HIGHLY LOPSIDED INFORMATION AND THE BOREL HIERARCHY

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Abstract. In a game where both contestants have perfect information, there is a strict limit on how perfect that information can be. By contrast, when one player is deprived of all information, the limit on the other player's information disappears, admitting a hierarchy of levels of lopsided perfection of information. We turn toward the question of when the player with super-perfect information has a winning strategy, and we exactly answer this question for a specific family of lopsided-information games which we call guessing games.

1. Introduction

Suppose Alice and Bob are playing an infinite game together and Alice has no information at all about what moves Bob makes. Formally this means Alice is only permitted to use strategies which do not depend on Bob's moves. Under this strong restriction, a strategy for Alice is really just a fixed move-sequence: loosely speaking, she decides all of her moves before the game ever begins. This opens the possibility for Bob to have "better than perfect" information. At the highest extreme, we could allow Bob to know Alice's entire move-sequence before he even makes his first move. Between that and what is normally called "perfect" information, there is a hierarchy of possible perfection. If Bob has perfect information in the traditional sense [3] of the word, then for every move he makes, he obtains a single Δ_0 fact about Alice's move-sequence. By contrast, we could allow that with every move he makes, one Δ_1 fact is revealed to Bob, or one Δ_2 fact, etc.

Suppose $S \subseteq \mathbb{N}^{\mathbb{N}}$ is a fixed subset of Baire space. The *guessing game* for S is as follows. Alice chooses a sequence $f : \mathbb{N} \to \mathbb{N}$, which may or may not be in S. The fact that this is chosen in advance corresponds to Alice having no information about Bob's moves. Now Bob tries to "guess" (formally: play 1 or 0) whether or not f is in S. The terms of Alice's sequence are revealed to him one-by-one and he gets to revise his guess with each revelation, and he wins if his guesses converge to the correct answer, otherwise Alice wins. If Bob has a winning strategy, S is said to be guessable. In an earlier paper [1], it is shown that S is guessable if and only if $S \in \mathbf{\Delta}^0_2$, the boldface pointclass of the Borel hierarchy. We will generalize this result to higher-order guessing games which correspond to one player having more and more lopsided information.

The way in which Alice is forced, by lack of information, to choose her moves before the game begins, bears some resemblance to an auxiliary game invented by

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Donald Martin [4] in which, at a certain point, one player plays a quasi-strategy and in so doing locks himself into always playing within that quasi-strategy.

2. Zeroth-Order Guessability

In this section we will formally introduce guessability. The basic definition does not clearly generalize to higher orders, so an equivalent definition will be proved (using some basic first-order logic) which generalizes more smoothly.

Definition 1. Suppose $S \subseteq \mathbb{N}^\mathbb{N}$. A function $G : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is said to *guess* S (and we say G is a *guesser* for S) if and only if, for every $f \in \mathbb{N}^\mathbb{N}$,

$$\lim_{n\to\infty} G(f(0), \ldots, f(n)) = \begin{cases} 1, \text{ if } f \in S; \\ 0, \text{ if } f \notin S. \end{cases}$$

If any such G exists, we say S is *guessable*.

Lemma 2. Let $S \subseteq \mathbb{N}^\mathbb{N}$. Suppose Alice and Bob are playing natural numbers, that Bob has perfect information (in the usual sense) but Alice has no information, and that Bob wins if either

(1) Alice's move-sequence is in S and Bob's moves are eventually always 1, or
(2) Alice's move-sequence is not in S, and Bob's moves are eventually always 0.

Then Bob has a winning strategy if and only if S is guessable.

Proof. If Bob has a winning strategy, then define $G(f(0), \ldots, f(n))$ to be the nth move Bob makes according to that strategy, assuming Alice's move-sequence begins with $(f(0), \ldots, f(n))$. Conversely, if S is guessable, say with guesser G, a winning strategy for Bob's nth move is to play $G(f(0), \ldots, f(n))$ where $f(i)$ denotes Alice's ith move. $\square$

In [1] we showed that the guessable sets are precisely the $\mathbf{\Delta^0_2}$ sets; this will also be a special case of a later theorem in the present paper.

