$, then there exists (a unique choice of) $x \in \mathcal{T}_k$ and $i_1, \ldots, i_j \in \{1, \ldots, \#\mathcal{S}_{k+1}\}$ satisfying*

$$\nu_{k+1}(\mathcal{B}) \leq \mathcal{L}(x) \prod_{l=1}^{j} \exp S_{n_{k+1}} \psi(x_{i_l}^{k+1}) M_{k+1}^{N_{k+1}-j}.$$

The following lemma is a consequence of lemma 5.1.8.

Lemma 5.1.13. *Let $x \in \mathcal{T}_k$ and $i_1, \ldots, i_j$ be as before. Then*

$$\begin{aligned} \mathcal{L}(x) \prod_{l=1}^{j} \exp S_{n_{k+1}} \psi(x_{i_l}^{k+1}) \leq \exp\{S_n \psi(q) \quad &+ \quad 2n Var(\psi, 2\varepsilon) \\ &+ \quad \|\psi\| (\sum_{i=1}^{k} N_i m_i + j m_{k+1})\}. \end{aligned}$$

The following lemma reflects the restriction on the number of points that can be contained in $\mathcal{B} \cap \mathcal{T}_{k+p}$.

Lemma 5.1.14. *For any $p \geq 1$, suppose $\mu_{k+p}(\mathcal{B}) > 0$. Let $x \in \mathcal{T}_k$ and $i_1, \ldots, i_j$ be as before. We have*

$$\nu_{k+p}(\mathcal{B}) \leq \mathcal{L}(x) \prod_{l=1}^{j} \exp S_{n_{k+1}} \psi(x_{i_l}^{k+1}) M_{k+1}^{N_{k+1}-j} M_{k+2}^{N_{k+2}} \ldots M_{k+p}^{N_{k+p}}.$$

Lemma 5.1.15.

$$\mu_{k+p}(\mathcal{B}) \leq \frac{1}{\kappa_k M_{k+1}^{j}} \exp \left\{ S_n \psi(q) + 2n Var(\psi, 2\varepsilon) + \|\psi\| (\sum_{i=1}^{k} N_i m_i + j m_{k+1}) \right\}.$$

Let $C := h_\mu + \int \varphi d\mu$. The following lemma is implied by lemma 5.1.4.

Lemma 5.1.16. *For sufficiently large n, $\kappa_k M_{k+1}^{j} \geq \exp((C - 2\gamma)n)$*

Combining the previous two lemmas gives us

Lemma 5.1.17. *For sufficiently large n,*

$$\limsup_{l\to\infty} \mu_l(B_n(q, \frac{\varepsilon}{2})) \le \exp\{-n(C - 2Var(\psi, 2\varepsilon) - 3\gamma) + \sum_{i=0}^{n-1} \psi(f^i q)\}.$$

Applying the Generalised Pressure Distribution Principle, we have

$$P_F(\psi, \varepsilon) \ge C - 2\mathrm{Var}(\psi, 2\varepsilon) - 3\gamma.$$

Recall that ε was chosen sufficiently small so $\mathrm{Var}(\psi, 2\varepsilon) < \gamma$. It follows that

$$P_{X(\varphi,\alpha)}(\psi, \varepsilon) \ge P_F(\psi, \varepsilon) \ge C - 5\gamma.$$

Since γ and ε were arbitrary, the proof of theorem 5.1.1 is complete.

5.2 Application to suspension flows

We apply theorem 5.1.1 to suspension flows. Let $f : X \mapsto X$ be a homeomorphism of a compact metric space (X, d). We consider a continuous roof function $\rho : X \mapsto (0, \infty)$. We define the suspension space to be

$$X_\rho = \{(x, s) \in X \times \mathbb{R} : 0 \le s \le \rho(x)\},$$

where $(x, \rho(x))$ is identified with $(f(x), 0)$ for all x. We define the flow $\Psi = \{g_t\}$ on X_ρ locally by $g_t(x, s) = (x, s + t)$. To a function $\Phi : X_\rho \mapsto \mathbb{R}$, we associate the function $\varphi : X \mapsto \mathbb{R}$ by $\varphi(x) = \int_0^{\rho(x)} \Phi(x, t)dt$. Since the roof function is continuous, when Φ is continuous, so is φ. We have (see lemma 4.3.1)

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \Phi(g_t(x, s))dt = \liminf_{n\to\infty} \frac{S_n\varphi(x)}{S_n\rho(x)},$$
$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \Phi(g_t(x, s))dt = \limsup_{n\to\infty} \frac{S_n\varphi(x)}{S_n\rho(x)}.$$

We consider

$$\begin{aligned} X_\rho(\Phi, \alpha) &:= \{(x, s) \in X_\rho : \lim_{T\to\infty} \frac{1}{T} \int_0^T \Phi(g_t(x, s))dt = \alpha\} \\ &= \{(x, s) : \lim_{n\to\infty} \frac{S_n\varphi(x)}{S_n\rho(x)} = \alpha, 0 \le s < \rho(x)\}. \end{aligned}$$

For $\mu \in \mathcal{M}_f(X)$, we define the measure μ_ρ by

$$\int_{X_\rho} \Phi d\mu_\rho = \int_X \varphi d\mu / \int \rho d\mu$$

for all $\Phi \in C(X_\rho)$, where φ is defined as above. We have Ψ-invariance of μ_ρ (ie. $\mu(g_t^{-1}A) = \mu(A)$ for all $t \ge 0$ and measurable sets A). The map $\mathcal{R} : \mathcal{M}_f(X) \mapsto \mathcal{M}_\Psi(X_\rho)$ given by $\mu \mapsto \mu_\rho$ is a bijection. Abramov's theorem [Abr], [PP1] states that $h_{\mu_\rho} = h_\mu / \int \rho d\mu$ and hence,

$$h_{top}(\Psi) = \sup\{h_\mu : \mu \in \mathcal{M}_\Psi(X_\rho)\} = \sup\left\{\frac{h_\mu}{\int \rho d\mu} : \mu \in \mathcal{M}_f(X)\right\},$$

where $h_{top}(\Psi)$ is the topological entropy of the flow. We use the notation $h_{top}(Z,\Psi)$ for topological entropy of a non-compact subset $Z \subset X_\rho$ with respect to Ψ (defined in 2.1.4).

Theorem 5.2.1. *Let (X,d) be a compact metric space and $f : X \mapsto X$ be a continuous map with specification. Let $\varphi, \psi \in C(X)$ and $\rho : X \mapsto (0,\infty)$ be continuous. Let $X(\varphi,\rho,\alpha) := \left\{x \in X : \lim_{n\to\infty} \frac{S_n\varphi(x)}{S_n\rho(x)} = \alpha\right\}$. For α such that $X(\varphi,\rho,\alpha) \neq \emptyset$, we have*

$$P_{X(\varphi,\rho,\alpha)}(\psi) = \sup\left\{h_\mu + \int \psi d\mu : \mu \in \mathcal{M}_f(X) \text{ and } \frac{\int \varphi d\mu}{\int \rho d\mu} = \alpha\right\}.$$

Proof. We require only a small modification to the proof of theorem 5.1.1. We modify lemma 5.1.2 so η_k satisfies $|\int \varphi d\mu / \int \rho d\mu - \int \varphi d\eta_k / \int \rho d\eta_k| < \delta_k$ and replace the family of sets defined at (5.5) by the following:

$$Y_{k,i} := \left\{x \in X : \left|\frac{S_n\varphi(x)}{S_n\rho(x)} - \frac{\int \varphi d\eta_i^k}{\int \rho d\eta_i^k}\right| < \delta_k \text{ for all } n \geq l_k\right\}$$

chosen to satisfy $\eta_i^k(Y_{k,i}) > 1-\gamma$ for every k. This is possible by the ratio ergodic theorem. The rest of the proof requires only superficial modifications. □

Theorem 5.2.2. *Let (X,d) be a compact metric space and $f : X \mapsto X$ be a homeomorphism with the specification property. Let $\rho : X \mapsto (0,\infty)$ be continuous. Let (X_ρ,Ψ) be the corresponding suspension flow over X. Let $\Phi : X_\rho \mapsto \mathbb{R}$ be continuous. We have*

$$h_{top}(X_\rho(\Phi,\alpha),\Psi) = \sup\left\{h_\mu : \mu \in \mathcal{M}_\Psi(X_\rho) \text{ and } \int \Phi d\mu = \alpha\right\}.$$

Proof. Let $Z \subset X$ be arbitrary and $Z_\rho := \{(x,s) : x \in Z, 0 \leq s < \rho(x)\}$. In theorem 4.3.3, we proved that if β is the unique solution to the equation $P_Z(-t\rho) = 0$, then $h_{top}(Z_\rho,\Psi) \geq \beta$. Thus, if h be the unique positive real number which satisfies $P_{X(\varphi,\rho,\alpha)}(-h\rho) = 0$, then $h_{top}(X_\rho(\Phi,\alpha),\Psi) \geq h$. By theorem 5.2.1,

$$\sup\left\{h_\mu - h\int \rho d\mu : \mu \in \mathcal{M}_f(X) \text{ and } \frac{\int \varphi d\mu}{\int \rho d\mu} = \alpha\right\} = 0.$$

Thus, if $\mu \in \mathcal{M}_f(X)$ satisfies $\frac{\int \varphi d\mu}{\int \rho d\mu} = \alpha$, then $h \geq \frac{h_\mu}{\int \rho d\mu}$ and

$$\begin{aligned} h &\geq \sup\left\{\frac{h_\mu}{\int \rho d\mu} : \mu \in \mathcal{M}_f(X), \frac{\int \varphi d\mu}{\int \rho d\mu} = \alpha\right\} \\ &= \sup\left\{h_\mu : \mu \in \mathcal{M}_\Psi(X_\rho) \text{ and } \int \Phi d\mu = \alpha\right\}. \end{aligned}$$

For the opposite inequality, we note that $h_{top}(Z,\Psi) \leq \underline{CP}_Z(0)$, where $\underline{CP}_Z(0)$ is defined with respect to the time-1 map of Ψ. Given $\gamma > 0$, we can adapt lemma 5.1.1 to find a set $Z \subset X_\rho$, $t_k \to \infty$ and $\varepsilon_k \to 0$ such that for $(x,s) \in X_\rho$, we have

$$\left|\frac{1}{T}\int_0^T \Phi(g_t(x,s))dt - \alpha\right| \leq \varepsilon_k \text{ for all } T \geq t_k$$

and $\underline{CP}_Z(0) \geq \underline{CP}_{X(\Phi,\alpha)}(0) - 4\gamma$. We repeat the argument of 5.1.1 to construct a suitable probability measure ν out of (n, ε) spanning sets for the time-1 map of the flow which satisfies $\int \int_0^1 \Phi(g_t x) dt d\nu = \alpha$ and $\underline{CP}_Z(0) - \gamma \leq h_\nu$. We use ν to define a flow invariant measure μ by

$$\int_{X_\rho} \zeta d\mu = \int_{X_\rho} \int_0^1 \zeta(g_t x) dt d\nu$$

for all $\zeta \in C(X_\rho)$ and note that $h_\mu = h_\nu$ and $\int \Phi d\mu = \alpha$. We obtain

$$h_{top}(X_\rho(\Phi, \alpha), \Psi) \leq \sup \left\{ h_\mu : \mu \in \mathcal{M}_\Psi(X_\rho) \text{ and } \int \Phi d\mu = \alpha \right\}.$$

$\square$

As a simple corollary, we note that if $\alpha = \int \Phi dm$, where m is a measure of maximal entropy for the flow, then $h_{top}(X_\rho(\Phi, \alpha), \Psi) = h_{top}(\Phi)$.

5.3 A Bowen formula for Hausdorff dimension of level sets of the Birkhoff average for certain interval maps

The following application was described to the author by Thomas Jordan. If f is a $C^{1+\alpha}$, uniformly expanding Markov map of the interval and $\varphi : [0,1] \mapsto \mathbb{R}$, then it was shown by Olsen [Ols] that

$$\text{dim}_H(X(\varphi, \alpha)) = \sup \left\{ \frac{h_\mu}{\int \log f' d\mu} : \int \varphi d\mu = \alpha \right\}. \tag{5.7}$$

In [JJÖP], the authors consider piecewise C^1 Markov maps of the interval with a finite number of parabolic fixed points x_i such that $f(x_i) = x_i$, $f'(x_i) = 1$ and $f'(x) > 1$ for $x \in [0,1] \setminus \bigcup_i x_i$. They show that (5.7) holds for $\alpha \in \mathcal{L}_\varphi \setminus [\min_i\{\varphi(x_i)\}, \max_i\{\varphi(x_i)\}]$. Simple examples in this category are provided by the Manneville-Pomeau family of maps $f_t(x) = x^t + x^{1+t}(\text{mod} 1)$ (where $t > 0$ is a fixed parameter), which have a single parabolic fixed point at 0. Henceforth, we let $\psi = \log f'$. Note that since ψ is non-negative, $s \mapsto P_{X(\varphi,\alpha)}(-s\psi)$ is decreasing (although possibly not strictly decreasing).

Theorem 5.3.1. *Suppose $s \mapsto P_{X(\varphi,\alpha)}(-s\psi)$ has a unique zero d and (5.7) holds true. Then $d = \text{dim}_H(X(\varphi, \alpha))$.*

Proof. By (5.7), if $\mu \in \mathcal{M}_f(X)$ and $\int \varphi d\mu = \alpha$, then

$$h_\mu - \text{dim}_H(X(\varphi, \alpha)) \int \psi d\mu \leq 0.$$

By theorem 5.1.1, $P_{X(\varphi,\alpha)}(-\text{dim}_H(X(\varphi, \alpha))\psi) \leq 0$. Thus $\text{dim}_H(X(\varphi, \alpha)) \geq d$.

Now suppose $\text{dim}_H(X(\varphi, \alpha)) < d$. Since $s \mapsto P_{X(\varphi,\alpha)}(-s\psi)$ is decreasing and has a unique zero, $P_{X(\varphi,\alpha)}(-\text{dim}_H(X(\varphi, \alpha))\psi) > 0$. By theorem 5.1.1, there exists μ with $\int \varphi d\mu = \alpha$ and

$h_\mu - \dim_H(X(\varphi,\alpha)) \int \psi d\mu > 0$. This implies that $\dim_H(X(\varphi,\alpha)) < h_\mu / \int \psi d\mu$, which contradicts (5.7). $\square$

We remark that by a slight modification to the proof, a more general statement is that if (5.7) holds and $d = \inf\{s : P_{X(\varphi,\alpha)}(-s\psi) = 0\}$, then $d = \dim_H(X(\varphi,\alpha))$.

We comment on the hypotheses of theorem 5.3.1. If there exists μ with $\int \varphi d\mu = \alpha$ and $\int \psi d\mu > 0$, then $s \mapsto P_{X(\varphi,\alpha)}(-s\psi)$ is strictly decreasing. Now suppose $\varphi = \psi = \log f'$. In the case of the Manneville-Pomeau family of maps, the only measure with $\int \psi d\mu = 0$ is the Dirac measure supported at 0, and so $s \mapsto P_{X(\varphi,\alpha)}(-s\psi)$ is decreasing for $\alpha \in \mathcal{L}_\varphi \setminus \{0\}$. By [JJÖP], (5.7) holds true for the same set of values and thus theorem 5.3.1 applies. We remark that for $\alpha = 0$, $P_{X(\log f',0)}(-s\psi) = 0$ for all $s \in \mathbb{R}$.

Chapter 6

Irregular sets for maps with the almost specification property and for the β-transformation

For a compact metric space (X, d), a continuous map $f : X \mapsto X$ and a continuous function $\varphi : X \mapsto \mathbb{R}$, we return to our study of the irregular set for φ,

$$\widehat{X}(\varphi, f) := \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \text{ does not exist} \right\}.$$

As a special case of the main result of chapter 4, we showed that when f has the specification property, $\widehat{X}(\varphi, f)$ is either empty or has full topological entropy. Here, we extend this result to the class of maps f which satisfies a property we call almost specification.

Pfister and Sullivan introduced the g-almost product property in [PS2], [PS1]. We have taken the liberty of renaming this property as the almost specification property (in fact, our definition is a priori slightly weaker). The most striking application of the almost specification property (to date) is that it applies to every β-shift. In sharp contrast, the set of β for which the β-shift has the specification property has zero Lebesgue measure [Buz], [Sch].

First main result of chapter 6. *When f satisfies the almost specification property, the irregular set is either empty or has full topological entropy.*

Second main result of chapter 6 . *The irregular set for an arbitrary β-transformation (or β-shift) is either empty or has full entropy $\log \beta$ and Hausdorff dimension 1.*

For a set of β of full Lebesgue measure, our second main result (stated formally as theorem 6.5.1 and theorem 6.5.2) is a corollary of our first main result (stated formally as theorem 6.4.1).

Some further analysis is required to extend the Hausdorff dimension part of the statement to the remaining null set of β (see theorem 6.5.3).

To undertake the proof of our first main result, we develop notions of almost spanning sets, strongly separated sets and a generalised version of the Katok formula for entropy. This should be of independent interest.

In chapter 4, we showed that when f has the specification property, the irregular set is either empty or has full topological pressure. The method of this chapter can be used to show that this more general result holds true in the almost specification setting. However, we restrict ourselves to the special case of entropy for clarity and brevity.

Furthermore, Pfister and Sullivan proved that the conditional variational principle for entropy of Takens and Verbitskiy holds for maps with the almost specification property (this corresponds to the special case $\psi = 0$ of our theorem 5.1.1). A synthesis of the techniques of this chapter and chapter 5 can be used to prove a full version of theorem 5.1.1 for maps with almost specification (i.e. a conditional variational principle for pressure). We choose not to write out the proof as all the necessary ideas are included in this chapter and chapter 5.

In §6.1, we define the almost specification property. In §6.3 we establish our general version of the Katok formula for entropy. In §6.4, we prove our first main result. In §6.5, we consider arbitrary β-shifts and β-transformations and establish our second main result.

6.1 The almost specification property

Pfister and Sullivan have introduced a property called the g-almost product property. We take the liberty of renaming this property the almost specification property. The almost specification property can be verified for every β-shift (see §6.5.1). The version we use here is a priori weaker than that in [PS2]. First we need an auxiliary definition.

Definition 6.1.1. *Let $\varepsilon_0 > 0$. A function $g : \mathbb{N} \times (0, \varepsilon_0) \mapsto \mathbb{N}$ is called a mistake function if for all $\varepsilon \in (0, \varepsilon_0)$ and all $n \in \mathbb{N}$, $g(n, \varepsilon) \leq g(n+1, \varepsilon)$ and*

$$\lim_{n \to \infty} \frac{g(n, \varepsilon)}{n} = 0.$$

Given a mistake function g, if $\varepsilon > \varepsilon_0$, we define $g(n, \varepsilon) := g(n, \varepsilon_0)$.

