SAMPLE EXAM SOLUTIONS

COSMIN ROMAN

1. AUTUMN 2000

(1) (a) $g(x) = 0 \Rightarrow 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$ so the domain of $\frac{f(x)}{g(x)}$ is all real numbers except $\frac{3}{2}$

(b) $f(4) - g(4) = (4^2 + 4) - (2 \cdot 4 - 3) = 20 - 5 = 15$

(c) $(f \circ g)(x) = f(g(x)) = f(2x-3) = (2x-3)^2+4 = 4x^2-12x+9+4 = 4x^2-12x+13$ (the idea here is to replace first the variable with the formula of g, and then replace in the formula of f all x with that g)

(d) $f(x)+g(x) = 0 \iff x^2+4+2x-3 = 0 \iff x^2+2x+1 = 0 \iff (x+1)^2 = 0 \iff x = -1$

(2) **A**s with all inequalities, best approach is to bring everything one side:

$$x^2 + 3x - 10 > 0$$

$$(x-2)(x+5) > 0$$

(we can either notice the above decomposition, or we get it using quadratic formula)

draw a table:

x	$-\infty$	-5		2	∞
(x-2)	$-$	-7	$-$	0	$+$
(x+5)	$-$	0	$+$	7	$+$
(x+2)(x+4)	$+$	0	$-$	0	$+$

conclusion: the left hand side is strictly positive on the domain:

$$(-\infty, -5) \cup (2, \infty)$$

(3) **S**ince $f(3) = 0$ it means that $x - 3$ divides $f = x^3 - 13x + 12$, so let's divide, using synthetic division, f by $x - 3$:

	1	0	-13	12
		3	9	-12
3 |	1	3	-4	0

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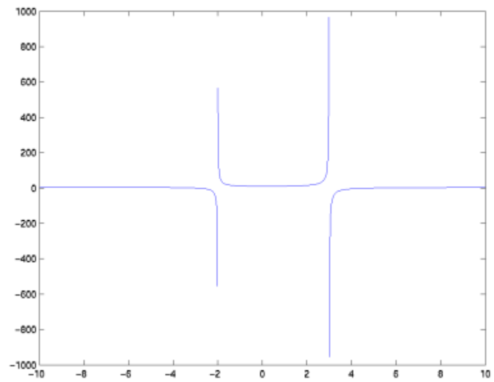


FIGURE 1. this is how graph in problem 4 should look like

Hence $f(x) \div (x-2) = x^2+3x-4$. Either we notice that $x^2-2x-8 = (x-1)(x+4)$ or we use quadratic formula:

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot (-4)}}{2} = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm \sqrt{25}}{2} =$$

$$= \frac{-3 \pm 5}{2}$$

hence we produce 1 and -4, which tell us too that we have $x^2 - 2x - 8 = (x-1)(x-(-4)) = (x-1)(x+4)$, and so we get that:

$$x^3 - 13x + 12 = (x-3)(x-1)(x+4)$$

(don't forget that $x - 3$ factor!!)

(4) (a) $x + 2 = 0 \Rightarrow x = -2$; $x - 3 = 0 \Rightarrow x = 3$
(b) same degree top and bottom (2) hence we do have a horizontal asymptote, at $y = 4$ (coefficients of x otherwise are 1)
(c) x-intercepts stand for zeroes of the function, so they are $x = 5$ and $x = -3$; y-intercept you obtain for $x = 0$: $y = \frac{4(-5)(3)}{(2)(-3)} = 10$
(d)

(5) **H**ere are the correspondences:
- $1^{st} \to$ B (the asymptotes are matched for B and F, but $f(0) < 0$, so F is out)
- $2^{nd} \to$ A (again, asymptotes are matched for A and E, but only A is decreasing)
- $3^{rd} \to$ F (B was already used - and $f(0) = 4$)
- $4^{th} \to$ D (is the only one with both asympototes at -2 ... also, $y < 0$ when $x < -2$)
- $5^{th} \to$ E (A is used ... and f is indeed increasing)

(6) **B**ring both logs in a single side:

$$1 = \log(x) + \log(x+3)$$
$$1 = \log(x(x+3))$$
$$10^1 = 10^{\log(x(x+3))}$$
$$10 = x(x+3) = x^2 + 3x$$
$$0 = x^2 + 3x - 10 = (x-2)(x+5)$$

Hence we get $x = 2$ and $x = -5$; notice, though, that $x = -5$ is not usable, since trying to plug it back into the equation gives us $\log(-5)$ which does not exist - on the other hand, $x = 2$ has no problems. That's our solution, hence.