For an alternate characterization of guessability which will generalize more easily, we will need some technical machinery from basic logic.

Definition 3.

- By $\mathscr{L}_{\max}$ is meant the first-order language which has a constant symbol $\overline{n}$ for every $n \in \mathbb{N}$; an n-ary function symbol $\tilde{w}$ for every function $w : \mathbb{N}^n \to \mathbb{N}$; an n-ary predicate symbol $\tilde{p}$ for every subset $p \subseteq \mathbb{N}^n$; a special unary function symbol $\mathbf{f}$; and, for every function $G : \mathbb{N}^n \times \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$, an $(n+1)$-ary function symbol $G \circ \mathbf{f}$.
- For any $f : \mathbb{N} \to \mathbb{N}$, the structure $\mathscr{M}_f$ for the language $\mathscr{L}_{\max}$ is defined as follows. It has universe $\mathbb{N}$. It interprets $\overline{n}$, $\tilde{w}$, and $\tilde{p}$ in the obvious ways. It interprets $\mathbf{f}$ as f. And for every $G : \mathbb{N}^n \times \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$, $\mathscr{M}_f$ interprets $G \circ \mathbf{f}$ as the function
$$(G \circ \mathbf{f})^{\mathscr{M}_f}(m_1, \ldots, m_n, m) = G(m_1, \ldots, m_n, f(0), \ldots, f(m)).$$
- If ϕ is an $\mathscr{L}_{\max}$-sentence and $\sigma \in \mathbb{N}^{<\mathbb{N}}$, say that ϕ is *determined* by σ if for every $f, g \in \mathbb{N}^\mathbb{N}$ which extend σ, $\mathscr{M}_f \models \phi$ iff $\mathscr{M}_g \models \phi$.

Lemma 4. If ϕ is a quantifier-free $\mathscr{L}_{\max}$-sentence and $f : \mathbb{N} \to \mathbb{N}$, then there is some k big enough that $(f(0), \ldots, f(k))$ determines ϕ.

Proof. First, we claim that if t is an $\mathscr{L}_{\max}$-term with no free variables then there is a k large enough that $t^{\mathscr{M}_g} = t^{\mathscr{M}_f}$ whenever $g \in \mathbb{N}^{\mathbb{N}}$ extends $(f(0), \ldots, f(k))$. This is a straightforward induction on the complexity of t. Most cases are omitted, but just for one example, suppose t is $(G \circ \mathbf{f})(u_0, \ldots, u_n)$ where $G : \mathbb{N}^n \times \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ and where $u_0, \ldots, u_n$ are simpler terms. By induction, find $k_0, \ldots, k_n$ such that each $u_i^{\mathscr{M}_g} = u_i^{\mathscr{M}_f}$ whenever $g \in \mathbb{N}^{\mathbb{N}}$ extends $(f(0), \ldots, f(k_i))$. Let $k = \max\{k_0, \ldots, k_n, u_n^{\mathscr{M}_f}\}$. Suppose $g \in \mathbb{N}^{\mathbb{N}}$ extends $(f(0), \ldots, f(k))$. Then

$$\begin{aligned}
(G \circ \mathbf{f})(u_0, \ldots, u_n)^{\mathscr{M}_g} &= (G \circ \mathbf{f})^{\mathscr{M}_g}(u_0^{\mathscr{M}_g}, \ldots, u_n^{\mathscr{M}_g}) \\
&= G(u_0^{\mathscr{M}_g}, \ldots, u_{n-1}^{\mathscr{M}_g}, g(0), \ldots, g(u_n^{\mathscr{M}_g})) \\
&= G(u_0^{\mathscr{M}_f}, \ldots, u_{n-1}^{\mathscr{M}_f}, g(0), \ldots, g(u_n^{\mathscr{M}_f})) \\
&= G(u_0^{\mathscr{M}_f}, \ldots, u_{n-1}^{\mathscr{M}_f}, f(0), \ldots, f(u_n^{\mathscr{M}_f})) \\
&= (G \circ \mathbf{f})(u_0, \ldots, u_n)^{\mathscr{M}_f},
\end{aligned}$$

as desired.