We note that for fixed $k \in \mathbb{N}$ and $\lambda > 0$, if g is a mistake function, then so is h defined by $h(n, \varepsilon) = kg(n, \lambda\varepsilon)$.

Definition 6.1.2. *For $n, m \in \mathbb{N}, m < n$, we define the set of $(n, -m)$ index sets to be*

$$I(n, -m) := \{\Lambda \subseteq \{0, \ldots, n-1\}, \#\Lambda \geq n - m\}.$$

Let g be a mistake function and $\varepsilon > 0$. For n sufficiently large so that $g(n,\varepsilon) < n$, we define the set of (g,n,ε) index sets to be $I(g;n,\varepsilon) := I(n, -g(n,\varepsilon))$. Equivalently,

$$I(g;n,\varepsilon) := \{\Lambda \subseteq \{0,\ldots,n-1\}, \#\Lambda \geq n - g(n,\varepsilon)\}.$$

For a finite set of indices Λ, we define

$$d_\Lambda(x,y) = \max\{d(f^j x, f^j y) : j \in \Lambda\} \text{ and } B_\Lambda(x,\varepsilon) = \{y \in X : d_\Lambda(x,y) < \varepsilon\}.$$

Definition 6.1.3. *When $g(n,\varepsilon) < n$, we define a 'dynamical ball of radius ε and length n with $g(n,\varepsilon)$ mistakes'. Let*

$$\begin{aligned} B_n(g;x,\varepsilon) &:= \{y \in X : y \in B_\Lambda(x,\varepsilon) \text{ for some } \Lambda \in I(g;n,\varepsilon)\} \\ &= \bigcup_{\Lambda \in I(g;n,\varepsilon)} B_\Lambda(x,\varepsilon) \end{aligned}$$

Definition 6.1.4. *A continuous map $f : X \mapsto X$ satisfies the almost specification property if there exists a mistake function g such that for any $\varepsilon_1,\ldots,\varepsilon_k > 0$, there exist integers $N(g,\varepsilon_1),\ldots,N(g,\varepsilon_k)$ such that for any $x_1,\ldots,x_k$ in X and integers $n_i \geq N(g,\varepsilon_i)$,*

$$\bigcap_{j=1}^{k} f^{-\sum_{i=0}^{j-1} n_i} B_{n_j}(g;x_j,\varepsilon_j) \neq \emptyset,$$

where $n_0 = 0$.

Remark 6.1.1. The function g can be interpreted as follows. The integer $g(n,\varepsilon)$ tells us how many mistakes we are allowed to make when we use the almost specification property to ε shadow an orbit of length n. Henceforth, we assume for convenience and without loss of generality that $N(g,\varepsilon)$ is chosen so that $g(n,\varepsilon)/n < 0.1$ for all $n \geq N(g,\varepsilon)$.

Remark 6.1.2. Pfister and Sullivan use a slightly different definition of mistake function (which they call a blowup function). They do not allow g to depend on ε. An example of a function which is a mistake function under our definition but is not considered by Pfister and Sullivan is $g(n,\varepsilon) = \varepsilon^{-1}\log n$. Since we allow a larger class of mistake functions, the almost specification property defined here is slightly more general than the g-almost product property of Pfister and Sullivan.

We compare specification with almost specification. We recall that $f : X \mapsto X$ satisfies specification if for all $\varepsilon > 0$, there exists an integer $m = m(\varepsilon)$ such that for any collection $\{I_j = [a_j, b_j] \subset \mathbb{N} : j = 1,\ldots,k\}$ of finite intervals with $a_{j+1} - b_j \geq m(\varepsilon)$ for $j = 1,\ldots,k-1$ and any $x_1,\ldots,x_k$ in X, there exists a point $x \in X$ such that

$$d(f^{p+a_j}x, f^p x_j) < \varepsilon \text{ for all } p = 0,\ldots,b_j - a_j \text{ and every } j = 1,\ldots,k.$$

Pfister and Sullivan showed that the specification property implies the almost specification property [PS2] using ANY blow-up function g. To see the relation between the two concepts, we note that if f has specification and we set $g(n,\varepsilon) = m(\varepsilon)$ for all n larger than $m(\varepsilon)$ and set $N(g,\varepsilon) = m(\varepsilon)+1$, then for any $x_1, \dots, x_k$ in X and integers $n_i \geq N(g,\varepsilon)$, we have

$$\bigcap_{j=1}^{k} f^{-\sum_{i=0}^{j-1} n_i} B_{n_j}(g; x_j, \varepsilon) \neq \emptyset.$$

The trick required to replace ε by $\varepsilon_1, \dots, \varepsilon_k$ can be found in [PS2].

6.2 Technique

To prove our first main theorem of the chapter, we modify the strategy laid out in chapter 3. We require a generalised version of the Katok entropy formula in order to successfully generalise the method of proof described above. We briefly explain why.

The key basic fact required for the construction in chapter 4 which does not generalise is lemma 4.2.2. We give an example to demonstrate why. First, we assume that f has specification. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be $(n, 4\varepsilon)$ separated sets. For each pair $(x,y) \in \mathcal{S}_1 \times \mathcal{S}_2$, let us use the specification property to define a point $z := z(x,y)$ such that $d_n(z,x) < \varepsilon$ and $d_n(f^{n+m(\varepsilon)}z, y) < \varepsilon$. Define

$$Y = \{z(x,y) : x \in \mathcal{S}_1, y \in \mathcal{S}_2\}.$$

Suppose that $z_1 = z(x_1,y_1), z_2 = z(x_2,y_2) \in Y$ and $(x_1,y_1) \neq (x_2,y_2)$. Either $x_1 \neq x_2$, in which case

$$d_n(z_1,z_2) > d_n(x_1,x_2) - d_n(x_1,z_1) - d_n(x_1,z_2) > 2\varepsilon,$$

or $y_1 \neq y_2$, in which case $d_n(f^{n+m(\varepsilon)}z_1, f^{n+m(\varepsilon)}z_2) > 2\varepsilon$. In particular, $z_1 \neq z_2$. Thus $\#Y = \#\mathcal{S}_1\#\mathcal{S}_2$. This kind of argument is essential for our entropy estimates.

We now see what happens when we attempt the same construction only assuming that f has the almost specification property. For each pair $(x,y) \in \mathcal{S}_1 \times \mathcal{S}_2$, the almost specification property guarantees the existence of a point $z(x,y)$ such that

$$z(x,y) \in B_n(g; x, \varepsilon) \cap f^{-n} B_n(g; y, \varepsilon),$$

where g is a suitable mistake function and we assume that n was chosen to be sufficiently large. Define

$$Y' = \{z(x,y) : x \in \mathcal{S}_1, y \in \mathcal{S}_2\},$$

where $z(x,y)$ is a choice of point in $B_n(g;x,\varepsilon) \cap f^{-n} B_n(g;y,\varepsilon)$. Let $z_1 = z(x_1,y_1)$, $z_2 = z(x_2,y_2) \in Y'$ with $(x_1,y_1) \neq (x_2,y_2)$. We have no guarantee that $z_1 \neq z_2$. This is because it is possible that

$d(f^i x_1, f^i x_2) > 4\varepsilon$ at only one index $i \in \{0, \ldots, n-1\}$ and we cannot guarantee that $d(f^i z_1, f^i x_1) < \varepsilon$. Thus, a priori, we may have $\#Y' < \#\mathcal{S}_1 \#\mathcal{S}_2$.

To solve this problem, we develop a notion of a 'strongly separated set'. The idea is that $\mathcal{S}$ is $(n, -m, 4\varepsilon)$ separated (where $m < n$) if for every set of indices $\Lambda \subset \{0, \ldots, n-1\}$ such that $\#\Lambda \geq n - m$ we have

$$\max\{d(f^i x, f^i y) : i \in \Lambda\} > 4\varepsilon.$$

In our example, if we replace $\mathcal{S}_1$ and $\mathcal{S}_2$ by $(n, -m, 4\varepsilon)$ separated sets where $m = 2g(n, \varepsilon) + 1$, we can guarantee that the set of points Y'' constucted as before using the almost specification property satisfies $\#Y'' = \#\mathcal{S}_1 \#\mathcal{S}_2$.

Thus to adapt our strategy to the almost specification setting, we prove a modification of the Katok entropy formula which uses 'strongly separated sets' in place of spanning sets.

6.3 A modified Katok entropy formula

The following definitions of 'strongly separated' and 'almost spanning' sets are inspired by Pfister and Sullivan and designed for use in the setting of maps with the almost specification property.

Definition 6.3.1. *Let $Z \subseteq X$. For $m < n$, a set S is $(n, -m, \varepsilon)$ separated for Z if $S \subset Z$ and for every $\Lambda \in I(n, -m)$, we have $d_\Lambda(x, y) > \varepsilon$ for every $x, y \in S$. We define a set S to be $(g; n, \varepsilon)$ separated if it is $(n, -g(n, \varepsilon), \varepsilon)$ separated. Equivalently, S is $(g; n, \varepsilon)$ separated if for every $x, y \in S$*

$$\#\{j \in \{0, \ldots n-1\} : d(f^j x, f^j y) > \varepsilon\} > g(n, \varepsilon).$$

We think of an $(n, -m, \varepsilon)$ separated set to be 'a set which remains (n, ε) separated when you permit m mistakes'. In particular, a set S which is $(g; n, \varepsilon)$ separated is (n, ε) separated in the usual sense. We define the natural dual notion of a $(g; n, \varepsilon)$ spanning set.

Definition 6.3.2. *For $m < n$, a set $S \subset Z$ is $(n, -m, \varepsilon)$ spanning for Z if for all $x \in Z$, there exists $y \in S$ and $\Lambda \in I(n, -m)$ such that $d_\Lambda(x, y) \leq \varepsilon$. Note that Λ depends on x and an (n, ε) spanning set is always $(n, -m, \varepsilon)$ spanning. We define a set S to be $(g; n, \varepsilon)$ spanning if it is $(n, -g(n, \varepsilon), \varepsilon)$ spanning.*

We think of an $(n, -m, \varepsilon)$ spanning set to be 'a set which requires up to m mistakes to be (n, ε) spanning'. Let

$$s_n(g; Z, \varepsilon) = \sup\{\#S : \text{ S is } (g; n, \varepsilon) \text{ separated for } Z\},$$

$$r_n(g; Z, \varepsilon) = \inf\{\#S : \text{ S is } (g; n, \varepsilon) \text{ spanning for } Z\},$$

$$s_n(Z,\varepsilon) = \sup\{\#S: \text{ S is } (n,\varepsilon) \text{ separated for } Z\},$$

$$r_n(Z,\varepsilon) = \inf\{\#S: \text{ S is } (n,\varepsilon) \text{ spanning for } Z\}.$$

Lemma 6.3.1. *We have*

(1) $r_n(g;Z,\varepsilon) \le s_n(g;Z,\varepsilon) \le s_n(Z,\varepsilon)$*,*

(2) $s_n(2g;Z,2\varepsilon) \le r_n(g;Z,\varepsilon) \le r_n(Z,\varepsilon) \le s_n(Z,\varepsilon)$*.*

Proof. Suppose that S is a $(g;n,\varepsilon)$ separated set for Z of maximum cardinality such that S is not $(g;n,\varepsilon)$ spanning. We can find

$$z \in Z \setminus \bigcup_{\Lambda \in I(g;n,\varepsilon)} \bigcup_{x \in S} \overline{B_\Lambda(x,\varepsilon)} = \bigcap_{\Lambda \in I(g;n,\varepsilon)} \left(Z \setminus \bigcup_{x \in S} \overline{B_\Lambda(x,\varepsilon)} \right).$$

Since $d_\Lambda(x,z) > \varepsilon$ for all $x \in S$ and $\Lambda \in I(g;n,\varepsilon)$, then $S \cup \{z\}$ is a $(g;n,\varepsilon)$ separated set, which contradicts the maximality of S. Thus, every $(g;n,\varepsilon)$ separated set of maximal cardinality is $(g;n,\varepsilon)$ spanning.

For (2), suppose E is $(2g;n,2\varepsilon)$ separated and F is $(g;n,\varepsilon)$ spanning for Z. Define $\phi : E \mapsto F$ by choosing for each $x \in E$ some $\phi(x) \in F$ and some $\Lambda_x \in I(g;n,\varepsilon)$ such that $d_{\Lambda_x}(x,\phi(x)) \le \varepsilon$. Suppose $x \neq y$. Let $\Lambda = \Lambda_x \cap \Lambda_y$. Since $\Lambda \in I(2g;n,\varepsilon)$, we have $d_\Lambda(\phi(x),\phi(y)) > 0$ and thus $\phi(x) \neq \phi(y)$. Thus ϕ is injective and hence $|E| \le |F|$. □

Theorem 6.3.1 (Modified Katok entropy formula)**.** *Let* (X,d) *be a compact metric space,* $f : X \mapsto X$ *be a continuous map and* μ *be an ergodic invariant measure. For* $\gamma \in (0,1)$ *and any mistake function* g*, we have*

$$h_\mu = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log(\inf\{r_n(g;Z,\varepsilon) : Z \subset X, \mu(Z) \ge 1-\gamma\})$$

The formula remains true if we replace the $\liminf$ *by* $\limsup$ *and/or* $r_n(g;Z,\varepsilon)$ *by* $s_n(g;Z,\varepsilon)$*. The value taken by the* $\liminf$ *(or* $\limsup$*) is independent of the choice of mistake function* g*.*

Proof. Since $r_n(Z,\varepsilon) \ge r_n(g;Z,\varepsilon)$, it follows from the original Katok entropy formula that the expression on the right hand side is less than or equal to h_μ (this is the easier inequality to prove directly any way). To prove the opposite inequality, we give a method inspired by the proof of theorem A2.1 of [Pes].

For any $\eta > 0$, there exists δ, $0 < \delta \le \eta$, a finite Borel partition $\xi = \{C_1, \dots, C_m\}$ and a finite open cover $\mathcal{U} = \{U_1, \dots, U_k\}$ of X where $k \ge m$ with the following properties:

(1) Diam$(U_i) \le \eta$, Diam$(C_j) \le \eta$ for all $i = 1, \dots, m$, $j = 1, \dots, k$,

(2) $\overline{U}_i \subset C_i$ for all $i = 1, \dots, m$,

(3) $\mu(C_i \setminus U_i) \le \delta$ for all $i = 1, \dots, m$ and $\mu(\bigcup_{i=m+1}^k U_i) \le \delta$,

(4) $2\delta \log m \leq \eta$.

This is a consequence of the regularity of the measure μ. Fix η so $1-\gamma > \eta > 0$ and take the corresponding number δ, covering $\mathcal{U}$ and partition ξ. Fix $Z \subset X$ with $\mu(Z) > 1-\gamma$. Let $t_n(x)$ denote the number of l, $0 \leq l \leq n-1$ for which $f^l(x) \in \bigcup_{i=m+1}^k U_i$. Let $\xi_n = \bigvee_{i=0}^{n-1} f^{-i}\xi$ and $C_{\xi_n}(x)$ denote the member of the partition ξ_n to which x belongs.

Lemma 6.3.2. *There exists a set $A \subset Z$ and $N > 0$ with $\mu(A) \geq \mu(Z) - \delta$ such that for every $x \in A$ and $n \geq N$*

(1) $t_n(x) \leq 2\delta n$

(2) $\mu(C_{\xi_n}(x)) \leq \exp(-(h_\mu(f,\xi) - \delta)n)$

Proof. Let χ be the characteristic function of $\bigcup_{i=m+1}^k U_i$. We can write $t_n(x) = \sum_{i=0}^{n-1} \chi(f^i x)$. By Birkhoff's ergodic theorem and Egorov's theorem, we can find a set $A_1 \subset X$ with $\mu(A_1) \geq \mu(Z) - \frac{\delta}{2}$ such that for $x \in A_1$, we have uniform convergence

$$n^{-1}t_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \chi(f^i x) \to \int \chi d\mu = \mu(\bigcup_{i=m+1}^{k} U_i) \leq \delta.$$

Choose N_1 such that if $n \geq N_1$ and $x \in A_1$, then $t_n(x) \leq 2\delta n$.

By Shannon-Mcmillan-Brieman theorem and Egorov's theorem, we can find a set $A_2 \subset X$ with $\mu(A_2) \geq \mu(Z) - \frac{\delta}{2}$ such that for $x \in A_1$, we have uniform convergence

$$-\frac{1}{n}\log \mu(C_{\xi_n}(x)) \to h_\mu(f,\xi)$$

There exists N_2 such that if $n \geq N_2$ and $x \in A_2$, then $-\frac{1}{n}\log \mu(C_{\xi_n}(x)) \leq h(f,\xi) + \delta$. Set $A = A_1 \cap A_2$ and $N = \max\{N_1, N_2\}$ and the lemma is proved. □

Let ξ_n^* be the collection of elements C_{ξ_n} of the partition ξ_n for which $C_{\xi_n} \cap A \neq \emptyset$. Then for $n \geq N$, using property (2) of A,

$$\begin{aligned} \#\xi_n^* &\geq \sum_{C \in \xi_n^*} \mu(C) \exp\{n(h_\mu(f,\xi) - \delta)\} \\ &\geq \mu(A) \exp\{n(h_\mu(f,\xi) - \delta)\}\}. \end{aligned}$$

Let 2ε be a Lebesgue number for $\mathcal{U}$ and let S be $(g; n, \varepsilon)$ spanning for Z. We have $Z \subseteq \bigcup_{x\in S} \overline{B}_{\Lambda_x}(x,\varepsilon)$ for suitably chosen $\Lambda_x \in I(g; n, \varepsilon)$. Let us fix $B = \overline{B}_{\Lambda_x}(x,\varepsilon)$. Let ξ_Λ be the partition $\bigvee_{i\in\Lambda} f^{-i}\xi$. We estimate the number $p(B, \xi_{\Lambda_x})$ of elements of the partition ξ_{Λ_x} which have non-empty intersection with $A \cap B$.

Since 2ε is a Lebesgue number for $\mathcal{U}$, then $\overline{B}(f^j x, \varepsilon) \subset U_{i_j}$ for some $U_{i_j} \in \mathcal{U}$. If $i_j \in \{1, \ldots, m\}$ then $f^{-j}(U_{i_j}) \subset f^{-j}(C_{i_j})$. If $i_j \in \{m+1, \ldots, k\}$, then anything up to m sets of the form $f^{-j}(C_{i_j})$ may have non-empty intersection with $f^{-j}(U_{i_j})$. It follows, using property (1) of A, that

$$p(B, \xi_{\Lambda_x}) \leq m^{2\delta n} = \exp(2\delta n \log m)$$

The number $p(B, \xi_n)$ of elements of the partition ξ_n which have non-empty intersection with both A and B satisfies

$$p(B, \xi_n) \leq p(B, \xi_{\Lambda_x}) m^{g(n,\varepsilon)} \leq \exp\{(2\delta n + g(n, \varepsilon)) \log m\}.$$

It follows that

$$\#\xi_n^* \leq \sum_{x \in S} p(\overline{B}_{\Lambda_x}(x, \varepsilon), \xi_n) \leq \#S \exp\{(2\delta n + g(n, \varepsilon)) \log m\}.$$

Rearranging, we have

$$\frac{1}{n} \log \#S \geq h_\mu(f, \xi) - \delta - \left(2\delta + \frac{g(n, \varepsilon)}{n}\right) \log m.$$

Since $2\delta \log m < \eta$, Diam$(\xi) < \eta$, $\delta < \eta$, $\frac{g(n,\varepsilon)}{n} \to 0$, and η was arbitrary, we are done. □

As a corollary, we have a version of theorem 6.3.1 for topological entropy (which we do not use).