(7) (a) **W**ell ... sketch it - use your calculator for hints; intercepts: $x = 2$ cannot be used, so no y-intercept, but $2\ln(x)+2 = 0$ has a solution, namely $\ln(x) = -1 \Rightarrow x = e^{-1}$; asymptote - only vertical, the y-axis

(b) the inverse function:

$$2\ln(x) + 2 = y$$
$$\ln(x) = \frac{y-2}{2}$$
$$x = e^{\frac{y-2}{2}}$$

and switch x with y

(8) (a) $\cos(\varphi) < 0$ and $\sin(\varphi) < 0$ means third quadrant

(b) use your calculator and produce $\cos^{-1}(.2121) = 1.7845$; but this is just one of the reference angles; the other one (here it's the cosine case) is -1.7845
by checking, we see that the first reference angle is in the second quadrant, so is not our solution; the second one must be then the solution then.
finally, since we have to get all angles, it's just a matter of adding a multiple of 2π: $-1.7845 + 2\pi \cdot k$, where k is integer (can be negative)

(9) **L**eft hand side: nothing to be done

Right hand side: since we want to have sines, change the cosine into one

$$\frac{\cos^2(y)}{1+\sin(y)} = \frac{1-\sin^2(y)}{1+\sin(y)} =$$
$$= \frac{(1-\sin(y))(1+\sin(y))}{1+\sin(y)} = 1 - \sin(y)$$

They are the same!

(10)
- $A = 3$
- $B = \frac{2\pi}{\frac{\pi}{2}} = 4$

- shift=1

and so we have $3\sin(4(x+1)) = 6\sin(4x+4)$. So, lastly, we get $C = 4$.

(11) **F**ormula for $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. We need, hence, besides $\cos(\alpha)$ and $\sin(\beta)$, the other two; since we're talking about acute angles (first quadrant) and we know that $\sin^2(x) + \cos^2(x) = 1$ we get:

$$\sin(\alpha) = \sqrt{1-\cos^2(\alpha)} = \sqrt{1-\frac{9}{16}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$
$$\cos(\beta) = \sqrt{1-\sin^2(\beta)} = \sqrt{1-\frac{4}{25}} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}$$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \\ &= \frac{3}{4}\cdot\frac{\sqrt{21}}{5} - \frac{\sqrt{7}}{4}\cdot\frac{2}{5} = \\ &= \frac{3\sqrt{21}-2\sqrt{7}}{20}\end{aligned}$$

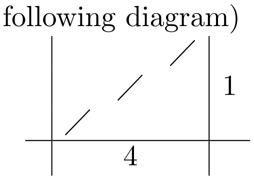
(12) (a) **W**e do know the formula for sine of sum:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

let's use it.

$$\begin{aligned}\sin(2x) &= \sin(x+x) = \sin(x)\cos(x) + \cos(x)\sin(x) = \sin(x)\cos(x) + \sin(x)\cos(x) = \\ &= 2\sin(x)\cos(x)\end{aligned}$$

(b) we know $\tan(\theta)$ but for $\sin(2\theta)$ we need sin and cos of θ. Construct the reference triangle (some hint is in the following diagram)



The hypothenusis is then, by Pythagora's theorem: $\sqrt{1^2+4^2} = \sqrt{17}$ and so we have:

$$\sin(\theta) = \frac{1}{\sqrt{17}}$$

and

$$\cos(\theta) = \frac{4}{\sqrt{17}}$$

In conclusion

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\frac{1}{\sqrt{17}}\cdot\frac{4}{\sqrt{17}} = \frac{8}{17}$$

(13) **L**et's find, first of all, the angles which make cos equal $\frac{1}{2}$: we have first reference angle to be 60; since we have a cosine, the second reference angle is -60. They will build for us two sequences:

- $\ldots, 60-360, 60, 60+360, 60+720, \ldots$
- $\ldots, -60-360, -60, -60+360, -60+720, \ldots$

