The Topology of Fiber Bundles
Lecture Notes

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Introduction

Fiber Bundles and more general fibrations are basic objects of study in many areas of mathematics. A fiber bundle with base space B and fiber F can be viewed as a parameterized family of objects, each "isomorphic" to F, where the family is parameterized by points in B. For example a vector bundle over a space B is a parameterized family of vector spaces V_x, one for each point $x \in B$. Given a Lie group G, a principal G - bundle over a space B can be viewed as a parameterized family of spaces F_x, each with a free, transitive action of G (so in particular each F_x is homeomorphic to G). A covering space is also an example of a fiber bundle where the fibers are discrete sets. Sheaves and "fibrations" are generalizations of the notion of fiber bundles and are fundamental objects in Algebraic Geometry and Algebraic Topology, respectively.

Fiber bundles and fibrations encode topological and geometric information about the spaces over which they are defined. Here are but a few observations on their impact in mathematics.

- A structure such as an orientation, a framing, an almost complex structure, a spin structure, and a Riemannian metric are all constructions on the the tangent bundle of a manifold.
- The exact sequence in homotopy groups, and the Leray - Serre spectral sequence for homology groups of a fibration have been basic tools in Algebraic Topology for nearly half a century.
- Understanding algebraic sections of algebraic bundles over a projective variety is a basic goal in algebraic geometry.
- K - theory, a type of classification of vector bundles over a topological space is at the same time an important homotopy invariant of the space, and a quantity for encoding index information about elliptic differential operators.
- The Yang - Mills partial differential equations are defined on the space of connections on a principal bundle over a Riemannian two dimensional or four dimensional manifold. The properties of the solution space have had a great impact on our understanding of four dimensional Differential Topology in the last fifteen years.

In these notes we will study basic topological properties of fiber bundles and fibrations. We will study their definitions, and constructions, while considering many examples. A main goal of these notes is to develop the topology needed to classify principal bundles, and to discuss various models of their classifying spaces. We will compute the cohomology of the classifying spaces of $O(n)$ and $U(n)$, and use them to study K - theory. These calculations will also allow us to describe characteristic classes for these bundles which we use in a variety of applications including the study of framings, orientations, almost complex structures, *Spin* and $Spin_{\mathbb{C}}$ - structures, vector fields, and immersions of manifolds. We will also discuss the Chern-Weil method of defining characteristic classes (via the curvature of a connection), and show how all $U(n)$ - characteristic classes can be defined this way. We also use these techniques to consider the topological implications when a bundle admits a flat connection. Finally, we study the algebraic topology of fibrations. This includes the exact sequence in homotopy groups of a fibration, general obstruction theory, including the interpretation of characteristic classes as obstructions to the existence of cross sections, and the construction and properties of Eilenberg - MacLane spaces. We then study the spectral sequence of a filtration and the Leray - Serre spectral sequence for a fibration. A variety of applications are given, including the Hurewicz theorem in homotopy theory, and a calculation of the homology of certain loop spaces and Lie groups.

None of the results in these notes are new. This is meant to be an exposition of classical topics that are of importance to students in any area of topology and geometry. There are several good references for many of the topics covered here, in particular the classical text of Steenrod on fiber bundles (describing the classification theorem) [**39**], the book of Milnor and Stasheff on characteristic classes of vector bundles [**31**], the texts of Whitehead [**42**] and Mosher and Tangora [**32**] for the results in homotopy theory. When good references are available we may not include the details of all the proofs. These notes grew out of a graduate topology course I gave at Stanford University during the Spring term, 1998. I am very grateful to the students in that course for comments on earlier versions of these notes.

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CHAPTER 1

Locally Trival Fibrations

In this chapter we define our basic object of study: locally trivial fibrations, or "fiber bundles". We discuss many examples, including covering spaces, vector bundles, and principal bundles. We also describe various constructions on bundles, including pull-backs, sums, and products. We then study the homotopy invariance of bundles, and use it in several applications.

Throughout all that follows, all spaces will be Hausdorff and paracompact.

1. Definitions and examples

Let B be connected space with a basepoint $b_0 \in B$, and $p : E \to B$ be a continuous map.

Definition 1.1. The map $p : E \to B$ is a **locally trivial fibration, or fiber bundle, with fiber F** if it satisfies the following properties:

(1) $p^{-1}(b_0) = F$
(2) $p : E \to B$ is surjective
(3) For every point $x \in B$ there is an open neighborhood $U_x \subset B$ and a "fiber preserving homeomorphism" $\Psi_{U_x} : p^{-1}(U_x) \to U_x \times F$, that is a homeomorphism making the following diagram commute:

$$\begin{tikzcd} p^{-1}(U_x) \arrow[r, "\Psi_{U_x}", "\cong"'] \arrow[d, "p"'] & U_x \times F \arrow[d, "proj"] \\ U_x \arrow[r, phantom, "="] & U_x \end{tikzcd}$$

Some examples:

- The projection map $X \times F \longrightarrow X$ is the *trivial* fibration over X with fiber F.
- Let $S^1 \subset \mathbb{C}$ be the unit circle with basepoint $1 \in S^1$. Consider the map $f_n : S^1 \to S^1$ given by $f_n(z) = z^n$. Then $f_n : S^1 \to S^1$ is a locally trivial fibration with fiber a set of n distinct points (the n^{th} roots of unity in S^1).
- Let $exp : \mathbb{R} \to S^1$ be given by

$$exp(t) = e^{2\pi i t} \in S^1.$$

Then exp is a locally trivial fibration with fiber the integers $\mathbb{Z}$.

- Recall that the n - dimensional real projective space $\mathbb{RP}^n$ is defined by
$$\mathbb{RP}^n = S^n / \sim$$
where $x \sim -x$, for $x \in S^n \subset \mathbb{R}^{n+1}$.

  Let $p : S^n \to \mathbb{RP}^n$ be the projection map. This is a locally trivial fibration with fiber the two point set.
- Here is the complex analogue of the last example. Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$. Recall that the complex projective space $\mathbb{CP}^n$ is defined by
$$\mathbb{CP}^n = S^{2n+1} / \sim$$
where $x \sim ux$, where $x \in S^{2n+1} \subset \mathbb{C}^n$, and $u \in S^1 \subset \mathbb{C}$. Then the projection $p : S^{2n+1} \to \mathbb{CP}^n$ is a locally trivial fibration with fiber S^1.
- Consider the Moebeus band $M = [0,1] \times [0,1] / \sim$ where $(t,0) \sim (1-t,1)$. Let C be the "center circle" $C = \{(1/2, s) \in M\}$ and consider the projection
$$\begin{aligned} p &: M \to C \\ (t,s) &\to (1/2, s). \end{aligned}$$
This map is a locally trivial fibration with fiber $[0,1]$.

Given a fiber bundle $p : E \to B$ with fiber F, the space B is called the *base space* and the space E is called the total space. We will denote this data by a triple (F, E, B).

Definition 1.2. A map (or "morphism") of fiber bundles $\Phi : (F_1, E_1, B_1) \to (F_2, E_2, B_2)$ is a pair of basepoint preserving continuous maps $\bar{\phi} : E_1 \to E_2$ and $\phi : B_1 \to B_2$ making the following diagram commute:
$$\begin{tikzcd} E_1 \arrow[r, "\bar{\phi}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "\phi"'] & B_2 \end{tikzcd}$$
Notice that such a map of fibrations determines a continuous map of the fibers, $\phi_0 : F_1 \to F_2$.

A map of fiber bundles $\Phi : (F_1, E_1, B_1) \to (F_2, E_2, B_2)$ is an *isomorphism* if there is an inverse map of fibrations $\Phi^{-1} : (F_2, E_2, B_2) \to (F_1, E_1, B_1)$ so that $\Phi \circ \Phi^{-1} = \Phi^{-1} \circ \Phi = 1$.

Finally we say that a fibration (F, E, B) is *trivial* if it isomorphic to the trivial fibration $B \times F \to B$.

Exercise. Verify that all of the above examples of fiber bundles are all nontrivial except for the first one.

The notion of a locally trivial fibration is quite general and includes examples of many types. For example you may have already noticed that **covering spaces** are examples of locally trivial fibrations. In fact one may simply define a covering space to be a locally trivial fibration with discrete fiber. Two other very important classes of examples of locally trivial fiber bundles are *vector bundles* and *principal bundles.* We now describe these notions in some detail.

1.1. Vector Bundles.

DEFINITION 1.3. An n- dimensional vector bundle over a field $\mathbf{k}$ is a locally trivial fibration $p : E \to B$ with fiber an n - dimensional $\mathbf{k}$ - vector space V satisfying the additional requirement that the local trivializations

$$\psi : p^{-1}(U) \to U \times V$$

induce $\mathbf{k}$ - linear transformations on each fiber. That is, restricted to each $x \in U$, ψ defines a $\mathbf{k}$ - linear transformation (and thus isomorphism)

$$\psi : p^{-1}(x) \xrightarrow{\cong} \{x\} \times V.$$

It is common to denote the data (V, E, B) defining an n - dimensional vector bundle by a Greek letter, e.g ζ.

A "map" or "morphism" of vector bundles $\Phi : \zeta \to \xi$ is a map of fiber bundles as defined above, with the added requirement that when restricted to each fiber, $\bar{\phi}$ is a $\mathbf{k}$ - linear transformation.

Examples

- Given an n - dimensional $\mathbf{k}$ vector space V, then $B \times V \to B$ is the corresponding trivial bundle over the base space B. Notice that since all n - dimensional trivial bundles over B are isomorphic, we denote it (or more precisely, its isomorphism class) by ϵ_n.
- Consider the " Moebeus line bundle" μ defined to the the one dimensional real vector bundle ("line bundle") over the circle given as follows. Let $E = [0,1] \times \mathbb{R}/ \sim$ where $(0,t) \sim (1,-t)$. Let C be the "middle" circle $C = \{(s,0) \in E\}$. Then μ is the line bundle defined by the projection

$$\begin{aligned} p &: E \to C \\ (s,t) &\to (s,0). \end{aligned}$$

- Define the real line bundle γ_1 over the projective space $\mathbb{RP}^n$ as follows. Let $x \in S^n$. Let $[x] \in \mathbb{RP}^n = S^n/\sim$ be the class represented by x. Then $[x]$ determines (and is determined by) the line through the origin in $\mathbb{R}^{n+1}$ going through x. It is well defined since both representatives of $[x]$ (x and $-x$) determine the same line. Thus $\mathbb{RP}^n$ can be thought of as the space of lines through the origin in $\mathbb{R}^{n+1}$. Let $E = \{([x], v) : [x] \in \mathbb{RP}^n, v \in [x]\}$. Then γ_1 is the line bundle defined by the projection

$$p : E \to \mathbb{RP}^n$$
$$([x], v) \to [x].$$

Exercise. Verify that the $\mathbb{RP}^1$ is a homeomorphic to a circle, and the line bundle γ_1 over $\mathbb{RP}^1$ is isomorphic to the Moebeus line bundle μ.

- By abuse of notation we let γ_1 also denote the *complex* line bundle over $\mathbb{CP}^n$ defined analogously to the *real* line bundle γ_1 over $\mathbb{RP}^n$ above.
- Let $Gr_k(\mathbb{R}^n)$ (respectively $Gr_k(\mathbb{C}^n)$) be the space whose points are k - dimensional subvector spaces of $\mathbb{R}^n$ (respectively $\mathbb{C}^n$). These spaces are called *"Grassmannian"* manifolds, and are topologized as follows. Let $V_k(\mathbb{R}^n)$ denote the space of *injective* linear transformations from $\mathbb{R}^k$ to $\mathbb{R}^n$. Let $V_k(\mathbb{C}^n)$ denote the analogous space of injective linear transformations $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$. These spaces are called *"Stiefel manifolds"*, and can be thought of as spaces of $n \times k$ matrices of rank k. These spaces are given topologies as subspaces of the appropriate vector space of matrices. To define $Gr_k(\mathbb{R}^n)$ and $Gr_k(\mathbb{C}^n)$, we put an equivalence relation on $V_k(\mathbb{R}^n)$ and $V_k(\mathbb{C}^n)$ by saying that two transformations A and B are equivalent if they have the same image in $\mathbb{R}^n$ (or $\mathbb{C}^n$). If viewed as matrices, then $A \sim B$ if and only if there is an element $C \in GL(k, \mathbb{R})$ (or $GL(k, \mathbb{C})$) so that $A = BC$. Then the equivalence classes of these matrices are completely determined by their image in $\mathbb{R}^n$ (or $\mathbb{C}^n$), i.e the equivalence class is determined completely by a k - dimensional subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$). Thus we define

$$Gr_k(\mathbb{R}^n) = V_k(\mathbb{R}^n)/\sim \quad \text{and} \quad Gr_k(\mathbb{C}^n) = V_k(\mathbb{C}^n)/\sim$$

with the corresponding quotient topologies.

Consider the vector bundle γ_k over $Gr_k(\mathbb{R}^n)$ whose total space E is the subspace of $Gr_k(\mathbb{R}^n) \times \mathbb{C}^n$ defined by

$$E = \{(W, \omega) : W \in Gr_k(\mathbb{R}^n) \,\text{and}\, \omega \in W \subset \mathbb{R}^n\}.$$

Then γ_k is the vector bundle given by the natural projection

$$E \to Gr_k(\mathbb{R}^n)$$
$$(W, \omega) \to W$$

For reasons that will become more apparent later in these notes, the bundles γ_k are called the "universal" or "canonical" k - dimensional bundles over the Grassmannians.

Exercises

1. Verify that γ_k is a k -dimensional real vector bundle over $Gr_k(\mathbb{R}^n)$.

2. Define the analogous bundle (which by abuse of notation we also call γ_k) over $Gr_k(\mathbb{C}^n)$. Verify that it is a k-dimensional complex vector bundle over $Gr_k(\mathbb{C}^n)$.
3. Verify that $\mathbb{RP}^{n-1} = Gr_1(\mathbb{R}^n)$ and that the line bundle γ_1 defined above is the universal bundle. Do the analogous exercise with $\mathbb{CP}^{n-1}$ and $Gr_1(\mathbb{C}^n)$.

- Notice that the universal bundle γ_k over the Grassmanians $Gr_k(\mathbb{R}^n)$ and $Gr_k(\mathbb{C}^n)$ come equipped with embeddings (i.e injective vector bundle maps) in the trivial bundles $Gr_k(\mathbb{R}^n)\times \mathbb{R}^n$ and $Gr_k(\mathbb{C}^n) \times \mathbb{C}^n$ respectively. We can define the orthogonal complement bundles $\gamma_k^\perp$ to be the $n-k$ dimensional bundles whose total spaces are given by

$$E_k^\perp = \{(W, \nu) \in Gr_k(\mathbb{R}^n) \times \mathbb{R}^n : \nu \perp W\}$$

and similarly over $Gr_k(\mathbb{C}^n)$. Observe that the natural projection to the Grassmannian defines $n-k$ dimensional vector bundles (over $\mathbb{R}$ and $\mathbb{C}$ respectively).

- Perhaps the most important class of vector bundles are tangent bundles over differentiable manifolds. To define the tangent bundle, for now we will assume we have a k - dimensional closed manifold M embedded in Euclidean space $\mathbb{R}^n$. (We will define the tangent bundle independently of such an embedding later in this chapter.) The tangent bundle τM has total space TM defined to be the subspace of $M \times \mathbb{R}^n$ given as follows.

$$TM = \{(x, v) \in M \times \mathbb{R}^n : v \,\text{is tangent to}\; M \;\text{at}\; x.\}$$

where v being tangent to M at x means there is a smooth curve

$$\alpha : (-\epsilon, \epsilon) \to M \subset \mathbb{R}^n$$

for some $\epsilon > 0$, satisfying the properties

$$\alpha(0) = x \quad \text{and} \quad \frac{d\alpha}{dt}(0) = v.$$

The natural projection

$$p : TM \to M$$
$$(x, v) \to x$$

defines the tangent bundle τM.

The fiber of the tangent bundle at $x \in M$, $p^{-1}(x)$ is called the *tangent space* of M at x, and is denoted by $T_x M$.

We say that a manifold is *parallelizable* if its tangent bundle is trivial. Parallelizable manifolds form an important class of manifolds, but as we will see below, not all manifolds are parallelizable.

Exercise. Verify that TM is indeed a k - dimensional vector bundle embedded in $M \times \mathbb{R}^n$.

- Let $M \subset \mathbb{R}^n$ be a k - dimensional closed manifold embedded in $\mathbb{R}^n$ as above. We define the $n-k$ dimensional *"normal"* bundle $\nu^{n-k}(M)$ to be the orthogonal complement of the tangent bundle. That is, $\nu^{n-k}(M)$ has total space $N^{n-k}(M)$ defined to be the subspace of $M \times \mathbb{R}^n$ given by

  $$N^{n-k}(M) = \{(x, u) \in M \times \mathbb{R}^n : u \perp T_x M \subset \mathbb{R}^n\}.$$

  The natural projection

  $$p : N^{n-k}(M) \to M$$
  $$(x, u) \to x$$

  defines the normal bundle $\nu^{n-k}(M)$. Given $x \in M$, the fiber at x, $p^{-1}(x)$ is called the normal space at x, and is denoted $N_x(M)$.

An important notion associated to vector bundles (and in fact all fibrations) is the notion of a (cross) section.

Definition 1.4. Given a fiber bundle

$$p : E \to B$$

a section s is a continuous map $s : B \to E$ such that $p \circ s = \mathit{identity} : B \to B$.

Notice that every vector bundle has a section, namely the *zero section*

$$z : B \to E$$
$$x \to 0_x$$

where 0_x is the origin in the vector space $p^{-1}(x)$. However most geometrically interesting sections have few zero's. Indeed as we will see later, an appropriate count of the number of zero's of a section of an n - dimensional bundle over an n - dimensional manifold is an important topological invariant of that bundle (called the "Euler number"). In particular an interesting geometric question is to determine when a vector bundle has a nowhere zero section, and if it does, how many linearly independent sections it has. (Sections $\{s_1, \cdots, s_m\}$ are said to be linearly independant if the vectors $\{s_1(x), \cdots, s_m(x)\}$ are linearly independent for every $x \in B$.) These questions are classical in the case where the vector bundle is the tangent bundle. A section of the tangent bundle is called a **vector field.**The question of how many linearly independent vector fields exist on the sphere S^n was answered by J.F. Adams [**2**] in the early 1960's using sophisticated techniques of homotopy theory.

Exercises (from [31])

1. Let $x \in S^n$, and $[x] \in \mathbb{RP}^n$ be the corresponding element. Consider the functions $f_{i,j} : \mathbb{RP}^n \to \mathbb{R}$ defined by $f_{i,j}([x]) = x_i x_j$. Show that these functions define a diffeomorphism between $\mathbb{RP}^n$ and the submanifold of $\mathbb{R}^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.

2. Use exercise 1 to show that $\mathbb{RP}^n$ is compact.

3. Prove that an n -dimensional vector bundle ζ has n - linearly independent sections if and only if ζ is trivial.

4. Show that the unit sphere S^n admits a nowhere zero vector field if n is odd. Show that the normal bundle of $S^n \subset \mathbb{R}^{n+1}$ is a trivial line bundle for all n.

5. If S^n admits a nowhere zero vector field show that the identity map of S^n is homotopic to the antipodal map. For n even show that the antipodal map of S^n is homotopic to the reflection

$$r(x_1, \cdots, x_{n+1}) = (-x_1, x_2, \cdots, x_{n+1});$$

and therefore has degree -1. Combining these facts, show that S^n is not parallelizable for n even, $n \geq 2$.

1.2. Lie Groups and Principal Bundles. Lie groups play a central role in bundle theory. In this section we give a basic description of Lie groups, their actions on manifolds (and other spaces), and one of the main objects of study in these notes, their principal bundles.

Definition 1.5. A Lie group is a topological group G which has the structure of a differentiable manifold. Moreover the multiplication map

$$G \times G \to G$$

and the inverse map

$$\begin{aligned} G &\to G \\ g &\to g^{-1} \end{aligned}$$

are required to be differentiable maps.

The following is an important basic property of the differential topology of Lie groups.

Theorem 1.1. *Let G be a Lie group. Then G is parallelizable. That is, its tangent bundle τG is trivial.*

PROOF. Let $1 \in G$ denote the identity element, and T_1G the tangent space of G at 1. If G is an n - dimensional manifold, T_1G is an n- dimensional vector space. We define a bundle isomorphism of the tangent bundle τG with the trivial bundle $G \times T_1(G)$, which, on the total space level is given by a map

$$\phi : G \times T_1G \longrightarrow TG$$

defined as follows. Let $g \in G$. Then multiplication by g on the right is a diffeomorphism

$$\times g : G \to G$$
$$x \to xg$$

Since $\times g$ is a diffeomorphism, its derivative is a linear isomorphism at every point:

$$Dg(x) : T_xG \xrightarrow{\cong} T_{xg}G.$$

We can now define

$$\phi : G \times T_1G \to TG$$

by

$$\phi(g, v) = Dg(1)(v) \in T_gG.$$

Clearly ϕ is a bundle isomorphism. □

Principal bundles are basically parameterized families of topological groups, and often Lie groups. In order to define the notion carefully we first review some basic properties of group actions.

Recall that a right action of topological group G on a space X is a map

$$\mu : X \times G \to X$$
$$(x, g) \to xg$$

satisfying the basic properties

(1) $x \cdot 1 = x$ for all $x \in X$
(2) $x(g_1g_2) = (xg_1)g_2$ for all $x \in X$ and g_1, $g_2 \in G$.

Notice that given such an action, every element g acts as a homeomorphism, since action by g^{-1} is its inverse. Thus the group action μ defines a map

$$\mu : G \to Homeo(X)$$

where $Homeo(X)$ denotes the group of homeomorphisms of X. The two conditions listed above are equivalent to the requirement that $\mu : G \to Homeo(X)$ be a group homomorphism.

If G is a Lie group and M is a smooth manifold with a right G - action. We say that the action is *smooth* if the homomorphism μ defined above factors through a homomorphism

$$\mu : G \to Diffeo(M)$$

where $Diffeo(M)$ is the group of diffeomorphisms of M.

Let X be a space with a right G - action. Given $x \in X$, let $xG = \{xg : g \in G\} \subset X$. This is called the *orbit* of x under the G - action. The isotropy subgroup of x, $Iso(x)$, is defined by $Iso(x) = \{g \in G : xg = x\}$ Notice that the map

$$G \to xG$$

defined by sending g to xg defines a homeomorphism from the coset space to the orbit

$$G/Iso(x) \xrightarrow{\cong} xG \subset X.$$

A group action on a space X is said to be *transitive* if the space X is the orbit of a single point, $X = xG$. Notice that if $X = x_0G$ for some $x_0 \in X$, then $X = xG$ for *any* $x \in X$. Notice furthermore that the transitivity condition is equivalent to saying that for any two points $x_1, x_2 \in X$, there is an element $g \in G$ such that $x_1 = x_2g$. Finally notice that if X has a transitive G - action, then the above discussion about isotropy subgroups implies that there exists a subgroup $H < G$ and a homeomorphism

$$G/H \xrightarrow{\cong} X.$$

Of course if X is smooth, G is a Lie group, and the action is smooth, then the above map would be a diffeomorphism.

A group action is said to be *(fixed point) free* if the isotropy groups of every point x are trivial,

$$Iso(x) = \{1\}$$

for all $x \in X$. Said another way, the action is free if and only if the only time there is an equation of the form $xg = x$ is if $g = 1 \in G$. That is, if for $g \in G$, the fixed point set $Fix(g) \subset X$ is the set

$$Fix(g) = \{x \in X : xg = x\},$$

then the action is free if and only if $Fix(g) = \emptyset$ for all $g \neq 1 \in G$.

We are now able to define principal bundles.

Definition 1.6. Let G be a topological group. A *principal G bundle* is a fiber bundle $p : E \to B$ with fiber $F = G$ satisfying the following properties.

(1) The total space E has a free, fiberwise right G action. That is, it has a free group action making the following diagram commute:

$$\begin{tikzcd} E \times G \arrow[r, "\mu"] \arrow[d, "p \times \epsilon"'] & E \arrow[d, "p"] \\ B \times \{1\} \arrow[r, phantom, "="] & B \end{tikzcd}$$

where ϵ is the constant map.

(2) The induced action on fibers

$$\mu : p^{-1}(x) \times G \to p^{-1}(x)$$

is free and transitive.

(3) There exist local trivializations

$$\psi : p^{-1}(U) \xrightarrow{\cong} U \times G$$

that are equivariant. That is, the following diagrams commute:

$$\begin{tikzcd}
p^{-1}(U) \times G \arrow[r, "\psi \times 1", "\cong"'] \arrow[d, "\mu"'] & U \times G \times G \arrow[d, "1\times \text{mult.}"] \\
p^{-1}(U) \arrow[r, "\cong", "\psi"'] & U \times G.
\end{tikzcd}$$

Notice that in a principal G - bundle, the group G acts freely on the total space E. It is natural to ask if a free group action suffices to induce a principal G - bundle. That is, suppose E is a space with a free, right G action, and define B to be the orbit space

$$B = E/G = E/\sim$$

where $y_1 \sim y_2$ if and only if there exists a $g \in G$ with $y_1 = y_2 g$ (i.e if and only if their orbits are equal: $y_1 G = y_2 G$). Define $p : E \to B$ to be the natural projection, $E \to E/G$. Then the fibers are the orbits, $p^{-1}([y]) = yG$. So for $p : E \to B$ to be a principal bundle we must check the local triviality condition. In general for this to hold we need the following extra condition.

Definition 1.7. A group action $E \times G \to E$ has slices if projection onto the orbit space

$$p : E \to E/G$$

has local sections. That is, around every $x \in E/G$ there is a neighborhood U and a continuous map $s : U \to E$ such that $p \circ s = id : U \to U$.

Proposition 1.2. *If E has a free G action with slices, then the projection map*

$$p : E \to E/G$$

is a principal G - bundle.

Proof. We need to verify the local triviality condition. Let $x \in E/G$. Let U be an open set around x admitting a section $s : U \to E$. Define a local trivialization

$$\psi : U \times G \to p^{-1}(U)$$

by $\psi(y, g) = s(y) \cdot g$. Clearly ψ is a local trivialization. $\square$

The following result is originally due to A. Gleason [**15**], and its proof can be found in Steenrod's book [**39**]. It is quite helpful in studying free group actions.

Theorem 1.3. *Let E be a smooth manifold, having a free, smooth G - action, where G is a compact Lie group. Then the action has slices. In particular, the projection map*

$$p : E \to E/G$$

defines a principal G - bundle.

The following was one of the early theorems in fiber bundle theory, appearing originally in H. Samelson's thesis. [**34**]

Corollary 1.4. *Let G be a Lie group, and let $H < G$ be a compact subgroup. Then the projection onto the orbit space*

$$p : G \to G/H$$

is a principal H - bundle.

Examples.

- The projection map $p : S^{2n+1} \to \mathbb{CP}^n$ is a principal S^1 - bundle.
- Let $V_k(\mathbb{R}^n)$ be the Stiefel manifold of rank k $n \times k$ matrices described above. Then the projection map

  $$p : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$$

  is a principal $GL(k, \mathbb{R})$ - bundle. Similarly the projection map

  $$p : V_k(\mathbb{C}^n) \to Gr_k(\mathbb{C}^n)$$

  is a principal $GL(k, \mathbb{C})$ - bundle.
- Let $V_k(\mathbb{R}^n)^O \subset \mathbb{R}$ denote those $n \times k$ matrices whose k - columns are orthonormal n - dimensional vectors. This is the Stiefel manifold of orthonormal k - frames in $\mathbb{R}^n$. Then the induced projection map

  $$p : V_k(\mathbb{R}^n)^O \to Gr_k(\mathbb{R}^n)$$

  is a principal $O(k)$ - bundle. Similarly, if $V_k(\mathbb{C}^n)^U$ is the space of orthonormal k - frames in $\mathbb{C}^n$ (with respect to the standard Hermitian inner product), then the projection map

  $$p : V_k(\mathbb{C}^n)^U \to Gr_k(\mathbb{C}^n)$$

  is a principal $U(n)$ - bundle.

- There is a diffeomorphism

$$\rho : U(n)/U(n-1) \xrightarrow{\cong} S^{2n-1}$$

and the projection map $U(n) \to S^{2n-1}$ is a principal $U(n-1)$ - bundle.

To see this, notice that $U(n)$ acts transitively on the unit sphere in $\mathbb{C}^n$ (i.e S^{2n-1}). Moreover the isotropy subgroup of the point $e_1 = (1, 0, \cdots, 0) \in S^{2n-1}$ are those elements $A \in U(n)$ which have first column equal to $e_1 = (1, 0, \cdots, 0)$. Such matrices also have first row $= (1, 0, \cdots, 0)$. That is, A is of the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

where A' is an element of $U(n-1)$. Thus the isotropy subgroup $Iso(e_1) \cong U(n-1)$ and the result follows.

Notice that a similar argument gives a diffeomorphism $SO(n)/SO(n-1) \cong S^{n-1}$.

- There is a diffeomorphism

$$\rho : U(n)/U(n-k) \xrightarrow{\cong} V_k(\mathbb{C}^n)^U.$$

The argument here is similar to the above, noticing that $U(n)$ acts transitively on $V_k(\mathbb{C}^n)^U$, and the isotopy subgroup of the $n \times k$ matrix

$$e = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

consist of matrices in $U(n)$ of them form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \vdots & & \ddots & & \vdots \\ 0 & 0 & 1 & \cdots & 0 & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ & & (0) & & & & (B) & & & \end{pmatrix}$$

where B is an $(n-k) \times (n-k)$ dimensional unitary matrix.

- A similar argument shows that there are diffeomorphisms

$$\rho : U(n)/\left(U(k) \times U(n-k)\right) \xrightarrow{\cong} Gr_k(\mathbb{C}^n)$$

and

$$\rho : O(n)/\left(O(k) \times O(n-k)\right) \xrightarrow{\cong} Gr_k(\mathbb{R}^n)$$

Principal bundles define other fiber bundles in the presence of group actions. Namely, suppose $p : E \to B$ be a principal G - bundle and F is a space with a cellular right group action. Then the product space $E \times F$ has the "diagonal" group action $(e, f)g = (eg, fg)$. Consider the orbit space, $E \times_G F = (E \times F)/G$. Then the induced projection map

$$p : E \times_G F \to B$$

is a locally trivial fibration with fiber F.

