1 Addendum

A Cantor-Bendixson-like process which detects $\mathbf{\Delta}^0_2$

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Abstract:

For each subset of Baire space, we define, in a way similar to a common proof of the Cantor-Bendixson Theorem, a sequence of decreasing subsets S_α of $\mathbb{N}^{<\mathbb{N}}$, indexed by ordinals. We use this to obtain two new characterizations of the boldface $\mathbf{\Delta}^0_2$ Borel pointclass. ADDENDUM: In January 2012 we learned that the notion of guessability appeared in an equivalent form, and even with the same name, in the doctoral dissertation of William Wadge [4]. As for the main result of this paper, Wadge proved one direction and gave a proof for the other direction which he attributed to Hausdorff. The proofs in this paper present an alternate means to those results.

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Please read the addendum in the above abstract for an important note on this paper's unoriginality.

The usual Cantor-Bendixson derivative "detects" countability, in the sense that the perfect kernel of $S \subseteq \mathbb{N}^{\mathbb{N}}$ (the result of applying the Cantor-Bendixson derivative repeatedly until a fixed point is reached) is empty if and only if S is countable ([3], page 34). In this paper, I will show a process which "detects" $\mathbf{\Delta}^0_2$: a process which depends on $S \subseteq \mathbb{N}^{\mathbb{N}}$ and which reaches a fixed point or kernel, a kernel which will be empty if and only if S is $\mathbf{\Delta}^0_2$.

Definition 1 *Suppose* $S \subseteq \mathbb{N}^{\mathbb{N}}$. *If* $X \subseteq \mathbb{N}^{<\mathbb{N}}$, *let* $[X]$ *denote the set of infinite sequences whose initial segments are all in* X.

- *Define* $S_\alpha \subseteq \mathbb{N}^{<\mathbb{N}}$ *for every ordinal* α *by induction as follows:* $S_0 = \mathbb{N}^{<\mathbb{N}}$, $S_\lambda = \cap_{\beta<\lambda} S_\beta$ *for any limit ordinal* λ. *And finally, for any ordinal* β *define*
$$S_{\beta+1} = \{x \in S_\beta \,:\, \exists x', x'' \in [S_\beta] \text{ such that } x \subseteq x',\ x \subseteq x'',\ x' \in S, \text{ and } x'' \notin S\}$$
- *Let* $\alpha(S)$ *be the minimal ordinal* α *such that* $S_\alpha = S_{\alpha+1}$.

- *Let* $S_\infty = S_{\alpha(S)}$ *(the* kernel *of the above process).*

Throughout the paper, S will denote a subset of $\mathbb{N}^\mathbb{N}$. If $f : \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$, I will use $f \restriction n$ to denote $(f(0), \ldots, f(n-1))$; $f \restriction 0$ will denote the empty sequence. My goal is to prove that the following are equivalent:

- S is $\mathbf{\Delta}^0_2$.
- $S_\infty = \emptyset$.
- $S = T \backslash [T_\infty]$ for some T.

The reader might wonder why I define S_α to lie in $\mathbb{N}^{<\mathbb{N}}$, rather than in $\mathbb{N}^\mathbb{N}$ as one might expect by extrapolating from the classical Cantor-Bendixson derivative. Why not define a new derivative

$$S^* = \{f \in S : f \text{ is a limit of points of } S\backslash\{f\} \text{ and also of } S^c\},$$

and then follow Cantor-Bendixson more directly? If we do this, we end up getting a kernel which does *not* detect $\mathbf{\Delta}^0_2$. For example, let $S = \{f : \forall n f(n) \neq 0\}$, a $\mathbf{\Delta}^0_2$ subset of $\mathbb{N}^\mathbb{N}$. It's easy to see $S^* = S$, whereas in order for our process to detect $\mathbf{\Delta}^0_2$, we would like for it to reduce S to $\emptyset$. The reader can check that $S_0 = \mathbb{N}^{<\mathbb{N}}$, S_1 is the set of finite sequences not containing 0, and $S_2 = \emptyset$.

Definition 2 *Say that* $S \subseteq \mathbb{N}^\mathbb{N}$ *is* guessable *if there is a function* $G : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ *such that for every* $f : \mathbb{N} \to \mathbb{N}$*,*

$$\lim_{n\to\infty} G(f \restriction n) = \begin{cases} 1, \text{ if } f \in S; \\ 0, \text{ otherwise.} \end{cases}$$

If so, we say G *is a* guesser *for* S*.*

Theorem 3 *A subset of* $\mathbb{N}^\mathbb{N}$ *is guessable if and only if it is* $\mathbf{\Delta}^0_2$*.*

This theorem is proved on page 11 of Alexander [1]. It is also a special case of the main theorem of Alexander [2].