Theorem 6.3.2. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a continuous map. We have*

$$h_{top}(f) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log r_n(g; X, \varepsilon)$$

The formula remains true if we replace the $\liminf$ *by* $\limsup$ *and/or* $r_n(g; X, \varepsilon)$ *by* $s_n(g; X, \varepsilon)$*. The value taken by the* $\liminf$ *(or* $\limsup$*) is independent of the choice of mistake function* g*.*

6.4 Main result

Theorem 6.4.1. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a continuous map with the almost specification property. Assume that $\varphi \in C(X)$ satisfies $\inf_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu < \sup_{\mu \in \mathcal{M}_f(X)} \int \varphi d\mu$. Let $\widehat{X}(\varphi, f)$ be the irregular set for φ, then $h_{top}(\widehat{X}(\varphi, f)) = h_{top}(f)$.*

We remark that $\widehat{X}(\varphi, f) \neq \emptyset$ is a sufficient condition on φ for the theorem to apply (see lemma 2.3.2).

Proof. Let us fix a small $\gamma > 0$, and take ergodic measures μ_1 and μ_2 such that

(1) $h_{\mu_i} > h_{top}(f) - \gamma$ for $i = 1, 2$,

(2) $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$.

That we are able to choose μ_i to be ergodic is a slightly subtle point. Let μ_1 be ergodic and satisfy $h_{\mu_1} > h_{top}(f) - \gamma/3$. Let $\nu \in \mathcal{M}_f(X)$ satisfy $\int \varphi d\mu_1 \neq \int \varphi d\nu$. Let $\nu' = t\mu_1 + (1-t)\nu$ where $t \in (0,1)$ is chosen sufficiently close to 1 so that $h_{\nu'} > h_{top}(f) - 2\gamma/3$. By [PS1], when f has the almost specification property, we can find a sequence of ergodic measures $\nu_n \in \mathcal{M}_f(X)$ such that $h_{\nu_n} \to h_{\nu'}$ and $\nu_n \to \nu'$ in the weak-$*$ topology. Therefore, we can choose a measure belonging to this sequence which we call μ_2 which satisfies $h_{\mu_2} > h_{top}(f) - \gamma$ and $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$. (We could avoid the use of the result from [PS1] by giving a self-contained proof along the lines of the 'modified construction' in §4.2.2. We do not do so in the interest of brevity.) Choose $\delta > 0$ sufficiently small so

$$\left| \int \varphi d\mu_1 - \int \varphi d\mu_2 \right| > 4\delta.$$

Let $\rho : \mathbb{N} \mapsto \{1,2\}$ be given by $\rho(k) = (k+1)(\text{mod} 2) + 1$. Choose a strictly decreasing sequence $\delta_k \to 0$ with $\delta_1 < \delta$ and a strictly increasing sequence $l_k \to \infty$ so the set

$$Y_k := \left\{ x \in X' : \left| \frac{1}{n} S_n \varphi(x) - \int \varphi d\mu_{\rho(k)} \right| < \delta_k \text{ for all } n \geq l_k \right\} \tag{6.1}$$

satisfies $\mu_{\rho(k)}(Y_k) > 1 - \gamma$ for every k.

The following lemma follows readily from proposition 6.3.1. The proof is similar to that of lemma 4.2.1.

Lemma 6.4.1. *Define mistake functions $h_k(n, \varepsilon) := 2g(n, \varepsilon/2^k)$. For any sufficiently small $\varepsilon > 0$, we can find a sequence $n_k \to \infty$ and a countable collection of finite sets $\mathcal{S}_k$ so that each $\mathcal{S}_k$ is a $(h_k; n_k, 4\varepsilon)$ separated set for Y_k and satisfies*

$$\# S_k \geq \exp(n_k(h_{top}(f) - 4\gamma)).$$

Furthermore, the sequence n_k can be chosen so that $n_k \geq l_k$, $n_k > N(h_k, \varepsilon)$ and $h_k(n_k, \varepsilon)/n_k \to 0$.

We choose ε sufficiently small and fix all the ingredients provided by lemma 6.4.1. Our strategy is to construct a certain fractal $F \subset \widehat{X}(\varphi, f)$, on which we can define a sequence of measures suitable for an application of the Entropy Distribution Principle (we use a version which is a special case of theorem 4.1.4).

Theorem 6.4.2 (Entropy Distribution Principle)**.** *Let $f : X \mapsto X$ be a continuous transformation. Let $Z \subseteq X$ be an arbitrary Borel set. Suppose there exists $\varepsilon > 0$ and $s \geq 0$ such that one can find a sequence of Borel probability measures μ_k, a constant $K > 0$ and an integer N satisfying*

$$\limsup_{k \to \infty} \mu_k(B_n(x, \varepsilon)) \leq K e^{-ns}$$

for every ball $B_n(x, \varepsilon)$ such that $B_n(x, \varepsilon) \cap Z \neq \emptyset$ and $n \geq N$. Furthermore, assume that at least one limit measure ν of the sequence μ_k satisfies $\nu(Z) > 0$. Then $h_{top}(Z, \varepsilon) \geq s$.

6.4.1 Construction of the fractal F

Let us choose a sequence with $N_0 = 0$ and N_k increasing to ∞ sufficiently quickly so that

$$\lim_{k\to\infty} \frac{n_{k+1}}{N_k} = 0, \lim_{k\to\infty} \frac{N_1 n_1 + \ldots + N_k n_k}{N_{k+1}} = 0. \tag{6.2}$$

Let $\underline{x}_i = (x_1^i, \ldots, x_{N_i}^i) \in S_i^{N_i}$. For any $(\underline{x}_1, \ldots, \underline{x}_k) \in S_1^{N_1} \times \ldots \times S_k^{N_k}$, by the almost specification property, we have

$$B(\underline{x}_1, \ldots \underline{x}_k) := \bigcap_{i=1}^{k} \bigcap_{j=1}^{N_i} f^{-\sum_{l=0}^{i-1} N_l n_l - (j-1)n_i} B_{n_i}(g; x_j^i, \frac{\varepsilon}{2^i}) \neq \emptyset.$$

We define F_k by

$$F_k = \{\overline{B(\underline{x}_1, \ldots, \underline{x}_k)} : (\underline{x}_1, \ldots \underline{x}_k) \in S_1^{N_1} \times \ldots \times S_k^{N_k}\}.$$

Note that F_k is compact and $F_{k+1} \subseteq F_k$. Define $F = \bigcap_{k=1}^{\infty} F_k$.

Lemma 6.4.2. *For any $p \in F$, the sequence $\frac{1}{t_k}\sum_{i=0}^{t_k-1} \varphi(f^i(p))$ diverges, where $t_k = \sum_{i=0}^{k} N_i n_i$.*

Proof. Choose $p \in F$ and let $p_k := f^{t_{k-1}}p$. Then there exists $(x_1^k, \ldots, x_{N_k}^k) \in S_k^{N_k}$ such that

$$p_k \in \bigcap_{j=1}^{N_k} f^{-(j-1)n_k} \overline{B_{n_k}}(g; x_j^k, \varepsilon/2_k).$$

For $c > 0$, let $\mathsf{Var}(\varphi, c) := \sup\{|\varphi(x) - \varphi(y)| : d(x,y) < c\}$. We have

$$S_{n_k N_k}\varphi(p_k) \leq \sum_{j=1}^{N_k} S_{n_k}\varphi(x_j^k) + n_k N_k \mathsf{Var}(\varphi, \varepsilon/2^k) + N_k g(n_k, \varepsilon/2^k)\|\varphi\|$$

and hence

$$\frac{1}{n_k N_k} S_{n_k N_k}\varphi(p_k) \leq \int \varphi d\mu_{\rho(k)} + \delta_k + \mathsf{Var}(\varphi, \varepsilon/2^k) + \frac{1}{n_k} g(n_k, \varepsilon/2^k).$$

It follows that

$$\left| \frac{1}{n_k N_k} S_{n_k N_k}\varphi(p_k) - \int \varphi d\mu_{\rho(k)} \right| \to 0.$$

We can use the fact that $\frac{n_k N_k}{t_k} \to 1$ to prove that

$$\left| \frac{1}{n_k N_k} S_{n_k N_k}\varphi(p_k) - \frac{1}{t_k} S_{t_k}\varphi(p) \right| \to 0,$$

and the result follows. □

6.4.2 Construction of a special sequence of measures μ_k

We first undertake an intermediate construction. For each $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_k) \in S_1^{N_1} \times \ldots \times S_k^{N_k}$, we choose (one) point $z = z(\underline{x})$ such that

$$z \in B(\underline{x}_1, \ldots \underline{x}_k).$$

Let $\mathcal{T}_k$ be the set of all points constructed in this way. We show that points constructed in this way are distinct and thus $\#\mathcal{T}_k = \#S_1^{N_1} \ldots \#S_k^{N_k}$.

Lemma 6.4.3. *Let $\underline{x}$ and $\underline{y}$ be distinct members of $S_1^{N_1} \times \ldots \times S_k^{N_k}$. Then $z_1 := z(\underline{x})$ and $z_2 := z(\underline{y})$ are distinct points. Thus $\#\mathcal{T}_k = \#S_1^{N_1} \ldots \#S_k^{N_k}$.*

Proof. Since $\underline{x} \neq \underline{y}$, there exists i, j so $x_j^i \neq y_j^i$. We have $\Lambda_1, \Lambda_2 \in I(g; n_i, \varepsilon/2^i)$ such that

$$d_{\Lambda_1}(x_j^i, f^a z_1) < \frac{\varepsilon}{2^i} \text{ and } d_{\Lambda_2}(y_j^i, f^a z_2) < \frac{\varepsilon}{2^i},$$

where $a = \sum_{l=0}^{i-1} N_l n_l + (j-1) n_i$. Let $\Lambda = \Lambda_1 \cap \Lambda_2$. Since $\Lambda \in I(2g; n_i, \varepsilon/2^i)$, we have $d_\Lambda(x_j^i, y_j^i) > 4\varepsilon$. Using these inequalities, we have $d_\Lambda(f^a z_1, f^a z_2) > 3\varepsilon$. $\square$

We now define the measures on F which yield the required estimates for the Entropy Distribution Principle. We define, for each k, an atomic measure centred on $\mathcal{T}_k$. Precisely, let

$$\nu_k := \sum_{z \in \mathcal{T}_k} \delta_z$$

We normalise ν_k to obtain a sequence of probability measures μ_k, ie. we let $\mu_k := \frac{1}{\#\mathcal{T}_k} \nu_k$.

Lemma 6.4.4. *Suppose μ is a limit measure of the sequence of probability measures μ_k. Then $\mu(F) = 1$.*

Proof. For any fixed l and all $p \geq 0$, $\mu_{l+p}(F_l) = 1$ since $\mu_{l+p}(F_{l+p}) = 1$ and $F_{l+p} \subseteq F_l$. Suppose $\mu = \lim_{k\to\infty} \mu_{l_k}$ for some $l_k \to \infty$, then $\mu(F_l) \geq \limsup_{k\to\infty} \mu_{l_k}(F_l) = 1$. It follows that $\mu(F) = \lim_{l\to\infty} \mu(F_l) = 1$. $\square$

In fact, the measures μ_k converge. However, by using our version of the Entropy Distribution Principle, we do not need to use this fact and so we omit the proof (which goes like lemma 5.4 of [TV2]).

Let $\mathcal{B} := B_n(q, \varepsilon)$ be an arbitrary ball which intersects F. Let k be the unique number which satisfies $t_k \leq n < t_{k+1}$. Let $j \in \{0, \ldots, N_{k+1} - 1\}$ be the unique number so

$$t_k + n_{k+1} j \leq n < t_k + n_{k+1}(j+1).$$

We assume that $j \geq 1$ and leave the details of the simpler case $j = 0$ to the reader. The following lemma reflects the restriction on the number of points that can be in $\mathcal{B} \cap \mathcal{T}_{k+p}$.

Lemma 6.4.5. *For $p \geq 1$, $\mu_{k+p}(\mathcal{B}) \leq (\#\mathcal{T}_k)^{-1}(\#\mathcal{S}_{k+1})^{-j}$*

Proof. First we show that $\mu_{k+1}(\mathcal{B}) \leq (\#\mathcal{T}_k)^{-1}(\#\mathcal{S}_{k+1})^{-j}$. We require an upper bound for the number of points in $\mathcal{T}_{k+1} \cap \mathcal{B}$. If $\mu_{k+1}(\mathcal{B}) > 0$, then $\mathcal{T}_{k+1} \cap \mathcal{B} \neq \emptyset$. Let $z = z(\underline{x}, \underline{x}_{k+1}) \in \mathcal{T}_{k+1} \cap \mathcal{B}$ where $\underline{x} \in S_1^{N_1} \times \ldots \times S_k^{N_k}$ and $\underline{x}_{k+1} \in S_{k+1}^{N_{k+1}}$. Let

$$\mathcal{A}_{\underline{x};x_1,\ldots,x_j} = \{z(\underline{x}, y_1, \ldots, y_{N_{k+1}}) \in \mathcal{T}_{k+1} : x_1 = y_1, \ldots, x_j = y_j\}.$$

We suppose that $z' = z(\underline{y}, \underline{y}_{k+1}) \in \mathcal{T}_{k+1} \cap \mathcal{B}$ and show that $z' \in \mathcal{A}_{x;x_1,\ldots,x_j}$. We have $d_n(z, z') < 2\varepsilon$ and we show that this implies $x_l = y_l$ for $l \in \{1, 2, \ldots, j\}$ (the proof that $\underline{x} = \underline{y}$ is similar). Suppose that $y_l \neq x_l$ and let $a_l = t_k + (l-1)(n_{k+1})$. There exists $\Lambda_1, \Lambda_2 \in I(g; n_{k+1}, \varepsilon/2^{k+1})$ such that

$$d_{\Lambda_1}(f^{a_l} z, x_l) < \frac{\varepsilon}{2^{k+1}} \text{ and } d_{\Lambda_2}(f^{a_l} z', y_l) < \frac{\varepsilon}{2^{k+1}}.$$

Let $\Lambda = \Lambda_1 \cap \Lambda_2$. Since $\Lambda \in I(2g; n_{k+1}, \varepsilon/2^{k+1})$, we have $d_\Lambda(x_l, y_l) > 4\varepsilon$. We have

$$\begin{aligned} d_n(z, z') &\geq d_\Lambda(f^{a_l} z, f^{a_l} z') \\ &\geq d_\Lambda(x_l, y_l) - d_\Lambda(f^{a_l} z, x_l) - d_\Lambda(f^{a_l} z', y_l) \geq 3\varepsilon, \end{aligned}$$

which is a contradiction. Thus, we have

$$\nu_{k+1}(\mathcal{B}) \leq \#\mathcal{A}_{x;x_1,\ldots,x_j} = (\#S_{k+1})^{N_{k+1}-j},$$

$$\mu_{k+1}(\mathcal{B}) \leq (\#\mathcal{T}_{k+1})^{-1}(\#S_{k+1})^{N_{k+1}-j} = (\#\mathcal{T}_k)^{-1}(\#\mathcal{S}_{k+1})^{-j}$$

Now consider $\mu_{k+p}(\mathcal{B})$. Arguing similarly to above, we have

$$\nu_{k+p}(\mathcal{B}) \leq \#\mathcal{A}_{x;x_1,\ldots,x_j}(\#\mathcal{S}_{k+2})^{N_{k+1}} \ldots (\#\mathcal{S}_{k+p})^{N_{k+p}}$$

The desired result follows from this inequality by dividing by $\#\mathcal{T}_{k+p}$. □

By lemma 6.4.1, we have

$$\begin{aligned} \#\mathcal{T}_k(\#\mathcal{S}_{k+1})^j &\geq \exp\{(h_{top}(f) - 4\gamma)(N_1 n_1 + N_2 n_2 + \ldots + N_k n_k + j n_{k+1})\} \\ &\geq \exp\{(h_{top}(f) - 4\gamma)n\}. \end{aligned}$$

Combining this with the previous lemma gives us

$$\limsup_{l \to \infty} \mu_l(B_n(q, \varepsilon)) \leq \exp\{-n(h_{top}(f) - 4\gamma)\}.$$

Applying the Entropy Distribution Principle, we have

$$h_{top}(F, \varepsilon) \geq h_{top}(f) - 4\gamma.$$

Since γ and ε were arbitrary and $F \subset \widehat{X}(\varphi, f)$, we have $h_{top}(\widehat{X}(\varphi, f)) = h_{top}(f)$. □

6.5 The β-transformation

In this section, let $X = [0, 1)$. For any fixed $\beta > 1$, we consider the β-transformation $f_\beta : X \mapsto X$ given by

$$f_\beta(x) = \beta x \text{ (mod1)}.$$

As reference for the basic properties of the β-transformation, we recommend the introduction of the thesis of Maia [Mai]. For $\beta \notin \mathbb{N}$, let $b = [\beta]$ and for $\beta \in \mathbb{N}$, let $b = \beta - 1$. We consider the partition into $b + 1$ intervals

$$J_0 = \left[0, \frac{1}{\beta}\right), J_1 = \left[\frac{1}{\beta}, \frac{2}{\beta}\right), \ldots, J_b = \left[\frac{b}{\beta}, 1\right).$$

For $x \in [0, 1)$, let $w(x) = (w_j(x))_{j=1}^{\infty}$ be the sequence given by $w_j(x) = i$ when $f^{j-1}x \in J_i$. We call $w(x)$ the greedy β-expansion of x and we have

$$x = \sum_{j=1}^{\infty} w_j(x)\beta^{-j}.$$

The β-shift $(\Sigma_\beta, \sigma_\beta)$ is the subshift defined by the closure of all such sequences in $\prod_{i=1}^{\infty}\{0, \ldots, b\}$. Let $w(\beta) = (w_j(\beta))_{j=1}^{\infty}$ denote the sequence which is the lexicographic supremum of all β-expansions. The sequence $w(\beta)$ satisfies

$$\sum_{j=1}^{\infty} w_j(\beta)\beta^{-j} = 1,$$

so we call $w(\beta)$ the β-expansion of 1. Parry showed that the set of sequences which belong to Σ_β can be characterised as

$$w \in \Sigma_\beta \iff \sigma^k(w) \leq w(\beta) \text{ for all } k \geq 1,$$

where $\leq$ is taken in the lexicographic ordering [Par]. Parry also showed that any sequence w which satisfies $\sigma^k(w) \leq w$ is the β-expansion of 1 for some $\beta > 1$. The β-shift contains every sequence which arises as a greedy β-expansion and an additional point for every x whose β-expansion is finite (i.e. when there exists j so $w_i(x) = 0$ for all $i \geq j$). Thus the map $\pi : \Sigma_\beta \mapsto [0, 1]$ defined by

$$\pi(\underline{w}) = \sum_{j=1}^{\infty} w_j\beta^{-j}$$

is one to one except at the countably many points for which the β-expansion is finite.