For example we have the following important class of fiber bundles.

Proposition 1.5. *Let G be a compact Lie group and $K < H < G$ closed subgroups. Then the projection map of coset spaces*

$$p : G/K \to G/H$$

is a locally trivial fibration with fiber H/K.

Proof. Observe that $G/K \cong G \times_H H/K$ where H acts on H/K in the natural way. Moreover the projection map $p : G/K \to G/H$ is the projection can be viewed as the projection

$$G/K = G \times_H H/K \to G/H$$

and so is the H/K - fiber bundle induced by the H - principal bundle $G \to G/H$ via the action of H on the coset space H/K. □

Example

We know by the above examples, that $U(2)/U(1) \cong S^3$, and that $U(2)/U(1) \times U(1) \cong Gr_1(\mathbb{C}^2) = \mathbb{CP}^1 \cong S^2$. Therefore there is a principal $U(1)$ - fibration

$$p : U(2)/U(1) \to U(2)/U(1) \times U(1),$$

or equivalently, a principal $U(1) = S^1$ fibration

$$p : S^3 \to S^2.$$

This fibration is the well known "Hopf fibration", and is of central importance in both geometry and algebraic topology. In particular, as we will see later, the map from S^3 to S^2 gives an nontrivial element in the homotopy group $\pi_3(S^2)$, which from the naive point of view is quite surprising. It says, that, in a sense that can be made precise, there is a "three dimensional hole" in S^2 that cannot be filled. Many people (eg. Whitehead, see [**43**]) refer to this discovery as the beginning of modern homotopy theory.

The fact that the Hopf fibration is a locally trivial fibration also leads to an interesting geometric observation. First, it is not difficult to see directly (and we will prove this later) that one can take the upper and lower hemispheres of S^2 to be a cover of S^2 over which the Hopf fibration is trivial. That is, there are local trivializations,

$$\psi_+ : D^2_+ \times S^1 \to p^{-1}(D^2_+)$$

and

$$\psi_- : D^2_- \times S^1 \to p^{-1}(D^2_-)$$

where D^2_+ and D^2_- are the upper and lower hemispheres of S^2, respectively. Putting these two local trivializations together yields the following classical result:

THEOREM 1.6. *The sphere S^3 is homeomorphic to the union of two solid tori $D^2 \times S^1$ whose intersection is their common torus boundary, $S^1 \times S^1$.*

As another example of fiber bundles induced by principal bundles, suppose that

$$\rho : G \to GL(n, \mathbb{R})$$

is a representation of a topological group G, and $p : E \to B$ is a principal G bundle. Then let $\mathbb{R}^n(\rho)$ denote the space $\mathbb{R}^n$ with the action of G given by the representation ρ. Then the projection

$$E \times_G \mathbb{R}^n(\rho) \to B$$

is a vector bundle.

Exercise.

Let $p : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$ be the principal bundle described above. Let $\mathbb{R}^n$ have the standard $GL(n, \mathbb{R})$ representation. Proved that the induced vector bundle

$$p : V_k(\mathbb{R}^n) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$$

is isomorphic to the universal bundle γ_k described in the last section.

In the last section we discussed sections of vector bundles and in particular vector fields. For principal bundles, the existence of a section (or lack thereof) completely determines the triviality of the bundle.

THEOREM 1.7. *A principal G - bundle $p : E \to B$ is trivial if and only if it has a section.*

PROOF. If $p : E \to B$ is isomorphic to the trivial bundle $B \times G \to B$, then clearly it has a section. So we therefore only need to prove the converse.

Suppose $s : B \to E$ is a section of the principal bundle $p : E \to B$. Define the map

$$\psi : B \times G \to E$$

by $\psi(b, g) = s(b)g$ where multiplication on the right by g is given by the right G - action of G on E. It is straightforward to check that ψ is an isomorphism of principal G - bundles, and hence a trivialization of E. $\square$

1.3. Clutching Functions and Structure Groups. Let $p : E \to B$ be a locally trivial fibration with fiber F. Cover the basespace B by a collection of open sets $\{U_\alpha\}$ equipped with local trivializations $\psi_\alpha : U_\alpha \times F \xrightarrow{\cong} p^{-1}(U_\alpha)$. Let us compare the local trivializations on the intersection: $U_\alpha \cap U_\beta$:

$$U_\alpha \cap U_\beta \times F \xrightarrow[\cong]{\psi_\beta} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow[\cong]{\psi_\alpha^{-1}} U_\alpha \cap U_\beta \times F.$$

For every $x \in U_\alpha \cap U_\beta$, $\psi_\alpha^{-1} \circ \psi_\beta$ determines a homeomorphism of the fiber F. That is, this composition determines a map $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to Homeo(F)$. These maps are called the *clutching functions* of the fiber bundle. When the bundle is a real n - dimensional vector bundle then the clutching functions are of the form

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}).$$

Similarly, complex vector bundles have clutching functions that take values in $GL(n, \mathbb{C})$.

If $p : E \to B$ is a G - principal - bundle, then the clutching functions take values in G:

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to G.$$

In general for a bundle $p : E \to B$ with fiber F, the group in which the clutching values take values is called the *structure group* of the bundle. If no group is specified, then the structure group is the homeomorphism group $Homeo(F)$. For example if the bundle is smooth, then we are requiring the structure group to be the subgroup of diffeomorphisms, $Diffeo(F)$.

The clutching functions and the associated structure group completely determine the isomorphism type of the bundle. Namely, given an open covering of a space B, and a compatible family of clutching functions $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$, and a space F upon which the group acts, we can form the space

$$E = \bigcup_\alpha U_\alpha \times F / \sim$$

where if $x \in U_\alpha \cap U_\beta$, then $(x, f) \in U_\alpha \times F$ is identified with $(x, f\phi_{\alpha,\beta}(x)) \in U_\beta \times F$. E is the total space of a locally trivial fibration over B with fiber F and structure group G. If the original data of clutching functions came from locally trivializations of a bundle, then notice that the construction of E above yields a description of the total space of the bundle. Thus we have a description of the total space of a fiber bundle completely in terms of the family of clutching functions.

Suppose ζ is an n - dimensional vector bundle with projection map $p : E \to B$ and local trivializations $\psi_\alpha : U_\alpha \times \mathbb{R}^n \to p^{-1}(U_\alpha)$. Then the clutching functions take values in the general linear group

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}).$$

So the total space E has the form $E = \bigcup_\alpha U_\alpha \times \mathbb{R}^n / \sim$ as above. We can then form the corresponding principal $GL(n, \mathbb{R})$ bundle with total space

$$E_{GL} = \bigcup_\alpha U_\alpha \times GL(n, \mathbb{R})$$

with the same clutching functions. That is, for $x \in U_\alpha \cap U_\beta$, $(x, g) \in U_\alpha \times GL(n, \mathbb{R})$ is identified with $(x, g \cdot \phi_{\alpha,\beta}(x)) \in U_\beta \times GL(n, \mathbb{R})$. The principal bundle

$$p : E_{GL} \to B$$

is called the *associated principal bundle* to the vector bundle ζ, or sometimes is referred to as the *associated frame* bundle.

Observe also that this process is reversable. Namely if $p : P \to X$ is a principal $GL(n, \mathbb{R})$ - bundle with clutching functions $\theta_{\alpha,\beta} : V_\alpha \cap V_\beta \to GL(n, \mathbb{R})$, then there is an associated vector bundle $p : P_{\mathbb{R}^n} \to X$ where

$$P_{\mathbb{R}^n} = \bigcup_\alpha V_\alpha \times \mathbb{R}^n$$

where if $x \in V_\alpha \cap V_\beta$, then $(x, v) \in V_\alpha \times \mathbb{R}^n$ is identified with $(x, v \cdot \theta_{\alpha,\beta}(x)) \in V_\beta \times \mathbb{R}^n$.

This correspondence between vector bundles and principal bundles proves the following result:

THEOREM 1.8. *Let $Vect_n^{\mathbb{R}}(X)$ and $Vect_n^{\mathbb{C}}(X)$ denote the set of isomorphism classes of real and complex n - dimensional vector bundles ovr X respectively. For a Lie group G let $Prin_G(X)$ denote the set of isomorphism classes of principal G - bundles. Then there are bijective correspondences*

$$Vect_n^{\mathbb{R}}(X) \xrightarrow{\cong} Prin_{GL(n,\mathbb{R})}(X)$$
$$Vect_n^{\mathbb{C}}(X) \xrightarrow{\cong} Prin_{GL(n,\mathbb{C})}(X).$$

This correspondence and theorem 1.6 allows for the following method of determining whether a vector bundle is trivial:

COROLLARY 1.9. *A vector bundle $\zeta : p : E \to B$ is trivial if and only if its associated principal $GL(n)$ - bundle $p : E_{GL} \to B$ admits a section.*

The use of clutching functions also allows us to describe the tangent bundle of a manifold, τM, independently of any embedding into Euclidean space. This is done as follows.

Recall that a smooth, n - dimensional manifold M admits a differentiable atlas, which consists of an open cover $\{U_\alpha\}$ of M and homeomorphisms to open sets in $\mathbb{R}^n$:

$$f_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha \subset \mathbb{R}^n.$$

For intersections, let $U_{\alpha,\beta} = f_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$. The smooth structure on M comes from the requirement that on the intersections, the compositions

$$U_{\alpha,\beta} \xrightarrow{f_\alpha^{-1}} U_\alpha \cap U_\beta \xrightarrow{f_\beta} U_{\beta,\alpha}$$

are required to be diffeomorphisms of open sets in $\mathbb{R}^n$. Using a fixed trivialization of the tangent bundle of $\mathbb{R}^n$, $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, (and hence of the tangent bundle of any open set in $\mathbb{R}^n$), then we can define, for $x \in U_\alpha \cap U_\beta$, the linear isomorphism $\phi_{\alpha,\beta}(x) : \mathbb{R}^n \to \mathbb{R}^n$ to be the derivative of the above composition

$$\phi_{\alpha,\beta}(x) : \mathbb{R}^n = T_{f_\alpha(x)} U_{\alpha,\beta} \xrightarrow{D(f_\beta \circ f_\alpha^{-1})(x)} T_{f_\beta(x)} U_{\alpha,\beta} = \mathbb{R}^n.$$

This defines clutching functions

$$\phi_{\alpha,\beta} = D(f_\beta \circ f_\alpha^{-1}) : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$$

from which we can form the tangent bundle

$$TM = \bigcup_\alpha U_\alpha \times \mathbb{R}^n / \sim$$

with the identifications as described above.

Exercise.

Verify that this definition of the tangent bundle is isomorphic to the one given in section 1.1 when the manifold is embedded in Euclidean space.

Clutching functions and structure groups are also useful in studying structures on principal bundles and their associated vector bundles.

Definition 1.8. Let $p : P \to B$ be a principal G - bundle, and let $H < G$ be a subgroup. P is said to have a reduction of its structure group to H if and only if P is isomorphic to a bundle whose clutching functions take values in H:

$$\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to H < G.$$

Exercise.

Let $P \to X$ be a principal G - bundle. Then P has a reduction of its structure group to $H < G$ if and only if there is a principal H - bundle $\tilde{P} \to X$ and an isomorphism of G bundles,

$$\begin{array}{ccc}
\tilde{P} \times_H G & \xrightarrow{\cong} & P \\
\Big\downarrow & & \Big\downarrow \\
X & = & X
\end{array}$$

DEFINITION 1.9. Let $H < GL(n, \mathbb{R})$. Then an H - structure on an n - dimensional vector bundle ζ is a reduction of the structure group of its associated $GL(n, \mathbb{R})$ - principal bundle to H.

Examples.

- A $\{1\} < GL(n, \mathbb{R})$ - structure on a vector bundle (or its associated principal bundle) is a trivialization or *framing* of the bundle. A framed manifold is a manifold with a framing of its tangent bundle.
- Given a $2n$ - dimensional real vector bundle ζ, an *almost complex structure* on ζ is a $GL(n, \mathbb{C}) < GL(2n, \mathbb{R})$ structure on its associated principal bundle. An almost complex structure on a manifold is an almost complex structure on its tangent bundle.

We now study two examples of vector bundle structures in some detail: Euclidean structures, and orientations.

Example 1: $\mathbf{O}(n)$ - structures and Euclidean structures on vector bundles.

Recall that a Euclidean vector space is a real vector space V together with a positive definite quadratic function

$$\mu : V \to \mathbb{R}.$$

Specifically, the statement that μ is quadratic means that it can written in the form

$$\mu(v) = \sum_i \alpha_i(v)\beta_i(v)$$

where each α_i and $\beta_i : V \to \mathbb{R}$ is linear. The statement that μ is positive definite means that

$$\mu(v) > 0 \quad \text{for } v \neq 0.$$

Positive definite quadratic functions arise from, and give rise to inner products (i.e symmetric bilinear pairings $(v, w) \to v \cdot w$) defined by

$$v \cdot w = \frac{1}{2}(\mu(v + w) - \mu(v) - \mu(w)).$$

Notice that if we write $|v| = \sqrt{v \cdot v}$ then $|v|^2 = \mu(v)$. So in particular there is a metric on V.

This notion generalizes to vector bundles in the following way.

DEFINITION 1.10. A Euclidean vector bundle is a real vector bundle $\zeta : p : E \to B$ together with a map

$$\mu : E \to \mathbb{R}$$

which when restricted to each fiber is a positive definite quadratic function. That is, μ induces a Euclidean structure on each fiber.

Exercise.

Show that an $O(n)$ - structure on a vector bundle ζ gives rise to a Euclidean structure on ζ. Conversely, a Euclidean structure on ζ gives rise to an $O(n)$ - structure.

Hint. Make the constructions directly in terms of the clutching functions.

DEFINITION 1.11. A smooth Euclidean structure on the tangent bundle $\mu : TM \to \mathbb{R}$ is called a *Riemannian* structure on M.

Exercises.

1. *Existence theorem for Euclidean metrics.* Using a partition of unity, show that any vector bundle over a paracompact space can be given a Euclidean metric.

2. *Isometry theorem.* Let μ and μ' be two different Euclidean metrics on the same vector bundle $\zeta : p : E \to B$. Prove that there exists a homeomorphism $f : E \to E$ which carries each fiber isomorphically onto itself, so that the composition $\mu \circ f : E \to \mathbb{R}$ is equal to μ'. (*Hint.* Use the fact that every positive definite matrix A can be xpressd uniquely as the square of a positive definite matrix $\sqrt{A}$. The power series expansion

$$\sqrt{(tI + X)} = \sqrt{t}(I + \frac{1}{2t}X - \frac{1}{8t^2}X^2 + - \cdots),$$

is valid providing that the characteristic roots of $tI + X = A$ lie between 0 and $2t$. This shows that the function $A \to \sqrt{A}$ is smooth.)

Example 2: S$L(n, \mathbb{R})$ - structures and orientations.

Recall that an orientation of a real n - dimensional vector space V is an equivalence class of basis for V, where two bases $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_n\}$ are equivalent (i.e determine the same orientation) if and only if the change of basis matrix $A = (a_{i,j})$, where $w_i = \sum_j a_{i,j} v_j$ has positive determinant, $det(A) > 0$. Let $Or(V)$ be the set of orientations of V. Notice that $Or(V)$ is a two point set.

For a vector bundle $\zeta : p : E \to B$, an orientation is a continuous choice of orientations of each fiber. Said more precisely, we may define the "orientation double cover" $Or(\zeta)$ to be the two - fold covering space

$$Or(\zeta) = E_{GL} \times_{GL(n,\mathbb{R})} Or(\mathbb{R}^n)$$

where E_{GL} is the associated principal bundle, and where $GL(n,\mathbb{R})$ acts on $Or(\mathbb{R}^n)$ by matrix multiplication on a basis representing the orientation.

Definition 1.12. ζ is *orientable* if the orientation double cover $Or(\zeta)$ admits a section. A choice of section is an orientation of ζ.

This definition is reasonable, in that a continuous section of $Or(\zeta)$ is a continuous choice of orientations of the fibers of ζ.

Recall that $SL(n,\mathbb{R}) < GL(n,\mathbb{R})$ and $SO(n) < O(n)$ are the subgroups consisting of matrices with positive determinants. The following is now straightforward.

Theorem 1.10. *An n - dimensional vector bundle ζ has an orientation if and only if it has a $SL(n,\mathbb{R})$ - structure. Similarly a Euclidean vector bundle is orientable if and only if it has a $SO(n)$ - structure. Choices of these structures are equivalent to choices of orientations.*

Finally, a manifold is said to be orientable if its tangent bundle τM is orientable.

2. Pull Backs and Bundle Algebra

In this section we describe the notion of the pull back of a bundle along a continuous map. We then use it to describe constructions on bundles such as direct sums, tensor products, symmetric and exterior products, and homomorphisms. We use direct sums to give a description of the tangent bundle of projective space in terms of line bundles. We then study group completions, and define the notion of K - theory. Finally we use exterior products of bundles to study differential forms, and introduce the notions of connections, and curvature.

2.1. Pull Backs. Let $p : E \to B$ be a fiber bundle with fiber F. Let $A \subset B$ be a subspace. The restriction of E to A, written $E_{|A}$ is simply given by

$$E_{|A} = p^{-1}(A).$$

The restriction of the projection $p : E_{|A} \to A$ is clearly still a locally trivial fibration with fiber F.

This notion generalizes from inclusions of subsets $A \subset B$ to general maps $f : X \to B$ in the form of the *pull back* bundle over X, $f^*(E)$. This bundle is defined by

$$f^*(E) = \{(x, u) \in X \times E : f(x) = p(u)\}.$$

PROPOSITION 1.11. *The map*

$$\begin{aligned} p_f : f^*(E) &\to X \\ (x, u) &\to x \end{aligned}$$

is a locally trivial fibration with fiber F. Furthermore if $\iota : A \hookrightarrow B$ is an inclusion of a subspace, then the pull-back $\iota^*(E)$ is equal to the restriction $E_{|A}$.

PROOF. Let $\{U_\alpha\}$ be a collection of open sets in B and $\psi_\alpha : U_\alpha \times F \to p^{-1}(U_\alpha)$ local trivializations of the bundle $p : E \to B$. Then $\{f^{-1}(U_\alpha)\}$ is an open cover of X, and the maps

$$\psi_\alpha(f) : f^{-1}(U_\alpha) \times F \to p_f^{-1}(f^{-1}(U_\alpha))$$

defined by $(x, y) \to (x, \psi_\alpha(f(x), y))$ are clearly local trivializations.

This proves the first statement in the proposition. The second statement is obvious. $\square$

We now use the pull back construction to define certain algebraic constructions on bundles.

Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be fiber bundles with fibers F_1 and F_2 respectively. Then the cartesian product

$$p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$$

is clearly a fiber bundle with fiber $F_1 \times F_2$. In the case when $B_1 = B_2 = B$, we can consider the pull back (or restriction) of this cartesian product bundle via the diagonal map

$$\Delta : B \hookrightarrow B \times B$$
$$x \to (x, x).$$

Then the pull-back $\Delta^*(E_1 \times E_2) \to B$ is a fiber bundle with fiber $F_1 \times F_2$, is defined to be the internal product, or *Whitney sum* of the fiber bundles E_1 and E_2. It is written

$$E_1 \oplus E_2 = \Delta^*(E_1 \times E_2).$$

Notice that if E_1 and E_2 are G_1 and G_2 principal bundles respectively, then $E_1 \oplus E_2$ is a principal $G_1 \times G_2$ - bundle. Similarly, if E_1 and E_2 are n and m dimensional vector bundles respectively, then $E_1 \oplus E_2$ is an $n+m$ - dimensional vector bundle. $E_1 \oplus E_2$ is called the *Whitney sum* of the vector bundles. Notice that the clutching functions of $E_1 \oplus E_2$ naturally lie in $GL(n, \mathbb{R}) \times GL(m, \mathbb{R})$ which is thought of as a subgroup of $GL(n+m, \mathbb{R})$ consisting of $(n+m) \times (n+m)$ - dimensional matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in GL(n, \mathbb{R})$ and $B \in GL(m, \mathbb{R})$.

We now describe other algebraic constructions on vector bundles. The first is a generalization of the fact that a given a subspace of a vector space, the ambient vector space splits as a direct sum of the subspace and the quotient space.

Let $\eta : E^\eta \to B$ be a k - dimensional vector bundle and $\zeta : E^\zeta \to B$ an n - dimensional bundle. Let $\iota : \eta \hookrightarrow \zeta$ be a linear embedding of vector bundles. So on each fiber ι is a linear embedding of a k - dimensional vector space into an n - dimensional vector space. Define ζ/η to be the vector bundle whose fiber at x is E^ζ_x/E^η_x.

Exercise.

Verify that ζ/η is an $n-k$ - dimensional vector bundle over B.

THEOREM 1.12. *There is a splitting of vector bundles*

$$\zeta \cong \eta \oplus \zeta/\eta.$$

PROOF. Give ζ a Euclidean structure. Define $\eta^\perp \subset \zeta$ to be the subbundle whose fiber at x is the orthogonal complement

$$E^{\eta^\perp}_x = \{v \in E^\zeta_x : v \cdot w = 0 \text{ for all } w \in E^\eta_x\}$$

Then clearly there is an isomorphism of bundles

$$\eta \oplus \eta^\perp \cong \zeta.$$

Moreover the composition

$$\eta^{\perp} \subset \zeta \to \zeta/\eta$$

is also an isomorphism. The theorem follows. $\square$

COROLLARY 1.13. *Let ζ be a Euclidean n - dimensional vector bundle. Then ζ has a $O(k) \times O(n-k)$ - structure if and only if ζ admits a k - dimensional subbundle $\eta \subset \zeta$.*

We now describe the dual of a vector bundle. So let $\zeta : E^{\zeta} \to B$ be an n - dimensional bundle. Its dual, $\zeta^* : E^{\zeta^*} \to B$ is the bundle whose fiber at $x \in B$ is the dual vector space $E_x^{\zeta^*} = Hom(E^{\zeta}, \mathbb{R})$. If

$$\{\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$$

are clutching functions for ζ, then

$$\{\phi^*_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$$

form the clutching functions for ζ^*, where $\phi^*_{\alpha,\beta}(x)$ is the adjoint (transpose) of $\phi_{\alpha,\beta}(x)$. The dual of a complex bundle is defined similarly.

Exercise.

Prove that ζ and ζ^* are isomorophic vector bundles. *HInt.* Give ζ a Euclidean structure.

Now let $\eta : E^{\eta} \to B$ be a k - dimensional, and as above, $\zeta : E^{\zeta} \to B$ an n - dimensional bundle. We define the tensor product bundle $\eta \otimes \zeta$ to be the bundle whose fiber at $x \in B$ is the tensor product of vector spaces, $E^{\eta}_x \otimes E^{\zeta}_x$. The clutching fucntions can be thought of as compositions of the form

$$\phi^{\eta\otimes\zeta}_{\alpha,\beta} : U_\alpha \cap U_\beta \xrightarrow{\phi^{\eta}_{\alpha,\beta} \times \phi^{\zeta}_{\alpha,\beta}} GL(k, \mathbb{R}) \times GL(n, \mathbb{R}) \xrightarrow{\otimes} GL(kn, \mathbb{R})$$

where the tensor product of two linear transformations $A : V_1 \to V_2$ and $B : W_1 \to W_2$ is the induced linear transformation $A \otimes B : V_1 \otimes W_1 \to V_2 \otimes W_2$.

With these two constructions we are now able to define the "homomorphism bundle", $Hom(\eta, \zeta)$. This will be the bundle whose fiber at $x \in B$ is the $k \cdot m$ - dimensional vector space of linear transformations

$$Hom(E^{\eta}_x, E^{\zeta}_x) \cong (E^{\eta}_x)^* \otimes E^{\zeta}_x.$$

So as bundles we can define

$$Hom(\eta, \zeta) = \eta^* \otimes \zeta.$$

Observation. A bundle homomorphsim $\theta : \eta \to \zeta$ assigns to every $x \in B$ a linear transformation of the fibers, $\theta_x : E^{\eta}_x \to E^{\zeta}_x$. Thus a bundle homomorphism can be thought of as a section of the bundle $Hom(\eta, \zeta)$. That is, there is a bijection between the space of sections, $\Gamma(Hom(\eta, \zeta))$ and the space of bundle homomorphisms, $\{\theta : \eta \to \zeta\}$.

2.2. The tangent bundle of Projective Space. We now use these constructions to identify the tangent bundle of projective spaces, $\tau\mathbb{RP}^n$ and $\tau\mathbb{CP}^n$. We study the real case first.

Recall the canonical line bundle, $\gamma_1 : E^{\gamma_1} \to \mathbb{RP}^n$. If $[x] \in \mathbb{RP}^n$ is viewed as a line in $\mathbb{R}^{n+1}$, then the fiber $E^{\gamma_1}_{[x]}$ is the one dimensional space of vectors in the line $[x]$. Thus γ_1 has a natural embedding into the trivial $n+1$ - dimensional bundle $\epsilon : \mathbb{RP}^n \times \mathbb{R}^{n+1} \to \mathbb{RP}^n$ via

$$E^{\gamma_1} = \{([x], u) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : u \in [x]\} \hookrightarrow \mathbb{RP}^n \times \mathbb{R}^{n+1}.$$

Let $\gamma_1^\perp$ be the n - dimensional orthogonal complement bundle of this embedding.

THEOREM 1.14. *There is an isomorphism of the tangent bundle with the homomorphism bundle*

$$\tau\mathbb{RP}^n \cong Hom(\gamma_1, \gamma_1^\perp)$$

PROOF. Let $p : S^n \to \mathbb{RP}^n$ be the natural projection. For $x \in S^n$, recall that the tangent space of S^n can be described as

$$T_x S^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : x \cdot v = 0\}.$$

Notice that $(x, v) \in T_x S^n$ and $(-x, -v) \in T_{-x} S^n$ have the same image in $T_{[x]}\mathbb{RP}^n$ under the derivative $Dp : TS^n \to T\mathbb{RP}^n$. Since p is a local diffeomorphism, $Dp(x) : T_x S^n \to T_{[x]}\mathbb{RP}^n$ is an isomorphism for every $x \in S^n$. Thus $T_{[x]}\mathbb{RP}^n$ can be identified with the space of pairs

$$T_{[x]}\mathbb{RP}^n = \{(x, v), (-x, -v) : x, v \in \mathbb{R}^{n+1},\ |x| = 1,\ x \cdot v = 0\}.$$

If $x \in S^n$, let $L_x = [x]$ denote the line through $\pm x$ in $\mathbb{R}^{n+1}$. Then a pair $(x, v), (-x, -v) \in T_{[x]}\mathbb{RP}^n$ is uniquely determined by a linear transformation

$$\ell : L_x \to L^\perp$$
$$\ell(tx) = tv.$$

Thus $T_{[x]}\mathbb{RP}^n$ is canonically isomorphic to $Hom(E_x^{\gamma_1}, E_x^{\gamma_1^\perp})$, and so

$$\tau\mathbb{RP}^n \cong Hom(\gamma_1, \gamma_1^\perp),$$

as claimed. □

The following description of the $\tau\mathbb{RP}^n \oplus \epsilon_1$ will be quite helpful to us in future calculations of characteristic classes.

THEOREM 1.15. *The Whiney sum of the tangent bundle and a trivial line bundle, $\tau\mathbb{RP}^n \oplus \epsilon_1$ is isomorphic to the Whitney sum of $n+1$ copies of the canonical line bundle γ_1,*

$$\tau\mathbb{RP}^n \oplus \epsilon_1 \cong \oplus_{n+1}\gamma_1.$$

PROOF. Consider the line bundle $Hom(\gamma_1, \gamma_1)$ over $\mathbb{RP}^n$. This line bundle is trivial since it has a canonical nowhere zero section

$$\iota(x) = 1 : E^{\gamma_1}_{[x]} \to E^{\gamma_1}_{[x]}.$$

We therefore have

$$\begin{aligned}
\tau\mathbb{RP}^n \oplus \epsilon_1 &\cong \tau\mathbb{RP}^n \oplus Hom(\gamma_1, \gamma_1) \\
&\cong Hom(\gamma_1, \gamma_1^\perp) \oplus Hom(\gamma_1, \gamma_1) \\
&\cong Hom(\gamma_1, \gamma_1^\perp \oplus \gamma_1) \\
&\cong Hom(\gamma_1, \epsilon_{n+1}) \\
&\cong \oplus_{n+1}\gamma_1^* \\
&\cong \oplus_{n+1}\gamma_1
\end{aligned}$$

as claimed. □

The following are complex analogues of the above theorems and are proved in the same way.

THEOREM 1.16.

$$\tau\mathbb{CP}^n \cong_{\mathbb{C}} Hom_{\mathbb{C}}(\gamma_1, \gamma_1^\perp)$$

and

$$\tau\mathbb{CP}^n \oplus \epsilon_1 \cong \oplus_{n+1}\gamma_1^*,$$

where $\cong_{\mathbb{C}}$ *and* $Hom_{\mathbb{C}}$ *denote isomorphisms and homomorphisms of complex bundles, respectively.*

Note. γ^* is not isomorphic as *complex* vector bundles to γ_1. It is isomorphic to γ_1 with the *conjugate* complex structure. We will discuss this phenomenon more later.

2.3. K - theory. Let $Vect^*(X) = \oplus_{n\geq 0} Vect^n(X)$ where, as above, $Vect^n(X)$ denotes the set of isomorphism classes of n - dimensional complex bundles over X. $Vect^*_{\mathbb{R}}(X)$ denotes the analogous set of real vector bundles. In both these cases $Vect^0(X)$ denotes, by convention, the one point set, representing the unique zero dimensional vector bundle.

Now the Whitney sum operation induces pairings

$$Vect^n(X) \times Vect^m(X) \xrightarrow{\oplus} Vect^{n+m}(X)$$

which in turn give $Vect^*(X)$ the structure of an abelian monoid. Notice that it is indeed abelian because given vector bundles η and ζ we have an obvious isomorphism

$$\eta \oplus \zeta \cong \zeta \oplus \eta.$$

The “zero” in this monoid structure is the unique element of $Vect^0(X)$.