From this, the lemma follows by induction on the complexity of ϕ. □

Corollary 5. If ϕ is a quantifier-free $\mathscr{L}_{\max}$-sentence then

$$[\phi] := \{f \in \mathbb{N}^{\mathbb{N}} : \mathscr{M}_f \models \phi\}$$

is a clopen subset of Baire space.

Proof. Openness is by Lemma 4. Closure follows since $[\phi]^c = [\neg\phi]$. □

If $f : \mathbb{N} \to \mathbb{N}$ and if ϕ is an $\mathscr{L}_{\max}$-sentence, write $f(\phi)$ for the number

$$f(\phi) = \begin{cases} 1 \text{ if } \mathscr{M}_f \models \phi; \\ 0 \text{ otherwise.} \end{cases}$$

In other words, $f(\phi) = 1$ if and only if $f \in [\phi]$. As an example, if ϕ is $\forall x \, \exists y \, \mathbf{f}(y) = x$ then $[\phi]$ is the set of surjections and $f(\phi) = 1$ if and only if f is surjective.

Proposition 6. Suppose $S \subseteq \mathbb{N}^{\mathbb{N}}$. Then S is guessable if and only if there exists a countable set Σ of symbols of $\mathscr{L}_{\max}$, a listing $\phi_0, \phi_1, \ldots$ of all the quantifier-free sentences of $\mathscr{L}_{\max} \cap \Sigma$, and a function $G : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ such that for every $f : \mathbb{N} \to \mathbb{N}$,

$$\lim_{n\to\infty} G(f(\phi_0), \ldots f(\phi_n)) = \begin{cases} 1, \text{ if } f \in S; \\ 0, \text{ if } f \notin S. \end{cases}$$

Proof. ($\Rightarrow$) Suppose S is guessable, say with guesser $G_0 : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$. Let Σ be the symbol-set containing $\mathbf{f}$ and $\overline{n}$ for every $n \in \mathbb{N}$. Let $\phi_0, \phi_1, \ldots$ be any listing of the quantifier-free $\mathscr{L}_{\max} \cap \Sigma$ sentences. Define $G : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ as follows. Suppose $(p_0, \ldots, p_n) \in \{0,1\}^{<\mathbb{N}}$. Say a formula ϕ *appears* if $\phi = \phi_i$ for some $i \leq n$ and $p_i = 1$. Find a maximum-length sequence $(n_0, \ldots, n_k)$ such that for each $i = 0, \ldots, k$, the formula $\mathbf{f}(\bar{i}) = \overline{n_i}$ appears, and let $G(p_0, \ldots, p_n) = G_0(n_0, \ldots, n_k)$; if $(n_0, \ldots, n_k)$ is not uniquely determined or no such nonempty sequence exists, let $G(p_0, \ldots, p_n) = 0$.

This witnesses the theorem's conclusion. Let $f \in S$. Since G_0 guesses S, find n_0 so big that $\forall n \geq n_0$, $G_0(f(0), \ldots, f(n)) = 1$. Since $\phi_0, \phi_1, \ldots$ is an exhaustive list, there is some k_0 so big that $(\phi_0, \ldots, \phi_{k_0})$ includes all the sentences $\mathbf{f}(\bar{i}) = \overline{f(i)}$ for $i = 1, \ldots, n_0$. For each such i, $f(\mathbf{f}(\bar{i}) = \overline{f(i)}) = 1$, so each formula $\mathbf{f}(\bar{i}) = \overline{f(i)}$

appears in the definition of $G(f(\phi_0),\ldots,f(\phi_k))$ for any $k>k_0$. So for any $k>k_0$, $G(f(\phi_0),\ldots,f(\phi_k))=G_0(f(0),\ldots,f(n))$ for some $n\geq n_0$, so equals 1. So $G(f(\phi_0),\ldots,f(\phi_k))\to 1$ as $k\to\infty$, as desired. A similar argument goes for the case $f\notin S$.