Σ_β is typically not a shift of finite type (nor even a shift with specification) and the set of all β-shifts gives a natural and interesting class of subshifts. In the next section, we decribe in detail the known results on specification properties for the β-shift. The key fact for our analysis is that every β-shift has the almost specification property [PS1]. We have (p.179 of [Wal]) that $h_{top}(\sigma_\beta) = \log \beta$.

Theorem 6.5.1. *For $\beta > 1$, let $f_\beta : X \mapsto X$ be the β-transformation, $f_\beta(x) = \beta x(\mathit{mod}1)$. Let $\varphi \in C([0, 1])$ and assume that the irregular set for φ is non-empty (ie. $\widehat{X}(\varphi, f_\beta) \neq \emptyset$), then $h_{top}(\widehat{X}(\varphi, f_\beta)) = \log \beta$.*

Proof. Let Σ'_β denote the set of sequences which arise as β-expansions. Recall that $\Sigma_\beta \setminus \Sigma'_\beta$ is a countable set and the restriction of π to Σ'_β is a homeomorphism satisfying $\pi \circ \sigma_\beta = f_\beta \circ \pi$. Thus, if $Z \in \Sigma_\beta$ and $Z' := Z \cap \Sigma'_\beta$, we have

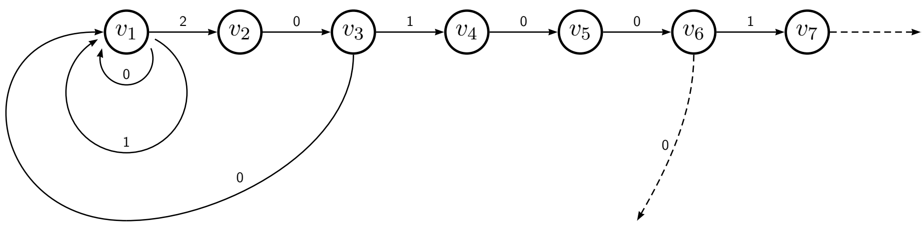
$$h_{top}(Z, \sigma_\beta) = h_{top}(Z', \sigma_\beta) = h_{top}(\pi(Z'), f_\beta).$$

Suppose $\varphi : [0,1) \mapsto \mathbb{R}$ satisfies $\widehat{X}(\varphi, f_\beta) \neq \emptyset$. Let $x \in \widehat{X}(\varphi, f_\beta)$ and let $w(x)$ be its β-expansion. We let $\overline{\varphi} \in C(\Sigma_\beta)$ be the unique continuous function which satisfies $\overline{\varphi} = \varphi \circ \pi$ on Σ'_β (this exists because we assumed φ to be continuous on $[0,1]$). It is clear that $w(x) \in \widehat{\Sigma_\beta}(\overline{\varphi}, \sigma_\beta)$. Since the dynamical system $(\Sigma_\beta, \sigma_\beta)$ satisfies the almost specification property, it follows from theorem 6.5.1 that $h_{top}(\widehat{\Sigma_\beta}(\overline{\varphi}, \sigma_\beta)) = \log \beta$. Since $\pi(\widehat{\Sigma_\beta}(\overline{\varphi}, \sigma_\beta) \cap \Sigma'_\beta) = \widehat{X}(\varphi, f_\beta)$, it follows that $h_{top}(\widehat{X}(\varphi), f_\beta) = \log \beta$. $\square$

6.5.1 β-transformations and specification properties

There is a simple presentation of Σ_β by a labelled graph $\mathcal{G}_\beta$. See [PS1] and [BH] for reference. We describe the construction of $\mathcal{G}_\beta$ when the β-expansion of 1 is not eventually periodic. We refer the reader to [PS1] for the slightly different construction required when the β-expansion of 1 is eventually periodic (in this case, Σ_β is a sofic shift [BM] and therefore has specification).

Let $v_1, v_2, \ldots$ be a countable set of vertices. We draw a directed edge from v_i to v_{i+1} and label it with the value $w_i(\beta)$ for all $i \geq 1$. If $w_i(\beta) \geq 1$, we draw a directed edge from v_i to v_1 labelled with the value 0. If $b = 1$, the construction is complete. If $b > 1$, then for all $j \in \{2, \ldots, b\}$ and all $w_i(\beta) \geq j$, we draw a directed edge from v_i to v_1 labelled with the value $j-1$. Note that if $w_i(\beta) = 0$, the only edge which starts at v_i is the edge from v_i to v_{i+1} labelled by 0, and if $w_i(\beta) \neq 0$ there is always an edge from v_i to v_1. We have $w \in \Sigma_\beta$ iff w labels an infinite path of directed edges of $\mathcal{G}_\beta$ which starts at the vertex v_1. The following figure depicts part of the graph $\mathcal{G}_\beta$ for a value of β satisfying $(w_j(\beta))_{j=1}^6 = (2, 0, 1, 0, 0, 1)$.



An arbitrary subshift Σ on $b+1$ symbols is a closed shift-invariant subset of $\prod_{i=1}^{\infty}\{0, \ldots, b\}$. We define v to be admissible word of length $n \geq 1$ for Σ if there exists $x \in \Sigma$ such that $v = (x_1, \ldots, x_n)$. The specification property of definition 2.2.1 can easily be seen to be equivalent to the following property in the case of an arbitrary subshift.

Definition 6.5.1. *A subshift Σ has the specification property if there exists $M > 0$ such that for any two admissible words w_1 and w_2, there exists a word w of length less than M such that $w_1 w w_2$ is an admissible word.*

We now return to the β-shift Σ_β. Define

$$z_n(\beta) = \min\{i \geq 0 : w_{n+i}(\beta) \neq 0\}.$$

Equivalently, $z_n(\beta) + 1$ is the minimum number of edges required to travel from v_n to v_1. The β-shift fails to have the specification property iff 'blocks of consecutive zeroes in the β expansion of 1 have unbounded length', ie. if $z_n(\beta)$ is unbounded [BM]. Consider concatenations of the word $c_n := (w_1(\beta), \ldots, w_n(\beta))$ with some other admissible word v. We can see from the graph $\mathcal{G}_\beta$ that the length of the shortest word w such that $c_n w v$ is an admissible word is $z_n(\beta)$ (the word w is a block of zeroes of length $z_n(\beta)$). Now for $x \in \Sigma_\beta$, we define $z_n(x)$ to be the length of the shortest word w required so that for any admissible word v, $(x_1, \ldots, x_n) w v$ is an admissible word. Note that for all $x \in \Sigma_\beta$, $z_n(x) \leq z_n(\beta)$. Thus Σ_β has specification iff $z_n(\beta)$ is bounded. Buzzi shows that the set of β for which this situation occurs has Lebesgue measure 0.

Pfister and Sullivan [PS1] used the graph $\mathcal{G}_\beta$ to observe that every β-shift has the almost specification property. Their strategy is to 'jump ship' on the last non-zero entry of an admissible word. More precisely, every β-shift has the following property. Given any admissible word w, there is a word w' which differs from w only by one symbol, such that $w'v$ is admissible for every admissible word v. The modified word w' is given by replacing the last non-zero entry of the word w by a 0. This property is best seen from inspection of the graph $\mathcal{G}_\beta$ and is the content of proposition 5.1 of [PS1]. It can easily be seen that this property implies the almost specification property.

6.5.2 Hausdorff dimension of the irregular set for the β-shift

We give an elementary direct proof of the relationship between topological entropy and Hausdorff dimension. We use the metric $d_\beta(x, y) = \frac{1}{\beta^n}$ where n is the smallest integer such that $x_{n+1} \neq y_{n+1}$. Let $C_n(x) = \{y \in \Sigma_\beta : x_i = y_i$ for $i = 0, \ldots, n-1\}$. We have

$$\text{Diam}(C_n(w(\beta))) = \frac{1}{\beta^{n+z_n(\beta)}},$$

$$\frac{1}{\beta^n} \geq \text{Diam}(C_n(x)) \geq \frac{1}{\beta^{n+z_n(\beta)}}.$$

For Hausdorff dimension, we recall our notation

$$H(Z, \alpha, \delta) = \inf\{\sum \delta_i^\alpha : Z \subseteq \bigcup_i B(x_i, \delta_i), \delta_i \leq \delta\},$$

$H(Z,\alpha) = \lim_{\delta\to 0} H(Z,\alpha,\delta)$ and $\text{Dim}_H(Z) = \inf\{\alpha : H(Z,\alpha) = 0\}$. We sometimes write $\text{Dim}_H(Z,d)$ in place of $\text{Dim}_H(Z)$ when we wish to emphasise the dependence on the metric d. We note that the map $x \mapsto w(x)$ is bi-Lipshitz with respect to the metric d_β and thus for $Z \subset [0,1)$, $\text{Dim}_H(Z) = \text{Dim}_H(\pi^{-1}(Z), d_\beta)$.

Define

$$z(\beta) := \limsup_{n\to\infty} z_n(\beta)/n.$$

Remark 6.5.1. $z(\beta)$ may be arbitrarily large or even ∞. To see this, let (a_n) be an increasing sequence. Let (b_n) be the sequence given by $b_1 = 1$, followed by a block of consecutive zeroes of length a_1, followed by $b_{a_1+2} = 1$, followed by a block of consecutive zeroes of length a_2, followed by $b_{a_1+a_2+3} = 1$, and so forth. Let β satisfy $w(\beta) = (b_n)$. We can choose a_n to grow as fast as we like.

Lemma 6.5.1. *For arbitrary $Z \subset \Sigma_\beta$, we have $\log\beta \mathit{Dim}_H(Z) \leq h_{top}(Z)$, and when $z(\beta) < 1$,*

$$\frac{1}{1+z(\beta)} h_{top}(Z) \leq \log\beta \mathit{Dim}_H(Z).$$

Proof. Recall that for shift spaces, topological entropy admits a simplified definition, which was described in §2.1.3. We use the notation from §2.1.3. That $\log\beta \text{Dim}_H(Z) \leq h_{top}(Z)$ is a standard argument which follows from the fact that $\text{Diam}(C_n(x)) \leq \frac{1}{\beta^n}$. For the other inequality, we fix $\varepsilon > 0$ and choose N sufficiently large so that for $n > N$, $z_n(\beta)/n < z(\beta) + \varepsilon$. For $n > N$, we have

$$\text{Diam}(C_n(x)) \geq \frac{1}{\beta^{n(1+z(\beta)+\varepsilon)}}$$

Let $\gamma_n = \beta^{-n(1+z(\beta)+\varepsilon)}$. Take a cover of Z by metric balls $B(x_i, \delta_i)$ with $\delta_i < \gamma_N$. Let n_i be the unique integer such that

$$\frac{1}{\beta^{(n_i-1)(1+z(\beta)+\varepsilon)}} \geq \delta_i > \frac{1}{\beta^{n_i(1+z(\beta)+\varepsilon)}}.$$

Then $\Gamma = \{C_{n_i}(x_i)\}$ covers Z and

$$\sum \delta_i^\alpha > \sum \exp(-\alpha n_i(1+z(\beta)+\varepsilon)\log\beta) = Q(Z, \alpha(1+z(\beta)+\varepsilon)\log\beta, \Gamma).$$

Taking infimums, we have $M(Z, \alpha(1+z(\beta)+\varepsilon)\log\beta, N) \leq H(Z,\alpha,\gamma_n)$. It follows that

$$M(Z, \alpha(1+z(\beta))\log\beta) \leq \lim_{n\to\infty} H(Z,\alpha,\gamma_n) = H(Z,\alpha)$$

and the inequality follows.

□

We will also use the following elementary lemma which can be proved similarly to the above.

Lemma 6.5.2. *For $Z \subset \Sigma_\beta$, if $z_n(x)$ is bounded for $x \in Z$ (ie. there exists $C > 0$ such that $\sup_{x \in Z} \sup_n z_n(x) < C$), then $\log \beta Dim_H(Z) = h_{top}(Z)$.*

In [PS1], Pfister and Sullivan sketch an argument which shows that the set of β for which $z(\beta) = 0$ has full Lebesgue measure. Thus for these β, lemma 6.5.1 tells us that $\log \beta \mathrm{Dim}_H(Z) = h_{top}(Z)$ for any set Z. In particular, it follows from theorem 6.5.1 that if $\varphi \in C(\overline{X})$ and $\widehat{X}(\varphi, f_\beta) \neq \emptyset$, then $\mathrm{Dim}_H(\widehat{X}(\varphi, f_\beta)) = 1$. In conclusion, this discussion proves the following theorem.

Theorem 6.5.2. *There is a set of β of full Lebesgue measure, such that*

(1) If $\varphi \in C(\Sigma_\beta)$ and $\widehat{\Sigma}_\beta(\varphi, \sigma_\beta) \neq \emptyset$, then $Dim_H(\widehat{\Sigma}_\beta(\varphi, \sigma_\beta)) = 1$,

(2) If $\varphi \in C([0,1])$ and $\widehat{X}(\varphi, f_\beta) \neq \emptyset$, then $Dim_H(\widehat{X}(\varphi, f_\beta)) = 1$.

6.5.3 An alternative approach which covers the case $z(\beta) > 0$

We describe a method of proof which shows that

Theorem 6.5.3. *For every $\beta > 1$,*

(1) If $\varphi \in C(\Sigma_\beta)$ and $\widehat{\Sigma}_\beta(\varphi, \sigma_\beta) \neq \emptyset$, then $Dim_H(\widehat{\Sigma}_\beta(\varphi, \sigma_\beta)) = 1$,

(2) If $\varphi \in C([0,1])$ and $\widehat{X}(\varphi, f_\beta) \neq \emptyset$, then $Dim_H(\widehat{X}(\varphi, f_\beta)) = 1$.

The method described does not use the almost specification property, and provides an alternative proof of our main results in the case of the β-shift. The key quoted result in this method is a version of theorem 6.5.1 in the special case of n-step Markov shifts. We note that the 'almost specification' method of proof applies in far greater generality and a self-contained version of the proof described below would be comparable in length.

Proof. We give the proof for the β-shift version of the statement (part of the proof of theorem 6.5.1 can be used to extend the result to the β-transformation). Recall that any sequence (a_n) on a finite number of symbols which satisfies $\sigma^k(a_n) \leq (a_n)$ for all $k \geq 0$ arises as $w(\beta)$ for some $\beta > 1$. Fix $\beta > 1$ and write $w_i := w_i(\beta)$. Let $\beta(n)$ be the simple β-number which corresponds to the sequence $(w_1, w_2, \ldots, w_n, 0, 0, 0, \ldots)$. An elementary argument [Par] shows that $\beta(n) \to \beta$. It is clear that $\Sigma_{\beta(n)}$ can be considered to be a subsystem of Σ_β (the subshift $\Sigma_{\beta(n)}$ corresponds to the set of labels of edges of infinite paths that only visit the first n vertices of $\mathcal{G}_\beta$).

Now suppose $\varphi \in C(\Sigma_\beta)$ is a function for which the irregular set is non-empty. Then there exists $x, y \in \Sigma_\beta$ such that

$$\lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) \neq \lim_{n\to\infty} \frac{1}{n} S_n\varphi(y).$$

Let $\delta > 0$ and $N_1 \in \mathbb{N}$ be such that for $n \geq N_1$,

$$\left| \frac{1}{n} S_n\varphi(x) - \frac{1}{n} S_n\varphi(y) \right| > 4\delta.$$

Pick N_2 sufficiently large that

$$\sup\{|\varphi(w) - \varphi(v)| : w, v \in \Sigma_\beta, w_i = v_i \text{ for } i = 1, \dots, N_2\} < \delta.$$

For any $n \geq N = \max\{N_1, N_2\}$, let us choose $x' \in C_N(x) \cap \Sigma_{\beta(n)}$ and $y' \in C_N(y) \cap \Sigma_{\beta(n)}$. We have for all $m \geq N$,

$$\left| \frac{1}{m} S_m \varphi(x') - \frac{1}{m} S_m \varphi(y') \right| > 2\delta.$$

Thus the restriction of φ to $\Sigma_{\beta(n)}$ does not have trivial spectrum of Birkhoff averages and by lemma 2.3.2, our main theorem gives us

$$h_{top}(\widehat{\Sigma}_{\beta(n)}(\varphi, \sigma_{\beta(n)})) = h_{top}(\sigma_{\beta(n)}) = \log \beta(n). \tag{6.3}$$

We remark that $\Sigma_{\beta(n)}$ is an n-step Markov shift (and thus has specification), so formula (6.3) also follows for Hölder continuous φ from theorem 9.3.2 of [Bar]. Note that $\widehat{\Sigma}_{\beta(n)}(\varphi, \sigma_{\beta(n)}) \subset \widehat{\Sigma}_\beta(\varphi, \sigma_\beta)$.

By lemma 6.5.2, any subset $Z \subset \Sigma_{\beta(n)}$ satisfies $\text{Dim}_H(Z, d_\beta) = h_{top}(Z)/\log \beta$. In particular, $\text{Dim}_H(\Sigma_{\beta(n)}, d_\beta) = \log(\beta(n))/\log \beta$. Thus

$$\text{Dim}_H(\widehat{\Sigma}_\beta(\varphi, \sigma_\beta), d_\beta) \geq \sup\left\{\text{Dim}_H(\widehat{\Sigma}_{\beta(n)}(\varphi, \sigma_{\beta(n)}), d_\beta)\right\} = \sup\left\{\frac{\log \beta(n)}{\log \beta}\right\} = 1.$$

$\square$

Remark 6.5.2. An almost sofic shift [LM] is defined to be a shift space Σ for which one can find a sequence of subshifts of finite type Σ_n such that $\Sigma_n \subset \Sigma$ and $\lim_{n\to\infty} h_{top}(\Sigma_n, \sigma) = h_{top}(\Sigma, \sigma)$. By our previous reasoning, every β-shift is almost sofic. We remark that the proof of theorem 6.5.3 shows that if (Σ, σ) is an almost sofic shift, $\varphi \in C(\Sigma)$ and $\widehat{\Sigma}(\varphi, \sigma) \neq \emptyset$, then $h_{top}(\widehat{\Sigma}(\varphi, \sigma)) = h_{top}(\sigma)$.