Given an abelian monoid, A, there is a construction due to Grothendieck of its *group completion* $K(A)$. Formally, $K(A)$ is the smallest abelian group equipped with a homomorphism of monoids, $\iota : A \to K(A)$. It is smallest in the sense if G is any abelian group and $\phi : A \to G$ is any homomorphism of monoids, then there is a unique extension of ϕ to a map of abelian groups $\bar{\phi} : K(A) \to G$ making the diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & K(A) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \bar{\phi}} \\ G & = & G \end{array}$$

This formal property, called the *universal property*, characterizes $K(A)$, and can be taken to be the definition. However there is a much more explicit description. Basically the group completion $K(A)$ is obtained by formally adjoining inverses to the elements of A. That is, an element of $K(A)$ can be thought of as a formal difference $\alpha - \beta$, where $\alpha,\ \beta \in A$. Strictly speaking we have the following definition.

Definition 1.13. Let $F(A)$ be the free abelian group generated by the elements of A, and let $R(A)$ denote the subgroup of $F(A)$ generated by elements of the form $a \oplus b - (a+b)$ where $a,\ b \in A$. Here "$\oplus$" is the group operation in the free abelian group and "$+$" is the addition in the monoid structure of A. We then define the Grothendieck group completion $K(A)$ to be the quotient group

$$K(A) = F(A)/R(A).$$

Notice that an element of $K(A)$ is of the form

$$\theta = \sum_i n_i a_i - \sum_j m_j b_j$$

where the n_i's and m_j's are positive integers, and each a_i and $b_j \in A$. That is, by the relations in $R(A)$, we may write

$$\theta = \alpha - \beta$$

where $\alpha = \sum_i n_i a_i \in A$, and $\beta = \sum_j m_j b_j \in A$.

Notice also that the composition $\iota : A \subset F(A) \to F(A)/R(A) = K(A)$ is a homomorphism of monoids, and clearly has the universal property described above. We can now make the following definition.

Definition 1.14. Given a space X, its complex and real (or orthogonal) K - theories are defined to be the Grothendieck group completions of the abelian monoids of isomorphism classes of vector bundles:

$$K(X) = K(Vect^*(X))$$

$$KO(X) = K(Vect^*_{\mathbb{R}}(X)$$

An element $\alpha = \zeta - \eta \in K(X)$ is often referred to as a "virtual vector bundle" over X.

Notice that the discusion of the tangent bundles of projective spaces above (section 2.2) can be interpreted in K -theoretic language as follows:

PROPOSITION 1.17. *As elements of $K(\mathbb{CP}^n)$, we have the equation*

$$[\tau \mathbb{CP}^n] = (n+1)[\gamma_1^*] - [1]$$

where $[m] \in K(X)$ refers to the class represented by the trivial bundle of dimension m. Similarly, in the orthogonal K - theory $KO(\mathbb{RP}^n)$ we have the equation

$$[\tau \mathbb{RP}^n] = (n+1)[\gamma_1] - [1].$$

Notice that for a point, $Vect^*(pt) = \mathbb{Z}^+$, the nonnegative integers, since there is precisely one vector bundle over a point (i.e vector space) of each dimension. Thus

$$K(pt) \cong KO(pt) \cong \mathbb{Z}.$$

Notice furthermore that by taking tensor products there are pairings

$$Vect^m(X) \times Vect^n(X) \xrightarrow{\otimes} Vect^{mn}(X).$$

The following is verified by a simple check of definitions.

PROPOSITION 1.18. *The tensor product pairing of vector bundles gives $K(X)$ and $KO(X)$ the structure of commutative rings.*

Now given a bundle ζ over Y, and a map $f : X \to Y$, we saw in the previous section how to define the pull-back, $f^*(\zeta)$ over X. This defines a homomorphism of abelian monoids

$$f^* : Vect^*(Y) \to Vect^*(X).$$

After group completing we have the following:

PROPOSITION 1.19. *A continuous map $f : X \to Y$ induces ring homomorphisms,*

$$f^* : K(Y) \to K(X)$$

and

$$f^* : KO(Y) \to KO(X).$$

In particular, consider the inclusion of a basepoint $x_0 \hookrightarrow X$. This induces a map of rings, called the augmentation,

$$\epsilon : K(X) \to K(x_0) \cong \mathbb{Z}.$$

This map is a split surjection of rings, because the constant map $c : X \to x_0$ induces a right inverse of ϵ, $c^* : \mathbb{Z} = K(x_0) \to K(X)$. Notice that the augmentation can be viewed as the "dimension" map in that when restricted to the monoid $Vect^*(X)$, then $\epsilon : Vect^m(X) \to \{m\} \subset \mathbb{Z}$. That is, on an element $\zeta - \eta \in K(X)$, $\epsilon(\zeta - \eta) = dim(\zeta) - dim(\eta)$. We then define the reduced K -theory as follows.

Definition 1.15. The reduced K - theory of X, denoted $\tilde{K}(X)$ is defined to be the kernel of the augmentation map

$$\tilde{K}(X) = ker\{\epsilon : K(X) \to \mathbb{Z}\}$$

and so consists of classes $\zeta - \eta \in K(X)$ such that $dim(\zeta) = dim(\eta)$. The reduced orthogonal K - theory, $\tilde{KO}(X)$ is defined similarly.

The following is an immediate consequence of the above observations:

Proposition 1.20. *There are natural splittings of rings*

$$K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$$
$$KO(X) \cong \tilde{KO}(X) \oplus \mathbb{Z}.$$

Clearly then the reduced K - theory is the interesting part of K - theory. Notice that a bundle $\zeta \in Vect^n(X)$ determines the element $[\zeta] - [n] \in \tilde{K}(X)$, where $[n]$ is the K - theory class of the trivial n - dimensional bundle.

The definitions of K - theory are somewhat abstract. The following discussion makes it clear precisely what K - theory measures in the case of compact spaces.

Definition 1.16. Let ζ and η be vector bundles over a space X. ζ and η are said to be *stably isomorphic* if for some m and n, there is an isomorphism

$$\zeta \oplus \epsilon_n \cong \eta \oplus \epsilon_m$$

where, as above, ϵ_k denotes the trivial bundle of dimension k. We let $\mathcal{S}Vect(X)$ denote the set of stable isomorphism classes of vector bundles over X.

Notice that $\mathcal{S}Vect(X)$ is also an abelian monoid under Whitney sum, and that since any two trivial bundles are stably isomorphic, and that adding a trivial bundle to a bundle does not change the stable isomorphic class, then any trivial bundle represents the zero element of $\mathcal{S}Vect(X)$.

THEOREM 1.21. *Let X be a compact space, then $\mathcal{S}Vect(X)$ is an abelian group and is isomorphic to the reduced K -theory,*

$$\mathcal{S}Vect(X) \cong \tilde{K}(X).$$

PROOF. A main component of the proof is the following result, which we will prove in the next chapter when we study the classification of vector bundles.

THEOREM 1.22. *Every vector bundle over a compact space can be embedded in a trivial bundle. That is, if ζ is a bundle over a compact space X , then for sufficiently large $N > 0$, there is bundle embedding*

$$\zeta \hookrightarrow \epsilon_N.$$

We use this result in the following way in order to prove the above theorem. Let ζ be a bundle over a compact space X. Then by this result we can find an embedding $\zeta \hookrightarrow \epsilon_N$. Let $\zeta^{\perp}$ be the orthogonal complement bundle to this embedding. So that

$$\zeta \oplus \zeta^{\perp} = \epsilon_N.$$

Since ϵ_N represents the zero element in $\mathcal{S}Vect(X)$, then as an equation in $\mathcal{S}Vect(X)$ this becomes

$$[\zeta] + [\zeta^{\perp}] = 0.$$

Thus every element in $\mathcal{S}Vect(X)$ is invertible in the monoid structure, and hence $\mathcal{S}Vect(X)$ is an abelian group.

To prove that $\mathcal{S}Vect(X)$ is isomorphic to $\tilde{K}(X)$, notice that the natural surjection of $Vect^*(X)$ onto $\mathcal{S}Vect(X)$ is a morphism of abelian monoids, and since $\mathcal{S}Vect(X)$ is an abelian group, this surjection extends linearly to a surjective homomorphism of abelian groups,

$$\rho : K(X) \to \mathcal{S}Vect(X).$$

Since $[\epsilon_n] = [n] \in K(X)$ maps to zero in $\mathcal{S}Vect(X)$ under ρ, this map factors through a surjective homomorphism from reduced K - theory, which by abuse of notation we also call ρ,

$$\rho : \tilde{K}(X) \to \mathcal{S}Vect(X).$$

To prove that ρ is a injective (and hence an isomorphism), we will construct a left inverse to ρ. This is done by considering the composition

$$Vect^*(X) \xrightarrow{\iota} K(X) \to \tilde{K}(X)$$

which is given by mapping an n - dimensional bundle ζ to $[\zeta]-[n]$. This map clearly sends two bundles which are stably isomorphic to the same class in $\tilde{K}(X)$, and hence factors through a homomorphism

$$j : \mathcal{S}Vect(X) \to \tilde{K}(X).$$

By checking its values on bundles, it becomes clear that the composition $j \circ \rho : \tilde{K}(X) \to \mathcal{S}Vect(X) \to \tilde{K}(X)$ is the identity map. This proves the theorem. $\square$

We end this section with the following observation. As we said above, in the next chapter we will study the classification of bundles. In the process we will show that homotopic maps induce isomorphic pull - back bundles, and therefore homotopy equivalences induce bijections, via pulling back, on the sets of isomorphism classes of bundles. This tells us that K -theory is a "homotopy invariant" of topological spaces and continuous maps between them. More precisely, the results of the next chapter will imply the following important properties of K - theory.

THEOREM 1.23. *Let $f : X \to Y$ and $g : X \to Y$ be homotopic maps. then the pull back homomorphisms are equal*

$$f^* = g^* : K(Y) \to K(X)$$

and

$$f^* = g^* : KO(Y) \to KO(X).$$

This can be expressed in categorical language as follows: (Notice the similarity of role K - theory plays in the following theorem to cohomology theory.)

THEOREM 1.24. *The assignments $X \to K(X)$ and $X \to KO^*(X)$ are contravariant functors from the category of topological spaces and homotopy classes of continuous maps to the category of rings and ring homomorphisms.*

2.4. Differential Forms. In the next two sections we describe certain differentiable constructions on bundles over smooth manifolds that are basic in geometric analysis. We begin by recalling some "multilinear algebra".

Let V be a vector space over a field k. Let $T(V)$ be the associated tensor algebra

$$T(V) = \oplus_{n \geq 0} V^{\otimes n}$$

where $V^0 = k$. The algebra structure is comes from the natural pairings

$$V^{\otimes n} \otimes V^{\otimes m} \xrightarrow{=} V^{\otimes (n+m)}.$$

Recall that the *exterior algebra*

$$\Lambda(V) = T(V)/\mathcal{A}$$

where $\mathcal{A} \subset T(V)$ is the two sided ideal generated by $\{a \otimes b + b \otimes a : \, a, \, b \in V\}$.

The algebra $\Lambda(V)$ inherits the grading from the tensor algebra, $\Lambda(V) = \oplus_{n \geq 0} \Lambda^k(V)$, and the induced multiplication is called the "wedge product", $u \wedge v$. Recall that if V is an n - dimensional vector space, $\Lambda^k(V)$ is an $\binom{n}{k}$ - dimensional vector space.

Assume now that V is a real vector space. An element of the dual space, $(V^{\otimes n})^* = Hom(V^{\otimes n}, \mathbb{R})$ is a multilinear form $V \times \cdots \times V \to \mathbb{R}$. An element of the dual space $(\Lambda^k(V))^*$ is an alternating form, i.e a multilinear function θ so that

$$\theta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = sgn(\sigma)\theta(v_1, \cdots, v_k)$$

where $\sigma \in \Sigma_k$ is any permutation.

Let $\mathcal{A}^k(V) = (\Lambda^k(V))^*$ be the space of alternating k - forms. Let $U \subset \mathbb{R}^n$ be an open set. Recall the following definition.

Definition 1.17. A differential k - form on the open set $U \subset \mathbb{R}^n$ is a smooth function

$$\omega : U \to \mathcal{A}^k(\mathbb{R}^n).$$

By convention, 0 -forms are just smooth functions, $f : U \to \mathbb{R}$. Notice that given such a smooth function, its differential, df assigns to a point $x \in U \subset \mathbb{R}^n$ a linear map on tangent spaces, $df(x) : \mathbb{R}^n = T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R} = \mathbb{R}$. That is, $df : U \to (\mathbb{R}^n)^*$, and hence is a one form on U.

Let $\Omega^k(U)$ denote the space of k - forms on the open set U. Recall that any k -form $\omega \in \Omega^k(U)$ can be written in the form

$$\omega(x) = \sum_I f_I(x)dx_I \tag{2.1}$$

where the sum is taken over all sequences of length k of integers from 1 to n, $I = (i_1, \cdots, i_k)$, $f_I : U \to \mathbb{R}$ is a smooth function, and where

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Here dx_i denotes the differential of the function $x_i : U \subset \mathbb{R}^n \to \mathbb{R}$ which is the projection onto the i^{th} - coordinate.

Recall also that there is an exterior derivative,

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

defined by

$$d(fdx_I) = df \wedge dx_I == \sum_{j=1}^{k} \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

A simple calculation shows that $d^2(\omega) = d(d\omega) = 0$, using the symmetry of second order partial derivatives.

These constructions can be extended to arbitrary manifolds in the following way. Given an n - dimensional smooth manifold M, let $\Lambda^k(\tau(M))$ be the $\binom{n}{k}$ - dimensional vector bundle whose fiber at $x \in M$ is the k - fold exterior product, of the tangent space, $\Lambda^k(T_xM)$.

Exercise.

Define clutching functions of $\Lambda^k(\tau(M))$ in terms of clutching functions of the tangent bundle, $\tau(M)$

Definition 1.18. A differential k-form on M is a section of the dual bundle,

$$\Lambda^k(\tau(M))^* \cong \Lambda^k(\tau^*(M)) \cong Hom(\Lambda^k(\tau(M)), \epsilon_1).$$

That is, the space of k -forms is given by the space of sections,

$$\Omega^k(M) = \Gamma(\Lambda^k(\tau^*(M))).$$

So a k -form $\omega \in \Omega^k(M)$ assigns to $x \in M$ an alternating k form on its tangent space,

$$\omega(x) : T_xM \times \cdots \times T_xM \to \mathbb{R}.$$

and hence given a local chart with a local coordinate system, then locally ω can be written in the form (2.1).

Since differentiation is a local operation, we may extend the definition of the exterior derivative of forms on open sets in $\mathbb{R}^n$ to all n - manifolds,

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

In particular, the zero forms are the space of functions, $\Omega^0(M) = C^\infty(M; \mathbb{R})$, and for $f \in \Omega^0(M)$, then $df \in \Omega^1(M) = \Gamma(\tau(M)^*)$ is the 1 -form defined by the differential,

$$df(x) : T_xM \to T_{f(x)}\mathbb{R} = \mathbb{R}.$$

Now as above, $d^2(\omega) = 0$ for any form ω. Thus we have a cochain complex, called the *deRham* complex,

$$\begin{aligned} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \\ \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0. \end{aligned} \tag{2.2}$$

Recall that a k - form ω with $d\omega = 0$ is called a *closed* form. A k - form ω in the image of d, i.e $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$ is called an *exact* form. The quotient vector space of closed forms modulo exact forms defined the "deRham cohomology" group:

Definition 1.19.

$$H^k_{deRham}(M) = \{\text{closed } k \text{ - forms}\}/\{\text{exact } k \text{ - forms}\}.$$

The famous *de Rham theorem* asserts that these cohomology groups are isomorphic to singular cohomology with $\mathbb{R}$ - coefficients. To see the relationship, let $C_k(M)$ be the space of k - dimensional singular chains on M, (i.e the free abelian group generated by smooth singular simplices $\sigma : \Delta^k \to M$), and let

$$C^k(M;\mathbb{R}) = Hom(C_k(M), \mathbb{R})$$

be the space of real valued singular cochains. Notice that a k -form ω gives rise to a k - dimensional singular cochain in that it acts on a singular simplex $\sigma : \Delta^k \to M$ by

$$\langle \omega, \sigma \rangle = \int_\sigma \omega.$$

This defines a homomorphism

$$\gamma : \Omega^k(M) \to C^k(M;\mathbb{R})$$

for each k.

Exercise. Prove that γ is a map of cochain complexes. That is,

$$\gamma(d\omega) = \delta\gamma(\omega)$$

where $\delta : C^k(M;\mathbb{R}) \to C^{k+1}(M;\mathbb{R})$ is the singular coboundary operator. *Hint.* Use Stokes' theorem.

We refer the reader to [**5**] for a proof of the *deRham Theorem:*

THEOREM 1.25. *The map of cochain complexes,*

$$\gamma : \Omega^*(M) \to C^*(M;\mathbb{R})$$

is a chain homotopy equivalence. Therefore it induces an isomorphism in cohomology

$$H^*_{deRham}(M) \xrightarrow{\cong} H^*(M;\mathbb{R}).$$

2.5. Connections and Curvature. In modern geometry, differential topology, and geometric analysis, one often needs to study not only smooth functions on a manifold, but more generally, spaces of smooth sections of a vector bundle $\Gamma(\zeta)$. (Notice that sections of bundles are indeed a generalization of smooth functions in that the space of sections of the n - dimensional trivial bundle over a manifold M, $\Gamma(\epsilon_n) = C^\infty(M;\mathbb{R}^n) = \oplus_n C^\infty(M;\mathbb{R})$.) Similarly, one needs to study differential forms that take values in vector bundles. These are defined as follows.

DEFINITION 1.20. Let ζ be a smooth bundle over a manifold M. A differential k - form with values in ζ is defined to be a smooth section of the bundle of homomorphisms, $Hom(\Lambda^k(\tau(M)), \zeta) = \Lambda^k(\tau(M)^*) \otimes \zeta$.

We write the space of k -forms with values in ζ as

$$\Omega^k(M;\zeta) = \Gamma(\Lambda^k(\tau(M)^* \otimes \zeta).$$

The zero forms are simply the space of sections, $\Omega^0(M;\zeta) = \Gamma(\zeta)$. Notice that if ζ is the trivial bundle $\zeta = \epsilon_n$, then one gets standard forms,

$$\Omega^k(M;\epsilon_n) = \Omega^k(M) \otimes \mathbb{R}^n = \oplus_n \Omega^k(M).$$

Even though spaces of forms with values in a bundle are easy to define, there is no canonical analogue of the exterior derivative. There do however exist differential operators

$$D : \Omega^k(M;\zeta) \to \Omega^{k+1}(M;\zeta)$$

that satisfy familiar product formulas. These operators are called *covariant derivatives* (or *connections*) and are related to the notion of a connection on a principal bundle, which we now define and study.

Let G be a compact Lie group. Recall that the tangent bundle τG has a canonical trivialization

$$\begin{aligned} \psi : G \times T_1 G &\to TG \\ (g, v) &\to D(\ell_g)(v) \end{aligned}$$

where for any $g \in G$, $\ell_g : G \to G$ is the map given by left multiplication by g, and $D(\ell_g) : T_h G \to T_{gh} G$ is its derivative. r_g and $D(r_g)$ will denote the analogous maps corresponding to right multiplication.

The differential of right multiplication on G defines a right action of G on the tangent bundle τG. We claim that the trivialization ψ is equivariant with respect to this action, if we take as the right action of G on $T_1 G$ to be the *adjoint action:*

$$\begin{aligned} T_1 G \times G &\to T_1 G \\ (v, g) &\to D(\ell_{g^{-1}})(v) D(r_g). \end{aligned}$$

Exercise. Verify this claim.

As is standard, we identify $T_1 G$ with the Lie algebra $\mathfrak{g}$. This action is referred to as the *adjoint representation* of the Lie group G on its Lie algebra $\mathfrak{g}$. Now let

$$p : P \to M$$

be a smooth principal G -bundle over a manifold M. This adjoint representation induces a vector bundle $ad(P)$,

$$ad(P) : P \times_G \mathfrak{g} \to M. \tag{2.3}$$

This bundle has the following relevance. Let $p^*(\tau M) : p^*(TM) \to P$ be the pull - back over the total space P of the tangent bundle of M. We have a surjective map of bundles

$$\tau P \to p^*(\tau M).$$

Define $T_F P$ to be the kernel bundle of this map. So the fiber of $T_F P$ at a point $y \in P$ is the kernel of the surjective linear transformation $Dp(y) : T_y P \to T_{p(y)} M$. Notice that the right action of G on the total space of the principal bundle P defines an action of G on the tangent bundle τP, which restricts to an action of G on $T_F P$. Furthermore, by recognizing that the fibers are equivariantly homeomorphic to the Lie group G, the following is a direct consequence of the above considerations:

Proposition 1.26. *$T_F P$ is naturally isomorphic to the pull - back of the adjoint bundle,*

$$T_F P \cong p^*(ad(P)).$$

Thus we have an exact sequence of G - equivariant vector bundles over P:

$$0 \to p^*(ad(P)) \to \tau P \xrightarrow{Dp} p^*(\tau M) \to 0. \tag{2.4}$$

Recall that short exact sequences of bundles split as Whitney sums. A connection is a G - equivariant splitting of this sequence:

Definition 1.21. A **connection** on the principal bundle P is a G - equivariant splitting

$$\omega_A : \tau P \to p^*(ad(P))$$

of the above sequence of vector bundles. That is, ω_A defines a G - equivariant isomorphism

$$\omega_A \oplus Dp : \tau P \to p^*(ad(P)) \oplus p^*(\tau M).$$

The following is an important description of the space of connections on P, $\mathcal{A}(P)$.

Proposition 1.27. *The space of connections on the principal bundle P, $\mathcal{A}(P)$, is an affine space modeled on the infinite dimensional vector space of one forms on M with values in the bundle $ad(P)$, $\Omega^1(M; ad(P))$.*

Proof. Consider two connections ω_A and ω_B,

$$\omega_A, \omega_B : \tau P \to p^*(ad(P).$$

Since these are splittings of the exact sequence 2.4, they are both the identity when restricted to $p^*(ad(P)) \hookrightarrow \tau P$. Thus their difference, $\omega_A - \omega_B$ is zero when restricted to $p^*(ad(P))$. By the exact sequence it therefore factors as a composition

$$\omega_A - \omega_B : \tau P \to p^*(\tau M) \xrightarrow{\alpha} p^*(ad(P))$$

for some bundle homomorphism $\alpha : p^*(\tau M) \to p^*(ad(P))$. That is, for every $y \in P$, α defines a linear transformation

$$\alpha_y : p^*(TM)_y \to p^*(ad(P))_y.$$

Hence for every $y \in P$, α defines (and is defined by) a linear transformation

$$\alpha_y : T_{p(y)}M \to ad(P)_{p(y)}.$$

Furthermore, the fact that both ω_A and ω_B are *equivariant* splittings says that $\omega_A - \omega_B$ is equivariant, which translates to the fact that α_y only depends on the orbit of y under the G - action. That is,

$$\alpha_y = \alpha_{yg} : T_{p(y)}M \to ad(P)_{p(y)}$$

for every $g \in G$. Thus α_y only depends on $p(y) \in M$. Hence for every $x \in M$, α defnes, and is defined by, a linear transformation

$$\alpha_x : T_x M \to ad(P)_x.$$

Thus α may be viewed as a section of the bundle of homormorphisms, $Hom(\tau M, ad(P))$, and hence is a one form,

$$\alpha \in \Omega^1(M; ad(P)).$$

Thus any two connections on P differ by an element in $\Omega^1(M; ad(P))$ in this sense.

Now reversing the procedure, an element $\beta \in \Omega^1(M; ad(P))$ defines an equivariant homomorphism of bundles over P,

$$\beta : p^*(\tau M) \to p^*(ad(P)).$$

By adding the composition

$$\tau P \xrightarrow{Dp} p^*(\tau M) \xrightarrow{\beta} p^*(ad(P))$$

to any connection (equivariant splitting)

$$\omega_A : \tau P \to p^*(ad(P))$$

one produces a new equivariant splitting of τP, and hence a new connection. The proposition follows. $\square$

Remark. Even though the space of connections $\mathcal{A}(P)$ is affine, it is not, in general a vector space. There is no "zero" in $\mathcal{A}(P)$ since there is no pre-chosen, canonical connection. The one exception to this, of course, is when P is the trivial G - bundle,

$$P = M \times G \to M.$$

In this case there is an obvious equvariant splitting of τP, which serves as the "zero" in $\mathcal{A}(P)$. Moreover in this case the adjoint bundle $ad(P)$ is also trivial,

$$ad(P) = M \times \mathfrak{g} \to M.$$

Hence there is a canonical identification of the space of connections on the trivial bundle with $\Omega^1(M;\mathfrak{g}) = \Omega^1(M)\otimes\mathfrak{g}$.

Let $p : P \to M$ be a principal G - bundle and let $\omega_A \in \mathcal{A}(P)$ be a connection.

The curvature F_A of ω_A is a two form

$$F_A \in \Omega^2(M; ad(P))$$

which measures to what extent the splitting ω_A commutes with the braket operation on vector fields. More precisely, let X and Y be vector fields on M. The connection ω_A defines an equivariant splitting of τP and hence defines a "horizontal" lifting of these vector fields, which we denote by $\tilde{X}$ and $\tilde{Y}$ respectively.

Definition 1.22. The curvature $F_A \in \Omega^2(M; ad(P))$ is defined by

$$F_A(X,Y) = \omega_A[\tilde{X},\tilde{Y}].$$

For those unfamiliar with the bracket operation on vector fields, we refer you to [**38**]

Another important construction with connections is the associated *covariant derivative* which is defined as follows.

Definition 1.23. The covariant derivative induced by the connection ω_A

$$D_A : \Omega^0(M; ad(P)) \to \Omega^1(M; ad(P))$$

is defined by

$$D_A(\sigma)(X) = [\tilde{X},\sigma].$$

where X is a vector field on M.

The notion of covariant derivative, and hence connection, extends to vector bundles as well. Let $\zeta : p : E^\zeta \to M$ be a finite dimensional vector bundle over M.

Definition 1.24. A connection on ζ (or a covariant derivative) is a linear transformation

$$D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$$

that satisfies the *Leibnitz rule*

$$D_A(f\phi) = df\otimes\phi + fD_A(\phi) \tag{2.5}$$

for any $f \in C^\infty(M;\mathbb{R})$ *and any* $\phi \in \Omega^0(M;\zeta)$.

Now we can model the space of connections on a vector bundle, $\mathcal{A}(\zeta)$ similarly to how we modeled the space of connections on a principal bundle $\mathcal{A}(P)$. Namely, given any two connections D_A and D_B on ζ and a function $f \in C^\infty(M;\mathbb{R})$, one can take the convex combination

$$f \cdot D_A + (1-f) \cdot D_B$$

and obtain a new connection. From this it is not difficult to see the following. We leave the proof as an exercise to the reader.

Proposition 1.28. *The space of connections on the vector bundle ζ, $\mathcal{A}(\zeta)$ is an affine space modeled on the vector space of one forms $\Omega^1(M; End(\zeta))$, where $End(\zeta)$ is the bundle of endomorphisms of ζ.*

Let X be a vector field on M and D_A a connection on the vector bundle ζ. The covariant derivative in the direction of X, which we denote by $(D_A)_X$ is an operator on the space of sections of ζ,

$$(D_A)_X : \Omega^0(M;\zeta) \to \Omega^0(M;\zeta)$$

defined by

$$(D_A)_X(\sigma) = \langle D_A(\phi); X\rangle.$$

One can then define the curvature $F_A \in \Omega^2(M; End(\zeta))$ by defining its action on a pair of vector fields X and Y to be

$$F_A(X,Y) = (D_A)_X(D_A)_Y - (D_A)_Y(D_A)_X - (D_A)_{[X,Y]}. \tag{2.6}$$

To interpret this formula notice that a - priori $F_A(X,Y)$ is a second order differential operator on the space of sections of ζ. However a direct calculation shows that for $f \in C^\infty(M;\mathbb{R})$ and $\sigma \in \Omega^0(M;\zeta)$, then

$$F_A(X,Y)(f\sigma) = fF_A(X,Y)(\sigma)$$

and hence $F_A(X,Y)$ is in fact a zero - order operator on $\Omega^0(M;\zeta)$. But a zero order operator on the space of sections of ζ is a section of the endomorphism bundle $End(\zeta)$. Thus F_A assigns to any pair of vector fields X and Y a section of $End(\zeta)$. Moreover it is straightforward to check that this assignment is tensorial in X and Y (i.e $F_A(fX,Y) = F_A(X,fY) = fF_A(X,Y)$). Thus F_A is an element of $\Omega^2(M; End(\zeta))$. The curvature measures the lack of commutativity in second order partial covariant derivatives.

Given a connection on a bundle ζ the linear mapping $D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$ extends to a *deRham* type sequence,

$$\Omega^0(M;\zeta) \xrightarrow{D_A} \Omega^1(M;\zeta) \xrightarrow{D_A} \Omega^2(M;\zeta) \xrightarrow{D_A} \cdots$$

where for $\sigma \in \Omega^p(M;\zeta)$, $D_A(\sigma)$ is the $p+1$ -form defined by the formula

$$D_A(\sigma)(X_0, \cdots, X_p) = \sum_{j=0}^{p} (-1)^j (D_A)_{X_j}(\sigma(X_0, \cdots, \hat{X}_j, \cdots, X_p)) + \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p). \tag{2.7}$$

We observe that unlike with the standard deRham exterior derivative (which can be viewed as a connection on the trivial line bundle), it is not generally true that $D_A \circ D_A = 0$. In fact we have the following, whose proof is a direct calculation that we leave to the reader.

Proposition 1.29.

$$D_A \circ D_A = F_A : \Omega^0(M;\zeta) \to \Omega^2(M;\zeta)$$

where in this context the curvature F_A is interpreted as a assigning to a section $\sigma \in \Omega^0(M;\zeta)$ the 2 - form $F_A(\sigma)$ which associates to vector fields X and Y the section $F_A(X,Y)(\sigma)$ as defined in (2.6).

