SOLUTIONS CHAPTER 7.1

MATH 549 AU00

$\boxed{3}$

Proof. (a) Let's choose an arbitrary division of the interval [0,1], namely $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$. For this, let's compute the upper and lower Riemann sum. We distinguish to cases: there is an $a_i = \frac{1}{2}$, or there isn't.

Case I: $a_i = \frac{1}{2}$, for some i. On all intervals $[a_{k-1}, a_k]$ with $k \leq i$ the lowest and the highest value coincide, both being 0. On all intervals $[a_k, a_{k+1}]$ with $k > i$ the lowest and the highest values coincide again, both being 1. On the interval $[a_i, a_{i+1}] = [\frac{1}{2}, a_{i+1}]$ the lowest value is 0 (which you get in $a_i = \frac{1}{2}$) and the highest is 1, which you get everywhere but a_i. These being said, let's write down the two sums:

$$U(P_n; g) =$$

$$= 0*(a_1-a_0)+0*(a_2-a_1)+\ldots 0*(a_i-a_{i-1})+1*(a_{i+1}-a_i)+\ldots 1*(a_n-a_{n-1}) =$$

$$= 0 + a_{i+1} - a_i + a_{i+2} - a_{i+1} + \ldots a_n - a_{n-1} = a_n - a_i = 1 - \frac{1}{2} = \frac{1}{2}$$

(telescopic sum).

$$L(P_n; g) =$$

$$= 0*(a_1-a_0)+0*(a_2-a_1)+\ldots 0*(a_i-a_{i-1})+0*(a_{i+1}-a_i)+\ldots 1*(a_n-a_{n-1}) =$$

$$= 0 + a_{i+2} - a_{i+1} + a_{i+3} - a_{i+2} + \ldots a_n - a_{n-1} = a_n - a_{i+1} = 1 - a_{i+1}$$

Since the goal is to take the sup of all possible $1 - a_{i+1}$'s, with $a_{i+1} > \frac{1}{2}$, this is obviously gotten for a_{i+1} getting "closest" to $\frac{1}{2}$, hence we're talking about $1 - \frac{1}{2} = \frac{1}{2}$.

Hence for this case both upper and lower sum are equal to each other and to $\frac{1}{2}$.

Case II: $a_i < \frac{1}{2} < a_{i+1}$. On all intervals $[a_{k-1}, a_k]$ with $k \leq i$ the lowest and the highest value coincide, both being 0. On all intervals $[a_k, a_{k+1}]$ with $k > i+1$ the lowest and the highest values coincide again, both being 1. On the interval $[a_i, a_{i+1}]$ the lowest value is 0 (which you get in $[a_i, \frac{1}{2}]$) and the highest is 1 (which you get in $[\frac{1}{2}, a_{i+1}]$). These being said, let's write down the two sums:

$$U(P_n; g) =$$

$$= 0*(a_1-a_0)+0*(a_2-a_1)+\ldots 0*(a_i-a_{i-1})+1*(a_{i+1}-a_i)+\ldots 1*(a_n-a_{n-1}) =$$

$$= 0 + a_{i+1} - a_i + a_{i+2} - a_{i+1} + \ldots a_n - a_{n-1} = a_n - a_i = 1 - a_i$$

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$$L(P_n; g) =$$
$$= 0*(a_1-a_0)+0*(a_2-a_1)+\dots 0*(a_i-a_{i-1})+0*(a_{i+1}-a_i)+\dots 1*(a_n-a_{n-1}) =$$
$$= 0 + a_{i+2} - a_{i+1} + a_{i+3} - a_{i+2} + \dots a_n - a_{n-1} = a_n - a_{i+1} = 1 - a_{i+1}$$

Looking for the sup of all lower sums we get again

$$\sup_{a_{i+1}>\frac{1}{2}}(1 - a_{i+1}) = 1 - \frac{1}{2} = \frac{1}{2}$$

and for the inf of all upper sums we get

$$\inf_{a_i<\frac{1}{2}}(1 - a_i) = 1 - \frac{1}{2} = \frac{1}{2}$$

So in this case we get again equality for the upper and lower sums, and equality to $\frac{1}{2}$, which proves the problem.

(b)If you look at the upper and lower sums for the partitions above, observe that we can rewrite them easily.