Remark 6.5.3. Pfister and Sullivan [PS1] consider the relationship between topological entropy and Billingsley dimension Dim_ν (with respect to a reference measure ν). We remark that when ν is equivalent to Lebesgue measure, then $\text{Dim}_\nu = \text{Dim}_H$. Every β-transformation has an invariant measure ν_β which is equivalent to Lebesgue (and is the measure of maximal entropy). It is thus a corollary of theorem 6.5.3 that if $\varphi \in C(\overline{X})$ and $\widehat{X}(\varphi, f_\beta) \neq \emptyset$, then $\text{Dim}_{\nu_\beta}(\widehat{X}(\varphi, f_\beta)) = 1$.

Chapter 7

Defining pressure via a conditional variational principle

We now give an alternative definition of topological pressure for arbitrary (non-compact, non-invariant) Borel subsets of metric spaces. We focus our attention on the case when the ambient metric space is compact. The current approach is to define pressure as a characteristic of dimension type, as used in chapter 4 and chapter 5. This approach was introduced by Bowen [Bow4] and generalised by Pesin and Pitskel [PP2]. The entropy version in particular is very well established in the literature as a dimension characteristic and plays an important role in dimension theory. One can also define the upper and lower capacity topological pressure (see §2.1.5). This definition involves the minimum cardinality of spanning sets and resembles the usual definition of topological pressure in the compact invariant setting. As we saw in §5.1.1, the capacity topological pressure has its uses (see also remark 7.1.3).

It would be desirable if topological pressure for arbitrary sets satisfied a variational principle analogous to the classical variational principle

$$P_X^{classic}(\varphi) = \sup\left\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_f(X)\right\}.$$

A variational principle for the pressure of Pesin and Pitskel does exist but only applies to invariant sets satisfying a certain condition which is very difficult to check (see theorem 7.1.1). No general variational principle is known in the non-compact or non-invariant case for the upper or lower capacity topological pressure (although the relativised variational principle of Ledrappier and Walters involves the consideration of upper capacity topological pressure, see remark 7.1.3). We propose a new notion of pressure, which by its very definition satisfies a suitable variational principle. We study the new definition directly, deriving many desirable properties satisfied by the previous notions of pressure. We study the relationship between the definitions and give interesting examples where the definitions differ or coincide. The new pressure has the advantage that its properties are significantly easier to derive than that of the dimension-like version and we seem to pay no price in terms of desirable properties. Since the new pressure is defined via a conditional variational principle, it is by its very nature adapted to the study of thermodynamic properties.

In §7.1, we state our definition and set up our notation. In §7.2, we study the properties of our new topological pressure when the ambient space is compact. In §7.3, we study the relationship between the different definitions. In §7.4, we consider some interesting examples. In §7.5, we study our new topological pressure when the ambient space is non-compact. In §7.6, we prove an elementary result which we use in §7.2 (this should serve as a nice digestif to round off the thesis).

7.1 The new definition

Let (X, d) be a compact metric space and $f : X \mapsto X$ a continuous map. Let $C(X)$ be the space of continuous real-valued functions on X. Let $Z \subset X$ be an arbitrary Borel set. Let $\mathcal{M}_f(X)$ denote the space of f-invariant Borel probability measures on X and $\mathcal{M}_f^e(X)$ denote those which are ergodic. If Z is f-invariant, let $\mathcal{M}_f(Z)$ denote the subset of $\mathcal{M}_f(X)$ for which the measures μ satisfy the additional property $\mu(Z) = 1$. Let $\mathcal{M}_f^e(Z) := \mathcal{M}_f(Z) \cap \mathcal{M}_f^e(X)$. We define the (empirical) probability measures

$$\delta_{x,n} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)}$$

where δ_x denotes the Dirac δ-measure supported on x. We define $\mathcal{V}(x)$ to be the set of limit points for $\delta_{x,n}$, namely:

$$\mathcal{V}(x) = \{\mu \in \mathcal{M}_f(X) : \delta_{x,n_k} \to \mu \text{ for some } n_k \to \infty\}.$$

We state the new definition which will be the object of our study.

Definition 7.1.1. *Let Z be an arbitrary non-empty Borel set and $\varphi \in C(X)$. Define*

$$P_Z^*(\varphi) = \sup\left\{h_\mu + \int_X \varphi d\mu \, : \, \mu \in \mathcal{V}(x) \text{ for some } x \in Z\right\}.$$

We set $P_\emptyset^*(\varphi) = \inf_{x\in X} \varphi(x)$. If $\varphi \equiv 0$, than we may denote $P_Z^*(0)$ by $h_{top}^*(Z)$.

Notation. As before, we denote the topological pressure of φ on Z defined as a dimension characteristic using the definition of Pesin (see §2.1.1) by $P_Z(\varphi)$ and $h_{top}(Z) := P_Z(0)$. The new topological pressure of definition 7.1.1 and quantities associated with it will always carry an asterisk, eg. $P_Z^*(\varphi)$, $h_{top}^*(Z)$.

Remark 7.1.1. An alternative natural definition to make is as follows:

$$P_Z^{\#}(\varphi) = \sup\left\{h_\mu + \int_X \varphi d\mu \, : \, \mu = \lim_{n\to\infty} \delta_{x,n} \text{ for some } x \in Z\right\}.$$

If no such measures exist, then we set $P_Z^{\#}(\varphi) = \inf_{x \in X} \varphi(x)$. One obvious relationship is $P_Z^*(\varphi) \geq P_Z^{\#}(\varphi)$. We take the point of view that $P_Z^*(\varphi)$ is the better quantity to study because it captures more information about Z than $P_Z^{\#}(\varphi)$. Furthermore, the relationship between $P_Z^*(\varphi)$ and $P_Z(\varphi)$ is better than the relationship between $P_Z^{\#}(\varphi)$ and $P_Z(\varphi)$ (see remark 7.3.3). Theorem 7.4.1 gives an example of a set Z for which $h_{top}^*(Z) = h_{top}(Z) = h_{top}(f)$ but $P_Z^{\#}(0) = 0$.

Remark 7.1.2. When the ambient space X is non-compact, we can define $h_{top}^*(Z)$ as in definition 7.1.1, although we must insist that if $\bigcup_{x \in Z} \mathcal{V}(x) = \emptyset$, then $h_{top}^*(Z) = 0$. The definition of $P_Z^*(\varphi)$ requires a small modification in the non-compact setting and we study this situation further in §7.5.

Let us recall that the variational principle for $P_Z(\varphi)$ proved by Pesin and Pitskel.

Theorem 7.1.1 (Pesin and Pitskel)**.** *Let Z be f-invariant and $\mathcal{L}(Z) = \{x \in Z : \mathcal{V}(x) \cap \mathcal{M}_f(Z) \neq \emptyset\}$. Then $P_{\mathcal{L}(Z)}(\varphi) = \sup \{h_\mu + \int_Z \varphi d\mu\}$, where the supremum is taken over either $\mathcal{M}_f(Z)$ or $\mathcal{M}_f^e(Z)$.*

Remark 7.1.3. We note that in the context of fibred systems (i.e. (X_1, f_1) and (X_2, f_2) are dynamical systems and $\pi : X_1 \mapsto X_2$ continuous satisfies $\pi(X_1) = X_2$ and $\pi \circ f_1 = f_2 \circ \pi$), the relativized variational principle of Ledrappier and Walters [LW] involves the pressure of compact non-invariant sets (the fibres), and they use $\overline{CP}_Z(\varphi)$ rather than $P_Z(\varphi)$. We state the entropy version of the relativized variational principle: given $\nu \in \mathcal{M}_f(X_2)$,

$$\sup_{\mu : \mu \circ \pi^{-1} = \nu} h_\mu = h_\nu + \int_{X_2} \overline{CP}_{\pi^{-1}(x)}(0) d\nu(x).$$

7.1.1 The set of generic points

For an invariant measure μ, let G_μ denote its set of generic points

$$G_\mu = \{x \in X : \delta_{x,n} \to \mu\}.$$

We consider G_μ repeatedly in this chapter. If μ is ergodic, G_μ is non-empty and by Birkhoff's theorem $\mu(G_\mu) = 1$. Furthermore, if f satisfies definition 2.2.1 (specification), G_μ is non-empty for any invariant measure. This is proved in [DGS] when f satisfies definition 2.2.2 (Bowen specification). When $h_\mu > 0$, it is a corollary of the result $h_{top}(G_\mu) = h_\mu$ for any invariant measure. This was proved for maps with the g-almost product property in [PS2], and thus for maps with specification.

7.2 Properties of $P_Z^*(\varphi)$

Theorem 7.2.1. *The topological pressure of definition 7.1.1 satisfies:*

(1) $P_{Z_1}^*(\varphi) \leq P_{Z_2}^*(\varphi)$ if $Z_1 \subset Z_2 \subset X$,

(2) $P^*_Z(\varphi) = \sup\{P^*_Y(\varphi) : Y \in \mathcal{F}\}$ *where* $Z = \bigcup_{Y\in\mathcal{F}} Y$ *and* $\mathcal{F}$ *is a collection (countable or uncountable) of Borel subsets of* X*,*

(3) $P^*_Z(\varphi \circ f) = P^*_Z(\varphi)$*,*

(4) If ψ *is cohomologous to* φ*, then* $P^*_Z(\varphi) = P^*_Z(\psi)$*,*

(5) $P^*_Z(\varphi + \psi) \le P^*_Z(\varphi) + \beta(\psi)$*, where* $\beta(\psi) = \sup_{\mu\in\mathcal{M}_f(X)} \int_X \psi d\mu$*,*

(6) $P^*_Z((1-t)\varphi + t\psi) \le (1-t)P^*_Z(\varphi) + tP^*_Z(\psi)$*.*

(7) $|P^*_Z(\varphi) - P^*_Z(\psi)| \le \|\psi - \varphi\|_\infty$*,*

(8) $P^*_Z(\varphi) \ge \inf_{x\in X} \varphi(x)$*,*

(9) For every $k \in \mathbb{Z}$*,* $P^*_{f^k Z}(\varphi) = P^*_Z(\varphi)$*,*

(10) $P^*_{\bigcup_{k\in\mathbb{Z}} f^k Z}(\varphi) = P^*_{\bigcup_{k\in\mathbb{N}} f^{-k} Z}(\varphi) = P^*_{\bigcup_{k\in\mathbb{N}} f^k Z}(\varphi) = P^*_Z(\varphi)$*.*

Proof. Since $\bigcup_{x\in Z_1} \mathcal{V}(x) \subseteq \bigcup_{x\in Z_2} \mathcal{V}(x)$, the first statement is immediate. The second statement is true because $\bigcup_{x\in Z} \mathcal{V}(x) \subseteq \bigcup_{Y\in\mathcal{F}} \bigcup_{x\in Y} \mathcal{V}(x)$. It is a standard result that $\mathcal{V}(x) \subseteq \mathcal{M}_f(X)$ (see for example [Wal]) and thus $\int_X \varphi d\mu = \int_X \varphi \circ f d\mu$ for $\mu \in \mathcal{V}(x)$. The third statement follows. If ψ is cohomologous to φ, then there exists a continuous function h so $\psi = \varphi + h - h \circ f$ and so $\int_X \varphi d\mu = \int_X \psi d\mu$. The fourth statement follows. We leave (5) and (6) as easy exercises. (7) follows from the fact that for $\mu \in \mathcal{V}(x)$,

$$h_\mu + \int \varphi d\mu \le h_\mu + \int \psi d\mu + \|\psi - \varphi\|_\infty.$$

(8) follows from the fact that $h_\mu + \int \varphi d\mu \ge \inf_{x\in X} \varphi(x)$. (9) is true since $\mathcal{V}(x) = \bigcup_{\{y : y = f^k x\}} \mathcal{V}(y)$ for all $x \in Z$ and we can apply (2). (10) follows from (9) and (2). □

$P^*_Z(\varphi)$ is a topological invariant of dynamical systems in the following sense:

Theorem 7.2.2. *Let* (X_i, d_i) *be compact metric spaces and* $f_i : X_i \mapsto X_i$ *be continuous maps for* $i = 1, 2$*. Let* $\pi : X_1 \mapsto X_2$ *be a homeomorphism satisfying* $\pi \circ f_1 = f_2 \circ \pi$*. Then for any continuous* $\varphi : X_2 \mapsto \mathbb{R}$ *and Borel* $Z \subset X_2$*, we have* $P^*_Z(\varphi) = P^*_{\pi^{-1}(Z)}(\varphi \circ \pi)$*.*

Proof. For $\psi \in C(X_2)$ and $\mu \in \mathcal{M}_{f_2}(X_2)$, let $\tilde{\psi} := \psi \circ \pi$ and $\tilde{\mu} := \mu \circ \pi$. Let $\mu \in \bigcup_{x\in Z} \mathcal{V}(x)$. Then $\mu = \lim_{n_k\to\infty} \delta_{x,n_k}$ for some $x \in Z$, $n_k \to \infty$. Let $y \in X_1$ satisfy $\pi(y) = x$. For an arbitrary function $\psi \in C(X_1)$,

$$\begin{aligned}
\int \psi d\tilde{\mu} &= \int \psi \circ \pi^{-1} d\mu \\
&= \lim_{n_k\to\infty} \frac{1}{n_k} S_{n_k} \psi \circ \pi^{-1}(x) \\
&= \lim_{n_k\to\infty} \frac{1}{n_k} S_{n_k} \psi(y) \\
&= \lim_{n_k\to\infty} \int \psi d\delta_{n_k,y}.
\end{aligned}$$

Since this is true for all $\psi \in C(X_1)$, we have $\tilde{\mu} \in \mathcal{V}(y)$. Thus $\mu \in \bigcup_{x \in Z} \mathcal{V}(x) \Rightarrow \tilde{\mu} \in \bigcup_{y \in \pi^{-1}(Z)} \mathcal{V}(y)$. Since $h_{\tilde{\mu}} + \int \tilde{\varphi} d\tilde{\mu} = h_\mu + \int \varphi d\mu$, then $P^*_{\pi^{-1}(Z)}(\tilde{\varphi}) \geq P^*_Z(\varphi)$. Reversing the previous argument gives the desired equality. □

The proof shows that if π were only assumed to be a continuous surjective map, we would obtain the inequality $P^*_Z(\varphi) \leq P^*_{\pi^{-1}(Z)}(\varphi \circ \pi)$. We now verify that in the compact, invariant case $P^*_Z(\varphi)$ agrees with the classical topological pressure.

Theorem 7.2.3. *If Z is compact and f-invariant, then $P^*_Z(\varphi) = P^{classic}_Z(\varphi)$.*

Proof. By compactness of Z, $\mathcal{M}_f(Z)$ is compact and thus $\bigcup_{x \in Z} \mathcal{V}(x) \subseteq \mathcal{M}_f(Z)$. The inequality $P^*_Z(\varphi) \leq P^{classic}_Z(\varphi)$ follows immediately. For the opposite inequality, let $\mu \in \mathcal{M}_f(Z)$ be ergodic. Taking any point x in G_μ, we have $\mathcal{V}(x) = \mu$. We conclude that $\mathcal{M}^e_f(Z) \subseteq \bigcup_{x \in Z} \mathcal{V}(x)$ and the desired inequality follows from the classical variational principle. □

The following result is clear from the definition.

Theorem 7.2.4. *Suppose Z contains a periodic point x with period n. Then*

$$P^*_{\{x\}}(\varphi) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \text{ and } P^*_Z(\varphi) \geq \frac{1}{n} \sum_{i=o}^{n-1} \varphi(f^i x).$$

We now consider the set of generic points G_μ. Bowen (for entropy [Bow4]) and Pesin (for pressure [PP2]) showed that $P_{G_\mu}(\varphi) = h_\mu + \int \varphi d\mu$. In fact, it was this property that motivated Bowen's original dimensional definition of topological entropy. We see that similar properties holds for the new topological pressure.

Theorem 7.2.5. *For any invariant measure, $P^*_{G_\mu}(\varphi) = h_\mu + \int \varphi d\mu$. Let Z be a Borel set with $Z \cap G_\mu \neq \emptyset$, then $P^*_Z(\varphi) \geq h_\mu + \int \varphi d\mu$. Now assume that μ is an equilibrium measure for φ, then $P^*_{G_\mu}(\varphi) = P^{classic}_X(\varphi)$. In particular, let m be a measure of maximal entropy and $Z \cap G_m \neq \emptyset$. Then $h^*_{top}(Z) = h_{top}(f)$.*

The proof follows immediately from the definitions. Let us remark that if a measure of maximal entropy is fully supported then $h^*_{top}(U) = h_{top}(f)$ for every open set U.

It is informative to consider the pressure of a single point.

Theorem 7.2.6. *Let $x \in G_\mu$. Then $P^*_{\{x\}}(\varphi) = h_\mu + \int \varphi d\mu$ and $P_{\{x\}}(\varphi) = \int \varphi d\mu$. Thus $P^*_{\{x\}}(\varphi) = P_{\{x\}}(\varphi)$ iff $h_\mu = 0$.*

Proof. The first statement is clear. The second follows from the formula for pressure at a point $P_{\{x\}}(\varphi) = \liminf_{n \to \infty} \frac{1}{n} S_n \varphi(x)$ (see §7.6). Since $x \in G_\mu$, $\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \to \int \varphi d\mu$ for every continuous φ. □

Theorem 7.2.7. *Let $x \in X$. If $h_\mu > 0$ for some $\mu \in \mathcal{V}(x)$, then $P^*_{\{x\}}(\varphi) > P_{\{x\}}(\varphi)$.*

Proof. Suppose $\mu \in \mathcal{V}(x)$. Then for some $m_k \to \infty$, we have

$$\int \varphi d\mu = \lim_{k\to\infty} \frac{1}{m_k} S_{m_k}\varphi(x) \geq \liminf_{n\to\infty} \frac{1}{n} S_n\varphi(x) = P_{\{x\}}(\varphi).$$

Therefore, if $h_\mu > 0$, then $P^*_{\{x\}}(\varphi) \geq h_\mu + \int \varphi d\mu > P_{\{x\}}(\varphi)$. □

Remark 7.2.1. Theorem 7.2.6 provides us with a simple example which shows that $P_Z(\varphi)$ and $P^*_Z(\varphi)$ are not equal. In theorem 7.6.1, we verify that for $x \in G_\mu$, $\underline{CP}_{\{x\}}(\varphi) = \overline{CP}_{\{x\}}(\varphi) = \int \varphi d\mu$. Hence, theorem 7.2.6 shows that $P^*_Z(\varphi)$ cannot be equal to these quantities either.