Thus the curvature of a connection F_A can also be viewed as measuring the extent to which the covariant derivative D_A fails to form a cochain complex on the space of differential forms with values in the bundle ζ. However it is always true that the covariant derivative of the curvature tensor is zero. This is the well known *Bianchi identity* (see [**38**] for a complete discussion).

Theorem 1.30. *Let A be a connection on a vector bundle ζ. Then*

$$D_A F_A = 0.$$

We end this section by observing that if P is a principal G - bundle with a connection ω_A, then any representation of G on a finite dimensional vector space V induces a connection on the corresponding vector bundle

$$P \times_G V \to M.$$

We refer the reader to [**16**] and [**38**] for thorough discussions of the various ways of viewing connections. [**3**] has a nice, brief discussion of connections on principal bundles, and [**14**] and [**23**] have similarly concise discussions of connections on vector bundles.

2.6. The Levi - Civita Connection. Let M be a manifold equipped with a Riemannian structure. Recall that this is a Euclidean structure on its tangent bundle. In this section we will show how this structure induces a connection, or covariant derivative, on the tangent bundle. This connection is called the *Levi - Civita* connection associated to the Riemannian structure. Our treatment of this topic follows that of Milnor and Stasheff [**31**]

Let $D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$ be a connection (or covariant derivative) on an n - dimensional vector bundle ζ. Its curvature is a two- form with values in the endomorphism bundle

$$F_A \in \Omega^2(M; End(\zeta))$$

The endomorphism bundle can be described alternatively as follows. Let E_ζ be the principal $GL(n,\mathbb{R})$ bundle associated to ζ. Then of course $\zeta = E_\zeta \otimes_{GL(n,\mathbb{R})} \mathbb{R}^n$. The endomorphism bundle can then be described as follows. The proof is an easy exercise that we leave to the reader.

Proposition 1.31.

$$End(\zeta) \cong ad(\zeta) = E_\zeta \times_{GL(n,\mathbb{R})} M_n(\mathbb{R})$$

where $GL(n,\mathbb{R})$ acts on $M_n(\mathbb{R})$ by conjugation,

$$A \cdot B = ABA^{-1}.$$

Let ω be a differential p - form on M with values in $End(\zeta)$,

$$\omega \in \Omega^p(M; End(\zeta)) \cong \Omega^p(M; ad(\zeta)) = \Omega^p(M; E_\zeta \times_{GL(n,\mathbb{R})} M_n(\mathbb{R})).$$

Then on a coordinate chart $U \subset M$ with local trivialization $\psi : \zeta_{|_U} \cong U \times \mathbb{C}^n$ for ζ, (and hence the induced coordinate chart and local trivialization for $ad(\zeta)$), ω can be viewed as an $n \times n$ matrix of p -forms on M. We write

$$\omega = (\omega_{i,j}).$$

Of course this description depends on the coordinate chart and local trivialization chosen, but at any $x \in U$, then by the above proposition, two trivializations yield conjugate matrices. That is, if $(\omega_{i,j}(x))$ and $(\omega'_{i,j}(x))$ are two matrix descriptions of $\omega(x)$ defined by two different local trivializations of $\zeta_{|_U}$, then there exists an $A \in GL(n,\mathbb{C})$ with

$$A(\omega_{i,j}(x))A^{-1} = (\omega'_{i,j}(x)).$$

Now suppose the bundle ζ is equipped with a Euclidean structure. As seen earlier in this chapter this is equivalent to its associated principal $GL(n,\mathbb{R})$ - bundle E_ζ having a reduction to the structure group $O(n)$. We let $E_{O(n)} \to M$ denote this principal $O(n)$ - bundle.

Now the Lie algebra $\mathfrak{o}(n)$ of $O(n)$ (i.e the tangent space $T_1(O(n))$) is a subspace of the Lie algebra of $GL(n,\mathbb{R})$, i.e

$$\mathfrak{o}(n) \subset M_n(\mathbb{R}).$$

The following is well known (see, for example[**35**])

PROPOSITION 1.32. *The Lie algebra $\mathfrak{o}(n) \subset M_n(\mathbb{R})$ is the subspace consisting of skew symmetric $n \times n$ - matrices. That is, $A \in \mathfrak{o}(n)$ if and only if*

$$A^t = -A$$

where A^t is the transpose.

So if ζ has a Euclidean structure, we can form the adjoint bundle

$$ad^O(\zeta) = E_{O(n)} \times_{O(n)} \mathfrak{o}(n) \subset E_\zeta \times_{GL(n,\mathbb{R})} M_n(\mathbb{R}) = ad(\zeta)$$

where, again $O(n)$ acts on $\mathfrak{o}(n)$ by conjugation.

Now suppose D_A is an orthogonal connection on ζ. That is, it is induced by a connection on the principal $O(n)$ - bundle $E_{O(n)} \to M$. The following is fairly clear, and we leave its proof as an exercise.

COROLLARY 1.33. *If D_A is an orthogonal connection on a Euclidean bundle ζ, then the curvature F_A lies in the space of $\mathfrak{o}(n)$ valued two forms*

$$F_A \in \Omega^2(M; ad^O(\zeta)) \subset \Omega^2(M; ad(\zeta)) = \Omega^2(M; End(\zeta)).$$

Furthermore, on a coordinate chart $U \subset M$ with local trivialization $\psi : \zeta_{|_U} \cong U \times \mathbb{C}^n$ that preserves the Euclidean structure, we may write the form F_A as a skew - symmetric matrix of two forms,

$$F_{A_{|_U}} = (\omega_{i,j}) \quad i, j = 1, \cdots, n$$

where each $\omega_{i,j} \in \Omega^2(M)$ and $\omega_{i,j} = -\omega_{j,i}$. In fact the connection D_A itself can be written as skew symmetric matrix of one forms

$$D_{A_{|_U}} = (\alpha_{i,j})$$

where each $\alpha_{i,j} \in \Omega^1(M)$.

We now describe the notion of a "symmetric" connection on the cotangent bundle of a manifold, and then show that if the manifold is equipped with a Riemannian structure (i.e there is a Euclidean structure on the (co) - tangent bundle), then there is a unique symmetric, orthogonal connection on the cotangent bundle.

DEFINITION 1.25. A connection D_A on the cotangent bundle τ^*M is *symmetric* (or *torsion free*) if the composition

$$\Gamma(\tau^*) = \Omega^0(M; \tau^*) \xrightarrow{D_A} \Omega^1(M; \tau^*) = \Gamma(\tau^* \otimes \tau^*) \xrightarrow{\wedge} \Gamma(\Lambda^2 \tau^*)$$

is equal to the exterior derivative d.

In terms of local coordinates $x_1, \cdots, x_n$, if we write

$$D_A(dx_k) = \sum_{i,j} \Gamma^k_{i,j} dx_i \otimes dx_j \tag{2.8}$$

(the functions $\Gamma^k_{i,j}$ are called the "Christoffel symbols"), then the requirement that D_A is symmetric is that the image $\sum_{i,j} \Gamma^k_{i,j} dx_i \otimes dx_j$ be equal to the exterior derivative $d(dx_k) = 0$. This implies that the Christoffel symbols $\Gamma^k_{i,j}$ must be symmetric in i and j. The following is straightforward to verify.

LEMMA 1.34. *A connection D_A on τ^* is symmetric if and only if the covariant derivative of the differential of any smooth function*

$$D_A(df) \in \Gamma(\tau^* \otimes \tau^*)$$

is a symmetric tensor. That is, if $\psi_1, \cdots, \psi_n$ form a local basis of sections of τ^*, and we write the corresponding local expression

$$D_A(df) = \sum_{i,j} a_{i,j}\, \psi_i \otimes \psi_j$$

then $a_{i,j} = a_{j,i}$.

We now show that the (co)-tangent bundle of a Riemannian metric has a preferred connection.

THEOREM 1.35. *The cotangent bundle $\tau^* M$ of a Riemannian manifold has a unique orthogonal, symmetric connection. (It is orthogonal with respect to the Euclidean structure defined by the Riemannian metric.)*

PROOF. Let U be an open neighborhood in M with a trivialization

$$\psi : U \times \mathbb{R}^n :\longrightarrow \tau^*_{|_U}$$

which preserves the Euclidean structure. ψ defines n orthonormal sections of $\tau^*_{|_U}$, $\psi_1, \cdots, \psi_n$. The ψ_j's constitute an orthonormal basis of one forms on M. We will show that there is one and only one skew-symmetric matrix $(\alpha_{i,j})$ of one forms such that

$$d\psi_k = \sum \alpha_{k,j} \wedge \psi_j.$$

We can then define a connection D_A on $\tau^*_{|_U}$ by requiring that

$$D_A(\psi_k) = \sum \alpha_{k,j} \otimes \psi_j.$$

It is then clear that D_A is the unique symmetric connection which is compatible with the metric. Since the local connections are unique, they glue together to yield a unique global connection with this property.

In order to prove the existence and uniqueness of the skew symmetric matrix of one forms $(\alpha_{i,j})$ we need the following combinatorial observation.

Any $n \times n \times n$ array of real valued functions $A_{i,j,k}$ can be written uniquely as the sum of an array $B_{i,j,k}$ which is symmetric in i, j, and an array $C_{i,j,k}$ which is skew symmetric in j, k. To see this, consider the formulas

$$B_{i,j,k} = \frac{1}{2}(A_{i,j,k} + A_{j,i,k} - A_{k,i,j} - A_{k,j,i} + A_{j,k,i} + A_{i,k,j})$$
$$C_{i,j,k} = \frac{1}{2}(A_{i,j,k} - A_{j,i,k} + A_{k,i,j} + A_{k,j,i} - A_{j,k,i} - A_{i,k,j})$$

Uniqueness would follow since if an array $D_{i,j,k}$ were both symmetric in i, j and skew symmetric in j, k, then one would have

$$D_{i,j,k} = D_{j,i,k} = -D_{j,k,i} = -D_{k,j,i} = D_{k,i,j} = D_{i,k,j} = -D_{i,j,k}$$

and hence all the entries are zero.

Now choose functions $A_{i,j,k}$ such that

$$d\psi_k = \sum A_{i,j,k}\, \psi_i \wedge \psi_j$$

and set $A_{i,j,k} = B_{i,j,k} + C_{i,j,k}$ as above. It then follows that

$$d\psi_k = \sum C_{i,j,k}\, \psi_i \wedge \psi_j$$

by the symmetry of the $B_{i,j,k}$'s. Then we define the one forms

$$\alpha_{k,j} = \sum C_{i,j,k}\, \psi_i.$$

They clearly form the unique skew symmetric matrix of one forms with $d\psi_k = \sum \alpha_{k,j} \wedge \psi_j$. This proves the lemma. $\square$

This preferred connection on the (co)tangent bundle of a Riemannian metric is called the Levi - Civita connection. Statements about the curvature of a metric on a manifold are actually statements about the curvature form of the Levi - Civita connection associated to the Riemannian metric. For example, a "flat metric" on a manifold is a Riemannian structure whose corresponding Levi-Civita connection has zero curvature form. As is fairly clear, these connections form a central object of study in Riemannian geometry.

CHAPTER 2

Classification of Bundles

In this chapter we prove Steenrod's classification theorem of principal G - bundles, and the corresponding classification theorem of vector bundles. This theorem states that for every group G, there is a "classifying space" BG with a well defined homotopy type so that the homotopy classes of maps from a space X, $[X, BG]$, is in bijective correspondence with the set of isomorphism classes of principal G - bundles, $Prin_G(X)$. We then describe various examples and constructions of these classifying spaces, and use them to study structures on principal bundles, vector bundles, and manifolds.

1. The homotopy invariance of fiber bundles

The goal of this section is to prove the following theorem, and to examine certain applications such as the classification of principal bundles over spheres in terms of the homotopy groups of Lie groups.

Theorem 2.1. *Let $p : E \to B$ be a fiber bundle with fiber F, and let $f_0 : X \to B$ and $f_1 : X \to B$ be homotopic maps.Then the pull - back bundles are isomorphic,*

$$f_0^*(E) \cong f_1^*(E).$$

The main step in the proof of this theorem is the basic *Covering Homotopy Theorem* for fiber bundles which we now state and prove.

Theorem 2.2. **Covering Homotopy theorem.** *Let $p_0 : E \to B$ and $q : Z \to Y$ be fiber bundles with the same fiber, F, where B is normal and locally compact. Let h_0 be a bundle map*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{h}_0} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[h_0]{} & Y \end{array}$$

Let $H : B \times I \to Y$ be a homotopy of h_0 (i.e $h_0 = H_{|_{B\times\{0\}}}$.) Then there exists a covering of the homotopy H by a bundle map

$$\begin{array}{ccc} E \times I & \xrightarrow{\tilde{H}} & Z \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle q} \\ B \times I & \xrightarrow[H]{} & Y. \end{array}$$

PROOF. We prove the theorem here when the base space B is compact. The natural extension is to when B has the homotopy type of a CW - complex. The proof in full generality can be found in Steenrod's book [**39**].

The idea of the proof is to decompose the homotopy H into homotopies that take place in local neighborhoods where the bundle is trivial. The theorem is obviously true for trivial bundles, and so the homotopy H can be covered on each local neighborhood. One then must be careful to patch the coverings together so as to obtain a global covering of the homotopy H.

Since the space X is compact, we may assume that the pull - back bundle $H^*(Z) \to B \times I$ has locally trivial neighborhoods of the form $\{U_\alpha \times I_j\}$, where $\{U_\alpha\}$ is a locally trivial covering of B (i.e there are local trivializations $\phi_{\alpha,\beta} : U_\alpha \times F \to p^{-1}(U_\alpha)$), and $I_1, \cdots, I_r$ is a finite sequence of open intervals covering $I = [0,1]$, so that each I_j meets only I_{j-1} and I_{j+1} nontrivially. Choose numbers

$$0 = t_0 < t_1 < \cdots < t_r = 1$$

so that $t_j \in I_j \cap I_{j+1}$. We assume inductively that the covering homotopy $\tilde{H}(x,t)$ has been defined $E \times [0, t_j]$ so as to satisfy the theorem over this part.

For each $x \in B$, there is a pair of neighborhoods (W, W') such that for $x \in W$, $\bar{W} \subset W'$ and $\bar{W}' \subset U_\alpha$ for some U_α. Choose a finite number of such pairs (W_i, W_i'), $(i = 1, \cdots, s)$ covering B. Then the Urysohn lemma implies there is a map $u_i : B \to [t_j, t_{j+1}]$ such that $u_i(\bar{W}_i) = t_{j+1}$ and $u_j(B - W_i') = t_j$. Define $\tau_0(x) = t_j$ for $x \in B$, and

$$\tau_i(x) = max(u_1(x), \cdots, u_i(x)), \quad x \in B, \quad i = 1, \cdots, s.$$

Then

$$t_j = \tau_0(x) \leq \tau_1(x) \leq \cdots \leq t_s(x) = t_{j+1}.$$

Define B_i to be the set of pairs (x,t) such that $t_j \leq t \leq \tau_i(x)$. Let E_i be the part of $E \times I$ lying over B_i. Then we have a sequence of total spaces of bundles

$$E \times t_j = E_0 \subset E_1 \subset \cdots \subset E_s = E \times [t_j, t_{j+1}].$$

We suppose inductively that $\tilde{H}$ has been defined on E_{i-1} and we now define its extension over E_i.

By the definition of the τ's, the set $B_i - B_{i-1}$ is contained in $W_i' \times [t_j, t_{j+1}]$; and by the definition of the W's, $\bar{W'}_i \times [t_j, t_{j+1}] \subset U_\alpha \times I_j$ which maps via H to a locally trivial neighborhood, say V_k, for $q : Z \to Y$. Say $\phi_k : V_k \times F \to q^{-1}(V_k)$ is a local trivialization. In particular we can define $\rho_k : q^{-1}(V_k) \to F$ to be the inverse of ϕ_k followed by the projection onto F. We now define

$$\tilde{H}(e,t) = \phi_k(H(x,t), \rho(\tilde{H}(e, \tau_{i-1}(x)))$$

where $(e,t) \in E_i - E_{i-1}$ and $x = p(e) \in B$.

It is now a straightforward verification that this extension of $\tilde{H}$ is indeed a bundle map on E_i. This then completes the inductive step. □

We now prove theorem 2.1 using the covering homotopy theorem.

Proof. Let $p : E \to B$, and $f_0; X \to B$ and $f_1 : X \to B$ be as in the statement of the theorem. Let $H : X \times I \to B$ be a homotopy with $H_0 = f_0$ and $G_1 = f_1$. Now by the covering homotopy theorem there is a covering homotopy $\tilde{H} : f_0^*(E) \times I \to E$ that covers $H : X \times I \to B$. By definition this defines a map of bundles over $X \times I$, that by abuse of notation we also call $\tilde{H}$,

$$\begin{array}{ccc} f_0^*(E) \times I & \xrightarrow{\tilde{H}} & H^*(E) \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow[=]{} & X \times I. \end{array}$$

This is clearly a bundle isomorphism since it induces the identity map on both the base space and on the fibers. Restricting this isomorphism to $X \times \{1\}$, and noting that since $H_1 = f_1$, we get a bundle isomorphism

$$\begin{array}{ccc} f_0^*(E) & \xrightarrow[\cong]{\tilde{H}} & f_1^*(E) \\ \downarrow & & \downarrow \\ X \times \{1\} & \xrightarrow[=]{} & X \times \{1\}. \end{array}$$

This proves theorem 2.1 □

We now derive certain consequences of this theorem.

Corollary 2.3. *Let $p : E \to B$ be a principal G - bundle over a connected space B. Then for any space X the pull back construction gives a well defined map from the set of homotopy classes of maps from X to B to the set of isomorphism classes of principal G - bundles,*

$$\rho_E : [X, B] \to Prin_G(X).$$

Definition 2.1. A principal G - bundle $p : EG \to BG$ is called *universal* if the pull back construction

$$\rho_{EG} : [X, BG] \to Prin_G(X)$$

is a bijection for every space X. In this case the base space of the universal bundle BG is called a *classifying space* for G (or for principal G - bundles).

The main goal of this chapter is to prove that universal bundles exist for every group G, and that the classifying spaces are unique up to homotopy type.

Applying theorem 2.1 to vector bundles gives the following, which was claimed at the end of chapter 1.

Corollary 2.4. *If $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic, they induce the same homomorphism of abelian monoids,*

$$\begin{aligned} f_0^* = f_1^* : Vect^*(Y) &\to Vect^*(X) \\ Vect^*_{\mathbb{R}}(Y) &\to Vect^*_{\mathbb{R}}(X) \end{aligned}$$

and hence of K theories

$$\begin{aligned} f_0^* = f_1^* : K(Y) &\to K(X) \\ KO(Y) &\to KO(X) \end{aligned}$$

Corollary 2.5. *If $f : X \to Y$ is a homotopy equivalence, then it induces isomorphisms*

$$\begin{aligned} f^* : Prin_G(Y) &\xrightarrow{\cong} Prin_G(X) \\ Vect^*(Y) &\xrightarrow{\cong} Vect^*(X) \\ K(Y) &\xrightarrow{\cong} K(X) \end{aligned}$$

Corollary 2.6. *Any fiber bundle over a contractible space is trivial.*

Proof. If X is contractible, it is homotopy equivalent to a point. Apply the above corollary. $\square$

The following result is a classification theorem for bundles over spheres. It begins to describe why understanding the homotopy type of Lie groups is so important in Topology.

Theorem 2.7. *There is a bijective correspondence between principal bundles and homotopy groups*

$$Prin_G(S^n) \cong \pi_{n-1}(G)$$

where as a set $\pi_{n-1}G = [S^{n-1}, x_0; G, \{1\}]$, *which refers to (based) homotopy classes of basepoint preserving maps from the sphere* S^{n-1} *with basepoint* $x_0 \in S^{n-1}$, *to the group* G *with basepoint the identity* $1 \in G$.

Proof. Let $p : E \to S^n$ be a G - bundle. Write S^n as the union of its upper and lower hemispheres,

$$S^n = D^n_+ \cup_{S^{n-1}} D^n_-.$$

Since D^n_+ and D^n_- are both contractible, the above corollary says that E restricted to each of these hemispheres is trivial. Morever if we fix a trivialization of the fiber of E at the basepoint $x_0 \in S^{n-1} \subset S^n$, then we can extend this trivialization to both the upper and lower hemispheres. We may therefore write

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G)$$

where θ is a clutching function defined on the equator, $\theta : S^{n-1} \to G$. That is, E consists of the two trivial components, $(D^n_+ \times G)$ and $(D^n_- \times G)$ where if $x \in S^{n-1}$, then $(x, g) \in (D^n_+ \times G)$ is identified with $(x, \theta(x)g) \in (D^n_- \times G)$. Notice that since our original trivializations extended a common trivialization on the basepoint $x_0 \in S^{n-1}$, then the trivialization $\theta : S^{n-1} \to G$ maps the basepoint x_0 to the identity $1 \in G$. The assignment of a bundle its clutching function, will define our correspondence

$$\Theta : Prin_G(S^n) \to \pi_{n-1}G.$$

To see that this correspondence is well defined we need to check that if E_1 is isomorphic to E_2, then the corresponding clutching functions θ_1 and θ_2 are homotopic. Let $\Psi : E_1 \to E_2$ be an isomorphism. We may assume this isomorphism respects the given trivializations of these fibers of these bundles over the basepoint $x_0 \in S^{n-1} \subset S^n$. Then the isomorphism Ψ determines an isomorphism

$$(D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G) \xrightarrow[\cong]{\Psi} (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G).$$

By restricting to the hemispheres, the isomorphism Ψ defines maps

$$\Psi_+ : D^n_+ \to G$$

and

$$\Psi_- : D^n_- \to G$$

which both map the basepoint $x_0 \in S^{n-1}$ to the identity $1 \in G$, and furthermore have the property that for $x \in S^{n-1}$,

$$\Psi_+(x)\theta_1(x) = \theta_2(x)\Psi_-(x),$$

or, $\Psi_+(x)\theta_1(x)\Psi_-(x)^{-1} = \theta_2(x) \in G$. Now by considering the linear homotopy $\Psi_+(tx)\theta_1(x)\Psi_-(tx)^{-1}$ for $t \in [0, 1]$, we see that $\theta_2(x)$ is homotopic to $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$, where the two zeros in this description refer to the origins of D^n_+ and D^n_- respectively, i.e the north and south poles of the sphere S^n. Now since Ψ_+ and Ψ_- are defined on connected spaces, their images lie in a connected component of the group G. Since their image on the basepoint $x_0 \in S^{n-1}$ are both the identity, there exist paths $\alpha_+(t)$ and $\alpha_-(t)$ in S^n that start when $t = 0$ at $\Psi_+(0)$ and $\Psi_-(0)$ respectively, and both end at $t = 1$ at the identity $1 \in G$. Then the homotopy $\alpha_+(t)\theta_1(x)\alpha_-(t)^{-1}$ is a homotopy from the map $\Psi_+(0)\theta_1(x)\Psi_-(0)^{-1}$ to the map $\theta_1(x)$. Since the first of these maps is homotopic to $\theta_2(x)$, we have that θ_1 is homotopic to θ_2, as claimed. This implies that the map $\Theta : Prin_G(S^n) \to \pi_{n-1}G$ is well defined.

The fact that Θ is surjective comes from the fact that every map $S^{n-1} \to G$ can be viewed as the clutching function of the bundle

$$E = (D^n_+ \times G) \cup_\theta (D^n_- \times G)$$

as seen in our discussion of clutching functions in chapter 1.

We now show that Θ is injective. That is, suppose E_1 and E_2 have homotopic clutching functions, $\theta_1 \simeq \theta_2 : S^{n-1} \to G$. We need to show that E_1 is isomorphic to E_2 As above we write

$$E_1 = (D^n_+ \times G) \cup_{\theta_1} (D^n_- \times G)$$

and

$$E_2 = (D^n_+ \times G) \cup_{\theta_2} (D^n_- \times G).$$

Let $H : S^{n-1} \times [-1, 1] \to G$ be a homotopy so that $H_1 = \theta_1$ and $H_1 = \theta_2$. Identify the closure of an open neighborhood $\mathcal{N}$ of the equator S^{n-1} in S^n with $S^{n-1} \times [-1, 1]$ Write $\mathcal{D}_+ = D^2_+ \cup \bar{\mathcal{N}}$ and $\mathcal{D}_- = D^2_- \cup \bar{\mathcal{N}}$ Then $\mathcal{D}_+$ and $\mathcal{D}_-$ are topologically closed disks and hence contractible, with

$$\mathcal{D}_+ \cap \mathcal{D}_- = \bar{\mathcal{N}} \cong S^{n-1} \times [-1, 1].$$

Thus we may form the principal G - bundle

$$E = \mathcal{D}_+ \times G \cup_H \mathcal{D}_- \times G$$

where by abuse of notation, H refers to the composition

$$\bar{\mathcal{N}} \cong S^{n-1} \times [-1, 1] \xrightarrow{H} G\cdot$$

We leave it to the interested reader to verify that E is isomorphic to both E_1 and E_2. This completes the proof of the theorem. □

2. Universal bundles and classifying spaces

The goal of this section is to study universal principal G - bundles, the resulting classification theorem, and the corresponding classifying spaces. We will discuss several examples including the universal bundle for any subgroup of the general linear group. We postpone the proof of the existence of universal bundles for all groups until the next section.

In order to identify universal bundles, we need to recall the following definition from homotopy theory.

DEFINITION 2.2. A space X is said to be *aspherical* if all of its homotopy groups are trivial,

$$\pi_n(X) = 0 \quad \text{for all } n \geq 0.$$

Equivalently, a space X is aspherical if every map from a sphere $S^n \to X$ can be extended to a map of its bounding disk, $D^{n+1} \to X$.

Note. A famous theorem of J.H.C. Whitehead states that if X has the homotopy type of a CW - complex, then X being aspherical is equivalent to X being contractible (see [**44**]).

The following is the main result of this section. It identifies when a principal bundle is universal.

THEOREM 2.8. *Let $p : E \to B$ be a principal G - bundle, where the total space E is aspherical. Then this bundle is universal* in the sense that if X is any space, the induced pull-back map

$$\psi : [X, B] \to Prin_G(X)$$
$$f \to f^*(E)$$

is a bijective correspondence.

For the purposes of these notes we will prove the theorem in the setting where the action of G on the total space E is *cellular.* That is, there is a CW - decomposition of the space E which, in an appropriate sense, is respected by the group action. There is very little loss of generality in these assumptions, since the actions of compact Lie groups on manifolds, and algebraic actions on projective varieties satisfy this property. For the proof of the theorem in its full generality we refer the reader to Steenrod's book [**39**], and for a full reference on equivariant CW - complexes and how they approximate a wide range of group actions, we refer the reader to [**24**]

In order to make the notion of cellular action precise, we need to define the notion of an *equivariant CW - complex,* or a G - CW - complex. The idea is the following. Recall that a CW - complex is a space that is made up out of disks of various dimensions whose interiors are disjoint. In particular it can be built up skeleton by skeleton, and the $(k+1)^{st}$ skeleton $X^{(k+1)}$ is constructed out of the k^{th} skeleton $X^{(k)}$ by attaching $(k+1)$ - dimensional disks via "attaching maps", $S^k \to X^{(k)}$.

A "G - CW - complex" is one that has a group action so that the orbits of the points on the interior of a cell are uniform in the sense that each point in a cell D^k has the same isotropy subgroup, say H, and the orbit of a cell itself is of the form $G/H \times D^k$. This leads to the following definition.

DEFINITION 2.3. A G - CW - complex is a space with G -action X which is topologically the direct limit of G - invariant subspaces $\{X^{(k)}\}$ called the equivariant skeleta,

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(k-1)} \subset X^{(k)} \subset \cdots X$$

where for each $k \geq 0$ there is a countable collection of k dimensional disks, subgroups of G, and maps of boundary spheres

$$\{D^k_j,\, H_j < G,\, \phi_j : \partial D^k_j \times G/H_j = S^{k-1}_j \times G/H_j \to X^{(k-1)} \quad j \in I_k\}$$

so that

(1) Each "attaching map" $\phi_j : S^{k-1}_j \times G/H_j \to X^{(k-1)}$ is G -equivariant, and

(2)
$$X^{(k)} = X^{k-1)} \bigcup_{\phi_j\, j \in I_j} (D^k_j \times G/H_j).$$

This notation means that each " disk orbit " $D^k_j \times G/H_j$ is attached to $X^{(k-1)}$ via the map $\phi_j : S^{k-1}_j \times G/H_j \to X^{(k-1)}$.

We leave the following as an exercise to the reader.

Exercise. Prove that when X is a G - CW complex the orbit space X/G has the an induced structure of a (non-equivariant) CW - complex.

Note. Observe that in a G -CW complex X with a *free* G action, all disk orbits are of the form $D^k \times G$, since all isotropy subgroups are trivial.

We now prove the above theorem under the assumption that the principal bundle $p : E \to B$ has the property that with respect to group action of G on E, then E has the structure of a G - CW - complex. The basespace is then given the induced CW - structure. The spaces X in the statement of the theorem are assumed to be of the homotopy type of CW - complexes.