($\Leftarrow$) Suppose Σ, $\phi_0,\phi_1,\ldots$, and $G_0:\{0,1\}^{<\mathbb{N}}\to\mathbb{N}$ are as in the theorem's conclusion. Define $G:\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}$ as follows. Given any sequence $(m_0,\ldots,m_n)\in\mathbb{N}^{<\mathbb{N}}$, let $k\leq n$ be maximal such that for every $i=0,\ldots,k$, ϕ_i is determined by $(m_0,\ldots,m_n)$ (if there is no such k, arbitrarily define $G(m_0,\ldots,m_n)=0$). Let $G(m_0,\ldots,m_n)=G_0(f(\phi_0),\ldots,f(\phi_k))$ for any $f:\mathbb{N}\to\mathbb{N}$ extending $(m_0,\ldots,m_n)$, well-defined since $(m_0,\ldots,m_n)$ determines $\phi_0,\ldots,\phi_k$.

Claim: G guesses S. Suppose $f\in S$. By hypothesis, there is some k_0 such that $\forall k\geq k_0$, $G_0(f(\phi_0),\ldots,f(\phi_k))=1$. By Lemma 4, we can find some j_0 such that $\phi_0,\ldots,\phi_{k_0}$ are all determined by $(f(0),\ldots,f(j_0))$. Then for any $j\geq j_0$, $G(f(0),\ldots,f(j))=G_0(f(\phi_0),\ldots,f(\phi_k))$ for some $k\geq k_0$. So for any such j, $G(f(0),\ldots,f(j))=1$, as desired. The case $f\notin S$ is similar. □

Corollary 7. Let $S\subseteq\mathbb{N}^{\mathbb{N}}$. Suppose Alice and Bob are playing natural numbers and neither can see the other's moves. But at the start of the game, Bob is allowed to choose a countable subset Σ of $\mathscr{L}_{\max}$ and a listing $\phi_0,\phi_1,\ldots$ of the quantifier-free Σ-sentences, and then, on his nth move, Bob is told whether or not ϕ_n holds of Alice's move-sequence (he is told this by a reliable third party, without Alice's knowledge). Suppose the winning conditions are as in Lemma 2. Then Bob has a winning strategy if and only if S is guessable.

Proof. If Bob has a winning strategy, let Σ and $\phi_0,\phi_1,\ldots$ be as dictated by that strategy. For any $(p_0,\ldots,p_n)\in\{0,1\}^{<\mathbb{N}}$, let $G(p_0,\ldots,p_n)$ be the move dictated by Bob's strategy assuming Bob is told that Alice's move-sequence satisfies ϕ_i for each $p_i=1$ $(i\leq n)$ and $\neg\phi_i$ for each $p_i=0$ $(i\leq n)$.

Conversely, suppose S is guessable. Let Σ, $\phi_0,\phi_1,\ldots$, and $G:\{0,1\}^{<\mathbb{N}}\to\mathbb{N}$ be as provided by Proposition 6. A winning strategy for Bob is to choose Σ, $\phi_0,\phi_1,\ldots$ at the beginning, and then always play $G(f(\phi_0),\ldots,f(\phi_n))$ where f is Alice's move-sequence; he can do this using the information he is given. □

3. Higher-Order Guessability

Proposition 6 provides a way to generalize guessability, giving us a foothold into a hierarchy of super-perfect information. The Σ_n and Π_n formulas of a language are defined inductively: $\Sigma_0=\Pi_0=\Delta_0$ is the set of quantifier-free formulas; having defined Σ_n and Π_n, let

$$\Sigma_{n+1}=\{\exists x\,\phi\,:\,\phi\in\Pi_n,\ x\text{ any variable}\}$$
$$\Pi_{n+1}=\{\forall x\,\phi\,:\,\phi\in\Sigma_n,\ x\text{ any variable}\}.$$

An $\mathscr{L}_{\max}$ formula is Δ_{n+1} if it is equivalent (over all the models $\mathscr{M}_f$) to some Σ_{n+1} formula and also to some Π_{n+1} formula of $\mathscr{L}_{\max}$.

