Ross Program 2012 Application Problems

This document is part of the application to the *Ross Mathematics Program*, and is posted at: www.math.ohio-state.edu/ross. This challenging six-week residential program for high school students will run from June 18 to July 27, 2012.

The deadline for applications is May 1, but spaces fill as applications arrive.
For adequate consideration of your application, submit it some time before the end of April.

Applicants should work on each of the problems below. We are interested in seeing how you approach unfamiliar math problems, not whether you can find answers by searching through web sites or books, or by asking experts.

Please submit your own work on *all* of these problems.

For each problem, explore the situation (with calculations, tables, pictures, etc.), observe the patterns, make some guesses, test the truth of those conjectures, and describe the progress you have made. Where were you led by your experimenting?

Include your thoughts even though you may not have completely solved the problem.
If you've seen one of the problems before (e.g. in a class or online), please include a reference with your solution.

Write up your problem solutions on paper (one side only). They can be handwritten or typed. Then mail your completed Application Form and the solutions to these problems, to the following address.

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Electronic submissions are discouraged.

Note:
Each *Ross Program* course concentrates deeply on one subject, unlike the problems here.
This Problem Set is an attempt to assess your general mathematical background and interests.

(1) The letters $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ represent seven positive whole numbers. The letters $b_1, b_2, b_3, b_4, b_5, b_6, b_7$ represent the same numbers but in a different order. Will the product

$$(a_1 - b_1)(a_2 - b_2)(a_3 - b_3)(a_4 - b_4)(a_5 - b_5)(a_6 - b_6)(a_7 - b_7)$$

always be an even number? Explain your conclusion.

(Note: A number is "even" if it equals $2n$ for some integer n. For example, -2, 0, 2, and 4 are even.)

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(2) Call a number "nice" if it can be expressed as a sum of two or more consecutive positive integers. For example, 5 and 6 are nice numbers because 5 = 2+3 and 6 = 1+2+3.

(a) Which numbers from 1 to 50 are not nice? What's the pattern for sizes beyond 50?

(b) *Explain* why the pattern you observed holds true generally.

(c) The number 15 is nice in several ways: 15 = 1+2+3+4+5 = 4+5+6 = 7+8.

List all the ways that 1000 can be expressed as a sum of consecutive positive integers.

- Given a number *n*, is there a simple method to count the number of ways *n* is nice? Explain.

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(3) A set of numbers has "the triple-sum property" (or TSP) if there exist three numbers in the set whose sum is also in the set. [Repetitions are allowed.]

For example, the set $U = \{2, 3, 7\}$ has TSP since $2 + 2 + 3 = 7$, while $V = \{2, 3, 10\}$ fails to have TSP.

(a) Suppose the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is separated into two parts, forming two subsets A and B.

<u>Prove</u>: Either A or B must have the triple-sum property.

[To begin, suppose that statement false, so there are sets A and B as above, each without TSP. If 1 lies in A then $3 = 1 + 1 + 1$ must be in B. Continue until you find an impossibility.]

(b) Is a similar result true when the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} is separated into two parts?

(4) Suppose 64 dots are arranged in a square 8×8 array, and each dot is colored red or blue.

(a) Prove that this array must contain a "monochromatic" rectangle. That is, no matter how the red and blue colors are assigned, there must be either a set of four red dots that form a rectangle or else a set of four blue dots that form a rectangle.

[Consider just the four corner points here, not the colors of dots inside that rectangle. Restrict attention to those rectangles with horizontal and vertical sides.]

(b) Does this result remain true for smaller rectangular arrays of dots? To begin, find a 4×5 array that admits no monochromatic rectangle. Must a monochromatic rectangle exist in a 5×5 array? In a 4×6 array?

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(5) My calculator has the usual keys for arithmetic and for "memory". After I dropped it in the bathtub and now several keys (like $\times$ and $\div$) don't work at all. The only keys that work are:

numbers keys, parentheses keys, recall memory keys, and the following three operation keys:

$+$ (plus key), $-$ (minus key), and x^{-1} (reciprocal key).

- Can you use that broken calculator to compute products?

That is, if A and B are stored in calculator memories, can the product AB be computed using only those few working keys? Stated in another way:

Is there a **formula** for AB in terms of A, B, parentheses, and those three operations?

Justify your answer.

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(6) A "lattice point" has integer coordinates. That is, $A = (m, n)$ is a lattice point when both m and n are integers. Let's call a point $P = (x, y)$ "generic" if all the distances from P to lattice points are different.

With some algebraic work, I checked that the point $S = (\sqrt{2}, \sqrt{3})$ is generic.

However, the point $T = (0, \pi)$ is not generic because it is equally distant from the lattice points $(1, 0)$ and $(-1, 0)$.

★ Is there some generic point with rational coordinates?

That is, if $Q = (r, s)$ for rational numbers r and s, must there exist two lattice points equidistant from Q?

As a first step, check that $R = \left(\frac{3}{4}, \frac{2}{5}\right)$ is not generic. (Find lattice points A, B equidistant from R.)

Can you use those ideas to answer the general question?

(7) Andy and Bob play a game using a long straight row of squares, alternating turns. When it's Andy's turn, he writes an A in one of the blank squares. When Bob takes a turn, he writes a B in some blank square. (Once a letter is written in a square, neither player can use that square again.)

A player wins the game when his initial is written in 4 equally-spaced squares. For example, suppose the following board is the result of several turns:

		B	B		A	B	A				A	B			A
									^						

Andy can win by writing A in the indicated square. (Four A's with spacing 2)
Bob can win by writing B in that same square. (Four B's with spacing 3)

If Andy goes first, find a strategy Andy can use that guarantees that he wins.
How many moves must Andy make to get 4 in a row, no matter what moves Bob makes? (Can Andy always win in just 4 moves?) Justify your answer.

- How many squares are needed in the game board to allow Andy's strategy to work?

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(8) What numbers can be expressed as an alternating-sum of an increasing sequence of powers of 2 ?
To form such a sum, choose a subset of the sequence 1, 2, 4, 8, 16, 32, 64, . . . (these are the powers of 2). List the numbers in that subset in increasing order (no repetitions allowed), and combine them with *alternating* plus and minus signs. For example,

$1 = -1 + 2$; $2 = -2 + 4$; $3 = 1 - 2 + 4$;
$4 = -4 + 8$; $5 = 1 - 4 + 8$; $6 = -2 + 8$; etc.

[But $5 = -1 - 2 + 8$ is not a valid expression here because the signs are not alternating.]

(a) Is every positive integer expressible in this fashion? If so, give a convincing proof.

(b) A number might have more than one expression of this type. For instance

$$3 = 1 - 2 + 4 \text{ and } 3 = -1 + 4.$$

Given a number n, how many different ways are there to write n in this way?
Explain why your answer is correct.

(9) Which of the problems here did you enjoy the most? Why?

We hope you enjoyed working on these problems!

Information about this summer mathematics program is available on the web at www.math.ohio-state.edu/ross or by email at ross@math.ohio-state.edu .