Proof. We first prove that the pull - back map

$$\psi : [X, B] \to Prin_G(X)$$

is surjective. So let $q : P \to X$ be a principal G - bundle, with P a G - CW - complex. We prove there is a G - equivariant map $h : P \to E$ that maps each orbit pG homeomorphically onto its image, $h(y)G$. We prove this by induction on the equivariant skeleta of P. So assume inductively that the map h has been constructed on the $(k-1)$ - skeleton,

$$h_{k-1} : P^{(k-1)} \to E.$$

Since the action of G on P is free, all the k - dimensional disk orbits are of the form $D^k \times G$. Let $D^k_j \times G$ be a disk orbit in the G-CW - structure of the k - skeleton $P^{(k)}$. Consider the disk $D^k_j \times \{1\} \subset D^k_j \times G$. Then the map h_{k-1} extends to $D^k_j \times \{1\}$ if and only if the composition

$$S^{k-1}_j \times \{1\} \subset S^{k-1}_j \times G \xrightarrow{\phi_j} P^{(k-1)} \xrightarrow{h_{k-1}} E$$

is null homotopic. But since E is aspherical, any such map is null homotopic and extends to a map of the disk, $\gamma : D_j^k \times \{1\} \to E$. Now extend γ equivariantly to a map $h_{k,j} : D_j^k \times G \to E$. By construction $h_{k,j}$ maps the orbit of each point $x \in D_j^k$ equivariantly to the orbit of $\gamma(x)$ in E. Since both orbits are isomorphic to G (because the action of G on both P and E are free), this map is a homeomorphism on orbits. Taking the collection of the extensions $h_{k,j}$ together then gives an extension

$$h_k : P^{(k)} \to E$$

with the required properties. This completes the inductive step. Thus we may conclude we have a G - equivariant map $h : P \to E$ that is a homeomorphism on the orbits. Hence it induces a map on the orbit space $f : P/G = X \to E/G = B$ making the following diagram commute

$$\begin{array}{ccc} P & \xrightarrow{h} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

Since h induces a homeomorphism on each orbit, the maps h and f determine a homeomorphism of principal G - bundles which induces an equivariant isomorphism on each fiber. This implies that h induces an isomorphism of principal bundles to the pull - back

$$\begin{array}{ccc} P & \xrightarrow[\cong]{h} & f^*(E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[=]{} & X. \end{array}$$

Thus the isomorphism class $[P] \in Prin_G(X)$ is given by $f^*(E)$. That is, $[P] = \psi(f)$, and hence

$$\psi : [X, B] \to Prin_G(X)$$

is surjective.

We now prove ψ is injective. To do this, assume $f_0 : X \to B$ and $f_1 : X \to B$ are maps so that there is an isomorphism

$$\Phi : f_0^*(E) \xrightarrow{\cong} f_1^*(E).$$

We need to prove that f_0 and f_1 are homotopic maps. Now by the cellular approximation theorem (see [**37**]) we can find cellular maps homotopic to f_0 and f_1 respectively. We therefore assume without loss of generality that f_0 and f_1 are cellular. This, together with the assumption that E is a G - CW complex, gives the pull back bundles $f_0^*(E)$ and $f_1^*(E)$ the structure of G -CW complexes.

Define a principal G - bundle $\mathcal{E} \to X \times I$ by

$$\mathcal{E} = f_0^*(E) \times [0, 1/2] \cup_\Phi f_1^*(E) \times [1/2, 1]$$

where $v \in f_0^*(E) \times \{1/2\}$ is identified with $\Phi(v) \in f_1^*(E) \times \{1/2\}$. $\mathcal{E}$ also has the structure of a G - CW - complex.

Now by the same kind of inductive argument that was used in the surjectivity argument above, we can find an equivariant map $H : \mathcal{E} \to E$ that induces a homeomorphism on each orbit, and that extends the obvious maps $f_0^*(E) \times \{0\} \to E$ and $f_1^*(E) \times \{1\} \to E$. The induced map on orbit spaces

$$F : \mathcal{E}/G = X \times I \to E/G = B$$

is a homotopy between f_0 and f_1. This proves the correspondence Ψ is injective, and completes the proof of the theorem. □

The following result establishes the homotopy uniqueness of universal bundles.

Theorem 2.9. *Let $E_1 \to B_1$ and $E_2 \to B_2$ be universal principal G - bundles. Then there is a bundle map*

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{h}"] \arrow[d] & E_2 \arrow[d] \\ B_1 \arrow[r, "h"'] & B_2 \end{tikzcd}$$

so that h is a homotopy equivalence.

Proof. The fact that $E_2 \to B_2$ is a universal bundle means, by 2.8 that there is a "classifying map" $h : B_1 \to B_2$ and an isomorphism $\tilde{h} : E_1 \to h^*(E_2)$. Equivalently, $\tilde{h}$ can be thought of as a bundle map $\tilde{h} : E_! \to E_2$ lying over $h : B_1 \to B_2$. Similarly, using the universal property of $E_1 \to B_1$, we get a classifying map $g : B_2 \to B_1$ and an isomorphism $\tilde{g} : E_2 \to g^*(E_1)$, or equivalently, a bundle map $\tilde{g} : E_2 \to E_1$. Notice that the composition

$$g \circ f : B_1 \to B_2 \to B_1$$

is a map whose pull back,

$$\begin{aligned} (g \circ f)^*(E_1) &= g^*(f^*(E_1)) \\ &\cong g^*(E_2) \\ &\cong E_1. \end{aligned}$$

That is, $(g \circ f)^*(E_1) \cong id^*(E_1)$, and hence by 2.8 we have $g \circ f \simeq id : B_1 \to B_1$. Similarly, $f \circ g \simeq id : B_2 \to B_2$. Thus f and g are homotopy inverses of each other. □

Because of this theorem, the basespace of a universal principal G - bundle has a well defined homotopy type. We denote this homotopy type by BG, and refer to it as the *classifying space* of the group G. We also use the notation EG to denote the total space of a universal G - bundle.

We have the following immediate result about the homotopy groups of the classifying space BG.

Corollary 2.10. *For any group G, there is an isomorphism of homotopy groups,*

$$\pi_{n-1}G \cong \pi_n(BG).$$

Proof. By considering 2.7 and 2.8 we see that both of these homotopy groups are in bijective correspondence with the set of principal bundles $Prin_G(S^n)$. To realize this bijection by a group homomorphism, consider the "suspension" of the group G, ΣG obtained by attaching two cones on G along the equator. That is,

$$\Sigma G = G \times [-1,1]/\sim$$

where all points of the form $(g,1)$, $(h,-1)$, or $(1,t)$ are identified to a single point.

Notice that this suspension construction can be applied to any space with a basepoint, and in particular $\Sigma S^{n-1} \cong S^n$.

Consider the principal G bundle E over ΣG defined to be trivial on both cones with clutching function $id : G \times \{0\} \xrightarrow{=} G$ on the equator. That is, if $C_+ = G \times [0,1]/\sim \subset \Sigma G$ and $C_- = G \times [-1,0] \subset \Sigma E$ are the upper and lower cones, respectively, then

$$E = (C_+ \times G) \cup_{id} (C_- \times G)$$

where $((g,0),h) \in C_+ \times G$ is identified with $((g,0)gh \in C_- \times G$. Then by 2.8 there is a classifying map

$$f : \Sigma G \to BG$$

such that $f^*(EG) \cong E$.

Now for any space X, let ΩX be the *loop space* of X,

$$\Omega X = \{\gamma : [-1,1] \to X \text{ such that } \gamma(-1) = \gamma(1) = x_0 \in X\}$$

where $x_0 \in X$ is a fixed basepoint. Then the map $f : \Sigma G \to BG$ determines a map (its adjoint)

$$\bar{f} : G \to \Omega BG$$

defined by $\bar{f}(g)(t) = f(g,t)$. But now the loop space ΩX of any connected space X has the property that $\pi_{n-1}(\Omega X) = \pi_n(X)$ (see the exercise below). We then have the induced group homomorphism

$$\pi_{n-1}(G) \xrightarrow{\bar{f}_*} \pi_{n-1}(\Omega BG) \xrightarrow{\cong} \pi_n(BG)$$

which induces the bijective correspondence described above. $\square$

Exercises. 1. Prove that for any connected space X, there is an isomorphism

$$\pi_{n-1}(\Omega X) \cong \pi_n(X).$$

2. Prove that the composition

$$\pi_{n-1}(G) \xrightarrow{\bar{f}_*} \pi_{n-1}(\Omega BG) \xrightarrow{\cong} \pi_n(BG)$$

in the above proof yields the bijection associated with identifying both $\pi_{n-1}(G)$ and $\pi_n(BG)$ with $Prin_G(S^n)$.

We recall the following definition from homotopy theory.

Definition 2.4. An Eilenberg - MacLane space of type (G, n) is a space X such that

$$\pi_k(X) = \begin{cases} G & \text{if k = n} \\ 0 & \text{otherwise} \end{cases}$$

We write $K(G, n)$ for an Eilenberg - MacLane space of type (G, n). Recall that for $n \geq 2$, the homotopy groups $\pi_n(X)$ are abelian groups, so in this $K(G, n)$ only exists

Corollary 2.11. *Let π be a discrete group. Then the classifying space $B\pi$ is an Eilenberg - MacLane space $K(\pi, 1)$.*

Examples.

- $\mathbb{R}$ has a free, cellular action of the integers $\mathbb{Z}$ by

$$(t, n) \to t + n \quad t \in \mathbb{R},\, n \in \mathbb{Z}.$$

  Since $\mathbb{R}$ is contractible, $\mathbb{R}/\mathbb{Z} = S^1 = B\mathbb{Z} = K(\mathbb{Z}, 1)$.
- The inclusion $S^n \subset S^{n+1}$ as the equator is clearly null homotopic since the inclusion obviously extends to a map of the disk. Hence the direct limit space

$$\varinjlim_n S^n = \cup_n S^n = S^\infty$$

  is aspherical. Now $\mathbb{Z}_2$ acts freely on S^n by the antipodal map, as described in chapter one. The inclusions $S^n \subset S^{n+1}$ are equivariant and hence there is an induced free action of $\mathbb{Z}_2$ on S^∞. Thus the projection map

$$S^\infty \to S^\infty/\mathbb{Z}_2 = \mathbb{RP}^\infty$$

  is a universal principal $\mathbb{Z}_2 = O(1)$ - bundle, and so

$$\mathbb{RP}^\infty = BO(1) = B\mathbb{Z}_2 = K(\mathbb{Z}_2, 1)$$

  .
- Similarly, the inclusion of the unit sphere in $\mathbb{C}^n$ into the unit sphere in $\mathbb{C}^{n+1}$ gives an the inclusion $S^{2n-1} \subset S^{2n+1}$ which is null homotopic. It is also equivariant with respect to the free $S^1 = U(1)$ - action given by (complex) scalar multiplication. Then the limit $S^\infty = \cup_n S^{2n+1}$ is aspherical with a free S^1 action. We therefore have that the projection

$$S^\infty \to S^\infty/S^1 = \mathbb{CP}^\infty$$

is a principal $S^1 = U(1)$ bundle. Hence we have

$$\mathbb{CP}^\infty = BS^1 = BU(1).$$

Moreover since S^1 is a $K(\mathbb{Z}, 1)$, then we have that

$$\mathbb{CP}^\infty = K(\mathbb{Z}, 2).$$

- The cyclic groups $\mathbb{Z}_n$ are subgroups of $U(1)$ and so they act freely on S^∞ as well. Thus the projection maps

$$S^\infty \to S^\infty/\mathbb{Z}_n$$

is a universal principal $\mathbb{Z}_n$ bundle. The quotient space $S^\infty/\mathbb{Z}_n$ is denoted $L^\infty(n)$ and is referred to as the infinite $\mathbb{Z}_n$ - lens space.

These examples allow us to give the following description of line bundles and their relation to cohomology. We first recall a well known theorem in homotopy theory. This theorem will be discussed further in chapter 4. We refer the reader to [**42**] for details.

THEOREM 2.12. *Let G be an abelian group. Then there is a natural isomorphism*

$$\phi : H^n(K(G,n);G) \xrightarrow{\cong} Hom(G,G).$$

Let $\iota \in H^n(K(G,n);G)$ be $\phi^{-1}(id)$. This is called the fundamental class. Then if X has the homotopy type of a CW - complex, the mapping

$$[X, K(G,n)] \to H^n(X;G)$$
$$f \to f^*(\iota)$$

is a bijective correspondence.

With this we can now prove the following:

THEOREM 2.13. *There are bijective correspondences which allow us to classify complex line bundles,*

$$Vect^1(X) \cong Prin_{U(1)}(X) \cong [X, BU(1)] = [X, \mathbb{CP}^\infty] \cong [X, K(\mathbb{Z},2)] \cong H^2(X;\mathbb{Z})$$

where the last correspondence takes a map $f : X \to \mathbb{CP}^\infty$ to the class

$$c_1 = f^*(c) \in H^2(X),$$

where $c \in H^2(\mathbb{CP}^\infty)$ is the generator. In the composition of these correspondences, the class $c_1 \in H^2(X)$ corresponding to a line bundle $\zeta \in Vect^1(X)$ is called the first Chern class of ζ (or of the corresponding principal $U(1)$ - bundle).

Proof. These correspondences follow directly from the above considerations, once we recall that $Vect^1(X) \cong Prin_{GL(1,\mathbb{C})}(X) \cong [X, BGL(1,\mathbb{C})]$, and that $\mathbb{CP}^\infty$ is a model for $BGL(1,\mathbb{C})$ as well as $BU(1)$. This is because, we can express $\mathbb{CP}^\infty$ in its homogeneous form as

$$\mathbb{CP}^\infty = \varinjlim_n (\mathbb{C}^{n+1} - \{0\})/GL(1,\mathbb{C}),$$

and that $\varinjlim_n (\mathbb{C}^{n+1} - \{0\})$ is an aspherical space with a free action of $GL(1,\mathbb{C}) = \mathbb{C}^*$. □

There is a similar theorem classifying real line bundles:

Theorem 2.14. *There are bijective correspondences*

$$Vect^1_{\mathbb{R}}(X) \cong Prin_{O(1)}(X) \cong [X, BO(1)] = [X, \mathbb{RP}^\infty] \cong [X, K(\mathbb{Z}_2, 1)] \cong H^1(X; \mathbb{Z}_2)$$

where the last correspondence takes a map $f : X \to \mathbb{RP}^\infty$ to the class

$$w_1 = f^*(w) \in H^1(X; \mathbb{Z}_2),$$

where $w \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$ is the generator. In the composition of these correspondences, the class $w_1 \in H^1(X; \mathbb{Z}_2)$ corresponding to a line bundle $\zeta \in Vect^1_{\mathbb{R}}(X)$ is called the first Stiefel - Whitney class of ζ (or of the corresponding principal $O(1)$ - bundle).

More Examples.

- Let $V_n(\mathbb{C}^N)$ be the Stieflel - manifold studied in the last chapter. We claim that the inclusion of vector spaces $\mathbb{C}^N \subset \mathbb{C}^{2N}$ as the first N - coordinates induces an inclusion $V_n(\mathbb{C}^N) \hookrightarrow V_n(\mathbb{C}^{2N})$ which is null homotopic. To see this, let $\iota : \mathbb{C}^n \to \mathbb{C}^{2N}$ be a fixed linear embedding, whose image lies in the last N - coordinates in $\mathbb{C}^{2N}$. Then given any $\rho \in V_n(\mathbb{C}^N) \subset V_n(\mathbb{C}^{2N})$, then $t \cdot \iota + (1-t) \cdot \rho$ for $t \in [0,1]$ defines a one parameter family of linear embeddings of $\mathbb{C}^n$ in $\mathbb{C}^{2N}$, and hence a contraction of the image of $V_n(\mathbb{C}^N)$ onto the element ι. Hence the limiting space $V_n(\mathbb{C}^\infty)$ is aspherical with a free $GL(n,\mathbb{C})$ - action. Therefore the projection

$$V_n(\mathbb{C}^\infty) \to V_n(\mathbb{C}^\infty)/GL(n,\mathbb{C}) = Gr_n(\mathbb{C}^\infty)$$

is a universal $GL(n,\mathbb{C})$ - bundle. Hence the infinite Grassmannian is the classifying space

$$Gr_n(\mathbb{C}^\infty) = BGL(n,\mathbb{C})$$

and so we have a classification

$$Vect^n(X) \cong Prin_{GL(n,\mathbb{C})}(X) \cong [X, BGL(n,\mathbb{C})] \cong [X, Gr_n(\mathbb{C}^\infty)].$$

- A simlar argument shows that the infinite unitary Stiefel manifold, $V_n^U(\mathbb{C}^\infty)$ is aspherical with a free $U(n)$ - action. Thus the projection

$$V_n^U(\mathbb{C}^\infty) \to V_n(\mathbb{C}^\infty)/U(n) = Gr_n(\mathbb{C}^\infty)$$

is a universal principal $U(n)$ - bundle. Hence the infinite Grassmanian $Gr_n(\mathbb{C}^\infty)$ is the classifying space for $U(n)$ bundles as well,

$$Gr_n(\mathbb{C}^\infty) = BU(n).$$

The fact that this Grassmannian is both $BGL(n, \mathbb{C})$ and $BU(n)$ reflects the fact that every n - dimensional complex vector bundle has a $U(n)$ - structure.
- We have similar universal $GL(n, \mathbb{R})$ and $O(n)$ - bundles:

$$V_n(\mathbb{R}^\infty) \to V_n(\mathbb{R}^\infty)/GL(n, \mathbb{R}) = Gr_n(\mathbb{R}^\infty)$$

and

$$V_n^O(\mathbb{R}^\infty) \to V_n^O(\mathbb{R}^\infty)/O(n) = Gr_n(\mathbb{R}^\infty).$$

Thus we have

$$Gr_n(\mathbb{R}^\infty) = BGL(n, \mathbb{R}) = BO(n)$$

and so this infinite dimensional Grassmannian classifies real n - dimensional vector bundles as well as principal $O(n)$ - bundles.

Now suppose $p : EG \to EG/G = BG$ is a universal G - bundle. Suppose further that $H < G$ is a subgroup. Then H acts freely on EG as well, and hence the projection

$$EG \to EG/H$$

is a universal H - bundle. Hence $EG/H = BH$. Using the infinite dimensional Stiefel manifolds described above, this observation gives us models for the classifying spaces for any subgroup of a general linear group. So for example if we have a subgroup (i.e a faithful representation) $H \subset GL(n, \mathbb{C})$, then

$$BH = V_n(\mathbb{C}^\infty)/H.$$

This observation also leads to the following useful fact.

PROPOSITION 2.15. . *Let $p : EG \to BG$ be a universal principal G - bundle, and let $H < G$. Then there is a fiber bundle*

$$BH \to BG$$

with fiber the orbit space G/H.

PROOF. This bundle is given by

$$G/H \to EG \times_G G/H \to EG/G = BG$$

together with the observation that $EG \times_G G/H = EG/H = BH$. □

3. Classifying Gauge Groups

In this section we describe the classifying space of the group of automorphisms of a principal G - bundle, or the *gauge group* of the bundle. We describe the classifying space in two different ways: in terms of the space of connections on the bundle, and in terms of the mapping space of the base manifold to the classifying space BG. These constructions are important in Yang - Mills theory, and we refer the reader to [**3**] and [**11**] for more details.

Let A be a connection on a principal bundle $P \longrightarrow M$ where M is a closed manifold equipped with a Riemannian metric. The Yang - Mills functional applied to A, $\mathcal{YM}(A)$ is the square of the L^2 norm of the curvature,

$$\mathcal{YM}(A) = \frac{1}{2}\int_M \|F_A\|^2 \, d(vol).$$

We view $\mathcal{YM}$ as a mapping $\mathcal{YM} : \mathcal{A}(P) \longrightarrow \mathbb{R}$. The relevance of the gauge group in Yang - Mills theory is that this is the group of symmetries of $\mathcal{A}$ that $\mathcal{YM}$ preserves.

DEFINITION 2.5. The gauge group $\mathcal{G}(P)$ of the principal bundle P is the group of bundle automorphisms of $P \longrightarrow M$. That is, an element $\phi \in \mathcal{G}(P)$ is a bundle isomorphism of P with itself lying over the identity:

$$\begin{array}{ccc} P & \xrightarrow[\cong]{\phi} & P \\ \downarrow & & \downarrow \\ M & \xrightarrow{=} & M. \end{array}$$

Equivalently, $\mathcal{G}(P)$ is the group $\mathcal{G}(P) = Aut_G(P)$ of G - equivariant diffeomorphisms of the space P.

The gauge group $\mathcal{G}(P)$ can be thought of in several equivalent ways. The following one is particularly useful.

Consider the conjugation action of the Lie group G on itself,

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longrightarrow ghg^{-1}. \end{aligned}$$

This left action defines a fiber bundle

$$Ad(P) = P \times_G G \longrightarrow P/G = M$$

with fiber G. We leave the following as an exercise for the reader.

PROPOSITION 2.16. *The gauge group of a principal bundle $P \longrightarrow M$ is naturally isomorphic (as topological groups) the the group of sections of $Ad(P)$, $C^\infty(M; Ad(P))$.*

The gauge group $\mathcal{G}(P)$ acts on the space of connections $\mathcal{A}(P)$ by the pull - back construction. More generally, if $f : P \to Q$ is any smooth map of principal G - bundles and A is a connection on Q, then there is a natural pull back connection $f^*(A)$ on Q, defined by pulling back the equivariant splitting of τQ to an equivariant splitting of τP in the obvious way. The pull - back construction for automorphisms $\phi : P \longrightarrow P$ defines an action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$.

We leave the proof of the following is an exercise for the reader.

PROPOSITION 2.17. *Let P be the trivial bundle $M \times G \to M$. Then the gauge group $\mathcal{G}(P)$ is given by the function space from M to G,*

$$\mathcal{G}(P) \cong C^\infty(M; G).$$

Furthermore if $\phi : M \to G$ is identified with an element of $\mathcal{G}(P)$, and $A \in \Omega^1(M; \mathfrak{g})$ is identified with an element of $\mathcal{A}(G)$, then the induced action of ϕ on G is given by

$$\phi^*(A) = \phi^{-1} A \phi + \phi^{-1} d\phi.$$

It is not difficult to see that in general the gauge group $\mathcal{G}(P)$ does not act freely on the space of connections $\mathcal{A}(P)$. However there is an important subgroup $\mathcal{G}_0(P) < \mathcal{G}(P)$ that does. This is the group of *based* gauge transformations. To define this group, let $x_0 \in M$ be a fixed basepoint, and let P_{x_0} be the fiber of P at x_0.

DEFINITION 2.6. The based gauge group $\mathcal{G}_0(P)$ is a subgroup of the group of bundle automorphisms $\mathcal{G}(P)$ which pointwise fix the fiber $P_{x_)}$. That is,

$$\mathcal{G}_0(P) = \{\phi \in \mathcal{G}(P) : \text{if}\, v \in P_{x_0} \text{then}\, \phi(v) = v\}.$$

THEOREM 2.18. *The based gauge group $\mathcal{G}_0(P)$ acts freely on the space of connections $\mathcal{A}(P)$.*

PROOF. Suppose that $A \in \mathcal{A}(P)$ is a fixed point of $\phi \in \mathcal{G}_0(P)$. That is, $\phi^*(A) = A$. We need to show that $\phi = 1$.

The equivariant splitting ω_A given by a connection A defines a notion of parallel transport in P along curves in M (see [**16**]) . It is not difficult to see that the statement $\phi^*(A) = A$ implies that application of the automorphism ϕ commutes with parallel transport. Now let $w \in P_x$ be a point in the fiber of an element $x \in M$. Given curve γ in M between the basepoint x_0 and x one sees that

$$\phi(w) = T_\gamma(\phi(T_{\gamma^{-1}}(w))$$

where T_γ is parallel transport along γ. But since $T_{\gamma^{-1}}(w) \in P_{x_0}$ and $\phi \in \mathcal{G}_0(P)$,

$$\phi(T_{\gamma^{-1}}(w)) = w.$$

Hence $\phi(w) = w$, that is, $\phi = 1$. □

Remark. Notice that this argument actually says that if $A \in \mathcal{A}(P)$ is the fixed point of *any* gauge transformation $\phi \in \mathcal{G}(P)$, then ϕ is determined by its action on a single fiber.

Let $\mathcal{B}(P)$ and $\mathcal{B}_0(P)$ be the orbit spaces of connections on P up to guage and based gauge equivalence respectively,

$$\mathcal{B}(P) = \mathcal{A}(P)/\mathcal{G}(P) \quad \mathcal{B}_0(P) = \mathcal{A}(P)/\mathcal{G}_0(P).$$

Now it is straightforward to check directly that the Yang - Mills functional in invariant under gauge transformations. Thus it yields maps

$$\mathcal{YM} : \mathcal{B}(P) \to \mathbb{R} \quad \text{and} \quad \mathcal{YM} : \mathcal{B}_0(P) \to \mathbb{R}.$$

It is therefore important to understand the homotopy types of these orbit spaces. Because of the freeness of the action of $\mathcal{G}_0(P)$, the homotopy type of the orbit space $\mathcal{G}_0(P)$ is easier to understand.

We end this section with a discussion of its homotopy type. Since the space of connections $\mathcal{A}(P)$ is affine, it is contractible. Moreover it is possible to show that the free action of the based gauge group $\mathcal{G}_0(P)$ has local slices (see [**11**]). Thus we have $\mathcal{B}_0(P) = \mathcal{A}(P)/\mathcal{G}_0(P)$ is the classifying space of the based gauge group,

$$\mathcal{B}_0(P) = B\mathcal{G}_0(P).$$

But the classifying spaces of the gauge groups are relatively easy to understand. (see [**3**].)

THEOREM 2.19. *Let $G \longrightarrow EG \longrightarrow BG$ be a universal principal bundle for the Lie group G (so that EG is aspherical). Let $y_0 \in BG$ be a fixed basepoint. Then there are homotopy equivalences*

$$B\mathcal{G}(P) \simeq Map^P(M, BG) \quad \textit{and} \quad \mathcal{B}_0(P) \simeq B\mathcal{G}_0(P) \simeq Map_0^P(M, BG)$$

where $Map(M, BG)$ is the space of all continuous maps from M to BG and $Map_0(M, BG)$ is the space of those maps that preserve the basepoints. The superscript P denotes the path component of these mapping spaces consisting of the homotopy class of maps that classify the principal G - bundle P.

Proof. Consider the space of all G - equivariant maps from P to EG, $Map^G(P, EG)$. The gauge group $\mathcal{G}(P) \cong Aut^G(P)$ acts freely on the left of this space by composition. It is easy to see that $Map^G(P, EG)$ is aspherical, and its orbit space is given by the space of maps from the G - orbit space of P $(= M)$ to the G - orbit space of EG (= BG),

$$Map^G(P, EG)/\mathcal{G}(P) \cong Map^P(M, BG).$$

This proves that $Map(M, BG) = B\mathcal{G}(P)$. Similarly $Map_0^G(P, EG)$, the space of G - equivariant maps that send the fiber P_{x_0} to the fiber EG_{y_0}, is an aspherical space with a free $\mathcal{G}_0(P)$ action, whose orbit space is $Map_0^P(M, BG)$. Hence $Map_0^P(M, BG) = B\mathcal{G}_0(P)$. □

4. Existence of universal bundles: the Milnor join construction and the simplicial classifying space

In the last section we proved a "recognition principle" for universal principal G bundles. Namely, if the total space of a principal G - bundle $p : E \to B$ is aspherical, then it is universal. We also proved a homotopy uniqueness theorem, stating among other things that the homotopy type of the base space of a universal bundle, i.e the classifying space BG, is well defined. We also described many examples of universal bundles, and particular have a model for the classifying space BG, using Stiefel manifolds, for every subgroup of a general linear group.

The goal of this section is to prove the general existence theorem. Namely, for every group G, there is a universal principal G - bundle $p : EG \to BG$. We will give two constructions of the universal bundle and the corresponding classifying space. One, due to Milnor [**30**] involves taking the "infinite join" of a group with itself. The other is an example of a simplicial space, called the simplicial bar construction. It is originally due to Eilenberg and MacLane [**12**]. These constructions are essentially equivalent and both yield G - CW - complexes. Since they are so useful in algebraic topology and combinatorics, we will also take this opportunity to introduce the notion of a general simplicial space and show how these classifying spaces are important examples.

4.1. The join construction. The "join" between two spaces X and Y, written $X * Y$ is the space of all lines connecting points in X to points in Y. The following is a more precise definition:

DEFINITION 2.7. The join $X * Y$ is defined by

$$X * Y = X \times I \times Y / \sim$$

where $I = [0, 1]$ is the unit interval and the equivalence relation is given by $(x, 0, y_1) \sim (x, 0, y_2)$ for any two points $y_1,\, y_2 \in Y$, and similarly $(x_1, 1, y) \sim (x_2, 1, y)$ for any two points $x_1,\, x_2 \in X$.

A point $(x, t, y) \in X * Y$ should be viewed as a point on the line connecting the points x and y. Here are some examples.

Examples.

- Let y be a single point. Then $X * y$ is the cone $CX = X \times I / X \times \{1\}$.
- Let $Y = \{y_1, y_2\}$ be the space consisting of two distinct points. Then $X * Y$ is the suspension ΣX discussed earlier. Notice that the suspension can be viewed as the union of two cones, with vertices y_1 and y_2 respectively, attached along the equator.
- **Exercise.** Prove that the join of two spheres, is another sphere,

$$S^n * S^m \cong S^{n+m+1}.$$

- Let $\{x_0, \cdots, x_k\}$ be a collection of $k + 1$ - distinct points. Then the k - fold join $x_0 * x_1 * \cdots * x_k$ is the convex hull of these points and hence is by the k - dimensional simplex Δ^k with vertices $\{x_0, \cdots, x_k\}$.

Observe that the space X sits naturally as a subspace of the join $X * Y$ as endpoints of line segments,

$$\begin{aligned} \iota : X &\hookrightarrow X * Y \\ x &\to (x, 0, y). \end{aligned}$$

Notice that this formula for the inclusion makes sense and does not depend on the choice of $y \in Y$. There is a similar embedding

$$\begin{aligned} j : Y &\hookrightarrow X * Y \\ y &\to (x, 1, y). \end{aligned}$$

LEMMA 2.20. *The inclusions $\iota : X \hookrightarrow X * Y$ and $j : Y \hookrightarrow X * Y$ are null homotopic.*

PROOF. Pick a point $y_0 \in Y$. By definition, the embedding $\iota : X \to X * Y$ factors as the composition

$$\begin{aligned} \iota : X &\hookrightarrow X * y_0 \subset X * Y \\ x &\to (x, 0, y_0). \end{aligned}$$

But as observed above, the join $X * y_0$ is the cone on X and hence contractible. This means that ι is null homotopic, as claimed. The fact that $j : Y \hookrightarrow X * Y$ is null homotopic is proved in the same way. $\square$

Now let G be a group and consider the iterated join

$$G^{*(k+1)} = G * G * \cdots * G$$

where there are $k+1$ copies of the group element. This space has a free G action given by the diagonal action

$$g \cdot (g_0, t_1, g_1, \cdots, t_k, g_k) = (gg_0, t_1, gg_1, \cdots, t_k, gg_k).$$

Exercise. **1.** Prove that there is a natural G - equivariant map

$$\Delta^k \times G^{k+1} \to G^{*(k+1)}$$

which is a homeomorphism when restricted to $\tilde{\Delta}^k \times G^{k+1}$ where $\tilde{\Delta}^k \subset \Delta^k$ is the interior. Here G acts on $\Delta^k \times G^{k+1}$ trivially on the simplex Δ^k and diagonally on G^{k+1}.