Definition 8. Let $S\subseteq\mathbb{N}^{\mathbb{N}}$, $m\in\mathbb{N}$. We say S is *mth-order guessable* if there exists a countable set Σ of $\mathscr{L}_{\max}$-symbols, a listing $\phi_0,\phi_1,\ldots$ of all Δ_m sentences of $\mathscr{L}_{\max}\cap\Sigma$, and a function $G:\{0,1\}^{<\mathbb{N}}\to\mathbb{N}$ such that for every $f:\mathbb{N}\to\mathbb{N}$,

$$\lim_{n\to\infty}G(f(\phi_0),\ldots,f(\phi_n))=\begin{cases}1,\text{ if } f\in S;\\ 0,\text{ if } f\notin S.\end{cases}$$

Thus, Proposition 6 can be restated as follows: "$S \subseteq \mathbb{N}^{\mathbb{N}}$ is guessable if and only if it is 0th-order guessable."

Lemma 9. Modify the game in Corollary 7 by changing "quantifier-free" to "Δ_m". Then Bob has a winning strategy if and only if S is mth-order guessable.

Proof. Immediate. □

The main theorem of the paper will be that mth-order guessability is equivalent to $\mathbf{\Delta}^0_2$ if $m = 0$ or to $\mathbf{\Delta}^0_{m+1}$ if $m \neq 0$. We will begin working toward that result now.

Definition 10. Let $\mathbf{\Delta}^{'}_2 = \mathbf{\Delta}^0_2$. For every $m > 2$, define $\mathbf{\Delta}^{'}_m$ as follows: a set $S \subseteq \mathbb{N}^{\mathbb{N}}$ is in $\mathbf{\Delta}^{'}_m$ if and only if S is a countable union of countable intersections of $\mathbf{\Delta}^0_{m-2}$ sets and also a countable intersection of countable unions of $\mathbf{\Delta}^0_{m-2}$ sets.

Lemma 11. If $m = 2$ then $\mathbf{\Delta}^{'}_m = \mathbf{\Delta}^0_2$. If $m > 2$ then $\mathbf{\Delta}^{'}_m = \mathbf{\Delta}^0_{m-1}$.

Proof. The $m = 2$ case is true by definition. Suppose $m > 2$. We claim $\mathbf{\Delta}^{'}_m \subseteq \mathbf{\Delta}^0_{m-1}$. Suppose S is $\mathbf{\Delta}^{'}_m$. Then $S = \cap_{i\in\mathbb{N}} \cup_{j\in\mathbb{N}} D_{ij} = \cup_{i\in\mathbb{N}} \cap_{j\in\mathbb{N}} E_{ij}$ where the D_{ij}, E_{ij} are $\mathbf{\Delta}^0_{m-2}$. In particular, the D_{ij} are $\mathbf{\Sigma}^0_{m-2}$, so for every i, $\cup_{j\in\mathbb{N}} D_{ij}$ is $\mathbf{\Sigma}^0_{m-2}$. Thus S is $\mathbf{\Pi}^0_{m-1}$. Similarly, since the E_{ij} are $\mathbf{\Pi}^0_{m-2}$, S is $\mathbf{\Sigma}^0_{m-1}$. So S is $\mathbf{\Delta}^0_{m-1}$.

Conversely, suppose S is $\mathbf{\Delta}^0_{m-1}$. Then $S = \cup_{i\in\mathbb{N}} \cap_{j\in\mathbb{N}} S_{ij} = \cap_{i\in\mathbb{N}} \cup_{j\in\mathbb{N}} P_{ij}$ where the S_{ij} are $\mathbf{\Sigma}^0_{m-3}$ (or the S_{ij} are basic-open if $m = 3$) and the P_{ij} are $\mathbf{\Pi}^0_{m-3}$ (or complements of basic-open if $m = 3$). So the P_{ij} and S_{ij} are $\mathbf{\Delta}^0_{m-2}$, which shows S is $\mathbf{\Delta}^{'}_m$. □

Say that $S \subseteq \mathbb{N}^{\mathbb{N}}$ is *defined* by an $\mathscr{L}_{\max}$ sentence ϕ if $S = [\phi]$. As an example, the set of surjections is defined by $\forall x\, \exists y\, \mathbf{f}(y) = x$.