But these are 2φ, so to get φ divide all elements there by 2:

- $\ldots, 30-180, 30, 30+180, 30+360, \ldots$
- $\ldots, -30-180, -30, -30+180, -30+360, \ldots$

Check now the angles there - we're supposed to have the angles within $[0, 360)$; we get as solutions: 30, 210, 150 and 330 (others are either negative, or bigger than 360)

(14)

$$\tan(x) = 2\sin(x)$$
$$\frac{\sin(x)}{\cos(x)} = 2\sin(x)$$

don't be tempted to divide by sin here ... sin can be zero, and this will give you division by zero . not allowed!

bring both terms in one side:

$$\frac{\sin(x)}{\cos(x)} - 2\sin(x) = 0$$
$$\sin(x)(\frac{1}{\cos(x)} - 2) = 0$$

implies either $\sin(x) = 0$ or $\frac{1}{\cos(x)} = 2 \iff \cos(x) = \frac{1}{2}$. Since we have such nice numbers, and we only consider the interval $[0, 2\pi)$, check on the unit circle.

- $\sin(x) = 0 \Rightarrow x = 0$ or $x = \pi$
- $\cos(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$; for the second angle we have to go around and get to the position of $-\frac{\pi}{3}$ (which we are not alowed to use, since it's negative), but we see it's the place of $2\pi - \frac{\pi}{3}$.

(15) (a) $x = r\cos(\theta) = 7\cos(\frac{2\pi}{3}) = 7 \cdot -\frac{1}{2} = -\frac{7}{2}$ $y = r\sin(\theta) = 7\sin(\frac{2\pi}{3}) = 7 \cdot \frac{\sqrt{3}}{2} = \frac{7\sqrt{3}}{2}$ so our point is $(-\frac{7}{2}, \frac{7\sqrt{3}}{2})$

(b) the goal is to get rid of r and θ - the sin is just itching to become a y, so let's multiply by r both sides the equality:

$$r^2 = -2r\sin(\theta) = -2y$$

but $r = \sqrt{x^2 + y^2} \iff r^2 = x^2 + y^2$ and so we have:

$$x^2 + y^2 = -2y$$

(c) just change x and y into their respective polar forms:

$$r\sin(\theta) = r^2\cos^2(\theta)$$

2. **spring 2001**

(1) (a) $g(x) = 0 \Rightarrow 7x - 2 = 0 \Rightarrow x = \frac{2}{7}$ so the domain of $\frac{f(x)}{g(x)}$ is all real numbers except $\frac{2}{7}$ (since we see that both $f(x)$ and $g(x)$ are defined everywhere - if, say, f were itself a fraction, or a square root, there would be more points taken out)
(b) $f(4) - g(4) = (4^2 - 3) - (7 \cdot 4 - 2) = 13 - 26 = -13$
(c) $(f \circ g)(x) = f(g(x)) = f(7x - 2) = (7x - 2)^2 - 3$ (the idea here is to replace first the variable with the formula of g, and then replace in the formula of f all x with that g)
(d) $g(x) = y \iff 7x - 2 = y$; solve for x: $x = \frac{y+2}{7}$

(2) **S**ince $f(2) = 0$ it means that $x - 2$ divides $f = x^3 - 4x^2 - 4x + 16$, so let's divide, using synthetic division, f by $x - 2$:

$$\begin{array}{r|rrrr} & 1 & -4 & -4 & 16 \\ & & 2 & -4 & -16 \\ \hline 2 & 1 & -2 & -8 & 0 \end{array}$$

Hence $f(x) \div (x-2) = x^2 - 2x - 8$. Either we notice that $x^2 - 2x - 8 = (x-4)(x+2)$ or we use quadratic formula:

$$x = \frac{2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-8)}}{2} = \frac{2 \pm \sqrt{4 + 32}}{2} = \frac{2 \pm \sqrt{36}}{2} =$$
$$= \frac{2 \pm 6}{2}$$

hence we produce 4 and -2, which tell us too that we have $x^2 - 2x - 8 = (x - 4)(x - (-2)) = (x - 4)(x + 2)$, and so we get that:

$$x^3 - 4x^2 - 4x + 16 = (x - 2)(x - 4)(x + 2)$$

(don't forget that $x - 2$ factor!!)

