SOLUTIONS CHAPTER 8.1

MATH 549 AU00

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Proof. Fix $x \neq 0$; then

$$\lim_{n\to\infty} \frac{x}{x+n} = \lim_{n\to\infty} \frac{1}{1+\frac{n}{x}} = \text{"}\frac{1}{\infty}\text{"} = 0$$

and when $x = 0$ then $\frac{x}{x+n} = \frac{0}{n} = 0$ regardless of n. Done. □

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Proof. Again fix $x \neq 0$, and we get:

$$\lim_{n\to\infty} \frac{nx}{1+nx} = \lim_{n\to\infty} \frac{1}{\frac{1}{nx}+1} = \frac{1}{0+1} = 1$$

and when $x = 0$ then $\frac{nx}{1+nx} = \frac{0}{1+0} = 0$. Hence the limit is $f(x) = 1$ when $x \neq 0$ and $f(0) = 0$. □

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Proof. We consider several cases:

$x = 0$: $\frac{0}{1+0} = 0$

$0 < x < 1$: $\lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0$

$x = 1$: $\frac{1}{1+1} = \frac{1}{2}$

$x > 1$: $\lim_{n\to\infty} \frac{x^n}{1+x^n} = \lim_{n\to\infty} \frac{1}{\frac{1}{x^n}+1} = \frac{1}{0+1} = 1$

hence the limit is as follows: $f(x) = 0$ for $0 \leq x < 1$; $f(1) = \frac{1}{2}$; $f(x) = 1$ for $x > 1$ (remember that $x^n \to 0$ when $0 < x < 1$ and $x^n \to \infty$ when $x > 1$, as $n \to \infty$). □

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Proof. Let $x = 0 \Rightarrow e^{-nx} = e^0 = 1$. Let now $x \neq 0 \Rightarrow nx \to_{n\to\infty} \infty \Rightarrow -nx \to -\infty \Rightarrow e^{-nx} \to_{n\to\infty}$ "$e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty}$" $= 0$. Hence limit is $f(0) = 1$ and $f(x) = 0$ for $x > 0$. □

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Proof. Take the bounded interval $[0,a]$; to show that $\frac{x}{x+n}$ converges uniformly to 0 on it we need to show that $\sup_{x\in[0,a]}(|\frac{x}{x+n}-0|)$ converges to 0 (that is, the uniform norm on $[0,a]$ of the sequence does that) - hence we need to find the sup (or in this case, since the interval is bounded, the max!) for $\frac{x}{x+n}=\frac{1}{1+\frac{n}{x}}$; but this function is increasing (why? because the denominator is decreasing, hence the fraction is doing the reverse thing, namely increases), so max you get in a, and it's equal to $\frac{a}{a+n}$. Hence the norm of $f_n - f = \frac{x}{x+n} - 0$ is $\leq \frac{a}{a+n} \to_{n\to\infty} 0$ (since $a \neq 0$); so the sequence converges uniformly to 0 on $[0,a]$.

Take now the unbounded interval $[0,\infty)$. In some sense it should be natural to have non-uniform convergence, since each function increases quite a lot (up to 1, actually) as x approaches ∞ ... and uniform convergence means that the sequence must be pretty close to its limit (which in this case it's 0). The best way to show non-uniform convergence is to find for each n a value x_n such that $f_n(x_n)$ stays constant with respect to n - for example, in our case I'll choose to have $f_n(x_n)=\frac{1}{2}$ for all n (why $\frac{1}{2}$? because it's a value between 0 and 1, which is the range of each f_n ... and any fixed value in the range, except 0, of course, will do the trick; as an exercise, try other values, like $\frac{1}{3}$, or $\frac{2}{3}$ ... or try bigger values and show it's impossible, like for 2, or 3) $\iff \frac{x_n}{x_n+n}=\frac{1}{2} \iff 2x_n = x_n + n \iff x_n = n$ (note that x_n is not necessarily fixed! actually in general it will wander pretty much towards the bad region of the domain - in our case the ∞ side). So we have $f_n(n)=\frac{1}{2}$, fixed, impossible if the sequence were converging uniformly (as said, if n is big enough, then all values of f_n should be arbitrarily close to 0, for x all the way to ∞ ... $\frac{1}{2}$ isn't!) □

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Proof. Same idea as above: on the interval $[0,b]$ the function $|\frac{x^n}{1+x^n}-0| = \frac{1}{\frac{1}{x^n}+1}$ is increasing, so the max you get in $x=b$ and is equal to $\frac{b^n}{1+b^n} \to_{n\to\infty} 0 \Rightarrow$ since the uniform norm goes to 0, the sequence converges uniformly.

On $[0,1]$ things change again (why? look at the limit itself - has a jump from 0 to $\frac{1}{2}$!), so let's choose again a value in the range of each f_n, let's say $\frac{1}{3}$ (should be less than $\frac{1}{2}$, isn't it?) and we get: $\frac{x^n}{x^n+1}=\frac{1}{3} \iff 3x^n = x^n+1 \iff x^n=\frac{1}{2} \iff x=\sqrt[n]{\frac{1}{2}}$ -so we have that $f_n(\sqrt[n]{\frac{1}{2}})=\frac{1}{3}$, which again proves the non-uniformity of the convergence. □

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Proof. Take $[a,\infty)$; the function $|e^{-nx}-0|$ (0 is the limit) is decreasing, so the max value is in a, and is equal to $e^{-na} \to_{n\to\infty} 0$ (since $a \neq 0$). But this means that the uniform norm of $|f_n - f| = e^{-nx}$ converges to 0, hence the convergence is uniform.

Take now the interval $[0, \infty)$ ... this must go bad, since the limit has a jump in 0! - use same trick ,namely choose a value in the range of all f_ns, which happens to be $(0, 1]$, and show you can produce it for all n: say it's $\frac{1}{2}$; $e^{-nx} = \frac{1}{2} \Rightarrow -nx = \ln(\frac{1}{2}) = -\ln(2) \Rightarrow x = \frac{\ln(2)}{n}$. Done! ($f_n(\frac{\ln(2)}{n}) = \frac{1}{2}$ - bad for uniformity!). $\square$