Our interest in defining Borel sets by formulas in a powerful language, as in the following lemma, is partially influenced by Vanden Boom [5] pp. 276–277. In [2] a similar result is obtained using a weaker but nonstandard language.

Lemma 12. Let $S \subseteq \mathbb{N}^{\mathbb{N}}$. For $n > 0$, S is $\mathbf{\Sigma}^0_n$ (resp. $\mathbf{\Pi}^0_n$, $\mathbf{\Delta}^0_n$) if and only if S is defined by a Σ_n (resp. Π_n, Δ_n) sentence of $\mathscr{L}_{\max}$.

Proof. Write $[f_0]$ for the collection of infinite extensions of a finite sequence $f_0 \in \mathbb{N}^{<\mathbb{N}}$.

($\Rightarrow$) Suppose S is $\mathbf{\Sigma}^0_n$. If n is even, write $S = \cup_{i_1\in\mathbb{N}} \cdots \cap_{i_n\in\mathbb{N}} [f_{i_1\cdots i_n}]^c$ where each $f_{i_1\cdots i_n} \in \mathbb{N}^{<\mathbb{N}}$ (we can assume the $f_{i_1\cdots i_n}$ are nonempty). If n is odd, write $S = \cup_{i_1\in\mathbb{N}} \cdots \cup_{i_n\in\mathbb{N}} [f_{i_1\cdots i_n}]$. Let $\ell : \mathbb{N}^n \to \mathbb{N}$ be defined by letting $\ell(i_1, \ldots, i_n)$ be the length of $f_{i_1\cdots i_n}$, minus 1. Define $\tau : \mathbb{N}^n \times \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ by $\tau(i_1, \ldots, i_n, a_1, \ldots, a_k) = 1$ if $(a_1, \ldots, a_k) = f_{i_1\cdots i_n}$, $\tau = 0$ everywhere else. Then for any $f : \mathbb{N} \to \mathbb{N}$, f extends $f_{i_1\cdots i_n}$ if and only if

$$\tau(i_1, \ldots, i_n, f(0), \ldots, f(\ell(i_1, \ldots, i_n))) = 1.$$

So if n is even, then S is defined by the $\mathscr{L}_{\max}$ sentence

$$\exists x_1 \cdots \forall x_n (\tau \circ \mathbf{f})(x_1, \ldots, x_n, \tilde{\ell}(x_1, \ldots, x_n)) = \overline{0}.$$

And if n is odd, then S is defined by the $\mathscr{L}_{\max}$ sentence

$$\exists x_1 \cdots \exists x_n (\tau \circ \mathbf{f})(x_1, \ldots, x_n, \tilde{\ell}(x_1, \ldots, x_n)) = \overline{1}.$$

The case $\mathbf{\Pi}^0_n$ case is similar, and the $\mathbf{\Delta}^0_n$ case follows.

($\Leftarrow$) Induction on n. For the base case, suppose S is defined by (say) the Σ_1 sentence $\exists x\,\phi$ where ϕ is quantifier-free. Corollary 5 ensures $[\phi(x|\bar{i})]$ is clopen for any $i\in\mathbb{N}$. Thus $S=\cup_{i\in\mathbb{N}}[\phi(x|\bar{i})]$ is open, so $\mathbf{\Sigma}^0_1$. Similarly for the Π_1 and Δ_1 cases. With the base case done, the induction case is straightforward. $\square$

Proposition 13. If $S\subseteq\mathbb{N}^\mathbb{N}$ is $\mathbf{\Delta}'_{m+2}$ then S is mth-order guessable.