Remark 7.2.2. We note that $P^*_Z(\varphi)$ is sensitive to the addition of a single point to the set Z. When $\varphi \neq 0$, the same is true of $P_Z(\varphi)$. However, in the case of entropy, we have a contrast between $h_{top}(Z)$, which remains the same under the addition of a countable set, and $h^*_{top}(Z)$, where a single point can carry full entropy.

For ergodic measures, an inverse variational principal holds.

Theorem 7.2.8. *Suppose μ is ergodic. Then*

(1) $h_\mu = \inf\{h^*_{top}(Z) : \mu(Z) = 1\}$,

(2) $h_\mu + \int \varphi d\mu = \inf\{P^*_Z(\varphi) : \mu(Z) = 1\}$.

Proof. We prove (2), then (1) follows as a special case. Suppose Z is a Borel set with $\mu(Z) = 1$. Since μ is assumed to be ergodic, $\mu(G_\mu) = 1$ and thus $Z \cap G_\mu \neq \emptyset$. It follows that $P^*_Z(\varphi) \geq h_\mu + \int \varphi d\mu$ and thus $\inf\{P^*_Z(\varphi) : \mu(Z) = 1\} \geq h_\mu + \int \varphi d\mu$. Since $P^*_{G_\mu}(\varphi) = h_\mu + \int \varphi d\mu$, we have an equality. □

The assumption that μ is ergodic is essential. For example, let $\mu = p\mu_1 + (1-p)\mu_2$ where μ_1, μ_2 are ergodic with $h_{\mu_1} \neq h_{\mu_2}$ and $p \in (0,1)$. If $\mu(Z) = 1$, then $\mu_1(Z) = 1$ and thus Z contains generic points for μ_1. Therefore, $h^*_{top}(Z) \geq h_{\mu_1}$. Repeating the argument for μ_2, we obtain $\inf\{h^*_{top}Z : \mu(Z) = 1\} \geq \max\{h_{\mu_1}, h_{\mu_2}\} > h_\mu = ph_{\mu_1} + (1-p)h_{\mu_2}$. In fact, since $\mu(G_{\mu_1} \cup G_{\mu_2}) = 1$ and $h^*_{top}(G_{\mu_1} \cup G_{\mu_2}) = \max\{h_{\mu_1}, h_{\mu_2}\}$, we have $\inf\{h^*_{top}Z : \mu(Z) = 1\} = \max\{h_{\mu_1}, h_{\mu_2}\}$.

We have a version of Bowen's equation.

Theorem 7.2.9. *Let φ be a strictly negative continuous function. Let $\psi : \mathbb{R} \mapsto \mathbb{R}$ be given by $\psi(t) := P^*_Z(t\varphi)$. Then the equation $\psi(t) = 0$ has a unique solution. The solution lies in $[0, \infty)$.*

Proof. Let $s > t$. Let $\mu \in \bigcup_{x\in Z} \mathcal{V}(x)$ and $C = \inf -\varphi(x) > 0$. We have

$$h_\mu + \int s\varphi d\mu = h_\mu + \int t\varphi d\mu - (s-t)\int -\varphi d\mu$$

and, since $\int -\varphi d\mu \in [C, \|\varphi\|_\infty]$,

$$h_\mu + \int s\varphi d\mu \le h_\mu + \int t\varphi d\mu - (s-t)C.$$

Therefore, $\psi(s) - \psi(t) \le -(s-t)C$ and so ψ is strictly decreasing. (Similarly, $\psi(s) - \psi(t) \ge -(s-t)\|\varphi\|_\infty$, so ψ is bi-Lipschitz.) Since $\psi(0) \ge 0$, $P^*_Z(t\varphi) = 0$ has a unique root. $\square$

Remark 7.2.3. We compare the properties derived here with those satisfied by $P_Z(\varphi)$. In theorem 7.2.1, properties (1), (3), (4), (6) and (7) hold for $P_Z(\varphi)$. Property (2) holds for $P_Z(\varphi)$ only when the union is at most countable. Properties (9) and (10) are known to hold for $P_Z(\varphi)$ when f is a homeomorphism. Theorems 7.2.2, 7.2.3, 7.2.8 and 7.2.9 hold for $P_Z(\varphi)$.

7.2.1 Equilibrium states for $P^*_Z(\varphi)$

Suppose a measure μ^* satisfies $P^*_Z(\varphi) = h_{\mu^*} + \int_X \varphi d\mu^*$ and $\mu^* \in \bigcup_{x\in Z} \mathcal{V}(x)$ for a (not necessarily invariant) Borel set Z. Then we call μ^* a $*$-equilibrium state for φ on Z. If μ^* satisfies $h^*_{top}(Z) = h_{\mu^*}$, we call μ^* a measure of maximal $*$-entropy. If Z is invariant, we call a measure μ that satisfies both $P_Z(\varphi) = h_\mu + \int_X \varphi d\mu$ and $\mu(Z) = 1$ simply an equilibrium state for φ on Z. The latter definition coincides with that of Pesin [Pes]. It is clear from the definition that if μ^* is a $*$-equilibrium state and μ is an equilibrium state for φ on Z, then

$$h_{\mu^*} + \int_X \varphi d\mu^* \ge h_\mu + \int_X \varphi d\mu.$$

Note that it is possible that $\mu^*(Z) = 0$. There are situations where the new definition seems more appropriate than the old. We describe a non-trivial example in 7.4.4 but first let us a consider a periodic point x of period $n > 1$. Then, for any function, $\delta_{x,n}$ is a $*$-equilibrium state on $\{x\}$. However, as $\{x\}$ is not invariant, the notion of equilibrium state is not defined.

7.3 The relationship between $P_Z(\varphi)$ and $P^*_Z(\varphi)$

In theorem 7.3.3, we show that the inequality $P_Z(\varphi) \le P^*_Z(\varphi)$ holds. Theorem 7.2.6 provides examples where $P_Z(\varphi) < P^*_Z(\varphi)$ and non-trivial examples can be constructed. $\S 7.4$ contains concrete examples where $P_Z(\varphi) = P^*_Z(\varphi)$ and we have the following:

Theorem 7.3.1. *For an f-invariant Borel set Z, let $\mathcal{G}(Z) = \bigcup_{\mu\in\mathcal{M}_f(Z)} G_\mu \cap Z$. Then $P_{\mathcal{G}(Z)}(\varphi) = P^*_{\mathcal{G}(Z)}(\varphi)$.*

Proof. Note that $\mathcal{L}(G(Z)) = G(Z)$. Applying theorem 7.1.1, we have $P_{\mathcal{G}(Z)}(\varphi) = \sup\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_f(\mathcal{G}(Z))\} = P^*_{\mathcal{G}(Z)}(\varphi)$. $\square$

Before embarking on a sketch proof that $P_Z(\varphi) \le P^*_Z(\varphi)$, we give a less sharp result, whose proof is straight forward given theorem 7.1.1.

Theorem 7.3.2. *If Z is an f-invariant Borel set, we have*

$$P_{\mathcal{L}(Z)}(\varphi) \le P^*_Z(\varphi) \le P^{classic}_{\overline{Z}}(\varphi) \text{ and } P_{\mathcal{L}(Z)}(\varphi) \le P^*_{\mathcal{L}(Z)}(\varphi).$$

Proof. We note that if $\mu \in \mathcal{M}^e_f(Z)$, then $\mu(Z \cap G_\mu) = 1$. Taking $x \in Z \cap G_\mu$, we have $\mathcal{V}(x) = \{\mu\}$ and thus $\mathcal{M}^e_f(Z) \subseteq \bigcup_{x \in Z} \mathcal{V}(x)$. Note that $x \in \mathcal{L}(Z)$ and so $\mathcal{M}^e_f(Z) \subseteq \bigcup_{x \in \mathcal{L}(Z)} \mathcal{V}(x)$. By theorem 7.1.1, the first and third inequalities follows. For the second inequality, we have $P^*_Z(\varphi) \le P^*_{\overline{Z}}(\varphi) = P^{classic}_{\overline{Z}}(\varphi)$. □

Example 7.4.4 shows that the second inequality may be strict (the sets $X(\varphi, \alpha)$ are dense but do not carry full entropy), and remark 7.4.3 shows that the third inequality may be strict. The first inequality of the following theorem is the main result of this section. We do not assume that Z is invariant.

Theorem 7.3.3. *Let Z be an arbitrary Borel set and $Y = \overline{\bigcup_{k \in \mathbb{N}} f^{-k} Z}$, then*

$$P_Z(\varphi) \le P^*_Z(\varphi) \le P^{classic}_Y(\varphi).$$

§7.3.2 constitutes a sketch proof of the first inequality. This result, although never stated before, follows from part of Pesin and Pitskel's proof of theorem 7.1.1, with only minor changes required. For a complete proof, we refer the reader to [PP2] or [Pes]. Here, we attempt to convey the key technical ingredients. The second inequality is trivial as Y is a closed invariant set containing Z.

7.3.1 Definition of Pesin and Pitskel's topological pressure using open covers

For the proof on which we are about to embark, it is more convenient to work with an alternative formulation of Pesin and Pitskel's topological pressure which is equivalent to that stated in §2.1.1. Let (X, d) be a compact metric space, $f : X \mapsto X$ be a continuous map and $\varphi \in C(X)$. Let $Z \subset X$ be a Borel subset. We take a finite open cover $\mathcal{U}$ of X and denote by $\mathcal{S}_m(\mathcal{U})$ the set of all strings $\mathbf{U} = \{(U_{i_0}, \ldots, U_{i_{m-1}}) : U_{i_j} \in \mathcal{U}\}$ of length $m = m(\mathbf{U})$. We define $\mathcal{S}(\mathcal{U}) = \bigcup_{m \ge 0} \mathcal{S}_m(\mathcal{U})$, where $S_0(\mathcal{U})$ consists of $\emptyset$. To a given string $\mathbf{U} = (U_{i_0}, \ldots, U_{i_{m-1}}) \in \mathcal{S}(\mathcal{U})$, we associate the set $\mathbf{X}(\mathbf{U}) = \{x \in X : f^j(x) \in U_{i_j} \text{ for all } j = 0, \ldots, m(\mathbf{U}) - 1\} = \bigcap_{j=0}^{m(\mathbf{U})-1} f^{-j} U_{i_j}$. We say that a collection of strings $\mathcal{G} \subset \mathcal{S}(\mathcal{U})$ covers Z if $Z \subset \bigcup_{\mathbf{U} \in \mathcal{G}} \mathbf{X}(\mathbf{U})$. Let $\alpha \in \mathbb{R}$. We make the following definitions:

$$Q(Z, \alpha, \mathcal{U}, \mathcal{G}, \varphi) = \sum_{\mathbf{U} \in \mathcal{G}} \exp\left(-\alpha m(\mathbf{U}) + \sup_{x \in \mathbf{X}(\mathbf{U})} \sum_{k=0}^{m(\mathbf{U})-1} \varphi(f^k(x))\right), \tag{7.1}$$

$$M(Z,\alpha,\mathcal{U},N,\varphi) = \inf_{\mathcal{G}} Q(Z,\alpha,\mathcal{U},\mathcal{G},\varphi), \tag{7.2}$$

where the infimum is taken over all finite or countable subcollections of strings $\mathcal{G} \subset \mathcal{S}(\mathcal{U})$ such that $m(\mathbf{U}) \geq N$ for all $\mathbf{U} \in \mathcal{G}$ and $\mathcal{G}$ covers Z. We set $\sup_{x\in\mathbf{X}(\mathbf{U})} \sum_{k=0}^{m(\mathbf{U})-1} \varphi(f^k(x)) = -\infty$ when $\mathbf{X}(\mathbf{U}) = \emptyset$. Define

$$m(Z,\alpha,\mathcal{U},\varphi) := \lim_{N\to\infty} M(Z,\alpha,\mathcal{U},N,\varphi). \tag{7.3}$$

There exists a critical value α_c with $-\infty \leq \alpha_c \leq +\infty$ such that $m(Z,\alpha,\mathcal{U},\varphi) = \infty$ for $\alpha < \alpha_c$ and $m(Z,\alpha,\mathcal{U},\varphi) = 0$ for $\alpha > \alpha_c$. Let $|\mathcal{U}| = \max\{\mathsf{Diam}(U_i) : U_i \in \mathcal{U}\}$.

Definition 7.3.1. *We define*

$$P_Z(\varphi,\mathcal{U}) := \inf\{\alpha : m(Z,\alpha,\mathcal{U},\varphi) = 0\} = \sup\{\alpha : m(Z,\alpha,\mathcal{U},\varphi) = \infty\} = \alpha_c,$$

Lemma 7.3.1.

$$P_Z(\varphi) = \lim_{|\mathcal{U}|\to 0} P_Z(\varphi,\mathcal{U}).$$

For the proof that $P_Z(\varphi)$ coincides with $\lim_{|\mathcal{U}|\to 0} P_Z(\varphi,\mathcal{U})$, we refer the reader to [PP2] or [Pes].

7.3.2 Sketch proof of $P_Z(\varphi) \leq P_Z^*(\varphi)$

Let $\mathcal{U} = \{U_1, \ldots, U_r\}$ be an open cover of X and $\varepsilon > 0$. Let

$$\mathsf{Var}(\varphi,\mathcal{U}) = \sup\{|\varphi(x) - \varphi(y)| : x, y \in U \text{ for some } U \in \mathcal{U}\}.$$

Let E be a finite set of cardinality n, and $\underline{a} = (a_0, \ldots, a_{k-1}) \in E^k$. Define the probability vector $\mu_{\underline{a}} = (\mu_{\underline{a}}(e_1), \ldots, \mu_{\underline{a}}(e_n))$ on E by

$$\mu_{\underline{a}}(e_i) = \frac{1}{k}(\text{the number of those } j \text{ for which } a_j = e_i).$$

Define

$$H(\underline{a}) = -\sum_{i=1}^{n} \mu_{\underline{a}}(e_i) \log \mu_{\underline{a}}(e_i).$$

In [Pes], the contents of the following lemma are proved under the assumption that $\mu \in \mathcal{V}(x) \cap \mathcal{M}_f(Z)$. However, the property $\mu(Z) = 1$ is not required. We omit the proof.

Lemma 7.3.2. *Given $x \in Z$ and $\mu \in \mathcal{V}(x)$, there exists a number $m > 0$ such that for any $n > 0$ one can find $N > n$ and a string $\mathbf{U} \in \mathcal{S}(\mathcal{U})$ of length N satisfying:*

(1) $x \in \mathbf{X}(\mathbf{U})$,

(2) $\sup_{x\in\mathbf{X}(\mathbf{U})} \sum_{k=0}^{N-1} \varphi(f^k(x)) \leq N\left(\int \varphi d\mu + \mathit{Var}(\varphi,\mathcal{U}) + \varepsilon\right)$,

(3) $\mathbf{U} = (U_0, \ldots, U_{N-1})$ contains a substring $\mathbf{U}'$ with the following properties: There exists $k \in \mathbb{N}$ with $N - m \le km \le N$ and $0 \le i_0 \le \ldots \le i_{k-1}$ so $a_0 = (U_{i_0}, \ldots, U_{i_0+m})$, $\ldots, a_{k-1} = (U_{i_{k-1}}, \ldots, U_{i_{k-1}+m})$ and $\mathbf{U}' = (a_0, \ldots, a_{k-1})$. Note that the length of $\mathbf{U}'$ is km. Writing $E = \{a_0, \ldots, a_{k-1}\}$ and $\underline{a} = (a_0, \ldots, a_{k-1})$, then

$$\frac{1}{m} H(\underline{a}) \le h_\mu + \varepsilon.$$

Given a number $m > 0$, denote by Z_m the set of points $x \in Z$ for which there exists a measure $\mu \in \mathcal{V}(x)$ so lemma 7.3.2 holds for this m. We have that $Z = \bigcup_{m>0} Z_m$. Denote by $Z_{m,u}$ the set of points $x \in Z_m$ for which there exists $\mu \in \mathcal{V}(x)$ so lemma 7.3.2 holds for this m and $\int \varphi d\mu \in [u-\varepsilon, u+\varepsilon]$. Set $c = \sup\{h_\mu + \int \varphi d\mu : \mu \in \bigcup_{x\in Z} \mathcal{V}(x)\}$. Note that if $x \in Z_{m,u}$, then the corresponding measure μ satisfies

$$h_\mu \le c - \int \varphi d\mu \le c - u + \varepsilon. \tag{7.4}$$

Suppose a finite set $\{u_1, \ldots, u_s\}$ forms an ε-net of the interval $[-\|\varphi\|, \|\varphi\|]$. Then

$$Z = \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{s} Z_{m,u_i}$$

and hence $P_Z(\varphi) \ge \sup_{m,i} P_{Z_{m,u_i}}(\varphi)$. It will suffice to prove that for arbitrary $m \in \mathbb{N}$ and $u \in \mathbb{R}$ that $P_{Z_{m,u}}(\varphi) \le c$.

For each $x \in Z_{m,u}$, we construct a string $\mathbf{U}_x$ and substring $\mathbf{U}'_x$ satisfying the conditions of lemma 7.3.2. Let $\mathcal{G}_{m,u}$ denote the collection of all such strings $\mathbf{U}_x$ and $\mathcal{G}^*_{m,u}$ denote the collection of all such substrings $\mathbf{U}'_x$. Choose N_0 so $m(\mathbf{U}_x) \ge N_0$ for all $\mathbf{U}_x \in \mathcal{G}_{m,u}$. Let $\mathcal{G}_{m,u,N}$ denote the subcollection of strings $\mathbf{U}_x \in \mathcal{G}_{m,u}$ with $m(\mathbf{U}) = N$ and $\mathcal{G}^*_{m,u,N}$ denote the correponding subcollection of substrings. Note that

$$\mathcal{G}_{m,u} = \bigcup_{N=N_0}^{\infty} \mathcal{G}_{m,u,N} \text{ and } \#\mathcal{G}_{m,u,N} \le \#\mathcal{U}^m \#\mathcal{G}^*_{m,u,N}.$$

We use the following lemma of Bowen [Bow6].