Case I: The upper sum changes, namely the 7 will be highest value for both $[a_{i-1}, a_i]$ and $[a_i, a_{i+1}]$:

$$U(P_n; g) =$$
$$= 0 * (a_1 - a_0) + 0 * (a_2 - a_1) + \cdots +$$
$$+7 * (a_i - a_{i-1}) + 7 * (a_{i+1} - a_i)+$$
$$+1 * (a_{i+2} - a_{i+1}) + \dots 1 * (a_n - a_{n-1}) =$$
$$= 0 + 7a_i - 7a_{i-1} + 7a_{i+1} - 7a_i + a_{i+2} - a_{i+1} + \dots a_n - a_{n-1} =$$
$$= -7a_{i-1} + 7a_{i+1} + a_n - a_{i+1} = 1 + 6a_{i+1} - 7a_{i-1}$$

Compute now

$$\inf_{a_{i-1}<\frac{1}{2}<a_{i+1}}(1 + 6a_{i+1} - 7a_{i-1}) = 1 + 6 * \frac{1}{2} - 7 * \frac{1}{2} = \frac{1}{2}$$

(a_{i+1} should be smallest and a_{i-1} should be biggest).

I leave the computation for the lower sum, and for case II to you - follow the above method. The answer should be that the same conclusion holds, regardless of having $g(\frac{1}{2})$ equal to 7 or 0 (or, in fact, any other number). □

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Proof. The main idea here is that whenever we are given two numbers, $a < b$, there is always a $q \in \mathbb{Q}$ and $r \in \mathbb{R} \setminus \mathbb{Q}$ with $a < q, r < b$ (density of both rationals and irrationals on the real line). Hence, if we choose any partition of $[0, 1]$, the lowest value on each subinterval is going to be (always) 0 (because, again, there always exists an irrational number between any pair of non-coinciding numbers). So the lower sum is 0.

For the upper sum we have the following: given the partition $0 = a_0 < a_1 < \cdots < a_n = 1$, the sup on each subinterval $[a_i, a_{i+1}]$ is going to be a_{i+1} (might not be the highest value, since a_{i+1} could be irrational), so the upper sum for this partition is going to be

$$U(P_n, h) = a_1(a_1 - a_0) + a_2(a_2 - a_1) + \cdots + a_n(a_n - a_{n-1})$$

We're looking for the inf of it; notice that it's (always - that is, for any partition) the same upper sum as for the function $f(x) = x$, hence the inf for h coincides with the inf for $f(x)$, hence it's

$$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$$

Since the lower sum and the upper sum are different, it means h is NOT integrable.

□

$\boxed{5}$

Proof. Same is in the example 7.1.7(b), we have that $f(x) = x^3$ is increasing on $[0, 1]$, so for the partition $P_n := (0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} = 1)$ we have lowest value on the subinterval $[\frac{k-1}{n}, \frac{k}{n}]$ is $m_k = (\frac{k-1}{n})^3$ and the highest value is $M_k = (\frac{k}{n})^3$. Each subinterval's length is $\frac{1}{n}$, so we have:

$$L(P_n, f) = \frac{1}{n}(0^3 + \frac{1^3}{n^3} + \cdots + \frac{(n-1)^3}{n^3} + \frac{n^3}{n^3}) =$$

$$= \frac{0^3 + 2^3 + \cdots + (n-1)^3}{n^4} = \frac{(\frac{1}{2}(n-1)n)^2}{n^4} = \frac{1}{4}\frac{(n-1)^2}{n^2}$$

which has as a sup $\frac{1}{4}$ (the bigger n gets, the closer $\frac{(n-1)^2}{n^2} < 1$ gets to 1).

Same trick for the upper sum (the difference is that we don't start at 0^3 but at 1^3, and hence we stop at n^3 ... which in turn means that

$$U(P_n, f) = \frac{1}{4}\frac{n^2(n+1)^2}{n^4} = \frac{1}{4}\frac{(n+1)^2}{n^2}$$

and so inf of the above is $\frac{1}{4}$ (because the bigger n gets, the closer $\frac{(n+1)^2}{n^2} > 1$ is to 1).

The two sums are equal to $\frac{1}{4}$ ... so we're done.

□

$\boxed{9}$

Proof. Assume $f(x_0) \neq 0$ for some $a < x_0 < b$. Hence $f(x_0) > 0$ (because $f \geq 0$). But f is continuous, so for $\epsilon = \frac{f(x_0)}{2} > 0$ there exists $\delta > 0$ such that, whenever $|x - x_0| < \delta \iff x \in (x_0 - \delta, x_0 + \delta)$ we have $|f(x) - f(x_0)| < \epsilon \iff -\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2} \iff 0 < \frac{f(x_0)}{2} = f(x_0) - \frac{f(x_0)}{2} < f(x) < f(x_0) + \frac{f(x_0)}{2} \Rightarrow$

$$\Rightarrow f(x) > \frac{f(x_0)}{2}$$

(in other words, for a well-chosen neighbourhood of x_0 the values of f don't get too far from the nonzero $f(x_0)$).