2. Use exercise 1 to prove that the iterated join $G^{*(k+1)}$ has the structure of a G - CW - complex.

Define $\mathcal{J}(G)$ to be the infinite join

$$\mathcal{J}(G) = \lim_{k \to \infty} G^{*(k+1)}$$

where the limit is taken over the embeddings $\iota : G^{*(k+1)} \hookrightarrow G^{*(k+2)}$ Since these embedding maps are G -equivariant, we have an induced G - action on $\mathcal{J}(G)$.

THEOREM 2.21. *The projection map*

$$p : \mathcal{J}(G) \to \mathcal{J}(G)/G$$

is a universal principal G - bundle.

PROOF. By the above exercise the space $\mathcal{J}(G)$ has the structure of a G - CW - complex with a free G - action. Therefore by the results of the last section the projection $p : \mathcal{J}(G) \to \mathcal{J}(G)/G$ is a principal G - bundle. To see that $\mathcal{J}(G)$ is aspherical, notice that since S^n is compact, any map $\alpha : S^n \to \mathcal{J}(G)$ is homotopic to one that factors through a finite join (that by abuse of notation we still call α), $\alpha : S^n \to G^{*(n+1)} \hookrightarrow \mathcal{J}(G)$. But by the above lemma the inclusion $G^{*(n+1)} \subset \mathcal{J}(G)$ is null homotopic, and hence so is α. Thus $\mathcal{J}(G)$ is aspherical. By the results of last section, this means that the projection $\mathcal{J}(G) \to \mathcal{J}(G)/G$ is a universal G - bundle. $\square$

4.2. Simplicial spaces and classifying spaces. We therefore now have a universal bundle for every topological group G. We actually know a fair amount about the geometry of the total space $EG = \mathcal{J}(G)$ which, by the above exercise can be described as the union of simplices, where the k - simplices are parameterized by $k+1$ -tuples of elements of G,

$$EG = \mathcal{J}(G) = \bigcup_k \Delta^k \times G^{k+1}/\sim$$

and so the classifying space can be described by

$$BG = \mathcal{J}(G)/G \cong \bigcup_k \Delta^k \times G^k/\sim$$

It turns out that in these constructions, the simplices are glued together along faces, and these gluings are parameterized by the $k+1$ - product maps $\partial_i : G^{k+2} \to G^{k+1}$ given by multiplying the i^{th} and $(i+1)^{st}$ coordinates.

Having this type of data (parameterizing spaces of simplices as well as gluing maps) is an example of an object known as a *"simplicial set"* which is an important combinatorial object in topology. We now describe this notion in more detail and show how these universal G - bundles and classifying spaces can be viewed in these terms.

Good references for this theory are [**9**], [**26**].

The idea of simplicial sets is to provide a combinatorial technique to study cell complexes built out of simplices; i.e simplicial complexes. A simplicial complex X is built out of a union of simplices, glued along faces. Thus if X_n denotes the indexing set for the n - dimensional simplices of X, then we can write

$$X = \bigcup_{n \geq 0} \Delta^n \times X_n/\sim$$

where Δ^n is the standard n - simplex in $\mathbb{R}^n$;

$$\Delta^n = \{(t_1, \cdots, t_n) \in \mathbb{R}^n : 0 \leq t_j \leq 1, \text{ and } \sum_{i=1}^n t_i \leq 1\}.$$

The gluing relation in this union can be encoded by set maps among the X_n's that would tell us for example how to identify an $n-1$ simplex indexed by an element of X_{n-1} with a particular face of an n - simplex indexed by an element of X_n. Thus in principal simplicial complexes can be studied purely combinatorially in terms of the sets X_n and set maps between them. The notion of a *simplicial set* makes this idea precise.

Definition 2.8. A *simplicial set* X_* is a collection of sets

$$X_n, \quad n \geq 0$$

together with set maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

for $0 \leq i, j \leq n$ called **face** and **degeneracy** maps respectively. These maps are required to satisfy the following compatibility conditions

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \quad \text{for } i < j \\ s_i s_j &= s_{j+1} s_i \quad \text{for } i < j \end{aligned}$$

and

$$\partial_i s_j = \begin{cases} s_{j-1} \partial_i & \text{for } i < j \\ 1 & \text{for } i = j,\, j+1 \\ s_j \partial_{i-1} & \text{for } i > j+1 \end{cases}$$

As mentioned above, the maps ∂_i and s_j encode the combinatorial information necessary for gluing the simplices together. To say precisely how this works, consider the following maps between the standard simplices:

$$\delta_i : \Delta^{n-1} \longrightarrow \Delta^n \quad \text{and} \quad \sigma_j : \Delta^{n+1} \longrightarrow \Delta^n$$

for $0 \leq i, j \leq n$ defined by the formulae

$$\delta_i(t_1, \cdots, t_{n-1}) = \begin{cases} (t_1, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}) & \text{for } i \geq 1 \\ (1 - \sum_{q=1}^{n-1} t_q, t_1, \cdots, t_{n-1}) & \text{for } i = 0 \end{cases}$$

and

$$\sigma_j(t_1, \cdots, t_{n+1}) = \begin{cases} (t_1, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_{n+1}) & \text{for } i \geq 1 \\ (t_2, \cdots, t_{n+1}) & \text{for } i = 0\ . \end{cases}$$

δ_i includes Δ^{n-1} in Δ^n as the i^{th} face, and σ_j projects, in a linear fashion, Δ^{n+1} onto its j^{th} face.

We can now define the space associated to the simplicial set X_* as follows.

Definition 2.9. The *geometric realization* of a simplicial set X_* is the space

$$\|X_*\| = \bigcup_{n \geq 0} \Delta^n \times X_n / \sim$$

where if $t \in \Delta^{n-1}$ and $x \in X_n$, then

$$(t, \partial_i(x)) \sim (\delta_i(t), x)$$

and if $t \in \Delta^{n+1}$ and $x \in X_n$ then

$$(t, s_j(x)) \sim (\sigma_j(t), x).$$

In the topology of $\|X_*\|$, each X_n is assumed to have the discrete topology, so that $\Delta^n \times X_n$ is a discrete set of n - simplices.

Thus $\|X_*\|$ has one n - simplex for every element of X_n, glued together in a way determined by the face and degeneracy maps.

Example. Consider the simplicial set $\mathbf{S}_*$ defined as follows. The set of n - simplices is given by

$$\mathbf{S}_n = \mathbb{Z}/(n+1), \text{ generated by an element } \tau_n.$$

The face maps are given by

$$\partial_i(\tau_n^r) = \begin{cases} \tau_{n-1}^r & \text{if } r \leq i \leq n \\ \tau_{n-1}^{r-1} & \text{if } 0 \leq i \leq r-1. \end{cases}$$

The degeneracies are given by

$$s_i(\tau_n^r) = \begin{cases} \tau_{n+1}^r & \text{if } r \leq i \leq n \\ \tau_{n+1}^{r+1} & \text{if } 0 \leq i \leq r-1. \end{cases}$$

Notice that there is one zero simplex, two one simplices, one of them the image of the degeneracy $s_0 : \mathbf{S}_0 \longrightarrow \mathbf{S}_1$, and the other nondegenerate (i.e not in the image of a degeneracy map). Notice also that all simplices in dimensions larger than one are in the image of a degeneracy map. Hence we have that the geometric realization

$$\|\mathbf{S}_*\| = \Delta^1/0 \sim 1 = S^1.$$

Let X_* be any simplicial set. There is a particularly nice and explicit way for computing the homology of the geometric realization, $H_*(\|X_*\|)$.

Consider the following chain complex. Define $C_n(X_*)$ to be the free abelian group generated by the set of n - simplices X_n. Define the homomorphism

$$d_n : C_n(X_*) \longrightarrow C_{n-1}(X_*)$$

by the formula

$$d_n([x]) = \sum_{i=0}^{n} (-1)^i \partial_i([x])$$

where $x \in X_n$.

Proposition 2.22. *The homology of the geometric realization $H_*(\|X_*\|)$ is the homology of the chain complex*

$$\longrightarrow \cdots \xrightarrow{d_{n+1}} C_n(X_*) \xrightarrow{d_n} C_{n-1}(X_*) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_0} C_0(X_*).$$

Proof. It is straightforward to check that the geometric realization $\|X_*\|$ is a CW - complex and that this is the associated cellular chain complex. □

Besides being useful computationally, the following result establishes the fact that all CW complexes can be studied simplicially.

Theorem 2.23. *Every CW complex has the homotopy type of the geometric realization of a simplicial set.*

Proof. Let X be a CW complex. Define the singular simplicial set of X , $\mathcal{S}(X)_*$ as follows. The n simplices $\mathcal{S}(X)_n$ is the set of singular n - simplices,

$$\mathcal{S}(X)_n = \{c : \Delta^n \longrightarrow X\}.$$

The face and degeneracy maps are defined by

$$\partial_i(c) = c \circ \delta_i : \Delta^{n-1} \longrightarrow \Delta^n \longrightarrow X$$

and

$$s_j(c) = c \circ \sigma_i : \Delta^{n+1} \longrightarrow \Delta^n \longrightarrow X.$$

Notice that the associated chain complex to $\mathcal{S}(X)_*$ as in 2.22 is the singular chain complex of the space X. Hence by 2.22 we have that

$$H_*(\|\mathcal{S}(X)\|) \cong H_*(X).$$

This isomorphism is actually realized by a map of spaces

$$E : \|\mathcal{S}(X)_*\| \longrightarrow X$$

defined by the natural evaluation maps

$$\Delta^n \times \mathcal{S}(X)_n \longrightarrow X$$

given by

$$(t, c) \longrightarrow c(t).$$

It is straightforward to check that the map E does induce an isomorphism in homology. In fact it induces an isomorphism in homotopy groups. We will not prove this here; it is more technical and we refer the reader to [M] for details. Note that it follows from the homological isomorphism by the Hurewicz theorem if we knew that X was simply connected. A map between spaces that induces an isomorphism in homotopy groups is called a *weak homotopy equivalence*. Thus *any* space is weakly homotopy equivalent to a CW - complex (i.e the geometric realization of its singular simplicial set). But by the Whitehead theorem, two CW complexes that are weakly homotopy equivalent are homotopy equivalent. Hence X and $\|\mathcal{S}(X)_*\|$ are homotopy equivalent. □

We next observe that the notion of simplicial set can be generalized as follows. We say that X_* is a **simplicial space** if it is a simplicial set (i.e it satisfies definition 2.8) where the sets X_n are topological spaces and the face and degeneracy maps

$$\partial_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_j : X_n \longrightarrow X_{n+1}$$

are continuous maps. The definition of the geometric realization of a simplicial space X_*, $\|X_*\|$, is the same as in 2.9 with the proviso that the topology of each $\Delta^n \times X_n$ is the product topology. Notice that since the "set of n - simplices" X_n is actually a space, it is not necessarily true that $\|X_*\|$ is a CW complex. However if in fact each X_n is a CW complex and the face and degeneracy maps are cellular, then $\|X_*\|$ does have a natural CW structure induced by the product CW - structures on $\Delta^n \times X_n$.

Notice that this simplicial notion generalizes even further. For example a **simplicial group** would be defined similarly, where each X_n would be a group and the face and degeneracy maps are group homomorphisms. Simplicial vector spaces, modules, etc. are defined similarly. The categorical nature of these definitions should by now be coming clear. Indeed most generally one can define a **simplicial object in a category** $\mathcal{C}$ using the above definition where now the X_n's are assumed to be objects in the category and the face and degenarcies are assumed to be morphisms. If the category $\mathcal{C}$ is a subcategory of the category of sets then geometric realizations can be defined as in 2.9 For example the geometric realization of a simplicial (abelian) group turns out to be a topological (abelian) group.(Try to verify this for yourself!)

We now use this simplicial theory to construct universal principal G - bundles and classifying spaces.

Let G be a topological group and let $\mathcal{E}G_*$ be the simplicial space defined as follows. The space of n - simplices is given by the $n+1$ - fold cartesian product

$$\mathcal{E}G_n = G^{n+1}.$$

The face maps $\partial_i : G^{n+1} \longrightarrow G^n$ are given by the formula

$$\partial_i(g_0, \cdots, g_n) = (g_0, \cdots, \hat{g}_i, \cdots, g_n).$$

The degeneracy maps $s_j : G^{n+1} \longrightarrow G^{n+2}$ are given by the formula

$$s_j(g_0, \cdots, g_n) = (g_0, \cdots, g_j, g_j, \cdots, g_n).$$

Exercise. Show that the geometric realization $\|\mathcal{E}G_*\|$ is aspherical. Hint. Let $\|\mathcal{E}G_*\|^{(n)}$ be the n^{th} - skeleton,

$$\|\mathcal{E}G_*\|^{(n)} = \bigcup_{p=0}^{n} \Delta^p \times G^{p+1}.$$

Then show that the inclusion of one skeleton in the next $\|\mathcal{E}G_*\|^{(n)} \hookrightarrow \|\mathcal{E}G_*\|^{(n+1)}$ is null - homotopic. One way of doing this is to establish a homeomorphism between $\|\mathcal{E}G_*\|^{(n)}$ and n - fold join $G * \cdots * G$. See [M] for details.

Notice that the group G acts freely on the right of $\|\mathcal{E}G_*\|$ by the rule

$$\|\mathcal{E}G_*\| \times G = \left(\bigcup_{p \geq 0} \Delta^p \times G^{p+1}\right) \times G \longrightarrow \|\mathcal{E}G_*\| \tag{4.1}$$

$$(t; (g_0, \cdots, g_p)) \times g \longrightarrow (t; (g_0 g, \cdots, g_p g)) \, .$$

Thus we can define $EG = \|\mathcal{E}G_*\|$. The projection map

$$p : EG \to EG/G = BG$$

is therefore a universal principal G - bundle.

This description gives the classfiying space BG an induced simplicial structure described as follows.

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Let $\mathcal{B}G_*$ be the simplicial space whose n - simplices are the cartesian product

$$\mathcal{B}G_n = G^n. \tag{4.2}$$

The face and degeneracy maps are given by

$$\partial_i(g_1, \cdots, g_n) = \begin{cases} (g_2, \cdots, g_n) & \text{for } i = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots g_n) & \text{for } 1 \leq i \leq n-1 \\ (g_1, \cdots, g_{n-1}) & \text{for } i = n. \end{cases}$$

The degeneracy maps are given by

$$s_j(g_1, \cdots, g_n) = \begin{cases} (1, g_1, \cdots, g_n) & \text{for } j = 0 \\ (g_1, \cdots g_j, 1, g_{j+1}, \cdots, g_n) & \text{for } j \geq 1. \end{cases}$$

The simplicial projection map

$$p : \mathcal{E}G_* \longrightarrow \mathcal{B}G_*$$

defined on the level of n - simplicies by

$$p(g_0, \cdots, g_n) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \cdots, g_{n-1} g_n^{-1})$$

is easily checked to commute with face and degeneracy maps and so induces a map on the level of geometric realizations

$$p : EG = \|\mathcal{E}G_*\| \longrightarrow \|\mathcal{B}G_*\|$$

which induces a homemorphism

$$BG = EG/G \xrightarrow{\cong} \|\mathcal{B}G_*\|.$$

Thus for any topological group this construction gives a simplicial space model for its classifying space. This is referred to as the **simplicial bar construction.** Notice that when G is discrete the bar construction is a CW complex for the classifying space $BG = K(G,1)$ and 2.22 gives a particularly nice complex for computing its homology. (The homology of a $K(G,1)$ is referred to as the homology of the group G.)

The n - chains are the group ring

$$C_n(\mathcal{B}G_*) = \mathbb{Z}[G^n] \cong \mathbb{Z}[G]^{\otimes n}$$

and the boundary homomorphisms

$$d_n : \mathbb{Z}[G]^{\otimes n} \longrightarrow \mathbb{Z}[G]^{\otimes n-1}$$

are given by

$$\begin{aligned} d_n(a_1 \otimes \cdots \otimes a_n) = (a_2 \otimes \cdots \otimes a_n) + &\sum_{i=1}^{n-1}(-1)^i(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &+ (-1)^n(a_1 \otimes \cdots \otimes a_{n-1}). \end{aligned}$$

This complex is called the **bar complex** for computing the homology of a group and was discovered by Eilenberg and MacLane in the mid 1950's.

We end this chapter by observing that the bar construction of the classifying space of a group did not use the full group structure. It only used the existence of an associative multiplication with unit. That is, it did not use the existence of inverse. So in particular one can study the classifying space BA of a monoid A. This is an important construction in algebraic - K - theory.

5. Some Applications

In a sense much of what we will study in the next chapter are applications of the classification theorem for principal bundles. In this section we describe a few immediate applications.

5.1. Line bundles over projective spaces. By the classification theorem we know that the set of isomorphism classes of complex line bundles over the projective space $\mathbb{CP}^n$ is given by

$$\begin{aligned} Vect^1(\mathbb{CP}^n) \cong Prin_{GL(1,\mathbb{C})}(\mathbb{CP}^n) \cong Prin_{U(1)}(\mathbb{CP}^n) \cong [\mathbb{CP}^n, BU(1)] = [\mathbb{CP}^n, \mathbb{CP}^\infty] \\ = [\mathbb{CP}^n, K(\mathbb{Z},2)] \cong H^2(\mathbb{CP}^n,\mathbb{Z}) \cong \mathbb{Z} \end{aligned}$$

Theorem 2.24. *Under the above isomorphism,*

$$Vect^1(\mathbb{CP}^n) \cong \mathbb{Z}$$

the n - fold tensor product of the universal line bundle $\gamma_1^{\otimes n}$ corresponds to the integer $n \geq 0$.

Proof. The classification theorem says that every line bundle ζ over $\mathbb{CP}^n$ is the pull back of the universal line bundle via a map $f_\zeta : \mathbb{CP}^n \to \mathbb{CP}^\infty$. That is,

$$\zeta \cong f_\zeta^*(\gamma_1).$$

The cohomology class corresponding to ζ, *the first chern class* $c_1(\zeta)$, is given by

$$c_1(\zeta) = f_\zeta^*(c) \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$$

where $c \in H^2(\mathbb{CP}^\infty) \cong \mathbb{Z}$ is the generator. Clearly $\iota^*(c) \in H^2(\mathbb{CP}^n)$ is the generator, where $\iota : \mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$ is natural inclusion. But $\iota^*(\gamma_1) = \gamma_1 \in Vect^1(\mathbb{CP}^n)$. Thus $\gamma_1 \in Vect^1(\mathbb{CP}^n) \cong \mathbb{Z}$ corresponds to the generator.

To see the effect of taking tensor products, consider the following "tensor product map"

$$BU(1) \times \cdots \times BU(1) \xrightarrow{\otimes} BU(1)$$

defined to be the unique map (up to homotopy) that classifies the external tensor product $\gamma_1 \otimes \cdots \otimes \gamma_1$ over $BU(1) \times \cdots \times BU(1)$. Using $\mathbb{CP}^\infty \cong Gr_1(\mathbb{C}^\infty)$ as our model for $BU(1)$, this tensor product map is given by taking k lines $\ell_1, \cdots, \ell_k$ in $\mathbb{C}^\infty$ and considering the tensor product line

$$\ell_1 \otimes \cdots \otimes \ell_k \subset \mathbb{C}^\infty \otimes \cdots \otimes \mathbb{C}^\infty \xrightarrow[\psi]{\cong} \mathbb{C}^\infty$$

where $\psi : \mathbb{C}^\infty \otimes \cdots \otimes \mathbb{C}^\infty \cong \mathbb{C}^\infty$ is a fixed isomorphism. The induced map

$$\tau : \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty \cong K(\mathbb{Z},2)$$

is determined up to homotopy by what its effect on H^2 is. Clearly the restriction to each factor is the identity map and so

$$\tau^*(c) = c_1 + \cdots + c_k \in H^2(\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty) = H^2(\mathbb{CP}^\infty) \oplus \cdots \oplus H^2(\mathbb{CP}^\infty) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$$

where c_i denotes the generator of H^2 of the i^{th} factor in the product. Therefore the composition

$$t_k : \mathbb{CP}^\infty \xrightarrow{\Delta} \mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty \xrightarrow{\tau} \mathbb{CP}^\infty$$

has the property that $t_k^*(c) = kc \in H^2(\mathbb{CP}^\infty)$. But also we have that on the bundle level,

$$t_k^*(\gamma_1) = \gamma_1^{\otimes k} \in Vect^1(\mathbb{CP}^\infty).$$

The theorem now follows. $\square$

We have a similar result for real line vector bundles over real projective spaces.

THEOREM 2.25. *The only nontrivial real line bundle over $\mathbb{RP}^n$ is the canonical line bundle γ_1.*

PROOF. We know that γ_1 is nontrivial because its restriction to $S^1 = \mathbb{RP}^1 \subset \mathbb{RP}^n$ is the Moebeus strip line bundle, which is nonorientable, and hence nontrivial. On the other hand, by the classification theorem,

$$Vect^1_{\mathbb{R}}(\mathbb{RP}^n) \cong [\mathbb{RP}^n, BGL(1,\mathbb{R})] = [\mathbb{RP}^n, \mathbb{RP}^\infty] = [\mathbb{RP}^n, K(\mathbb{Z}_2, 1)] \cong H^1(\mathbb{RP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Hence there is only one nontrivial line bundle over $\mathbb{RP}^n$. $\square$

.

5.2. Structures on bundles and homotopy liftings. The following theorem is a direct consequence of the classification theorem. We leave its proof as an exercise.

THEOREM 2.26. . *Let $p : E \to B$ be a principal G - bundle classified by a map $f : B \to BG$. Let $H < G$ be a subgroup. By the naturality of the construction of classifying spaces, this inclusion induces a map (well defined up to homotopy) $\iota : BH \to BG$. Then the bundle $p : E \to B$ has an H - structure (i.e a reduction of its structure group to H) if and only if there is a map*

$$\tilde{f} : B \to BH$$

so that the composition

$$B \xrightarrow{\tilde{f}} BH \xrightarrow{\iota} BG$$

is homotopic to $f : B \to BG$. In particular if $\tilde{p} : \tilde{E} \to B$ is the principal H - bundle classified by $\tilde{f}$, then there is an isomorphism of principal G bundles,

$$\tilde{E} \times_H G \cong E.$$

The map $\tilde{f} : B \to BH$ is called a "lifting" of the classifying map $f : B \to BG$. It is called a lifting because, as we saw at the end of the last section, the map $\iota : BH \to BG$ can be viewed as a fiber bundle, by taking our model for BH to be $BH = EG/H$. Then ι is the projection for the fiber bundle

$$G/H \to EG/H = BH \xrightarrow{\iota} EG/G = BG.$$

This bundle structure will allow us to analyze in detail what the obstructions are to obtaining a lift $\tilde{f}$ of a classifying map $f : B \to BG$. We will study this is chapter 4.

Examples.

- An orientation of a bundle classified by a map $f : B \to BO(k)$ is a lifting $\tilde{f} : B \to BSO(k)$. Notice that the map $\iota : BSO(k) \to BO(k)$ can be viewed as a two - fold covering map
$$\mathbb{Z}_2 = O(k)/SO(k) \to BSO(k) \xrightarrow{\iota} BO(k).$$
- An almost complex structure of a bundle classified by a map $f : B \to BO(2n)$ is a lifting $\tilde{f} : B \to BU(n)$. Notice we have a bundle
$$O(2n)/U(n) \to BU(n) \to BO(2n).$$

The following example will be particularly useful in the next chapter when we define characteristic classes and do calculations with them.

Theorem 2.27. *A complex bundle vector bundle ζ classified by a map $f : B \to BU(n)$ has a nowhere zero section if and only if f has a lifting $\tilde{f} : B \to BU(n-1)$. Similarly a real vector bundle η classified by a map $f : B \to BO(n)$ has a nowhere zero section if and only if f has a lifting $\tilde{f} : B \to BO(n-1)$. Notice we have the following bundles:*
$$S^{2n-1} = U(n)/U(n-1) \to BU(n-1) \to BU(n)$$
and
$$S^{n-1} = O(n)/O(n-1) \to BO(n-1) \to BO(n).$$

This theorem says that $BU(n-1)$ forms a sphere bundle (S^{2n-1}) over $BU(n)$, and similarly, $BO(n-1)$ forms a S^{n-1} - bundle over $BO(n)$. We identify these sphere bundles as follows.

Corollary 2.28. *The sphere bundles*
$$S^{2n-1} \to BU(n-1) \to BU(n)$$
and
$$S^{n-1} \to BO(n-1) \to BO(n)$$
are isomorphic to the unit sphere bundles of the universal vector bundles γ_n over $BU(n)$ and $BO(n)$ respectively.

Proof. We consider the complex case. The real case is proved in the same way. Notice that the model for the sphere bundle in the above theorem is the projection map

$$p : BU(n-1) = EU(n)/U(n-1) \to EU(n)/U(n) = BU(n).$$

But γ_n is the vector bundle $EU(n) \times_{U(n)} \mathbb{C}^n \to BU(n)$ which therefore has unit sphere bundle

$$S(\gamma_n) = EU(n) \times_{U(n)} S^{2n-1} \to BU(n) \tag{5.1}$$

where $S^{2n-1} \subset \mathbb{C}^n$ is the unit sphere with the induced $U(n)$ - action. But $S^{2n-1} \cong U(n)/U(n-1)$ and this diffeomorphism is equivariant with respect to this action. Thus the unit sphere bundle is given by

$$S(\gamma_n) = EU(n) \times_{U(n)} U(n)/U(n-1) \cong EU(n)/U(n-1) = BU(n-1)$$

as claimed. □

We observe that by using the Grassmannian models for $BU(n)$ and $BO(n)$, then their relation to the sphere bundles can be seen explicitly in the following way. This time we work in the real case.

Consider the embedding

$$\iota : Gr_{n-1}(\mathbb{R}^N) \hookrightarrow Gr_n(\mathbb{R}^N \times \mathbb{R}) = Gr_n(\mathbb{R}^{N+1})$$

defined by

$$(V \subset \mathbb{R}^N) \to (V \times \mathbb{R} \subset \mathbb{R}^N \times \mathbb{R}).$$

Clearly as $N \to \infty$ this map becomes a model for the inclusion $BO(n-1) \hookrightarrow BO(n)$. Now for $V \in Gr_{n-1}(\mathbb{R}^N)$ consider the vector $(0,1) \in V \times \mathbb{R} \subset \mathbb{R}^N \times \mathbb{R}$. This is a unit vector, and so is an element of the fiber of the unit sphere bundle $S(\gamma_n)$ over $V \times \mathbb{R}$. Hence this association defines a map

$$j : Gr_{n-1}(\mathbb{R}^N) \to S(\gamma_n)$$

which lifts $\iota : Gr_{n-1}(\mathbb{R}^N) \hookrightarrow Gr_n(\mathbb{R}^{N+1})$. By taking a limit over N we get a map $j : BO(n-1) \to S(\gamma_n)$.

To define a homotopy inverse $\rho : S(\gamma_n) \to BO(n-1)$, we again work on the finite Grassmannian level.

Let $(W, w) \in S(\gamma_n)$, the unit sphere bundle over $Gr_n(\mathbb{R}^K)$. Thus $W \subset \mathbb{R}^K$ is an n -dimensional subspace and $w \in W$ is a unit vector. Let $W_w \subset W$ denote the orthogonal complement to the vector w in W. Thus $W_w \subset W \subset \mathbb{R}^K$ is an $n-1$ - dimensional subspace. This association defines a map

$$\rho : S(\gamma_n) \to Gr_{n-1}(\mathbb{R}^K)$$

and by taking the limit over K, defines a map $\rho : S(\gamma_n) \to BO(n-1)$. We leave it to the reader to verify that $j : BO(n-1) \to S(\gamma_n)$ and $\rho : S(\gamma_n) \to BO(n-1)$ are homotopy inverse to each other.

5.3. Embedded bundles and K -theory. The classification theorem for vector bundles says that for every n - dimensional complex vector bundle ζ over X, there is a classifying map $f_\zeta : X \to BU(n)$ so that ζ is isomorphic to pull back, $f^*(\gamma_n)$ of the universal vector bundle. A similar statement holds for real vector bundles. Using the Grassmannian models for these classifying spaces, we obtain the following as a corollary.

THEOREM 2.29. *Every n - dimensional complex bundle ζ over a space X can be embedded in a trivial infinite dimensional bundle, $X \times \mathbb{C}^\infty$. Similarly, every n - dimensional real bundle η over X can be embedded in the trivial bundle $X \times \mathbb{R}^\infty$.*

PROOF. Let $f_\zeta : X \to Gr_n(\mathbb{C}^\infty) = BU(n)$ classify ζ. So $\zeta \cong f^*(\gamma_n)$. But recall that

$$\gamma_n = \{(V, v) \in Gr_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty \quad \text{such that}\, v \in V.\}$$

Hence γ_n is naturally embedded in the trivial bundle $Gr_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty$. Thus $\zeta \cong f^*(\gamma_n)$ is naturally embedded in $X \times \mathbb{C}^\infty$. The real case is proved similarly. $\square$

Notice that because of the direct limit topology on $Gr_n(\mathbb{C}^\infty) = \varinjlim Gr_n(\mathbb{C}^N)$, then if X is a compact space, any map $f : X \to Gr_n(\mathbb{C}^\infty)$ has image that lies in $Gr_n(\mathbb{C}^N)$ for some finite N. But notice that over this finite Grassmannian, $\gamma_n \subset Gr_n(\mathbb{C}^N) \times \mathbb{C}^N$. The following is then an immediate corollary. This result was used in chapter one in our discussion about K -theory.

