Knots and Graphs

Working Group [Summer 2007]

Instructor: ***Sergei Chmutov***

PROPOSAL

Background.

The proposed research projects involve knot theory. The central goal of that subject is a mathematical study of knots (the type of knots that can be physically tied in ropes). In the past twenty years the theory of knots, links and braids has grown tremendously. Surprisingly simple mathematical tools have been used to make great progress on many old questions.

The proposed project is a continuation of the working groups funded by the OSU Math Department's NSF-VIGRE grant during the past two summers. Professor Sergei Chmutov was the faculty advisor for these research projects.

- *Knots and Graphs*, Summer 2006,
  `http://www.math.ohio-state.edu/~chmutov/wor-gr-su06/wor-gr.htm`
- *Knot theory and Combinatorics*, Summer 2005,
  `http://www.math.ohio-state.edu/~chmutov/wor-gr-su05/wor-gr.htm`

In those projects undergraduate and beginning graduate students were involved in research projects in topology, combinatorics, and their mutual interaction. This year we propose to extend the previous projects and to initiate new projects in the same mathematical areas.

The most significant results obtained in those projects are reported in three research papers:

- M. Khoury, A. Rossi, *Polyak-Viro formulas for the coefficients of the Conway polynomial.* In preparation.
- S. Chmutov, J. Voltz, *Thistlethwaite's theorem for virtual links.* In preparation.
- M. Chmutov, S. Chmutov, Y. Rong, *Knight move in chromatic cohomology.* Accepted for publication in the *European Journal of Combinatorics.*

Undergraduate students presented their mathematical work at research conferences reporting their work on these projects. Those reports include:

1. M. Khoury, A. Rossi, *Polyak-Viro formulas for the coefficients of the Conway polynomial.* American Mathematical Society sectional Meeting #1025. Miami University, Oxford, OH. March 17, 2007.
2. J. Sharpnack, *Hamiltonicity of $\langle 2, 4, t\rangle$ Cayley Graphs.* The 2006 Shenandoah Undergraduate Mathematics and Statistics Conference. James Madison University, Harrisonburg, Virginia. October 28, 2006.

3. J. Wiser, *The Hamiltonicity of Cayley Graphs.* The 2006 Shenandoah Undergraduate Mathematics and Statistics Conference. James Madison University, Harrisonburg, Virginia. October 28, 2006.

4. A. Rossi, *An algorithm for producing arrow diagram formulas for the coefficients of the Conway polynomial.* The 2006 Shenandoah Undergraduate Mathematics and Statistics Conference. James Madison University, Harrisonburg, Virginia. October 28, 2006.

5. J. Voltz, *Thistlethwaite's theorem for virtual links.* The 2006 Shenandoah Undergraduate Mathematics and Statistics Conference. James Madison University, Harrisonburg, Virginia. October 28, 2006.

6. J. Voltz, *Thistlethwaite's theorem for virtual links.* The Young Mathematicians Conference. The Ohio State University, Columbus, OH. August 4, 2006.

7. M. Chmutov, *Categorifications.* Hudson River Undergraduate Mathematics Conference. Westfield State College, Westfield, Massachusetts. April 8, 2006.

8. M. Chmutov, *Torsion in the chromatic cohomology of graphs.* Knots in Washington XXI. George Washington University, Washington, DC. December 9, 2005.

9. M. Chmutov, *Ranks of chromatic graph homologies.* American Mathematical Society sectional Meeting #1009. Bard College. Annandale-on-Hudson, NY. October 8, 2005.

10. M. Chmutov, *The Knight Move in Chromatic Graph Cohomology.* The Young Mathematicians Conference. The Ohio State University, Columbus, OH. August 13, 2006.

The work of M. Khoury, A. Rossi, J. Voltz, and S. Ault from Summer 2006 was recognized by leading knot theorists such as L. Kauffman (University of Illinois at Chicago), M. Polyak (Technion - Israel Institute of Technology), O. Viro (Uppsala University, Sweden), N. Stoltzfus (Louisiana State University).

Proposed projects

We briefly describe seven projects for 2007. Since one to two students should work on each project, the optimal size of the entire group is 10 to 15 students. The whole group will meet twice a week to discuss general theories and techniques. The instructor will meet with each small research group at least one more time every week.