Proposition 4 *Suppose* S *is* $\mathbf{\Delta}^0_2$*. Then* $S_\infty = \emptyset$*.*

Proof Contrapositively, suppose $S_\infty \neq \emptyset$. I will show S is non-guessable, hence non-$\mathbf{\Delta}^0_2$ by Theorem 3. Assume not, and let $G : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be a guesser for S. I will build a sequence on whose initial segments G diverges, contrary to Definition 2. There is some $\sigma_0 \in S_\infty$. Now inductively suppose I've defined finite sequences $\sigma_0 \subset_{\neq} \cdots \subset_{\neq} \sigma_k$ in S_∞ such that for $0 < i \leq k$, $G(\sigma_i) \equiv i \bmod 2$. Since $\sigma_k \in S_\infty = S_{\alpha(S)} = S_{\alpha(S)+1}$, this means there are $\sigma', \sigma'' \in [S_\infty]$, extending σ_k, with $\sigma' \in S$, $\sigma'' \notin S$. Choose $\sigma \in \{\sigma', \sigma''\}$ with $\sigma \in S$ iff k is even. Then $\lim_{n\to\infty} G(\sigma \upharpoonright n) \equiv k+1 \bmod 2$. Let $\sigma_{k+1} \subset \sigma$ properly extend σ_k such that $G(\sigma_{k+1}) \equiv k+1 \bmod 2$. Note $\sigma_{k+1} \in S_\infty$ since $\sigma \in [S_\infty]$.

By induction, I've defined $\sigma_0 \subset_{\neq} \sigma_1 \subset_{\neq} \cdots$ such that for $i > 0$, $G(\sigma_i) \equiv i \bmod 2$. This contradicts Definition 2 since $\lim_{n\to\infty} G((\cup_i \sigma_i) \upharpoonright n)$ ought to converge. □

For the converse we need more machinery.

Definition 5 *If* $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $\sigma \notin S_\infty$, *then let* $\beta(\sigma)$ *denote the least ordinal such that* $\sigma \notin S_{\beta(\sigma)}$.

Note that whenever $\sigma \notin S_\infty$, $\beta(\sigma)$ is a successor ordinal.

Lemma 6 *Suppose* $\sigma \subseteq \tau$ *are finite sequences. If* $\tau \in S_\infty$ *then* $\sigma \in S_\infty$. *And if* $\sigma \notin S_\infty$, *then* $\beta(\tau) \leq \beta(\sigma)$.

Proof It's enough to show for any ordinal β if $\tau \in S_\beta$ then $\sigma \in S_\beta$. This is by induction on β, the limit case and $\beta = 0$ case being trivial. Assume β is successor. If $\tau \in S_\beta$, this means $\tau \in S_{\beta-1}$ and there are $\tau', \tau'' \in [S_{\beta-1}]$ extending τ with $\tau' \in S$, $\tau'' \notin S$. Since τ' and τ'' extend τ, and τ extends σ, τ' and τ'' extend σ, and since $\sigma \in S_{\beta-1}$ by induction, this shows $\sigma \in S_\beta$. □

Lemma 7 *Suppose* $f : \mathbb{N} \to \mathbb{N}$, $f \notin [S_\infty]$. *Then there is some* i *such that for all* $j \geq i$, $f \upharpoonright j \notin S_\infty$ *and* $\beta(f \upharpoonright j) = \beta(f \upharpoonright i)$. *Furthermore,* $f \in [S_{\beta(f\upharpoonright i)-1}]$.

Proof The first part of the lemma follows from Lemma 6 and the well-foundedness of *ORD*. For the second part we must show $f \upharpoonright k \in S_{\beta(f\upharpoonright i)-1}$ for every k. If $k \leq i$, then $f \upharpoonright k \in S_{\beta(f\upharpoonright i)-1}$ by Lemma 6. If $k \geq i$, then $\beta(f \upharpoonright k) = \beta(f \upharpoonright i)$ and so $f \upharpoonright k \in S_{\beta(f\upharpoonright i)-1}$ since it is in $S_{\beta(f\upharpoonright k)-1}$ by definition of β. □

Proposition 8 *If* $S_\infty = \emptyset$ *then* S *is* $\mathbf{\Delta}^0_2$.

Proof Assume $S_\infty = \emptyset$. I will define a function $G : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ which guesses S, which is sufficient by Theorem 3.