COROLLARY 2.30. *If X is compact, then every n - dimensional complex bundle zeta can be embedded in a trivial bundle $X \times \mathbb{C}^N$ for some N. The analogous result also holds for real vector bundles.*

Let $f : X \to BU(n)$ classify the n - dimensional complex vector bundle ζ. Then clearly the composition $f : X \to BU(n) \hookrightarrow BU(n+1)$ classifies the $n+1$ dimensional vector bundle $\zeta \oplus \epsilon_1$, where as before, ϵ_1 is the one dimensional trivial line bundle. This observation leads to the following.

PROPOSITION 2.31. *Let ζ_1 and ζ_2 be two n -dimensional vector bundles over X classified by f_1 and $f_2 : X \to BU(n)$ respectively. Then if we add trivial bundles, we get an isomorphism*

$$\zeta_1 \oplus \epsilon_k \cong \zeta_2 \oplus \epsilon_k$$

if and only if the compositions,

$$f_1,\, f_2 : X \to BU(n) \hookrightarrow BU(n+k)$$

are homotopic.

Now recall from the discussion of K - theory in chapter 1 that the set of *stable* isomorphism classes of vector bundles $\mathcal{S}Vect(X)$ is isomorphic to the reduced K - theory, $\tilde{K}(X)$, when X is compact. This proposition then implies the following important result, which displays how in the case of compact spaces, computing K -theory reduces to a specific homotopy theory calculation.

Definition 2.10. Let BU be the limit of the spaces

$$BU = \varinjlim_n BU(n).$$

Similarly,

$$BO = \varinjlim_n BO(n).$$

Theorem 2.32. *For X compact there are isomorphisms (bijective correspondences)*

$$\tilde{K}(X) \cong \mathcal{S}Vect(X) \cong [X, BU]$$

and

$$\tilde{K}O(X) \cong \mathcal{S}Vect_{\mathbb{R}}(X) \cong [X, BO].$$

5.4. Representations and flat connections. Recall the following classification theorem for covering spaces.

Theorem 2.33. . *Let X be a connected space. Then the set of isomorphism classes of connected covering spaces, $p : E \to X$ is in bijective correspondence with conjugacy classes of normal subgroups of $\pi_1(X)$. This correspondence sends a covering $p : E \to B$ to the image $p_*(\pi_1(E)) \subset \pi_1(X)$.*

Let $\pi = \pi_1(X)$ and let $p : E \to X$ be a connected covering space with $\pi_1(E) = N \lhd \pi$. Then the group of deck transformations of E is the quotient group π/N, and so can be thought of as a principal π/N - bundle. Viewed this way it is classified by a map $f_E : X \to B(\pi/N)$, which on the level of fundamental groups,

$$f_* : \pi = \pi_1(X) \to \pi_1(B\pi/N) = \pi/N$$

(5.2)

is just the projection on to the quotient space. In particular the universal cover $\tilde{X} \to X$ is the unique simply connected covering space. It is classified by a map

$$\gamma_X : X \to B\pi$$

which induces an isomorphism on the fundamental group.

Now let $\theta : \pi \to G$ be any group homomorphism. By the naturality of classifying spaces this induces a map on classifying spaces,

$$B\theta : B\pi \to BG.$$

This induces a principal G - bundle over X classified by the composition

$$X \xrightarrow{\gamma_X} B\pi \xrightarrow{B\theta} BG.$$

The bundle this map classifies is given by

$$\tilde{X} \times_\pi G \to X$$

where π acts on G via the homomorphism $\theta : \pi \to G$.

This construction defines a map

$$\rho : Hom(\pi_1(X), G) \to Prin_G(X).$$

Now if X is a smooth manifold then its universal cover $p : \tilde{X} \to X$ induces an isomorphism on tangent spaces,

$$Dp(x) : T_x\tilde{X} \to T_{p(x)}X$$

for every $x \in \tilde{X}$. Thus, viewed as a principal π - bundle, it has a canonical connection. Notice furthermore that this connection is *flat*, i.e its curvature is zero. (**Exercise.** Check this claim!) Moreover notice that any bundle of the form $\tilde{X} \times_\pi G \to X$ has an induced flat connection. In particular the image of $\rho : Hom(\pi_1(X), G) \to Prin_G(X)$ consists of principal bundles equipped with flat connections.

Notice furthermore that by taking $G = GL(n, \mathbb{C})$ the map ρ assigns to an n - dimensional representation an n - dimensional vector bundle with flat connection

$$\rho : Rep_n(\pi_1(X)) \to Vect_n(X).$$

By taking the sum over all n and passing to the Grothendieck group completion,we get a homomorphism of rings from the representation ring to K - theory,

$$\rho : R(\pi_1(X)) \to K(X).$$

An important question is what is the image of this map of rings. Again we know the image is contained in the classes represented by bundles that have flat connections. For $X = B\pi$, for π a finite group, the following is a famous theorem of Atiyah and Segal:

Let

$$\epsilon : R(\pi) \to \mathbb{Z} \quad \text{and} \quad \epsilon : K(B\pi) \to \mathbb{Z}$$

be the augmentation maps induced by sending a representation or a vector bundle to its dimension. Let $I \subset R(\pi)$ and $I \subset K(B\pi)$ denote the kernels of these augmentations, i.e the "augmentation ideals". Finally let $\bar{R}(\pi)$ and $\bar{K}(B\pi)$ denote the completions of these rings with respect to these ideals. That is,

$$\bar{R}(\pi) = \varprojlim_n R(\pi)/I^n \quad \text{and} \quad \bar{K}(B\pi) = \varprojlim_n K(B\pi)/I^n$$

where I^n is the product of the ideal I with itself n - times.

THEOREM 2.34. *(Atiyah and Segal)* [**4**] *For π a finite group, the induced map on the completions of the rings with respect ot the augmentation ideals,*

$$\rho : \bar{R}(\pi) \to \bar{K}(B\pi)$$

is an isomorphism.

CHAPTER 3

Characteristic Classes

In this chapter we define and calculate characteristic classes for principal bundles and vector bundles. Characteristic classes are the basic cohomological invariants of bundles and have a wide variety of applications throughout topology and geometry. Characteristic classes were introduced originally by E. Stiefel in Switzerland and H. Whitney in the United States in the mid 1930's. Stiefel, who was a student of H. Hopf intoduced in his thesis certain "characteristic homology classes" determined by the tangent bundle of a manifold. At about the same time Whitney studied general sphere bundles, and later introduced the general notion of a characteristic cohomology class coming from a vector bundle, and proved the product formula for their calculation.

In the early 1940's, L. Pontrjagin, in Moscow, introduced new characteristic classes by studying the Grassmannian manifolds, using work of C. Ehresmann from Switzerland. In the mid 1940's, after just arriving in Princeton from China, S.S Chern defined characteristic classes for complex vector bundles using differential forms and his calculations led a great clarification of the theory.

Much of the modern view of characteristic classes has been greatly influenced by the highly influential book of Milnor and Stasheff. This book was originally circulated as lecture notes written in 1957 and finally published in 1974. This book is one of the great textbooks in modern mathematics. These notes follow, in large part, their treatment of the subject. The reader is encouraged to consult their book for further details.

1. Preliminaries

Definition 3.1. Let G be a topological group (possibly with the discrete topology). Then a *characteristic class* for principal G - bundles is an assignment to each principal G - bundle $p : P \to B$ a cohomology class

$$c(P) \in H^*(B)$$

satisfying the following naturality condition. If

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{f}} & P_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[f]{} & B_2 \end{array}$$

is a map of principal G - bundles inducing an equivariant homeomorphism on fibers, then

$$f^*(c(P_2)) = c(P_1) \in H^*(B_1).$$

Remarks. 1. In this definition cohomology could be taken with any coefficients, including, for example, DeRham cohomology which has coefficients in the real numbers $\mathbb{R}$. The particular cohomology theory used is referred to as the "values" of the characteristic classes.

2. The same definition of characteristic classes applies to real or complex vector bundles as well as principal bundles.

The following is an easy consequence of the definition.

LEMMA 3.1. *Let c be a characteristic class for principal G - bundles so that c takes values in $H^q(-)$, for $q \geq 1$. Then if ϵ is the trivial G bundle,*

$$\epsilon = X \times G \to X$$

then $c(\epsilon) = 0$.

PROOF. The trivial bundle ϵ is the pull - back of the constant map to the one point space $e : X \to pt$ of the bundle $\nu = G \to pt$. Thus $c(\epsilon) = e^*(c(\nu))$. But $c(\nu) \in H^q(pt) = 0$ when $q > 0$. $\square$

The following observation is also immediate from the definition.

LEMMA 3.2. *Characteristic classes are invariant under isomorphism. More specifically, Let c be a characteristic class for principal G - bundles. Also let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be isomorphic principal G - bundles. Then*

$$c(E_1) = c(E_2) \in H^*(X).$$

Thus for a given space X, a characteristic class c can be viewed as a map

$$c : Prin_G(X) \to H^*(X).$$

3. The naturality property in the definition can be stated in more functorial terms in the following way.

Cohomology (with any coefficients) $H^*(-)$ is a contravariant functor from the category $ho\mathcal{T}op$ of topological spaces and homotopy classes of maps, to the category $\mathcal{A}b$ of abelian groups. By the results of chapter 2, the set of principal G - bundles $Prin_G(-)$ can be viewed as a contravariant functor from the category $ho\mathcal{T}op$ to the category of sets $\mathcal{S}ets$.

Definition 3.2. **(Alternative)** A characteristic class is a natural transformation c between the functors $Prin_G(-)$ and $H^*(-)$:

$$c : Prin_G(-) \rightsquigarrow H^*(-)$$

Examples.

(1) The *first Chern class* $c_(\zeta)$ is a characteristic class on principal $U(1)$ - bundles, or equivalently, complex line bundles. If ζ is a line bundle over X, then $c_1(\zeta) \in H^2(X;\mathbb{Z})$. As we saw in the last chapter, c_1 is a complete invariant of line bundles. That is to say, the map
$$c_1 : Prin_{U(1)}(X) \to H^2(X;\mathbb{Z})$$
is an isomorphism.

(2) The *first Stiefel - Whitney class* $w_1(\eta)$ is a characteristic class of two fold covering spaces (i.e a principal $\mathbb{Z}_2 = O(1)$ - bundles) or of real line bundles. If η is a real line bundle over a space X, then $w_1(\eta) \in H^1(X;\mathbb{Z}_2)$. Moreover, as we saw in the last chapter, the first Stiefel - Whitney class is a complete invariant of line bundles. That is, the map
$$w_1 : Prin_{O(1)}(X) \to H^1(X;\mathbb{Z}_2)$$
is an isomorphism.

We remark that the first Stiefel - Whitney class can be extended to be a characteristic class of real n - dimensional vector bundles (or principal $O(n)$ - bundles) for any n. To see this, consider the subgroup $SO(n) < O(n)$. As we saw in the last chapter, a bundle has an $SO(n)$ structure if and only if it is orientable. Moreover the induced map of classifying spaces gives a 2 - fold covering space or principal $O(1)$ - bundle,

$$\mathbb{Z}_2 = O(1) = O(n)/SO(n) \to BSO(n) \to BO(n).$$

This covering space defines, via its classifying map $w_1 : BO(n) \to BO(1) = \mathbb{RP}^\infty$ an element $w_1 \in H^1(BO(n);\mathbb{Z}_2)$ which is the first Stiefel - Whitney class of this covering space.

Now let η be any n - dimensional real vector bundle over X, and let

$$f_\eta : X \to BO(n)$$

be its classifying map.

Definition 3.3. The first Stiefel - Whitney class $w_1(\eta) \in H^1(X;\mathbb{Z}_2)$ is defined to be

$$w_1(\eta) = f_\eta^*(w_1) \in H^1(X;\mathbb{Z}_2)$$

The first Chern class c_1 of an n - dimensional complex vector bundle ζ over X is defined similarly, by pulling back the first Chern class of the principal $U(1)$ - bundle

$$U(1) \cong U(n)/SU(n) \to BSU(n) \to BU(n)$$

via the classifying map $f_\zeta : X \to BU(n)$.

The following is an immediate consequence of the above lemma and the meaning of $SO(n)$ and $SU(n)$ - structures.

THEOREM 3.3. *Given a complex n - dimensional vector bundle ζ over X, then $c_1(\zeta) \in H^2(X)$ is zero if and only if ζ has an $SU(n)$ -structure.*

Furthermore, given a real n - dimensional vector bundle η over X, then $w_1(\eta) \in H^1(X; \mathbb{Z}_2)$ is zero if and only if the bundle η has an $SO(n)$ - structure, which is equivalent to η being orientable.

We now use the classification theorem for bundles to describe the set of characteristic classes for principal G - bundles.

Let R be a commutative ring and let $Char_G(R)$ be the set of all characteristic classes for principal G bundles that take values in $H^*(-; R)$. Notice that the sum (in cohomology) and the cup product of characteristic classes is again a characteristic class. This gives $Char_G$ the structure of a ring. (Notice that the unit in this ring is the constant characteristic class $c(\zeta) = 1 \in H^0(X)$.

THEOREM 3.4. *There is an isomorphism of rings*

$$\rho : Char_G(R) \xrightarrow{\cong} H^*(BG; R)$$

PROOF. Let $c \in Char_G(R)$. Define

$$\rho(c) = c(EG) \in H^*(BG; R)$$

where $EG \to BG$ is the universal G - bundle over BG. By definition of the ring structure of $Char_G(R)$, ρ is a ring homomorphism.

Now let $\gamma \in H^q(BG; R)$. Define the characteristic class c_γ as follows. Let $p : E \to X$ be a principal G - bundle classified by a map $f_E : X \to BG$. Define

$$c_\gamma(E) = f_E^*(\gamma) \in H^q(X; R)$$

where $f_E^* : H^*(BG : R) \to H^*(X; R)$ is the cohomology ring homomorphism induced by f_E. This association defines a map

$$c : H^*(BG; R) \to Char_G(R)$$

which immediately seen to be inverse to ρ. $\square$

2. Chern Classes and Stiefel - Whitney Classes

In this section we compute the rings of unitary characteristic classes $Char_{U(n)}(\mathbb{Z})$ and $\mathbb{Z}_2$ - valued orthogonal characteristic classes $Char_{O(n)}(\mathbb{Z}_2)$. These are the characteristic classes of complex and real vector bundles and as such have a great number of applications. By 3.4 computing these rings of characteristic classes reduce to computing the cohomology rings $H^*(BU(n);\mathbb{Z})$ and $H^*(BO(n);\mathbb{Z}_2)$. The following is the main theorem of this section.

THEOREM 3.5. *a. The ring of $U(n)$ characteristic classes is a polynomial algebra on n - generators,*

$$Char_{U(n)}(\mathbb{Z}) \cong H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \cdots, c_n]$$

where $c_i \in H^{2i}(BU(n);\mathbb{Z})$ is known as the ith - Chern class.

b. The ring of $\mathbb{Z}_2$ - valued $O(n)$ characteristic classes is a polynomial algebra on n - generators,

$$Char_{O(n)}(\mathbb{Z}_2) \cong H^*(BO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \cdots, w_n]$$

where $w_i \in H^i(BO(n);\mathbb{Z}_2)$ is known as the ith - Stiefel - Whitney class.

This theorem will be proven by induction on n. For $n = 1$ $BU(1) = \mathbb{CP}^\infty$ and $BO(1) = \mathbb{RP}^\infty$ and so the theorem describes the ring structure in the cohomology of these projective spaces. To complete the inductive step we will study the sphere bundles

$$S^{n-1} \to BO(n-1) \to BO(n)$$

and

$$S^{2n-1} \to BU(n-1) \to BU(n)$$

described in the last chapter. In particular recall from 2.28 that in these fibrations, $BO(n-1)$ and $BU(n-1)$ are the unit sphere bundles $S(\gamma_n)$ of the universal bundle γ_n over $BO(n)$ and $BU(n)$ respectively. Let $D(\gamma_n)$ be the unit disk bundles of the universal bundles. That is, in the complex case,

$$D(\gamma_n) = EU(n) \times_{U(n)} D^{2n} \to BU(n)$$

and in the real case,

$$D(\gamma_n) = EO(n) \times_{O(n)} D^n \to BO(n)$$

where $D^{2n} \subset \mathbb{C}^n$ and $D^n \subset \mathbb{R}^n$ are the unit disks, and therefore have the induced unitary and orthogonal group actions.

Here is one easy observation about these disk bundles.

PROPOSITION 3.6. *The projection maps*

$$p : D(\gamma_n) = EU(n) \times_{U(n)} D^{2n} \to BU(n)$$

and

$$D(\gamma_n) = EO(n) \times_{O(n)} D^n \to BO(n)$$

are homotopy equivalences.

PROOF. Both of these bundles have zero sections $\mathcal{Z} : BU(n) \to D(\gamma_n)$ and $\mathcal{Z} : BO(n) \to D(\gamma_n)$. In both the complex and real cases, we have $p \circ \mathcal{Z} = 1$. To see that $\mathcal{Z} \circ p \simeq 1$ consider the homotopy $H : D(\gamma_n) \times I \to D(\gamma_n)$ defined by $H(v, t) = tv$. □

We will use this result when studying the cohomology exact sequence of the pair $(D(\gamma_n), S(\gamma_n))$:

$$\begin{aligned} \cdots \to H^{q-1}(S(\gamma_n)) &\xrightarrow{\delta} H^q(D(\gamma_n), S(\gamma_n)) \to H^q(D(\gamma_n)) \to H^q(S(\gamma_n)) \\ &\xrightarrow{\delta} H^{q+1}(D(\gamma_n), S(\gamma_n)) \to H^{q+1}(D(\gamma_n)) \to \cdots \end{aligned} \tag{2.1}$$

Using the above proposition and 2.28 we can substitute $H^*(BU(n))$ for $H^*(D(\gamma_n))$, and $H^*(BU(n-1))$ for $H^*(S(\gamma_n))$ in this sequence to get the following exact sequence

(2.2)

$$\begin{aligned} \cdots \to H^{q-1}(BU(n-1)) &\xrightarrow{\delta} H^q(D(\gamma_n), S(\gamma_n)) \to H^q(BU(n)) \xrightarrow{\iota^*} H^q(BU(n-1)) \\ &\xrightarrow{\delta} H^{q+1}(D(\gamma_n), S(\gamma_n)) \to H^{q+1}(BU(n)) \to \cdots \end{aligned}$$

and we get a similar exact sequence in the real case

(2.3)

$$\begin{aligned} \cdots \to H^{q-1}(BO(n-1);\mathbb{Z}_2) &\xrightarrow{\delta} H^q(D(\gamma_n), S(\gamma_n);\mathbb{Z}_2) \to H^q(BO(n);\mathbb{Z}_2) \xrightarrow{\iota^*} \\ H^q(BO(n-1);\mathbb{Z}_2) &\xrightarrow{\delta} H^{q+1}(D(\gamma_n), S(\gamma_n);\mathbb{Z}_2) \to H^{q+1}(BO(n);\mathbb{Z}_2) \to \cdots \end{aligned}$$

These exact sequences will be quite useful for inductively computing the cohomology of these classifying spaces, but to do so we need a method for computing $H^*(D(\gamma_n), S(\gamma_n))$, or more generally, $H^*(D(\zeta), S(\zeta))$, where ζ is any Euclidean vector bundle and $D(\zeta)$ and $S(\zeta)$ are the associated unit disk bundles and sphere bundles respectively. The quotient space,

$$T(\zeta) = D(\zeta)/S(\zeta) \tag{2.4}$$

is called the *Thom space* of the bundle ζ. As the name suggests, this construction was first studied by R. Thom [**41**], and has been quite useful in both bundle theory and cobordism theory. Notice that on each fiber (say at $x \in X$) of the n - dimensional disk bundle ζ, the Thom space construction takes the unit n - dimensional disk modulo its boundary $n-1$ - dimensional sphere which therefore yields an n - dimensional sphere, with marked basepoint, say $\infty_x \in S^n(\zeta_x) = D^n(\zeta_x)/S^{n-1}(\zeta_x)$. The Thom space construction then identifies all the basepoints ∞_x to a single point. Notice that for a bundle over a point $\mathbb{R}^n \to pt$, the Thom space $T(\mathbb{R}^n) = D^n/S^{n-1} = S^n \cong \mathbb{R}^n \cup \infty$. More generally, notice that when the basespace X is compact, then the Thom space is simply the one point compactification of the total space of the vector bundle ζ,

$$T(\zeta) \cong \zeta^+ = \zeta \cup \infty \tag{2.5}$$

where we think of the extra point in this compactification as the common point at infinity assigned to each fiber. In order to compute with the above exact sequences, we will need to study the cohomology of Thom spaces. But before we do we examine the topology of the Thom spaces of product bundles. For this we introduce the "smash product" construction.

Let X and Y be spaces with basepoints $x_0 \in X$ and $y_0 \in Y$.

Definition 3.4. The *wedge* $X \vee Y$ is the "one point union",

$$X \vee Y = X \times y_0 \cup x_0 \times Y \subset X \times Y.$$

The *smash product* $X \wedge Y$ is given by

$$X \wedge Y = X \times Y/X \vee Y.$$

Observations. 1. The k be a field. Then the Kunneth formula gives

$$\tilde{H}^*(X \wedge Y; k) \cong \tilde{H}^*(X; k) \otimes \tilde{H}^*(Y; k).$$

2. Let V and W be vector spaces, and let V^+ and W^+ be their one point compactifications. These are spheres of the same dimension as the respective vector spaces. Then

$$V^+ \wedge W^+ = (V \times W)^+.$$

So in particular,

$$S^n \wedge S^m = S^{n+m}.$$

Proposition 3.7. *Let ζ be an n - dimensional vector bundle over a space X, and let η be an m - dimensional bundle over X. Let $\zeta \times \eta$ be the product $n+m$ - dimensional vector bundle over $X \wedge Y$. Then the Thom space of $\zeta \times \eta$ is given by*

$$T(\zeta \times \eta) \cong T(\zeta) \wedge T(\eta).$$

Proof. Notice that the disk bundle is given by

$$D(\zeta \times \eta) \cong D(\zeta) \times D(\eta)$$

and its boundary sphere bundle is given by

$$S(\zeta \times \eta) \cong S(\zeta) \times D(\eta) \cup D(\zeta) \times S(\eta).$$

Thus

$$\begin{aligned} T(\zeta \times \eta) = D(\zeta \times \eta)/S(\zeta \times \eta) &\cong D(\zeta) \times D(\eta)/\left(S(\zeta) \times D(\eta) \cup D(\zeta) \times S(\eta)\right) \\ &\cong D(\zeta)/S(\zeta) \wedge D(\eta)/S(\eta) \\ &\cong T(\zeta) \wedge T(\eta). \end{aligned}$$

□

We now proceed to study the cohomology of Thom spaces.

2.1. The Thom Isomorphism Theorem. We begin by describing a cohomological notion of orientability of an vector bundle ζ over a space X.

Consider the 2 - fold cover over X defined as follows. Let E_ζ be the principal $GL(n,\mathbb{R})$ bundle associated to ζ. Also let Gen_n be the set of generators of $H^n(S^n) \cong \mathbb{Z}$. So Gen_n is a set with two elements. Moreover the general linear group $GL(n,\mathbb{R})$ acts on $S^n = \mathbb{R}^n \cup \infty$ by the usual linear action on $\mathbb{R}^n$ extended to have a fixed point at $\infty \in S^n$. By looking at the induced map on cohomology, there is an action of $GL(n,\mathbb{R})$ on Gen_n. We can then define the double cover

$$\mathcal{G}(\zeta) = E_\zeta \times_{GL(n,\mathbb{R})} Gen_n \longrightarrow E_\zeta/GL(n,\mathbb{R}) = X.$$

Lemma 3.8. *The double covering $\mathcal{G}(\zeta)$ is isomorphic to the orientation double cover $Or(\zeta)$.*

Proof. Recall from chapter 1 that the orientation double cover $Or(\zeta)$ is given by

$$Or(\zeta) = E_\zeta \times_{GL(n,\mathbb{R})} Or(\mathbb{R}^n)$$

where $Or(\mathbb{R}^n)$ is the two point set consistingof orientations of the vector space $\mathbb{R}^n$. A matrix $A \in GL(n,\mathbb{R})$ acts on this set trivially if and only if the determinant $detA$ is positive. It acts nontrivially (i.e permutes the two elements) if and only if $detA$ is negative. Now the same is true of the action of $GL(n,\mathbb{R})$ on Gen_n. This is because $A \in GL(n,\mathbb{R})$ induces multiplication by the sign of $detA$ on $H^n(S^n)$. (Verify this as an exercise!)

Since $Or(\mathbb{R}^n)$ and Gen_n are both two point sets with the same action of $GL(n,\mathbb{R})$, the corresponding two fold covering spaces $Or(\zeta)$ and $\mathcal{G}(\zeta)$ are isomorphic. □

Corollary 3.9. *An orientation of an n - dimensional vector bundle ζ is equivalent to a section of $\mathcal{G}(\zeta)$ and hence defines a continuous family of generators*

$$u_x \in H^n(S^n(\zeta_x)) \cong \mathbb{Z}$$

for every $x \in X$. Here $S^n(\zeta_x)$ is the unit disk of the fiber ζ_x modulo its boundary sphere. $S^n(\zeta_x)$ is called the sphere at x.

Now recall that given a pair of spaces $A \subset Y$, there is a relative cup product in cohomology,

$$H^q(Y) \otimes H^r(Y, A) \xrightarrow{\cup} H^{q+r}(Y, A).$$

So in particular the relative cohomology $H^*((Y, A)$ is a (graded) module over the (graded) ring $H^*(Y)$.

In the case of a vector bundle ζ over a space X, we then have that $H^*(D(\zeta), S(\zeta)) = \tilde{H}^*(T(\zeta))$ is a module over $H^*(D(\zeta)) \cong H^*(X)$. So in particular, given any cohomology class in the Thom space, $\alpha \in H^r(T(\zeta))$ we get an induced homomorphism

$$H^q(X) \xrightarrow{\cup\alpha} H^{q+r}(T(\zeta)).$$

Our next goal is to prove the famous *Thom Isomorphism Theorem* which can be stated as follows.

Theorem 3.10. *Let ζ be an oriented n - dimensional real vector bundle over a connected space X. Let R be any commutative ring. The orientation gives generators $u_x \in H^n(S^n(\zeta_x); R) \cong R$. Then there is a unique class (called the Thom class) in the cohomology of the Thom space*

$$u \in H^n(T(\zeta); R)$$

so that for every $x \in X$, if

$$j_x : S^n(\zeta_x) \hookrightarrow D(\zeta)/S(\zeta) = T(\zeta)$$

is the natural inclusion of the sphere at x in the Thom space, then under the induced homomorphism in cohomology,

$$j_x^* : H^n(T(\zeta); R) \to H^n(S^n(\zeta_x); R) \cong R$$

$j_x^*(u) = u_x$.

Furthermore The induced cup product map

$$\gamma : H^q(X; R) \xrightarrow{\cup u} \tilde{H}^{q+n}(T(\zeta); R)$$

is an isomorphism for every $q \in \mathbb{Z}$. So in particular $\tilde{H}^r(T(\zeta); R) = 0$ for $r < n$.

If ζ is not an orientable bundle over X, then the theorem remains true if we take $\mathbb{Z}_2$ coefficients, $R = \mathbb{Z}_2$.

Proof. We prove the theorem for oriented bundles. We leave the nonorientable case (when $R = \mathbb{Z}_2$) to the reader. We also restrict our attention to the case $R = \mathbb{Z}$, since the theorem for general coefficients will follow immediately from this case using the universal coefficient theorem.

Case 1: ζ is the trivial bundle $X \times \mathbb{R}^n$.

In this case the Thom space $T(\zeta)$ is given by

$$T(\zeta) = X \times D^n / X \times S^{n-1}.$$

The projection of X to a point, $X \to pt$ defines a map

$$\pi : T(\zeta) = X \times D^n / X \times S^{n-1} \to D^n / S^{n-1} = S^n.$$

Let $u \in H^n(T(\zeta))$ be the image in cohomology of a generator,

$$\mathbb{Z} \cong H^n(S^n) \xrightarrow{\pi^*} H^n(T(\zeta)).$$

The fact that taking the cup product with this class

$$H^q(X) \xrightarrow{\cup u} H^{q+n}(T(\zeta)) = H^{q+n}(X \times D^n, X \times S^{n-1}) = H^{q+n}(X \times S^n, X \times pt)$$

is an isomorphism for every $q \in \mathbb{Z}$ follows from the universal coefficient theorem.

Case 2: X is the union of two open sets $X = X_1 \cup X_2$, where we know the Thom isomorphism theorem holds for the restrictions $\zeta_i = \zeta_{|X_i}$ for $i = 1, 2$ and for $\zeta_{1,2} = \zeta_{|X_1 \cap X_2}$.

We prove the theorem for X using the Mayer - Vietoris sequence for cohomology. Let $X_{1,2} = X_1 \cap X_2$.

$$\to H^{q-1}(T(\zeta_{1,2})) \to H^q(T(\zeta)) \to H^q(T(\zeta_1)) \oplus H^q(T(\zeta_2)) \to H^q((T(\zeta_{1,2})) \to \cdots$$

Looking at this sequence when $q < n$, we see that since

$$H^q(T(\zeta_{1,2})) = H^q(T(\zeta_1)) = H^q(T(\zeta_2)) = 0,$$

then by exactness we must have that $H^q(T(\zeta)) = 0$.

We now let $q = n$, and we see that by assumption, $H^n(T(\zeta_1)) \cong H^n(T(\zeta_2)) \cong H^n(T(\zeta_{1,2})) \cong \mathbb{Z}$, and that the Thom classes of each of the restriction maps $H^n(T(\zeta_1)) \to H^n(T(\zeta_{1,2}))$ and $H^n(T(\zeta_2)) \to H^n(T(\zeta_{1,2}))$ correspond. Moreover $H^{n-1}(T(\zeta_{1,2})) = 0$. Hence by the exact sequence, $H^n(T(\zeta)) \cong \mathbb{Z}$ and there is a class $u \in H^n(T(\zeta))$ that maps to the direct sum of the Thom classes in $H^n(T(\zeta_1)) \oplus H^n(T(\zeta_2))$.