Proof. Case 1: $m>0$. Then S is a $\cup\cap$ of $\mathbf{\Delta}^0_m$ sets and also a $\cap\cup$ of $\mathbf{\Delta}^0_m$ sets. Write $S=\cup_{i\in\mathbb{N}}\cap_{j\in\mathbb{N}}D_{ij}=\cap_{i\in\mathbb{N}}\cup_{j\in\mathbb{N}}E_{ij}$ where the D_{ij}, E_{ij} are $\mathbf{\Delta}^0_m$. By Lemma 12, we may find Δ_m sentences σ_{ij} defining each D_{ij}, and Δ_m sentences τ_{ij} defining each E_{ij}. Let Σ be the (countable) set of $\mathscr{L}_{\max}$ symbols appearing in the σ_{ij} and τ_{ij}. Let $\phi_0,\phi_1,\ldots$ be any listing of all the Δ_m sentences of $\mathscr{L}_{\max}\cap\Sigma$. We will find a $G:\{0,1\}^{<\mathbb{N}}\to\mathbb{N}$ which will witness the mth-order guessability of S.

Define G in terms of two functions $\mu,\nu:\{0,1\}^{<\mathbb{N}}\to\mathbb{N}$. Suppose $(p_0,\ldots,p_n)\in\{0,1\}^{<\mathbb{N}}$. Say that a sentence ϕ_i *appears* if $i\le n$ and $p_i=1$. Let $\mu(p_0,\ldots,p_n)$ be the minimum $x\in\mathbb{N}$ such that there is no $y\in\mathbb{N}$ such that $\neg\sigma_{xy}$ appears. Let $\nu(p_0,\ldots,p_n)$ be the minimum $x\in\mathbb{N}$ such that there is no $y\in\mathbb{N}$ such that τ_{xy} appears. Finally, let $G(p_0,\ldots,p_n)=1$ if $\mu(p_0,\ldots,p_n)<\nu(p_0,\ldots,p_n)$ and let $G(p_0,\ldots,p_n)=0$ otherwise.

We claim $\Sigma,\phi_0,\phi_1,\ldots,G$ witnesses the mth-order guessability of S. First, suppose $f\in S$. We must show $\lim_{n\to\infty}G(f(\phi_0),\ldots,f(\phi_n))=1$. Since $f\in S=\cup_{i\in\mathbb{N}}\cap_{j\in\mathbb{N}}D_{ij}$, we have $f\in\cap_{j\in\mathbb{N}}D_{ij}$ for some i. So $f\in D_{ij}$ for every j. Thus $\mathscr{M}_f\models\sigma_{ij}$ for every j, and thus $\neg\sigma_{ij}$ cannot *appear* in the definition of $\mu(f(\phi_0),\ldots,f(\phi_n))$ for any n. Thus μ is bounded above by i. We claim $\lim_{n\to\infty}\nu(f(\phi_0),\ldots,f(\phi_n))=\infty$, which will show that ν is eventually always above μ and thus that G converges to 1. It is enough to let $i\in\mathbb{N}$ be arbitrary and show $\nu(f(\phi_0),\ldots,f(\phi_n))\ne i$ for all n sufficiently large. Well, $S^c=\cup_{i\in\mathbb{N}}\cap_{j\in\mathbb{N}}E^c_{ij}$, and $f\notin S^c$, so for any arbitrary $i\in\mathbb{N}$, there is some j such that $f\in E_{ij}$, whence $\mathscr{M}_f\models\tau_{ij}$. Thus, for any n large enough that $\phi_0,\ldots,\phi_n$ includes τ_{ij}, τ_{ij} *appears* in the definition of $\nu(f(\phi_0),\ldots,f(\phi_n))$, so $\nu(f(\phi_0),\ldots,f(\phi_n))\ne i$. There is such a sufficiently large n, because τ_{ij} is Δ_m.

A similar argument shows that $\lim_{n\to\infty}G(f(\phi_0),\ldots,f(\phi_n))=0$ if $f\notin S$.

Case 2: $m=0$. This case is similar to Case 1, but instead of writing S as a $\cup\cap$ of $\mathbf{\Delta}^0_m$ sets, write it as a $\cup\cap$ of complements of basic open sets. And instead of writing S as a $\cap\cup$ of $\mathbf{\Delta}^0_m$ sets, write it as a $\cap\cup$ of basic open sets. Then take the τ_{ij} and σ_{ij} to be quantifier-free formulas in the obvious way. $\square$

Proposition 14. If $S\subseteq\mathbb{N}^\mathbb{N}$ is mth-order guessable, then S is $\mathbf{\Delta}'_{m+2}$.