(3) **A**s with all inequalities, best approach is to bring everything one side:

$$x^2 + 8 + 6x > 0$$

$$x^2 + 6x + 8 > 0$$

$$(x + 2)(x + 4) > 0$$

(same comment as for the previous problem - we can either notice the above decomposition, or we get it using quadratic formula)

draw a table:

x	$-\infty$	-4		-2	∞
(x+2)	$-$	-2	$-$	0	$+$
(x+4)	$-$	0	$+$	2	$+$
(x+2)(x+4)	$+$	0	$-$	0	$+$

conclusion: the left hand side is strictly positive on the domain:

$$(-\infty, -4) \cup (-2, \infty)$$

(4) **H**ere are the correspondences:
- $1^{st} \to$ C (since the graph is upside-down)
- $2^{nd} \to$ A (the graph looks squeezed, so it must be divided by something)
- $3^{rd} \to$ D (the graph is shifted up)
- $4^{th} \to$ E (upside down AND shifted)
- $5^{th} \to$ F (shifted right, hence we SUBSTRACT inside the function)

(5) **E**h, not much information from that graph (if it happens so in the exam ask for help, OK?)

I see the following:
- horizontal asymptote at $y = 3$
- vertical asymptote at $x = -2$ (???? - there seems to be a full interval missing there)
- y-intercept for $y = -3$
- x-intercept for $x = 3$

start with the simplest components:
- vertical asymptote means
$$\frac{1}{x+2}$$
- x-intercept means
$$\frac{x-3}{x+2}$$
- y-intercept requires some fine tuning of the above formula - we need to multiply it by some value, say M, and plug in 0 for x; expect to get -3
$$M \cdot \frac{-3}{2} = -3$$
$$M = 2$$
we get
$$2\frac{x-3}{x+2}$$

at this point graph it on your calculator ... it should look right

the only thing that is disturbing here is the fact that the horizontal asymptote doesn't match! but since the graph isn't very accurate I guess we can get away with that by replacing the 3 with a 2 above in our assessing the function's properties (I'll check if that's allowed)

(6) (a) **W**ell ... sketch it - use your calculator
(b) best thing is actually finding the inverse function:

$$\log(x+5) + 2 = y \iff \log(x+5) = y - 2 \iff x + 5 = 10^{y-2}$$
$$x = 10^{y-2} - 5$$

and now replacing x with y and viceversa we get

$$y = 10^{x-2} - 5$$

let's look to see where the function is defined: $x - 2 > 0 \Rightarrow x > 2$, so the first and third alternatives fall; also the second, since it goes in the negative domain. Conclusion: the fourth is the right one.

(7) **F**ormula for $\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$

we see that we need $\sin(\theta)$ and $\cos(\varphi)$. Acute angles means that both those values should be positive (acute = first quadrant), and we get them from the fact that $\sin^2(\alpha) + \cos^2(\alpha) = 1$.

$$\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{1 - \frac{9}{16}} = \sqrt{\frac{16-9}{16}} = \frac{\sqrt{7}}{4}$$

$$\cos(\varphi) = \sqrt{1 - \sin^2(\varphi)} = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{9-1}{9}} = \frac{\sqrt{8}}{3}$$

Use the above formula now:

$$\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) =$$

$$= \frac{3}{4} \cdot \frac{\sqrt{8}}{3} - \frac{\sqrt{7}}{4} \cdot \frac{1}{3} =$$

$$= \frac{3\sqrt{8} - \sqrt{7}}{12}$$

(8) (a) $\cos(\varphi) > 0$ and $\sin(\varphi) < 0$ means fourth quadrant

(b) use your calculator and produce $\cos^{-1}(.2626) = 1.3050$; but this is just one of the reference angles; the other one (here it's the cosine case) is -1.3050
by checking, we see that the first reference angle is in the first quadrant, so is not our solution; the second one must be then the solution then.
finally, since we have to get all angles, it's just a matter of adding a multiple of 2π: $-1.3050 + 2\pi \cdot k$, where k is integer (can be negative)

(9) (a) **R**ewrite the function a bit: $2\sin(\pi x - \frac{\pi}{3}) = 2\sin(\pi(x - \frac{1}{3}))$; hence
- amplitude=2
- period $=\frac{2\pi}{\pi} = 2$
- shift$=-\frac{1}{3}$ and so phase shift$=\frac{1}{3}$.