Lemma 7.3.3. *Fix $h > 0$. Let $R(k,h,E) = \{\underline{a} \in E^k : H(\underline{a}) \le h\}$. Then*

$$\limsup_{k\to\infty} \frac{1}{k} \log \#(R(k,h,E)) \le h.$$

Set $h = c-u+\varepsilon$. It follows from (7.4) and the third statement of lemma 7.3.2 that if $x \in Z_{m,u}$ has an associated string $\mathbf{U}_x$ of length N, then its substring $\mathbf{U}'_x$ is contained in $R(k, m(h+\varepsilon), \mathcal{U}^m)$ where k satisfies $N > km \ge N - m$. Therefore, $\#\mathcal{G}^*_{m,u,N}$ does not exceed $\#R(k, m(h+\varepsilon), \mathcal{U}^m)$, and thus $\#\mathcal{G}_{m,u,N} \le \#\mathcal{U}^m \#(R(k, m(h+\varepsilon), \mathcal{U}^m))$. Applying lemma 7.3.3, we obtain

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{N} \log \#\mathcal{G}_{m,u,N} &\le \limsup_{k\to\infty} \frac{1}{mk} \log \#\mathcal{U}^m \#(R(k, m(h+\varepsilon), \mathcal{U}^m)) \\ &\le h + \varepsilon. \end{aligned}$$

Since the collection of strings $\mathcal{G}_{m,u}$ covers the set $Z_{m,u}$, we use property (2) of lemma 7.3.2 to get

$$\begin{aligned} Q(Z_{m,u}, \lambda, \mathcal{U}, \mathcal{G}_{m,u}, \varphi) &= \sum_{N=N_0}^{\infty} \sum_{\mathbf{U} \in \mathcal{G}_{m,u,N}} \exp\left\{-\lambda N + \sup_{x \in \mathbf{X}(\mathbf{U})} \sum_{k=0}^{N-1} \varphi(f^k(x))\right\} \\ &\leq \sum_{N=N_0}^{\infty} \#\mathcal{G}_{m,u,N} \exp\left\{N\left(-\lambda + \mathsf{Var}(\varphi, \mathcal{U}) + \int \varphi d\mu + \varepsilon\right)\right\}. \end{aligned}$$

Choose N_0 sufficiently large so for $N \geq N_0$, we have $\#\mathcal{G}_{m,u,N} \leq \exp(N(h + 2\varepsilon))$ and thus

$$M(Z_{m,u}, \lambda, \mathcal{U}, N_0, \varphi) \leq \sum_{N=N_0}^{\infty} \exp\left\{N\left(h - \lambda + \mathsf{Var}(\varphi, \mathcal{U}) + \int \varphi d\mu + 3\varepsilon\right)\right\}.$$

Let $\beta = \exp\left(h - \lambda + \mathsf{Var}(\varphi, \mathcal{U}) + \int \varphi d\mu + 3\varepsilon\right)$. If $\lambda > c + \mathsf{Var}(\varphi, \mathcal{U}) + 5\varepsilon$, then $0 < \beta < 1$. Thus,

$$M(Z_{m,u}, \lambda, \mathcal{U}, N_0, \varphi) \leq \frac{\beta^{N_0}}{1 - \beta},$$

$$m(Z_{m,u}, \lambda, \mathcal{U}, \varphi) \leq \lim_{N_0 \to \infty} \frac{\beta^{N_0}}{1 - \beta} = 0.$$

It follows that $\lambda \geq P_{Z_{m,u}}(\varphi, \mathcal{U})$. Since we can choose λ arbitrarily close to $c + \mathsf{Var}(\varphi, \mathcal{U}) + 5\varepsilon$, it follows that

$$P_{Z_{m,u}}(\varphi, \mathcal{U}) \leq c + \mathsf{Var}(\varphi, \mathcal{U}) + 5\varepsilon.$$

We are free to choose ε arbitrarily small, so on taking the limit $|\mathcal{U}| \to 0$, we have $P_{Z_{m,u}}(\varphi) \leq c$, as required. It follows that $P_Z(\varphi) \leq c$.

Remark 7.3.1. In [PP2], it is shown that if $\mu \in \mathcal{M}_f(X)$ and $\mu(Z) = 1$ then $P_Z(\varphi) \geq h_\mu + \int \varphi d\mu$. Thus, if Z is a set satisfying $\mu(Z) = 1$ for all $\mu \in \bigcup_{x \in Z} \mathcal{V}(x)$, then $P_Z(\varphi) = P_Z^*(\varphi)$.

Remark 7.3.2. If $P_Z(\varphi) < P_Z^*(\varphi)$, then we see a phenomenon similar to example 7.4.4, where probability measures μ with $\mu(Z) < 1$ or even $\mu(Z) = 0$ capture information about the set Z. This may seem unusual but example 7.4.4 motivates the utility of this point of view.

Remark 7.3.3. We can adapt the proof to obtain the inequality $P_{\mathcal{G}(Z)}(\varphi) \leq P_Z^{\#}(\varphi)$. The argument would differ in the paragraph above lemma 7.3.3. We would construct strings $\mathbf{U}_x$ and $\mathbf{U}'_x$ only for those $x \in \mathcal{G}(Z)$ rather than every $x \in Z$.

Remark 7.3.4. We can view the result of this section as an inequality for $P_Z(\varphi)$. We state this explicitly without reference to definition 7.1.1. Let Z be a Borel subset (not necessarily invariant) of a compact metric space (X, d). Then

$$P_Z(\varphi) \leq \sup\{h_\mu + \int \varphi d\mu : \mu = \lim_{n_k \to \infty} \delta_{x, n_k} \text{ for some } x \in Z, n_k \to \infty\}.$$

7.4 Examples

Here are some interesting examples for which $P_Z(\varphi)$ and $P_Z^*(\varphi)$ coincide.

7.4.1 North-South map

The following example was suggested by Pesin. Let $X = S^1$, f be the North-South map and $Z = S^1 \setminus \{S\}$. (By the North-South map, we mean the map $f = g^{-1} \circ h \circ g$ where g is the stereographic projection from a point N onto the tangent line at S, where S is the antipodal point of N, and $h : \mathbb{R} \mapsto \mathbb{R}$ is $h(x) = x/2$.) One can verify that if $x \in S^1 \setminus \{N, S\}$, then $\mathcal{V}(x) = \delta_S$ and it is clear that $\mathcal{V}(\{N\}) = \delta_N$. Using this and the fact that $h_{\delta_S} = h_{\delta_N} = 0$, we have

$$P_Z^*(\varphi) = \max\{\int \varphi d\delta_S, \int \varphi d\delta_N\} = \max\{\varphi(N), \varphi(S)\}$$

To calculate $P_Z(\varphi)$, one can use $P_Z(\varphi) = \max\{P_{\{N\}}(\varphi), P_{Z\setminus\{N\}}(\varphi)\}$. Using the formula for pressure at a point or Pesin's variational principle, $P_{\{N\}}(\varphi) = \varphi(N)$. One can verify that $P_{Z\setminus\{N\}}(\varphi) = \varphi(S)$. Thus, $P_Z(\varphi)$ and $P_Z^*(\varphi)$ coincide for all continuous φ.

Remark 7.4.1. Note that $\mathcal{L}(Z) = \{N\}$. If we choose φ so that $\varphi(S) > \varphi(N)$, we are furnished with an example where $P_Z(\varphi) > P_{\mathcal{L}(Z)}(\varphi)$, showing that we could not replace $P_{\mathcal{L}(Z)}(\varphi)$ by $P_Z(\varphi)$ in Pesin's variational principle (see theorem 7.1.1).

Remark 7.4.2. Our example shows that, in contrast to the compact case, the wandering set can contribute to the pressure (whether we consider $P_Z^*(\varphi)$ or $P_Z(\varphi)$). Let $\mathcal{NW}(X)$ be the non-wandering set of (X, f) and $\mathcal{W}(X) := X \setminus \mathcal{NW}(X)$. (Recall that $x \in \mathcal{NW}(X)$ if for any open set U containing x there exists N so $f^N(U) \cap U \neq \emptyset$.) For an arbitrary set $Y \subset X$, let $\mathcal{NW}(Y) = Y \cap \mathcal{NW}(X)$ and $\mathcal{W}(Y) = Y \cap \mathcal{W}(X)$. For the set Z of our example, $\mathcal{NW}(Z) = N$ (see §5.3 of [Wal]). Assuming that $\varphi(S) > \varphi(N)$, we have

$$P_{\mathcal{NW}(Z)}^*(\varphi) = \varphi(N) < \varphi(S) = P_Z^*(\varphi).$$

This contrasts with the compact case, where $P_{\mathcal{NW}(X)}^{classic}(\varphi) = P_X^{classic}(\varphi)$.

7.4.2 Irregular sets

Theorem 7.4.1. *Let (Σ, σ) be a topologically mixing subshift of finite type and $\hat{\Sigma}$ be the set*

$$\hat{\Sigma} := \Sigma \setminus \bigcup_{\mu \in \mathcal{M}_f(\Sigma)} G_\mu.$$

Then $h_{top}^*(\hat{\Sigma}) = h_{top}(\sigma)$ and $P_{\hat{\Sigma}}^*(\psi) = P_\Sigma^{classic}(\psi)$ for all $\psi \in C(X)$.

We remark that Barreira and Schmeling showed in [BS5] that $h_{top}(\hat{\Sigma}) = h_{top}(\sigma)$. It follows that $h_{top}(\hat{\Sigma}) = h_{top}^*(\hat{\Sigma})$. After an application of the classical variational principle, the proof of theorem 7.4.1 follows immediately from the next lemma in which, for simplicity, we assume Σ is a full shift.

Lemma 7.4.1. $\mathcal{M}_f^e(\Sigma) \subseteq \bigcup_{x\in\hat{\Sigma}} \mathcal{V}(x)$.

Proof. Let μ_1 be some ergodic measure. Let μ_2 be some other ergodic measure. Let $x \in G_{\mu_1}$, $y \in G_{\mu_2}$ and $N_k \to \infty$ sufficiently rapidly that $N_{k+1} > 2^{N_k}$. We can use the specification property of the shift to construct a point p so $\delta_{p,N_{2k}} \to \mu_1$ and $\delta_{p,N_{2k+1}} \to \mu_2$. Namely, let $w_{2i-1} = (x_1, \dots, x_{N_{2i-1}})$ and $w_{2i} = (y_1, \dots, y_{N_{2i}})$ for all $i \geq 1$. Let $p = w_1 w_2 w_3 \dots \in \Sigma$. Then $p \in \hat{\Sigma}$ and $\mu_1 \in \mathcal{V}(p)$. $\square$

We establish a result analogous to the main result of chapter 4.

Theorem 7.4.2. *Let (X, d) be a compact metric space and $f : X \mapsto X$ be a continuous map with the specification property. Assume $\varphi \in C(X)$ satisfies $\inf_{\mu\in\mathcal{M}_f(X)} \int \varphi d\mu < \sup_{\mu\in\mathcal{M}_f(X)} \int \varphi d\mu$. Let*

$$\widehat{X}(\varphi, f) := \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \text{ does not exist} \right\}.$$

Then $h^*_{top}(\widehat{X}(\varphi, f)) = h_{top}(f)$ and $P^*_{\widehat{X}(\varphi,f)}(\psi) = P^{classic}_X(\psi)$ for all $\psi \in C(X)$.

Combining this with the main result of chapter 4, we have $P_{\widehat{X}(\varphi,f)}(\psi) = P^*_{\widehat{X}(\varphi,f)}(\psi)$ when f has specification. The proof of theorem 7.4.2 follows immediately from the next lemma by the classical variational principle.

Lemma 7.4.2. $\mathcal{M}_f^e(X) \subseteq \bigcup_{x\in\widehat{X}_\varphi} \mathcal{V}(x)$.

Sketch proof. Let μ_1, μ_2 be ergodic measures with $\int \varphi d\mu_1 < \int \varphi d\mu_2$. Let x_i satisfy $\frac{1}{n} S_n\varphi(x_i) \to \int \varphi d\mu_i$ for $i = 1, 2$. Let $m_k := m(\varepsilon/2^k)$ be as in the definition of specification and $N_k \to \infty$ sufficiently rapidly that $N_{k+1} > \exp\{\sum_{i=1}^k (N_i + m_i)\}$. We define $z_i \in X$ inductively using the specification property. Let $t_1 = N_1$, $t_k = t_{k-1} + m_k + N_k$ for $k \geq 2$ and $\rho(k) := (k+1)(\text{mod}2) + 1$. Let $z_1 = x_1$. Let z_2 satisfy $d_{N_1}(z_2, z_1) < \varepsilon/4$ and $d_{N_2}(f^{N_1+m_2} z_2, x_2) < \varepsilon/4$. Let z_k satisfy $d_{t_{k-1}}(z_{k-1}, z_k) < \varepsilon/2^k$ and $d_{N_k}(f^{t_{k-1}+m_k} z_k, x_{s(k)}) < \varepsilon/2^k$. Let $B_n(x, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\}$. We can verify that $\overline{B}_{t_{k+1}}(z_{k+1}, \varepsilon/2^k) \subset \overline{B}_{t_k}(z_k, \varepsilon/2^{k-1})$. Define $p := \bigcap \overline{B}_{t_k}(z_k, \varepsilon/2^{k-1})$. For any $\psi \in C(X)$, we can show $\frac{1}{t_k} S_{t_k}\psi(p) \to \int \psi d\mu_{\rho(k)}$. Thus $\delta_{p,t_{2k-1}} \to \mu_1$, $\delta_{p,t_{2k}} \to \mu_2$ and so $\mu_1, \mu_2 \in \mathcal{V}(p)$. In particular, $p \in \widehat{X}(\varphi, f)$. $\square$

Remark 7.4.3. Using a similar construction to the proof of lemma 7.4.1, we can show that the inequality $P_{\mathcal{L}(Z)}(\varphi) \leq P^*_{\mathcal{L}(Z)}(\varphi)$ may be strict. Let (Σ, σ) be a Bernoulli shift. Let μ_1, μ_2 be ergodic measures with $h_{\mu_1} > h_{\mu_2}$. We can construct a point z so the sequence of measures $\delta_{z,n}$ does not converge and $\mathcal{V}(z) = \{\mu_1, \mu_2\}$. Let $Z = G_{\mu_2} \cup \{z\}$. We see that $\mathcal{L}(Z) = Z$ and, by theorem 7.1.1, $h_{top}(Z) = h_{\mu_2}$. However, $h^*_{top}(Z) = h_{\mu_1}$.

Remark 7.4.4. The proof of lemma 7.4.2 generalises in the expected way to the setting of maps f with the almost specification property. Thus, the statement of theorem 7.4.2 holds for continuous maps with the almost specification property. In particular, the statement of theorem 7.4.2 holds when f is the β-shift.

7.4.3 Levels sets of the Birkhoff average

We establish a result analogous to the main result of chapter 5.

Theorem 7.4.3. *Let (X, d) be a compact metric space, $f : X \mapsto X$ be a continuous map with the specification property and $\varphi, \psi \in C(X)$. For $\alpha \in \mathbb{R}$, let*

$$X(\varphi, \alpha) = \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \alpha \right\}.$$

Suppose $X(\varphi, \alpha) \neq \emptyset$, then

(1) $h^*_{top}(X(\varphi, \alpha)) = \sup\{h_\mu : \mu \in \mathcal{M}_f(X)$ and $\int \varphi d\mu = \alpha\}$,

(2) $P^*_{X(\varphi,\alpha)}(\psi) = \sup\{h_\mu + \int \psi d\mu : \mu \in \mathcal{M}_f(X)$ and $\int \varphi d\mu = \alpha\}$.

Combining this with the main result of chapter 5, we have $P_{X(\varphi,\alpha)}(\psi) = P^*_{X(\varphi,\alpha)}(\psi)$ when f has specification. The proof of theorem 7.4.3 follows from the next lemma.

Lemma 7.4.3. $\{\mu \in \mathcal{M}_f(X) : \int \varphi d\mu = \alpha\} = \{\mu \in \mathcal{V}(x) : x \in X(\varphi, \alpha)\}$.

Proof. Let $\mu \in \mathcal{M}_f(X)$ and $\int \varphi d\mu = \alpha$. Recall that $G_\mu \neq \emptyset$ and let $x \in G_\mu$. Then $\mathcal{V}(x) = \mu$, and so $\{\mu \in \mathcal{M}_f(X) : \int \varphi d\mu = \alpha\} \subseteq \{\mu \in \mathcal{V}(x) : x \in X(\varphi, \alpha)\}$. Conversely, if $\mu \in \mathcal{V}(x)$ for $x \in X(\varphi, \alpha)$ then there exists $n_k \to \infty$ so $\int \varphi d\mu = \lim_{n_k\to\infty} \int \varphi d\delta_{x,n_k} = \lim_{n_k\to\infty} \frac{1}{n_k} S_{n_k}\varphi(x) = \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha$. $\square$

7.4.4 Manneville-Pomeau maps

Manneville-Pomeau maps are the family of maps on $[0, 1]$ given by

$$f_s(x) = x + x^{1+s} (\text{mod} 1)$$

where $s \in (0, 1)$ is a fixed parameter value. Each of these maps is a topological factor of a full one-sided shift on 2 symbols and so satisfies the specification property. Takens and Verbitskiy have performed a multifractal analysis for the function $\varphi(x) = \log f_s'(x)$ (i.e. the multifractal analysis of pointwise Lyapunov exponents). We recall some results which can be found in [TV2]. One of the key results used for their multifractal analysis, restated in our new language, is

Theorem 7.4.4. *$f : X \mapsto X$ be a continuous map with the specification property, and $\varphi : X \mapsto \mathbb{R}$ a continuous function. Then*

$$(1)\ h^*_{top}(X(\varphi, \alpha)) \leq \inf_{q\in\mathbb{R}} \{P^{classic}_X(q\varphi) - q\alpha\}.$$

Furthermore, if f has upper semi-continuous entropy map then

$$(2)\ h^*_{top}(X(\varphi,\alpha)) = \inf_{q\in\mathbb{R}}\{P^{classic}_X(q\varphi) - q\alpha\}.$$

Since f_s is positively expansive, it has upper semi-continuous entropy map. There is an interval of values $\mathcal{I}$ (which turns out to be $(0, h_\mu)$ where μ is the absolutely continuous invariant measure for f_s) which has the following property. For $\alpha \in \mathcal{I}$, the infimum of theorem 7.4.4 (2) is attained uniquely at $q = -1$ and $P^{classic}_X(-\varphi) = 0$ (using results from [Urb] and [PS3]). Thus, $h^*_{top}(X(\varphi,\alpha)) = \alpha$ and if ν is an equilibrium measure for $-\varphi$ with $\int \varphi d\nu = \alpha$, then $h^*_{top}(X(\varphi,\alpha)) = h_\nu$. The set $\mathcal{A} = \{p\delta_0 + (1-p)\mu : p \in [0,1]\}$ consists of equilibrium measures for $-\varphi$ and Takens and Verbitsky show there is a unique measure satisfying $\mu_\alpha \in \mathcal{A}$ and $\int \varphi d\mu_\alpha = \alpha$. By lemma 7.4.3, $\mu_\alpha \in \mathcal{V}(x)$ for some $x \in X(\varphi,\alpha)$, and so μ_α is a $*$-equilibrium measure (for 0 on $X(\varphi,\alpha)$). However, even though $h_{top}(X(\varphi,\alpha)) = h_{\mu_\alpha}$, they show $\mu_\alpha(X(\varphi,\alpha)) = 0$, so μ_α is not an equilibrium measure (for 0 on $X(\varphi,\alpha)$) under the definition of Pesin.