Project 1. ***Virtual links.***

This continues the work of M. Khoury, A. Rossi, and J. Voltz from the last summer. The aim is to find new invariants of virtual links based on [ChVo, KR]. In [KR], M. Khoury and A. Rossi found two expressions for each coefficient of the Conway polynomial of a knot. For a classical knot, these expressions are equal, but we expect that similar expressions can be defined for virtual knots and in this case they would be different. This would lead to new invariants of virtual knots which vanish on

classical knots. We would like to to investigate this situation in detail and obtain new virtual invariants if possible. References [Ka3, CP] contain further information about virtual links.

Project 2. ***Parallel connection of ribbon graphs.***

This project was considered last summer but still remains unsolved.

The *tensor product* $G_1 \otimes G_2$ of two (ribbon) graphs G_1 and G_2 is defined as the result of replacing each edge of G_1 by a copy of the graph G_2. There is a known formula for the Tutte polynomial of $G_1 \otimes G_2$ in terms of the Tutte polynomials of G_1 and G_2, as described in [Hug, JVW, Wo]. This project aims to generalize that formula to the the Bollobás-Riordan polynomial of ribbon graphs. Some special cases of such formulas for ribbon graphs were found in [Mof, p.8-9].

New references [Br, DHH] concerning this problem may provide some hints toward a solution. This project is also related to project 6 concerning the Khovanov homology of links. See [Wa].

Project 3. ***Strong maps of matroids.***

This is also a continuation of a project from last summer.

For each ribbon graph, regarded as a graph embedded in a surface, there is a dual ribbon graph, embedded in the same surface. The matroids of these two graphs form a so-called *strong map* of matroids. For such a strong map, the Tutte polynomials has been generalized to a polynomial in three variables [LV1, LV2]. We propose a systematic investigation of the relations of this generalized Tutte polynomial with the Bollobás-Riordan polynomial of the initial ribbon graph. For an introduction to matroids see [Ox, Wh].

In a related direction, we plan consider a newly discovered generalized duality of ribbon graphs with respect to a subset of edges. Is there any strong map of the corresponding matroids? How are the generalized Tutte polynomial and the Bollobás-Riordan polynomial related in this case?

Project 4. ***Reconstruction of ribbon graphs***

This problem was posted by Professor N. Robertson as an interesting open question.

The *Reconstruction Conjecture* in graph theory says that graphs are determined uniquely by the list of their proper subgraphs. We plan to investigate this famous conjecture in the case of ribbon graphs. Even those ribbon graphs with one vertex provide an interesting starting point that is not yet resolved. The relevant references are [Ke, McK, NW].

Project 5. ***Lie algebra generalization of the Bollobás-Riordan polynomial.***

In [BR2, BR3], one motivation for introducing the Bollobás-Riordan polynomial of a ribbon graph was to find new *weight systems* for finite type knot invariants. The Bollobás-Riordan polynomial arose from consideration of the weight system for the Lie algebra $\mathfrak{gl}(N)$. However there are weight systems coming from other semisimple Lie algebras. The aim of this project is to generalize the Bollobás-Riordan polynomial by using other semisimple Lie algebras. As a first step we propose to do this for the Lie algebra $\mathfrak{so}(N)$.

Project 6. ***Khovanov and Khovanov-Rozansky homology of links.***

Several years ago M. Khovanov [Kho] discovered a homology theory for links whose graded Euler characteristic equals the Jones polynomial. More precisely, given a link L with Jones polynomial $J_L(q)$, he defined a sequence of vector spaces $\mathcal{H}^{i,j}(L)$ such that

$$J_L(q) = \sum_{i,j} (-1)^i q^j \dim(\mathcal{H}^{i,j}(L)) \ .$$

These ideas are explained in [BN] and [Vi]. This Khovanov homology provides a *categorification* of the Jones polynomial of the link. This construction was generalized by Khovanov and Rozansky to a categorification of the more general HOMFLY polynomial. The aim of this project is to figure how this specialization works on the level of categorification.

Project 7. ***Categorification of graph polynomials.***

The chromatic polynomial, the Tutte polynomial, and the Bollobás-Riordan polynomial of graphs were categorified by Y. Rong and his students. We propose to find categorifications of other generalizations of the chromatic polynomial, such as the chromatic symmetric function defined in [MMW, St].

References

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