Let $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we have $\sigma \notin S_\infty$ (since $S_\infty = \emptyset$) and so $\sigma \in S_{\beta(\sigma)-1} \backslash S_{\beta(\sigma)}$. Since $\sigma \notin S_{\beta(\sigma)}$, this means that for every two extensions σ', σ'' of σ in $[S_{\beta(\sigma)-1}]$, either σ'

and σ'' are both in S, or both are outside S. It might be that there *is* no extension of σ lying in $[S_{\beta(\sigma)-1}]$. In that case, arbitrarily define $G(\sigma) = 0$. But if there are such extensions, let $G(\sigma) = 1$ if all of those extensions are in S, and let $G(\sigma) = 0$ if all of those extensions are outside S.

I claim G guesses S. To see this, let $f \in S$. I will show $G(f \upharpoonright n) \to 1$ as $n \to \infty$. Since $f \notin [S_\infty]$, let i be as in Lemma 7. I claim $G(f \upharpoonright j) = 1$ whenever $j \geq i$. Fix $j \geq i$. We have $\beta(f \upharpoonright j) = \beta(f \upharpoonright i)$ by choice of i, and $f \in [S_{\beta(f\upharpoonright i)-1}] = [S_{\beta(f\upharpoonright j)-1}]$. By the previous paragraph, if any infinite sequence extends $f \upharpoonright j$ and lies in $[S_{\beta(f\upharpoonright j)-1}]$, then either all such sequences are in S, or all are outside S. One such sequence is f, and it is inside S, and therefore, all such sequences are inside S, whereby $G(f \upharpoonright j) = 1$ as desired.

Identical reasoning shows that if $f \notin S$ then $\lim_{n\to\infty} G(f \upharpoonright n) = 0$. So G guesses S, S is guessable, and by Theorem 3, S is $\mathbf{\Delta}^0_2$. □

Theorem 9 *S is $\mathbf{\Delta}^0_2$ if and only if $S_\infty = \emptyset$.*

Proof By combining Propositions 4 and 8. □

We will close by giving one more characterization of $\mathbf{\Delta}^0_2$.

Theorem 10 *S is $\mathbf{\Delta}^0_2$ if and only if $S = T\backslash[T_\infty]$ for some $T \subseteq \mathbb{N}^{\mathbb{N}}$.*

Proof By Theorem 9, if S is $\mathbf{\Delta}^0_2$ then $S = S\backslash[S_\infty]$. For the converse, it suffices to let S be arbitrary and prove $S\backslash[S_\infty]$ is $\mathbf{\Delta}^0_2$.

By Theorem 3, it is enough to exhibit a guesser $G : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ for $S\backslash[S_\infty]$. Let $\sigma \in \mathbb{N}^{<\mathbb{N}}$. If $\sigma \in S_\infty$, let $G(\sigma) = 0$. Otherwise, if σ has at least one infinite extension in $[S_{\beta(\sigma)-1}]$, and all such extensions are also in S, then let $G(\sigma) = 1$. In any other case, let $G(\sigma) = 0$.

We claim G guesses $S\backslash[S_\infty]$.

Case 1: $f \in S\backslash[S_\infty]$. By Lemma 7, find an i such that for all $j \geq i$ we have $f \upharpoonright j \notin S_\infty$ and $\beta(f \upharpoonright j) = \beta(f \upharpoonright i)$ and $f \in [S_{\beta(f\upharpoonright i)-1}]$. Thus for any $j \geq i$, $f \upharpoonright j$ does have one extension in $[S_{\beta(f\upharpoonright j)-1}]$, namely f itself, and f is in S. All other such extensions must also be in S, or else we would have $f \upharpoonright j \in S_{\beta(f\upharpoonright j)}$, violating the definition of β. So $G(f \upharpoonright j) = 1$, showing that $G(f \upharpoonright n) \to 1$ as $n \to \infty$.

Case 2: $f \in [S_\infty]$. Then for every n, $f \upharpoonright n \in S_\infty$ and thus by definition $G(f \upharpoonright n) = 0$.

Case 3: $f \notin S$ and $f \notin [S_\infty]$. As in Case 1, find i such that for all $j \geq i$, $f \upharpoonright j \notin S_\infty$ and $\beta(f \upharpoonright j) = \beta(f \upharpoonright i)$, and $f \in [S_{\beta(f\upharpoonright j)-1}]$. For any $j \geq i$, $f \upharpoonright j$ has one extension in $[S_{\beta(f\upharpoonright j)-1}]$, namely f, and $f \notin S$; so by definition $G(f \upharpoonright j) = 0$. □

References

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