In fact, μ_α is the unique $*$-equilibrium measure. In Proposition 1 of [PSY], Pollicott, Sharp and Yuri show that ν is an equilibrium state for $-\varphi$ iff $\nu \in \mathcal{A}$ (they also give a nice proof that $P^{classic}_X(-\varphi) = 0$). It follows that if $\mu \notin \mathcal{A}$ and $\int \varphi d\mu = \alpha$, then $h_\mu < \alpha$. Combining this with the above discussion shows that μ_α is unique.

7.5 Topological pressure in a non-compact ambient space

We define $P^*_Z(\varphi)$ for an arbitrary set $Z \subset X$ and $\varphi \in C(X)$ when the ambient space X is non-compact. For the definition to make sense, we must exclude the consideration of measures μ such that both $h_\mu = \infty$ and $\int \varphi d\mu = -\infty$.

Definition 7.5.1. *Let Z be an arbitrary Borel set and $\varphi \in C(X)$. Define*

$$P^*_Z(\varphi) = \sup\left\{h_\mu + \int_X \varphi d\mu : \mu \in \bigcup_{x\in Z}\mathcal{V}(x) \text{ and } \int_X \varphi d\mu > -\infty\right\}.$$

If $\bigcup_{x\in Z}\mathcal{V}(x) = \emptyset$, let $P^*_Z(\varphi) = \inf_{x\in X}\varphi(x)$. If $\bigcup_{x\in Z}\mathcal{V}(x) \neq \emptyset$ and $\{\mu \in \bigcup_{x\in Z}\mathcal{V}(x) : \int_X \varphi d\mu > -\infty\} = \emptyset$, then $P^*_Z(\varphi) = -\infty$.

The reason we set $P^*_Z(\varphi) = \inf_{x\in X}\varphi(x)$ when $\bigcup_{x\in Z}\mathcal{V}(x) = \emptyset$ is to ensure that the inequality $P^*_{Z_1}(\varphi) \leq P^*_{Z_2}(\varphi)$ holds for all $Z_1 \subseteq Z_2$. We remark that if φ is bounded below, then we have $\int_X \varphi d\mu > -\infty$ for all $\mu \in \mathcal{M}_f(X)$. Hence, if X is compact, definitions 7.5.1 and 7.1.1 agree.

Remark 7.5.1. Assume $h^*_{top}(Z) < \infty$. Then we do not have to restrict ourselves to measures with $\int_X \varphi d\mu > -\infty$ in the definition of $P^*_Z(\varphi)$. Either $P^*_Z(\varphi) = -\infty$ or the extra measures considered do not contribute to the supremum.

Remark 7.5.2. In the non-compact setting, dimensional definitions of pressure have the disadvantage that there are examples of metrizable spaces X (eg. countable state shifts) and metrics d_1, d_2 on X where $P_{Z,X_1}(\varphi) \neq P_{Z,X_2}(\varphi)$ (where $X_1 = (X, d_1)$ and $X_2 = (X, d_2)$) but d_1 and d_2 give rise to the same Borel structure on X and thus no variational principle can hold. However, $P^*_Z(\varphi)$ depends only on the Borel structure of X and is thus invariant under a change of topologically equivalent metric.

Remark 7.5.3. In [DJ], Dai and Jiang study a definition of topological entropy for non-compact spaces adapted to the problem of estimating the Hausdorff dimension of the space. Their definition is not a topological invariant, so is not equivalent to ours. They give an interesting discussion of the issues one faces when considering entropy as a measure of chaotic behaviour in the non-compact setting.

We now study some properties of $P^*_Z(\varphi)$ in the non-compact setting.

Theorem 7.5.1. *Let $P^*_{Z,Y}(\varphi)$ denote the pressure of φ on Z when $Z \subset Y$ and Y is considered as the ambient space in the definition. Let $K \subset X$ be compact and invariant and $Z \subset K$. Then $P^*_{Z,X}(\varphi) = P^*_{Z,K}(\varphi)$.*

Proof. It suffices to notice that if $\mu \in \bigcup_{x \in Z} \mathcal{V}(x)$, then $\mu \in \mathcal{M}_f(K)$ and $h_\mu(f|_K) = h_\mu(f)$. □

Theorem 7.5.2. *Let X be a separable metric space and $\varphi \in C(X)$. Then*

(1) $P^*_X(\varphi) = \sup\{P^*_{K,X}(\varphi) : K \subset X$ is compact$\}$,

(2) $P^*_X(\varphi) = \sup\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_f(X), \int \varphi d\mu > -\infty\}$.

Proof. For (1), we note that if K_n is a countable collection of compact sets that cover X, then $P^*_X(\varphi) = \sup\{P^*_{K_n,X}(\varphi)\}$ by basic properties of $P^*_X(\varphi)$. For (2), let c denote the value taken by the supremum. That $P^*_X(\varphi) \leq c$ is immediate. It suffices to consider only ergodic measures in the supremum. We note that since X is a separable space, if μ is ergodic then $\mu(G_\mu) = 1$. Thus, there exists x satisfying $\mathcal{V}(x) = \mu$, which shows that $P^*_X(\varphi) \geq c$. □

In [GS], Gurevich and Savchenko study two definitions of topological pressure adapted to non-compact spaces. We compare these with $P^*_Z(\varphi)$.

Definition 7.5.2. *Set $P^{int}(X, \varphi) = \sup\{P^{classic}_K(\varphi)\}$, where the supremum is over all subsets $K \subset X$ which are compact and invariant. Suppose X can be continuously embedded in a compact metric space $\hat{X}$ and f can be extended continuously to X. We set $P^{ext}(X, \varphi) = \inf\{P_{X,\hat{X}}(\varphi)\}$, where the infimum is over all such embeddings.*

Theorem 7.5.3. *For any X separable, $f : X \mapsto X$ and $\varphi \in C(X)$, we have $P^{int}(X, \varphi) \leq P^*_X(\varphi)$. When $P^{ext}(X, \varphi)$ is well defined, $P^*_X(\varphi) \leq P^{ext}(X, \varphi)$.*

Proof. The first inequality follows from the classical variational principle and (2) of theorem 7.5.2. Let $\hat{X}$ be a compact metric space satisfying the requirements of the definition and $\hat{\varphi}$ be the extension by continuity of φ to $\hat{X}$. By theorem 7.1.1 and (2) of theorem 7.5.2,

$$P_X^*(\varphi) = P_{\mathcal{L}(X),\hat{X}}(\hat{\varphi}) \leq P_{X,\hat{X}}(\hat{\varphi}).$$

Since $\hat{X}$ was arbitrary, we obtain the desired inequality. $\square$

Remark 7.5.4. Both inequalities of theorem 7.5.3 may be strict. As noted in [GS] and [HKR], let Y be a compact metric space and $f : Y \mapsto Y$ be a minimal homeomorphism with $h_{top}(f) > 0$. Let $\varphi = 0$. Let $X = Y \setminus \mathcal{O}(x)$, where $\mathcal{O}(x)$ is the orbit of an arbitrary $x \in Y$. There are no compact, invariant, non-empty subsets of X, so $P^{int}(X, 0) = 0$. However, $h^*_{top}(X) = \sup\{h_\mu : \mu \in \mathcal{M}_f(Y)\} = h_{top}(f)$. For the second inequality, we use an example similar to 7.4.1. Let $X = S^1 \setminus \{S\}$ with induced metric d from S^1, f is the North-South map and $\varphi(x) = d(x, \{N\})$. We have $P^*_X(\varphi) = \varphi(N) = 0$. We can verify that given any continuous embedding into $\hat{X}$ and any $y \in \hat{X} \setminus X$, $P_{X,\hat{X}}(\varphi) \geq \varphi(y) > 0$.

Remark 7.5.5. In [HNP], the authors compare various definitions of topological entropy for a non-compact space X and a continuous map $f : X \mapsto X$. One of these definitions is a natural generalisation of Adler-Konheim-McAndrew's original definition of entropy [AKM], which we denote by $h^{AKM}_{top}(f)$. Proposition 5.1 of [HNP] provides an example of a homeomorphism f of the open unit interval (equipped with a non-standard metric) for which $h^{AKM}_{top}(f) = \infty$ but $h^*_{top}(f) = 0$.

In [HKR], Handel, Kitchens and Rudolph give another definition of entropy for a non-compact metric space (X, d) and continuous $f : X \mapsto X$, which is invariant under a change of topologically equivalent metric and is a generalisation of $\overline{CP}_Z(0)$. Let $S(K, n, \varepsilon, d)$ denote the smallest cardinality of an (n, ε) spanning set for a compact set $K \subset X$ in the metric d. Let

$$h^d_{top}(X) := \sup\{\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S(K, n, \varepsilon, d) : K \subset X \text{ is compact }\}.$$

In fact, this definition first appeared in [Bow1]. The innovation of [HKR] is to define

$$h^{HKR}_{top}(X) := \inf\{h^{d'}_{top}(X) : d' \text{ is a metric topologically equivalent to d}\}.$$

They show that $h^{HKR}_{top}(X) \geq \sup\{h_\mu : \mu \in \mathcal{M}_f(X)\}$ and construct an example where the inequality is strict. Thus $h^{HKR}_{top}(X) \geq h^*_{top}(X)$ and it is possible that the two quantities may not coincide.

However, if X is locally compact, $f : X \mapsto X$ is uniformly continuous, Y is the one point compactification of X and $g : Y \mapsto Y$ is the extension by continuity of f, they show that $h^{HKR}_{top}(X) = h^*_{top}(X) = h_{top}(g)$.

7.5.1 Countable state shifts of finite type

We conclude by considering a topologically mixing countable state shift of finite type (Σ, σ). Following Sarig [Sar], we equip Σ with the metric $d(x,y) = r^{t(x,y)}$ where $t(x,y) = \inf(\{k : x_k \neq y_k\} \cup \infty)$ and $r \in (0,1)$. Let $P^G(\varphi)$ denote the Gurevic pressure as defined by Sarig [Sar] where φ is a locally Hölder function and $h^G(\sigma) := P^G(0)$. In [GS], the authors allow Σ to be equipped with more general metrics and study $P^{int}(\Sigma, \varphi)$ and $P^{ext}(\Sigma, \varphi)$ for $\varphi \in C(\Sigma)$. To rephrase corollary 1 of [Sar], Sarig showed that in his setting $P^G(\varphi) = P^{int}(\Sigma, \varphi)$.

Theorem 7.5.4. $h^*_{top}(\Sigma) = h^G(\sigma)$.

Proof. By corollary 1.7 of [GS], $P^{int}(\Sigma, 0) = P^{ext}(\Sigma, 0)$ in the metric d. The result follows from theorem 7.5.3. □

Theorem 7.5.5. *We have $P^*_\Sigma(\varphi) \geq P^G(\varphi)$. With the extra assumption $\sup_{x\in\Sigma} |\sum_{\sigma y = x} e^{\varphi(y)}| < \infty$, we have $P^*_\Sigma(\varphi) = P^G(\varphi) < \infty$. If $P^G(\varphi) = \infty$, then $P^*_\Sigma(\varphi) = \infty$.*

Proof. The first inequality is a rephrasing of theorem 7.5.3. Under the extra assumption, Sarig showed $P^G(\varphi) = \sup\{h_\mu + \int \varphi d\mu : \mu \in \mathcal{M}_\sigma(\Sigma), \int \varphi d\mu < \infty\} < \infty$. The supremum is equal to $P^*_\Sigma(\varphi)$ by theorem 7.5.2. □

7.6 Pressure at a point

In theorems 7.2.6, 7.2.7 and the remark afterwards, we considered the topological pressure on a point z. Here, we prove the formulae that we quoted for $P_{\{z\}}(\varphi)$, $\underline{CP}_{\{z\}}(\varphi)$ and $\overline{CP}_{\{z\}}(\varphi)$.

Theorem 7.6.1. *Let X be a compact metric space, $f : X \mapsto X$ and z be an arbitrary point. Then*

$$P_{\{z\}}(\varphi) = \underline{CP}_{\{z\}}(\varphi) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)),$$

$$\overline{CP}_{\{z\}}(\varphi) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)).$$

Remark 7.6.1. It follows from theorem 7.6.1 and the ergodic theorem that for any invariant measure μ, there is a set of full measure so that $P_{\{z\}}(\varphi) = \underline{CP}_{\{z\}}(\varphi) = \overline{CP}_{\{z\}}(\varphi)$. If μ is ergodic, this value is $\int \varphi d\mu$.

Remark 7.6.2. If z is a point for which the Birkhoff average of φ does not exist, then $P_{\{z\}}(\varphi) = \underline{CP}_{\{z\}}(\varphi) < \overline{CP}_{\{z\}}(\varphi)$.

The theorem is a consequence of the lemmas that follow and the relation $P_Z(\varphi) \leq \underline{CP}_Z(\varphi) \leq \overline{CP}_Z(\varphi)$ for any Borel set $Z \subset X$ (formula (11.9) of [Pes]).

Lemma 7.6.1. *Let (X, d) be a compact metric space, $\varphi : X \mapsto \mathbb{R}$ a continuous function, and $z \in X$. Then*

$$P_{\{z\}}(\varphi) \geq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)).$$

Proof. We work directly with our usual definition of Pesin and Pistskel topological pressure (see §2.1.1). Without loss of generality, it suffices to consider covers of $\{z\}$ by a single set $B_n(x, \varepsilon)$. Fix $\varepsilon > 0$, $N \in \mathbb{N}$ and $0 < \delta < \frac{1}{2}$. Choose α satisfying

$$\alpha < \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)) - \mathsf{Var}(\varphi, \varepsilon) - \delta. \tag{7.5}$$

Assume N was chosen sufficiently large so that for $m \geq N$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)) \geq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)) - \delta. \tag{7.6}$$

Choose $\Gamma = \{B_m(x, \varepsilon)\}$ such that $z \in B_m(x, \varepsilon)$, $m \geq N$ and

$$|Q(\{z\}, \alpha, \Gamma, \varphi) - M(\{z\}, \alpha, \varepsilon, N, \varphi)| \leq \delta.$$

We can prove that $\sum_{k=0}^{m-1} \varphi(f^k(z)) - m\mathsf{Var}(\varphi, \varepsilon) - m\alpha > 0$, which follows from (7.5) and (7.6). It follows that

$$\begin{aligned} M(\{z\}, \alpha, \varepsilon, N, \varphi) &\geq \exp\left\{-\alpha m + \sup_{y \in B_m(x,\varepsilon)} \sum_{k=0}^{m-1} \varphi(f^k(y))\right\} - \delta \\ &\geq \exp\left\{-\alpha m + \sum_{k=0}^{m-1} \varphi(f^k(z)) - m\mathsf{Var}(\varphi, \varepsilon)\right\} - \delta \\ &\geq 1 - \delta \geq \frac{1}{2}. \end{aligned}$$

So $M(\{z\}, \alpha, \varepsilon, N, \varphi) > 0$ and hence $P_{\{z\}}(\varphi, \varepsilon) \geq \alpha$. It follows that

$$P_{\{z\}}(\varphi, \varepsilon) \geq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)) - \mathsf{Var}(\varphi, \varepsilon) - \delta.$$

On taking the limit $\varepsilon \to 0$ and noting that δ was arbitrary, we obtain the desired result. $\square$

Lemma 7.6.2. $\underline{CP}_{\{z\}}(\varphi) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z))$.

Proof. It follows from the definition of $\underline{CP}_Z(\varphi)$ that

$$\underline{CP}_{\{z\}}(\varphi) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log\left(\inf_{x: z \in B_n(x,\varepsilon)} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\}\right).$$

For a fixed ε and $B_n(x, \varepsilon)$ which contains z,

$$\sum_{i=0}^{n-1} \varphi(f^i(x)) \leq \sum_{i=0}^{n-1} \varphi(f^i(z)) + n\gamma(\varepsilon).$$

It follows that

$$\underline{CP}_{\{z\}}(\varphi) \leq \lim_{\varepsilon \to 0} \liminf_{n\to\infty} \frac{1}{n}\{\sum_{i=0}^{n-1} \varphi(f^i(z)) + \gamma(\varepsilon)\}.$$

We obtain $\underline{CP}_{\{z\}}(\varphi) \leq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(z))$ and we can prove the reverse inequality in the same way. □

Lemma 7.6.3. $\overline{CP}_{\{z\}}(\varphi) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(z))$.

Proof. It follows from the definition of $\overline{CP}_Z(\varphi)$ that

$$\underline{CP}_{\{z\}}(\varphi) = \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \frac{1}{n} \log\left(\inf_{x:z\in B_n(x,\varepsilon)} \exp\left\{\sum_{i=0}^{n-1} \varphi(f^i(x))\right\}\right).$$

The rest of the proof proceeds in the same way as that of lemma 7.6.2. □

Future directions

We mention some questions of further interest which relate to the contents of the thesis.

There are some very interesting questions surrounding the almost specification property (see chapter 6). Many interesting results which are known for maps with the specification property should generalise to the class of maps with almost specification. For example, Bowen's results concerning uniqueness of equilibrium states for maps with the specification property [Bow5] should carry over to this more general setting.

Another obvious avenue of investigation is to see which other maps have almost specification. We have some ideas about this problem in the context of piecewise monotonic interval maps and for certain examples of shift spaces. However, this project has not come to fruition in time to be included in the thesis.

In §6.5.3, we found subshifts of finite type within the β-shift with entropy arbitrarily close to $\log \beta$. This suggests the investigation of a 'horseshoe' method of proof for results about the topological entropy of the irregular set. More precisely, we could study the class of systems (X, f) which contain subsystems (Y, f^n) which are topological factors of shifts of finite type, where $Y \subset X$ is compact and $n \in \mathbb{N}$. We call (Y, f^n) a horseshoe for (X, f). If the entropy of the horseshoe can be chosen to approximate that of the whole space arbitrarily well, then it should suffice to study the intersection of the irregular set with the horseshoe. We note that systems with specification do not necessarily contain any horseshoes, so this approach would not recover our current results. Also, theorems on the existence of horseshoes typically require smoothness of the system (see, for example, theorem S.5.9 of [KH]), whereas our current approach is a topological approach to a topological question. However, we do note that examples exist that do not have specification but where a 'horseshoe' approach could yield results. For example, a continuous interval map which is not mixing contains horseshoes but does not have specification (see corollary 15.2.10 of [KH]). Thus, the 'horseshoe' approach certainly has merit.

An idea from the thesis which we hope will prove useful is to only ask for specification to hold on an interesting invariant non-compact subset $X' \subset X$ (see definition 2.2.3). The idea could have applications for the study of non-uniformly hyperbolic systems. The interesting invariant set

X' to which we allude is the set of points which return infinitely often to a set on which the map is uniformly hyperbolic. We hope to pursue this in the future. A particular avenue of interest for this is the Rauzy-Veech map and Teichmüller flow [Buf], which are related systems of great current interest arising from geometry.

Finally, we hope to further investigate the properties of $P_Z^*(\psi)$ in some explicit examples.

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