(b) we get:
- $A = 6$
- $B = \frac{2\pi}{2} = \pi$
- shift=1

and so we have $6\sin(\pi(x + 1)) = 6\sin(\pi x + \pi)$. So, lastly, we get $C = \pi$.

(10) **L**eft hand side:

$$\frac{(1-\sin(x))(1+\sin(x))}{1-\cos^2(x)} =$$
$$= \frac{1-\sin^2(x)}{1-\cos^2(x)} =$$
$$= \frac{\cos^2(x)}{\sin^2(x)}$$

Right hand side:

$$\cot^2(x) = (\frac{\cos(x)}{\sin(x)})^2 =$$
$$= \frac{\cos^2(x)}{\sin^2(x)}$$

They are the same!

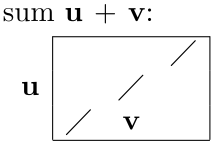
(11) (a) $|\mathbf{v}| = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$

(b) $2\mathbf{u} - \mathbf{v} = 2\langle -1, 2\rangle - \langle 3, -2\rangle = \langle -2, 4\rangle - \langle 3, -2\rangle = \langle -2-3, 4-(-2)\rangle = \langle -5, 6\rangle$

(c) unit vector which has same direction as **u** is

$$\frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{\sqrt{(-1)^2 + 2^2}}\langle -1, 2\rangle =$$
$$= \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$$

(12) **D**raw the two vectors and their sum, and you have something that looks like this where the dotted diagonal is the sum **u** + **v**:



The angle between **u** and the diagonal is, hence 22 degrees, and the diagonal's length is 32; let's view the whole thing in the upper right triangle formed there.

$$|\mathbf{u}| = 32\cos(22) = 32 \cdot 0.9271 = 29.6698$$

the upper side of the rectangle is actually **v**, so we also get

$$|\mathbf{v}| = 32\sin(22) = 32 \cdot 0.3746 = 11.9874$$

(13)

$$2\cos^2(\theta) = \sin(2\theta)$$
$$2\cos^2(\theta) = 2\sin(\theta)\cos(\theta)$$
$$2\cos^2(\theta) - 2\sin(\theta)\cos(\theta) = 0$$
$$2\cos(\theta)(\cos(\theta) - \sin(\theta)) = 0$$

which implies either $\cos(\theta) = 0$ or $\sin(\theta) = \cos(\theta)$. Since we're talking the interval $[0, \frac{\pi}{2}]$ - the first quadrant, that is - let's look at the unit circle: cos is 0 when $\theta = \frac{\pi}{2}$; sin=cos when $\theta = \frac{\pi}{4}$. Done.

(14) (a) $r = \sqrt{3^2 + 8^2} = \sqrt{9 + 64} = \sqrt{73} = 8.5440$ while $\tan(\theta) = \frac{8}{3}$ and so $\theta = \tan^{-1}(2.666) = 1.2120$; so the point in question is $(8.5440, 1.2120)$

(b) the best way to handle these transformation is to "complete" either the cos or sin to x or y, respectively. Hence, multiply both sides of the equation by r and get:

$$r^2 = 2r\cos(\theta)$$
$$r^2 = 2x$$

now use the fact that $r = \sqrt{x^2 + y^2}$ or, equivalently, $r^2 = x^2 + y^2$:

$$x^2 + y^2 = 2x$$

(c) $x = r\cos(theta)$ and $y = r\sin(theta)$; hence

$$r\cos(\theta) = r^2\sin^2(\theta)$$

(maybe simplify a r: $\cos(\theta) = r\sin^2(\theta)$)

(15) (a) $zw = (5+3i)(7-4i) = 35-20i+21i-12i^2 = 35+i-12\cdot(-1) = 35+12+i = 47+i$

(b) $r = \sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{1+3} = \sqrt{4} = 2$; $\tan(\theta) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$ and hence $\theta = -\frac{\pi}{3}$ (so, in fact, you can write the complex number as $2e^{-i\frac{\pi}{3}}$)

(c) $2e^{\frac{\pi}{6}i} = 2(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})) = 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \sqrt{3} + i$