Proof. Suppose S is mth-order guessable. There is a countable set Σ of $\mathscr{L}_{\max}$ symbols and a listing $\phi_0,\phi_1,\ldots$ of all the Δ_m sentences of $\mathscr{L}_{\max}\cap\Sigma$, and a function $G:\{0,1\}^{<\mathbb{N}}\to\mathbb{N}$ which witnesses the mth-order guessability of S. For any $f:\mathbb{N}\to\mathbb{N}$, $f\in S$ if and only if $G(f(\phi_0),\ldots,f(\phi_n))$ is eventually always 1. Thus we can write

$$\begin{aligned}S&=\bigcup_{i\in\mathbb{N}}\bigcap_{j>i}\{f\,:\,G(f(\phi_0),\ldots,f(\phi_j))=1\}\\&=\bigcup_{i\in\mathbb{N}}\bigcap_{j>i}\bigcup_{0\le a_1,\ldots,a_j\le 1\text{ and }G(\vec{a})=1}\bigcap_{0\le k\le j}\{f\,:\,f(\phi_k)=a_k\}.\end{aligned}$$

Each set $\{f : f(\phi_k) = a_k\}$ is $\mathbf{\Delta}^0_m$ if $m > 0$, or is clopen if $m = 0$: this is because if $a_k = 1$ then $\{f : f(\phi_k) = a_k\} = \{f : \mathscr{M}_f \models \phi_k\}$, and if $a_k = 0$ then $\{f : f(\phi_k) = a_k\} = \{f : \mathscr{M}_f \models \neg\phi_k\}$. Either way, we have a $\mathbf{\Delta}^0_m$ set by Lemma 12 (or a clopen set by Corollary 5, if $m = 0$). Since $\mathbf{\Delta}^0_m$ is closed under *finite* unions and intersections (as are the clopen sets), S is a countable union of countable intersections of $\mathbf{\Delta}^0_m$ sets (or of clopen sets if $m = 0$).

For the dual situation, note for any $f : \mathbb{N} \to \mathbb{N}$, saying $G(f(\phi_0), \ldots, f(\phi_n)) \to 1$ is equivalent to saying $G(f(\phi_0), \ldots, f(\phi_n)) = 1$ for *infinitely many* values of n, because $\lim_{n\to\infty} G(f(\phi_0), \ldots, f(\phi_n))$ must exist by Definition 8. Thus

$$S = \bigcap_{i\in\mathbb{N}} \bigcup_{j>i} \{f : G(f(\phi_0), \ldots, f(\phi_j)) = 1\}.$$

Thus S is a countable intersection of countable unions of $\mathbf{\Delta}^0_m$ sets (or of clopen sets if $m = 0$). Put together, S is $\mathbf{\Delta}'_{m+2}$. □

Theorem 15. The 0th-order guessable sets are exactly the $\mathbf{\Delta}^0_2$ sets, and for $m > 0$, the mth-order guessable sets are exactly the $\mathbf{\Delta}^0_{m+1}$ sets.

Proof. By combining Propositions 13 and 14, for any m, the mth-order guessable sets are exactly the $\mathbf{\Delta}'_{m+2}$ sets. The theorem now follows by Lemma 11. □

In the proof of Proposition 14 we actually proved slightly more than we needed, which leads to an unexpected standalone corollary.

Corollary 16. If $S = \cup_{i\in\mathbb{N}} \cap_{j\in\mathbb{N}} X_{ij} = \cap_{i\in\mathbb{N}} \cup_{j\in\mathbb{N}} Y_{ij}$, where the X_{ij} and Y_{ij} are $\mathbf{\Delta}^0_n$, $0 < n \in \mathbb{N}$, then there is a single family Z_{ij} of $\mathbf{\Delta}^0_n$ sets such that

$$S = \cup_{i\in\mathbb{N}} \cap_{j\in\mathbb{N}} Z_{ij} = \cap_{i\in\mathbb{N}} \cup_{j\in\mathbb{N}} Z_{ij}.$$

Proof. By Proposition 13, S is nth-order guessable. Proposition 14 gives Z_{ij}. □

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