Because f is continuous it means it's integrable, in particular it's integrable on $[x_0 - \delta, x_0 + \delta]$; together with the above fact we get that

$$\int_{x_0-\delta}^{x_0+\delta} f(x)\,dx > \frac{f(x_0)}{2} * ((x_0 + \delta) - (x_0 - \delta)) = \frac{f(x_0)}{2} * 2\delta =$$

$$= \delta f(x_0) > 0$$

Since this is only part of the whole integral, and since on $[a, x_0 - \delta]$ and on $[x_0 + \delta, b]$ the function is positive, we have that

$$\int_a^b f(x)\,dx = \int_a^{x_0-\delta} f(x)\,dx + \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx + \int_{x_0+\delta}^{b} f(x)\,dx >$$

$$> \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx > \delta f(x_0) > 0$$

hence NOT EQUAL to 0; contradiction!

So our assumption was wrong, therefore $f(x) = 0\ \forall x \in [a, b]$

□

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Proof. Let $P_n = (a = x_0 < x_1 < \cdots < x_n = b)$ be a partition for $[a, b]$. We have that

$$L(P_n, f_1 + f_2) =$$

$$= (x_1 - x_0) * [\inf_{x_0<x<x_1}(f_1(x) + f_2(x))] +$$

$$+ \cdots +$$

$$+ (x_n - x_{n-1}) * [\inf_{x_{n-1}<x<x_n}(f_1(x) + f_2(x))]$$

Based on Exercise 2.5.7 (check it out!) we have that $\inf_{x_i<x<x_{i+1}}(f_1(x) + f_2(x)) \geq \inf_{x_i<x<x_{i+1}}(f_1(x)) + \inf_{x_i<x<x_{i+1}}(f_2(x))$, so if we break each term into its two components we get (I'll only write the result, the intermediate computation is very easy, and I leave it as exercise)

$$L(P_n, f_1 + f_2) \geq L(P_n, f_1) + L(P_n, f_2)$$

Taking now the sup in the left hand side we get that, for any partition P_n, $L(f_1 + f_2) \geq L(P_n, f_1) + L(P_n, f_2)$

Take now two (different) partitions, P_n and Q_m. The combined partition R_p (the one that involves all points of P_n and all points of Q_m) is a refinement for both partitions, hence $L(R_p, f_1) \geq L(P_n, f_1)$ and $L(R_p, f_2) \geq L(Q_m, f_2) \Rightarrow L(R_p, f_1) + L(R_p, f_2) \geq L(P_n, f_1) + L(Q_m, f_2)$; but then $L(f_1 + f_2) \geq L(R_p, f_1) + L(R_p, f_2) \geq L(P_n, f_1) + L(Q_m, f_2)$. Taking now the sup separately for f_1 and for f_2 (which we just proved we can do - the previous inequality had the big disadvantage that it involved the VERY SAME partition for both funtions, which is a strong restriction!) we get that $L(f_1 + f_2) \geq L(f_1) + L(f_2)$. □

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Proof. Use Riemann's Criterion for Integrability, namely we start with an $\epsilon > 0$ and try to construct a partition P_ϵ that yields $U(P_\epsilon; f) - L(P_\epsilon; f) < \epsilon$ (it's obvious that f is bounded: the cosine is never bigger that 1 or less than -1).

We build the partition as follows: take the subinterval $[0, \frac{\epsilon}{4}]$ - this will be the first. Then, for the interval $[\frac{\epsilon}{4}, 1]$, since $f(x)$ is continuous, which implies it's integrable, we can always find a partition of $[\frac{\epsilon}{4}, 1]$, let's say $(\frac{\epsilon}{4} = x_0 < x_1 < \cdots < x_n = 1)$ on which the difference between the upper and the lower sum is less than $\frac{\epsilon}{2}$ (the criterion says "if and only if"!). Let now P_ϵ be the partition of $[0, 1]$ made out of $(0 < \frac{\epsilon}{4} < x_1 < x_2 < \cdots < x_n = 1)$. Computing the difference between the upper and the lower sum on this partition means to actually add the differences on the two instances: on the second part it's $< \frac{\epsilon}{2}$ by construction, and on the first (that is, on the first subinterval), since the highest value is 1 and lowest value is -1 (cosine!) we get that the difference is $< (1 - (-1)) * \frac{\epsilon}{4} = \frac{\epsilon}{2}$; adding these two yields $< \epsilon$, which is what we needed. Hence f is integrable.

□