Now for $q \geq n$ we compare the above Mayer - Vietoris sequence with the one of base spaces,

$$\to H^{q-1}(X_{1,2}) \to H^q(X) \to H^q(X_1) \oplus H^q(X_2) \to H^q(X_{1,2}) \to \cdots$$

This sequence maps to the one for Thom spaces by taking the cup product with the Thom classes. By assumption this map is an isomorphism on $H^*(X_i)$, $\;i = 1, 2$ and on $H^*(X_{1,2})$. Thus by the Five Lemma it is an isomorphism on $H^*(X)$. This proves the theorem in this case.

Case 3. X is covered by finitely many open sets X_i, $i = 1, \cdots, k$ so that the restrictions of the bundle to each X_i, ζ_i is trivial.

The proof in this case is an easy inductive argument (on the number of open sets in the cover), where the inductive step is completed using cases 1 and 2.

Notice that this case includes the situation when the basespace X is compact.

Case 4. General Case. We now know the theorem for compact spaces. However it is not necessarily true that the cohomology of a general space (i.e homotopy type of a C.W complex) is determined by the cohomology of its compact subspaces. However it is true that the homology of a space X is given by

$$H_*(X) \cong \varinjlim_K H_*(K)$$

where the limit is taken over the partially ordered set of compact subspaces $K \subset X$. Thus we want to first work in homology and then try to transfer our observations to cohomology.

To do this, recall that the construction of the cup product pairing actually comes from a map on the level of cochains,

$$C^q(Y) \otimes C^r(Y, A) \xrightarrow{\cup} C^{q+r}(Y, A)$$

and therefore has a dual map on the chain level

$$C_*(Y, A) \xrightarrow{\psi} C_*(Y) \otimes C_*(Y, A).$$

and thus induces a map in homology

$$\psi : H_k(Y, A) \to \oplus_{r \geq 0} H_{k-r}(Y) \otimes H_r(Y, A).$$

Hence given $\alpha \in H^r(Y, A)$ we have an induced map in homology (the "slant product")

$$/\alpha : H_k(Y, A) \to H_{k-r}(Y)$$

defined as follows. If $\theta \in H_k(Y, A)$ and

$$\psi(\theta) = \sum_j a_j \otimes b_j \in H_*(Y) \otimes H_*(Y, A)$$

then

$$/\alpha(\theta) = \sum_j \alpha(b_j) \cdot a_j$$

where by convention, if the degree of a homology class b_j is not equal to the degree of α, then $\alpha(b_j) = 0$.

Notice that this slant product is dual to the cup product map

$$H^q(Y) \xrightarrow{\cup \alpha} H^{q+r}(Y, A).$$

Again, by considering the pair $(D(\zeta), S(\zeta))$, and identifying $H_*(D(\zeta)) \cong H_*(X)$, we can apply the slant product operation to the Thom class, to define a map

$$/u : H_k(T(\zeta)) \to H_{k-n}(X).$$

which is dual to the Thom map $\gamma : H^q(X) \xrightarrow{\cup u} H^{q+n}(T(\zeta))$. Now since γ is an isomorphism in all dimensions when restricted to compact sets, then by the universal coefficient theorem, $/u : H_q(T(\zeta_{|K})) \to H_{q-n}(K)$ is an isomorphism for all q and for every compact subset $K \subset X$. By taking the limit over the partially ordered set of compact subsets of X, we get that

$$/u : H_q(T(\zeta)) \to H_{q-n}(X)$$

is an isomorphism for all q. Applying the universal coefficient theorem again, we can now conclude that

$$\gamma : H^k(X) \xrightarrow{\cup u} H^{k+n}(T(\zeta))$$

is an isomorphism for all k. This completes the proof of the theorem. □

We now observe that the Thom class of a product of two bundles is the appropriately defined product of the Thom classes.

LEMMA 3.11. *Let ζ and η be an n and m dimensional oriented vector bundles over X and Y respectively. Then the Thom class $u(\zeta \times \eta)$ is given by the tensor product: $u(\zeta\times\eta) \in H^{n+m}(T(\zeta\times\eta))$ is equal to*

$$\begin{aligned} u(\zeta) \otimes u(\eta) &\in H^n(T(\zeta)) \otimes H^m(T(\eta)) \\ &\cong H^{n+m}(T(\zeta) \wedge T(\eta)) \\ &= H^{n+m}(T(\zeta \times \eta)). \end{aligned}$$

In this description, cohomology is meant to be taken with $\mathbb{Z}_2$ - coefficients if the bundles are not orientable.

PROOF. $u(\zeta) \otimes u(\eta)$ restricts on each fiber $(x, y) \in X \times Y$ to

$$\begin{aligned} u_x \otimes u_y &\in H^n(S^n(\zeta_x)) \otimes H^m(S^m(\eta_y)) \\ &\cong H^{n+m}(S^n(\zeta_x) \wedge S^m(\eta_y)) \\ &= H^{n+m}(S^{n+m}(\zeta \times \eta)_{(x,y)})) \end{aligned}$$

which is the generator determined by the product orientation of $\zeta_x \times \eta_y$. The result follows by the uniqueness of the Thom class. □

We now use the Thom isomorphism theorem to define a characteristic class for oriented vector bundles, called the *Euler class*.

DEFINITION 3.5. The Euler class of an oriented, n dimensional bundle ζ, over a connected space X, is the n - dimensional cohomology class

$$\chi(\zeta) \in H^n(X)$$

defined to be the image of the Thom class $u(\zeta) \in H^n(T(\zeta))$ under the composition

$$H^n(T(\zeta)) = H^n(D(\zeta), S(\zeta)) \to H^n(D(\zeta)) \cong H^n(X).$$

Again, if ζ is not orientable, cohomology is taken with $\mathbb{Z}_2$ - coefficients.

Exercise. Verify that the Euler class is a characteristic class according to our definition.

The following is then a direct consequence of 3.11.

COROLLARY 3.12. *Let ζ and η be as in 3.11. Then the Euler class of the product is given by*

$$\chi(\zeta \times \eta) = \chi(\zeta) \otimes \chi(\eta) \in H^n(X) \otimes H^m(Y) \hookrightarrow H^{n+m}(X \times Y).$$

We will also need the following observation.

PROPOSITION 3.13. *Let η be an odd dimensional oriented vector bundle over a space X. Say dim $(\eta) = 2n+1$. Then its Euler class has order two:*

$$2\chi(\eta) = 0 \in H^{2n+1}(X).$$

PROOF. Consider the bundle map

$$\begin{aligned} \nu : \eta &\to \eta \\ v &\to -v. \end{aligned}$$

Since η is odd dimensional, this bundle map is an orientation reversing automorphism of η. This means that $\nu^*(u) = -u$, where $u \in H^{2n+1}(T(\eta))$ is the Thom class. By the definition of the Euler class this in turn implies that $\nu^*(\chi(\eta)) = -\chi(\eta)$. But since the Euler class is a characteristic class and ν is a bundle map, we must have $\nu^*(\chi(\eta)) = \chi(\eta)$. Thus $\chi(\eta) = -\chi(\eta)$. $\square$

2.2. The Gysin sequence. We now input the Thom isomorphism theorem into the cohomology exact sequence of the pair $D(\zeta), S(\zeta))$ in order to obtain an important calculational tool for computing the homology of vector bundles and sphere bundles.

Namely, let ζ be an oriented n - dimensional oriented vector bundle over a space X, and consider the exact sequence

$$\begin{aligned} \cdots \to H^{q-1}(S(\zeta)) &\xrightarrow{\delta} H^q(D(\zeta), S(\zeta)) \to H^q(D(\zeta)) \to H^q(S(\zeta)) \\ &\xrightarrow{\delta} H^{q+1}(D(\zeta), S(\zeta)) \to H^{q+1}(D(\zeta)) \to \cdots \end{aligned}$$

By identifying $H^*(D(\zeta), S(\zeta)) = \tilde{H}^*(T(\zeta))$ and $H^*(D(\zeta)) \cong H^*(X)$, this exact sequence becomes

$$\begin{aligned} \cdots \to H^{q-1}(S(\zeta)) &\xrightarrow{\delta} H^q(T(\zeta)) \to H^q(X) \to H^q(S(\zeta)) \\ &\xrightarrow{\delta} H^{q+1}(T(\zeta)) \to H^{q+1}(X) \to \cdots \end{aligned}$$

Finally, by inputting the Thom isomorphism, $H^{q-n}(X) \xrightarrow[\cong]{\cup u} H^q(T(\zeta))$ we get the following exact sequence known as the **Gysin sequence:**

$$\begin{aligned} \cdots \to H^{q-1}(S(\zeta)) &\xrightarrow{\delta} H^{q-n}(X) \xrightarrow{\chi} H^q(X) \to H^q(S(\zeta)) \\ &\xrightarrow{\delta} H^{q-n+1}(X) \xrightarrow{\chi} H^{q+1}(X) \to \cdots \end{aligned} \tag{2.6}$$

We now make the following observation about the homomorphism $\chi : H^q(X) \to H^{q+n}(X)$ in the Gysin sequence.

PROPOSITION 3.14. *The homomorphism $\chi : H^q(X) \to H^{q+n}(X)$ is given by taking the cup product with the Euler class,*

$$\chi : H^q(X) \xrightarrow{\cup\chi} H^{q+n}(X).$$

PROOF. The theorem is true for $q = 0$, by definition. Now in general, the map χ was defined in terms of the Thom isomorphism $\gamma : H^r(X) \xrightarrow{\cup u} H^{r+n}(T(\zeta))$, which, by definition is a homomorphism of graded $H^*(X)$ - modules. This will then imply that

$$\chi : H^q(X) \to H^{q+n}(X)$$

is a homomorphism of graded $H^*(X)$ - modules. Thus

$$\begin{aligned} \chi(\alpha) &= \chi(1 \cdot \alpha) \\ &= \chi(1) \cup \alpha \quad \text{since } \chi \text{ is an } H^*(X) \text{ - module homomorphism} \\ &= \chi(\zeta) \cup \alpha \end{aligned}$$

as claimed. $\square$

2.3. Proof of theorem 3.5. the goal of this section is to use the Gysin sequence to prove 3.5, which we begin by restating:

THEOREM 3.15. *a. The ring of $U(n)$ characteristic classes is a polynomial algebra on n - generators,*

$$Char_{U(n)}(\mathbb{Z}) \cong H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \cdots, c_n]$$

where $c_i \in H^{2i}(BU(n);\mathbb{Z})$ is known as the ith - Chern class.

b. The ring of $\mathbb{Z}_2$ - valued $O(n)$ characteristic classes is a polynomial algebra on n - generators,

$$Char_{O(n)}(\mathbb{Z}_2) \cong H^*(BO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \cdots, w_n]$$

where $w_i \in H^i(BO(n);\mathbb{Z}_2)$ is known as the ith - Stiefel - Whitney class.

PROOF. We start by considering the Gysin sequence, applied to the universal bundle γ_n over $BU(n)$. We input the fact that the sphere bundle $S(\gamma_n)$ is given by $BU(n-1)$ see 2.2:

(2.7)

$$\cdots \to H^{q-1}(BU(n-1)) \xrightarrow{\delta} H^{q-2n}(BU(n)) \xrightarrow{\cup\chi(\gamma_n)} H^q(BU(n)) \xrightarrow{\iota^*} H^q(BU(n-1))$$
$$\xrightarrow{\delta} H^{q-2n+1}(BU(n)) \xrightarrow{\cup\chi(\gamma_n)} H^{q+1}(BU(n)) \to \cdots$$

and we get a similar exact sequence in the real case

(2.8)

$$\cdots \to H^{q-1}(BO(n-1);\mathbb{Z}_2) \xrightarrow{\delta} H^{q-n}(BO(n);\mathbb{Z}_2) \xrightarrow{\cup\chi(\gamma_n)} H^q(BO(n);\mathbb{Z}_2) \xrightarrow{\iota^*}$$
$$H^q(BO(n-1);\mathbb{Z}_2) \xrightarrow{\delta} H^{q-n+1}(BO(n));\mathbb{Z}_2) \xrightarrow{\cup\chi(\gamma_n)} H^{q+1}(BO(n);\mathbb{Z}_2) \to \cdots$$

We use these exact sequences to prove the above theorem by induction on n. For $n = 1$ then sequence 2.7 reduces to the short exact sequences,

$$0 \to H^{q-2}(BU(1)) \xrightarrow[\cong]{\cup\chi(\gamma_1)} H^q(BU(1)) \to 0$$

for each $q \geq 2$. We let $c_1 \in H^2(BU(1)) = H^2(\mathbb{CP}^\infty)$ be the Euler class $\chi(\gamma_1)$. These isomorphisms imply that the ring structure of $H^*(BU(1))$ is that of a polynomial algebra on this single generator,

$$H^*(BU(1)) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[c_1]$$

which is the statement of the theorem in this case.

In the real case when $n = 1$ the Gysin sequence 2.8 reduces to the short exact sequences,

$$0 \to H^{q-1}(BO(1);\mathbb{Z}_2) \xrightarrow[\cong]{\cup\chi(\gamma_1)} H^q(BO(1);\mathbb{Z}_2) \to 0$$

for each $q \geq 1$. We let $w_1 \in H^(BO(1);\mathbb{Z}_2) = H^1(\mathbb{RP}^\infty;\mathbb{Z}_2)$ be the Euler class $\chi(\gamma_1)$. These isomorphisms imply that the ring structure of $H^*(BO(1);\mathbb{Z}_2)$ is that of a polynomial algebra on this single generator,

$$H^*(BO(1);\mathbb{Z}_2) = H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[w_1]$$

which is the statement of the theorem in this case.

We now inductively assume the theorem is true for $n-1$. That is,

$$H^*(BU(n-1)) \cong \mathbb{Z}[c_1, \cdots, c_{n-1}] \quad \text{and} \quad H^*(BO(n-1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_{n-1}].$$

We first consider the Gysin sequence 2.7, and observe that by exactness, for $q \leq 2(n-1)$, the homomorphism

$$\iota^* : H^q(BU(n)) \to H^q(BU(n-1))$$

is an isomorphism. That means there are unique classes, $c_1, \cdots, c_{n-1} \in H^*(BU(n))$ that map via ι^* to the classes of the same name in $H^*(BU(n-1))$. Furthermore, since ι^* is a ring homomorphism, every polynomial in $c_1, \cdots, c_{n-1}$ in $H^*(BU(n-1))$ is in the image under ι^* of the corresponding polynomial in the these classes in $H^*(BU(n))$. Hence by our inductive assumption,

$$\iota^* : H^*(BU(n)) \to H^*(BU(n-1)) = \mathbb{Z}[c_1, \cdots, c_{n-1}]$$

is a split surjection of rings. But by the exactness of the Gysin sequence 2.7 this implies that this long exact splits into short exact sequences,

$$0 \to H^{*-2n}(BU(n)) \xrightarrow{\cup\chi(\gamma_n)} H^*(BU(n)) \xrightarrow{\iota^*} H^*(BU(n-1)) \cong \mathbb{Z}[c_1, \cdots c_{n-1}] \to 0$$

Define $c_n \in H^{2n}(BU(n))$ to be the Euler class $\chi(\gamma_n)$. Then this sequence becomes

$$0 \to H^{*-2n}(BU(n)) \xrightarrow{\cup c_n} H^*(BU(n)) \xrightarrow{\iota^*} \mathbb{Z}[c_1, \cdots c_{n-1}] \to 0$$

which implies that $H^*(BU(n)) \cong \mathbb{Z}[c_1, \cdots, c_n]$. This completes the inductive step in this case.

In the real case now consider the Gysin sequence 2.8, and observe that by exactness, for $q < n-1$, the homomorphism

$$\iota^* : H^q(BO(n);\mathbb{Z}_2) \to H^q(BO(n-1);\mathbb{Z}_2)$$

is an isomorphism. That means there are unique classes, $w_1, \cdots, w_{n-2} \in H^*(BO(n);\mathbb{Z}_2)$ that map via ι^* to the classes of the same name in $H^*(BO(n-1);\mathbb{Z}_2)$.

In dimension $q = n-1$, the exactness of the Gysin sequence tells us that the homomorphism $\iota^* H^{n-1}(BO(n);\mathbb{Z}_2) \to H^{n-1}(BO(n-1);\mathbb{Z}_2)$ is injective. Also by exactness we see that ι^* is surjective if and only if $\chi(\gamma_n) \in H^n(BO(n);\mathbb{Z}_2)$ is nonzero. But to see this, by the universal property of γ_n, it suffices to prove that there exists some n -dimensional bundle ζ with Euler class $\chi(\zeta) \neq 0$. Now by 3.12, the Euler class of the product

$$\begin{aligned}\chi(\gamma_k \times \gamma_{n-k}) &= \chi(\gamma_k) \otimes \chi(\gamma_{n-k}) \in H^k(BO(k) \times BO(n-k);\mathbb{Z}_2)\\ &= w_k \otimes w_{n-k} \in H^*(BO(k);\mathbb{Z}_2) \otimes H^{n-k}(BO(n-k);\mathbb{Z}_2)\end{aligned}$$

which, by the inductive assumption is nonzero for $k \geq 1$. Thus $\chi(\gamma_n) \in H^n(BO(n); \mathbb{Z}_2)$ is nonzero, and we define it to be the n^{th} Stiefel - Whitney class

$$w_n = \chi(\gamma_n) \in H^n(BO(n); \mathbb{Z}_2).$$

As observed above, the nontriviality of $\chi(\gamma_n)$ implies that $\iota^* H^{n-1}(BO(n); \mathbb{Z}_2) \to H^{n-1}(BO(n-1); \mathbb{Z}_2)$ is an isomorphism, and hence there is a unique class $w_{n-1} \in H^{n-1}(BO(n-1); \mathbb{Z}_2)$ (as well as $w_1, \cdots w_{n-2}$) restricting to the inductively defined classes of the same names in $H^*(BO(n-1); \mathbb{Z}_2)$.

Furthermore, since ι^* is a ring homomorphism, every polynomial in $w_1, \cdots, w_{n-1}$ in $H^*(BO(n-1); \mathbb{Z}_2)$ is in the image under ι^* of the corresponding polynomial in the these classes in $H^*(BO(n); \mathbb{Z}_2)$. Hence by our inductive assumption,

$$\iota^* : H^*(BO(n); \mathbb{Z}_2) \to H^*(BO(n-1); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \cdots, w_{n-1}]$$

is a split surjection of rings. But by the exactness of the Gysin sequence 2.8 this implies that this long exact splits into short exact sequences,

$$0 \to H^{*-n}(BO(n); \mathbb{Z}_2) \xrightarrow{\cup w_n} H^*(BO(n); \mathbb{Z}_2) \xrightarrow{\iota^*} H^*(BO(n-1); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots w_{n-1}] \to 0$$

which implies that $H^*(BO(n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_n]$. This completes the inductive step and therefore the proof of the theorem. $\square$

3. The product formula and the splitting principle

Perhaps the most important calculational tool for characteristic classes is the Whitney sum formula, which we now state and prove.

THEOREM 3.16. *a. Let ζ and η be vector bundles over a space X. Then the Stiefel - Whitney classes of the Whitney sum bundle $\zeta \oplus \eta$ are given by*

$$w_k(\zeta \oplus \eta) = \sum_{j=0}^{k} w_j(\zeta) \cup w_{k-j}(\eta) \in H^k(X; \mathbb{Z}_2).$$

where by convention, $w_0 = 1 \in H^0(X; \mathbb{Z}_2)$.

b. If ζ and η are complex vector bundles, then the Chern classes of the Whitney sum bundle $\zeta \oplus \eta$ are given by

$$c_k(\zeta \oplus \eta) = \sum_{j=0}^{k} c_j(\zeta) \cup c_{k-j}(\eta) \in H^{2k}(X).$$

Again, by convention, $c_0 = 1 \in H^0(X)$.

Proof. We prove the formula in the real case. The complex case is done the same way.

Let ζ be an n - dimensional vector bundle over X, and let η be an m - dimensional bundle. Let $N = n+m$. Since we are computing $w_k(\zeta \oplus \eta)$, we may assume that $k \leq N$, otherwise this characteristic class is zero.

We prove the Whitney sum formula by induction on $N \geq k$. We begin with the case $N = k$. Since $\zeta \oplus \eta$ is a k - dimensional bundle, the k^{th} Stiefel - Whitney class, $w_k(\zeta \oplus \eta)$ is equal to the Euler class $\chi(\zeta \oplus \eta)$. We then have

$$\begin{aligned} w_k(\zeta \oplus \eta) &= \chi(\zeta \oplus \eta) \\ &= \chi(\zeta) \cup \chi(\eta) \quad \text{by } 3.12 \\ &= w_n(\zeta) \cup w_m(\eta). \end{aligned}$$

This is the Whitney sum formula in this case as one sees by inputting the fact that for a bundle ρ with $j > dim\,(\rho)$, $w_j(\rho) = 0$.

Now inductively assume that the Whitney sum formula holds for computing w_k for any sum of bundles whose sum of dimensions is $\leq N-1 \geq k$. Let ζ have dimension n and η have dimension m with $n+m = N$. To complete the inductive step we need to compute $w_k(\zeta \oplus \eta)$.

Suppose ζ is classified by a map $f_\zeta : X \to BO(n)$, and η is classified by a map $f_\eta : X \to BO(m)$. Then $\zeta \oplus \eta$ is classified by the composition

$$f_{\zeta \oplus \eta} : X \xrightarrow{f_\zeta \times f_\eta} BO(n) \times BO(m) \xrightarrow{\ \mu\ } BO(n+m)$$

where μ is the map that classifies the product of the universal bundles $\gamma_n \times \gamma_m$ over $BO(n) \times BO(m)$. Equivalently, μ is the map on classifying spaces induced by the inclusion of the subgroup $O(n) \times O(m) \hookrightarrow O(n+m)$. Thus to prove the theorem we must show that the map $\mu : BO(n) \times BO(m) \to BO(n+m)$ has the property that

$$\mu^*(w_k) = \sum_{j=0}^{k} w_j \otimes w_{k-j} \in H^*(BO(n);\mathbb{Z}_2) \otimes H^*(BO(m);\mathbb{Z}_2). \tag{3.1}$$

For a fixed $j \leq k$, let

$$p_j : H^k(BO(n) \times BO(m);\mathbb{Z}_2) \to H^j(BO(n);\mathbb{Z}_2) \otimes H^{k-j}(BO(m);\mathbb{Z}_2)$$

be the projection onto the summand. So we need to show that $p_j(\mu^*(w_k)) = w_j \otimes w_{k-j}$. Now since $n+m = N > k$, then either $j < n$ or $k-j < m$ (or both). We assume without loss of generality that $j < n$. Now by the proof of 3.5

$$\iota^* : H^j(BO(n);\mathbb{Z}_2) \to H^j(BO(j);\mathbb{Z}_2)$$

is an isomorophism. Moreover we have a commutative diagram:

$$\begin{array}{ccccc}
H^k(BO(N);\mathbb{Z}_2) & \xrightarrow{\mu^*} & H^k(BO(n)\times BO(m);\mathbb{Z}_2) & \xrightarrow{p_j} & H^j(BO(n);\mathbb{Z}_2)\otimes H^{k-j}(BO(m);\mathbb{Z}_2) \\
{\scriptstyle \iota^*}\downarrow & & & & \downarrow{\scriptstyle \iota^*\otimes 1} \\
H^k(BO(j+m);\mathbb{Z}_2) & \xrightarrow[\mu^*]{} & H^k(BO(j)\times BO(m);\mathbb{Z}_2) & \xrightarrow[p_j]{} & H^j(BO(j);\mathbb{Z}_2)\otimes H^{k-j}(BO(m);\mathbb{Z}_2).
\end{array}$$

Since $j < n$, $j + m < n + m = N$ and $\iota^*(w_k) = w_k \in H^k(BO(j+m);\mathbb{Z}_2)$. This fact and the commutativity of this diagram give,

$$\begin{aligned}
(\iota^*\otimes 1)\circ p_j\circ\mu^*(w_k) &= p_j\circ\mu^*\circ\iota^*(w_k) \\
&= p_j\circ\mu^*(w_k) \\
&= w_j\otimes w_{k-j} \quad \text{by the inductive assumption.}
\end{aligned}$$

Since $\iota^*\otimes 1$ is an isomorphism in this dimension, and since $\iota^*(w_j\otimes w_{k-j}) = w_j\otimes w_{k-j}$ we have that

$$p_j\circ\mu^*(w_k) = w_j\otimes w_{k-j}.$$

As remarked above, this suffices to complete the inductive step in the proof of the theorem. □

We can restate the Whitney sum formula in the following convenient way. For an n - dimensional bundle ζ, let

$$w(\zeta) = 1 + w_1(\zeta) + w_2(\zeta) + \cdots + w_n(\zeta) \in H^*(X;\mathbb{Z}_2)$$

This is called the *total Stiefel - Whitney class.* The total Chern class of a complex bundle is defined similarly.

The Whitney sum formula can be interpreted as saying these total characteristic classes have the "exponential property" that they take sums to products. That is, we have the following:

Corollary 3.17.

$$w(\zeta\oplus\eta) = w(\zeta)\cup w(\eta)$$

and

$$c(\zeta\oplus\eta) = c(\zeta)\cup c(\eta).$$

This implies that these characteristic classes are invariants of the stable isomorphism types of bundles:

Corollary 3.18. *If ζ and η are stably equivalent real vector bundles over a space X, then*

$$w(\zeta) = w(\eta) \in H^*(X;\mathbb{Z}_2),$$

Similarly if they are complex bundles,

$$c(\zeta) = c(\eta) \in H^*(X).$$

Proof. If ζ and η are stably equivalent, then

$$\zeta \oplus \epsilon^m \cong \eta \oplus \epsilon^r$$

for some m and r. So

$$w(\zeta \oplus \epsilon^m) = w(\eta \oplus \epsilon^r).$$

But by 3.17

$$w(\zeta \oplus \epsilon^m) = w(\zeta)w(\epsilon) = w(\zeta) \cdot 1 = w(\zeta).$$

Similarly $w(\eta \oplus \epsilon^r) = w(\eta)$. The statement follows. The complex case is proved in the same way. $\square$

By our description of K - theory in chapter 2, we have that these characteristic classes define invariants of K - theory.

Theorem 3.19. *The Chern classes c_i and the Stiefel - Whitney classes w_i define natural transformations*

$$c_i : K(X) \to H^{2i}(X)$$

and

$$w_i : KO(X) \to H^i(X; \mathbb{Z}_2).$$

The total characteristic classes

$$c : K(X) \to \bar{H}^*(X)$$

and

$$w : KO(X) \to \bar{H}^*(X; \mathbb{Z}_2)$$

are exponential in the sense that

$$c(\alpha + \beta) = c(\alpha)c(\beta) \quad \textit{and} \quad w(\alpha + \beta) = w(\alpha)w(\beta).$$

Here $\bar{H}^*(X)$ is the direct product $\bar{H}^*(X) = \prod_q H^q(X)$.

As an immediate application of these product formulas, we can deduce a "splitting principle" for characteristic classes. We now explain this principle.

Recall that an n - dimensional bundle ζ over X splits as a sum of n line bundles if and only if its associated principal bundle has an $O(1) \times \cdots \times O(1)$ - structure. That is, the classifying map $f_\zeta : X \to BO(n)$ lifts to the n -fold product, $BO(1)^n$. The analogous observation also holds for complex vector bundles. If we have such a lifting, then in cohomology, $f_\zeta^* : H^*(BO(n); \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$ factors through $\otimes_n H^*(BO(1); \mathbb{Z}_2)$.

The "splitting principle" for characteristic classes says that this cohomological property always happens.

To state this more carefully, recall that $H^*(BO(1);\mathbb{Z}_2) = \mathbb{Z}_2[w_1]$. Thus

$$H^*(BO(1)^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \cdots, x_n]$$

where $x_j \in H^1$ is the generator of the cohomology of the j^{th} factor in this product. Similarly,

$$H^*(BU(1)^n) \cong \mathbb{Z}[y_1, \cdots, y_n]$$

where $y_j \in H^2$ is the generator of the cohomology of the j^{th} factor in this product.

Notice that the symmetric group Σ_n acts on these polynomial algebras by permuting the generators. The subalgebra consisting of polynomials fixed under this symmetric group action is called the algebra of symmetric polynomials, $Sym[x_1, \cdots, x_n]$ or $Sym[y_1, \cdots, y_n]$.

THEOREM 3.20. (**Splitting Principle.**) The maps

$$\mu : BU(1)^n \to BU(n) \quad \text{and} \quad \mu : BO(1)^n \to BO(n)$$

induce injections in cohomology

$$\mu^* : H^*(BU(n)) \to H^*(BU(1)^n) \quad \text{and} \quad \mu^* : H^*(BO(n);\mathbb{Z}_2) \to H^*(BO(1)^n;\mathbb{Z}_2).$$

Furthermore the images of these monomorphisms are the symmetric polynomials

$$H^*(BU(n)) \cong Sym[y_1, \cdots, y_n] \quad \text{and} \quad H^*(BO(n);\mathbb{Z}_2) \cong Sym[x_1, \cdots, x_n].$$

PROOF. By the Whitney sum formula,

$$\mu^*(w_j) = \sum_{j_1+\cdots+j_n=j} w_{j_1} \otimes \cdots \otimes w_{j_n} \in H^*(BO(1);\mathbb{Z}_2) \otimes \cdots \otimes H^*(BO(1);\mathbb{Z}_2).$$

But $w_i(\gamma_1) = 0$ unless $i = 0, 1$. So

$$\mu^*(w_j) = \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j} \in \mathbb{Z}_2[x_1, \cdots, x_n].$$

This is the j^{th} - elementary symmetric polynomial, $\sigma_j(x_1, \cdots, x_n)$. Thus the image of $\mathbb{Z}_2[w_1, \cdots, w_n] = H^*(BO(n);\mathbb{Z}_2)$ is the subalgebra of $\mathbb{Z}_2[x_1, \cdots, x_n]$ generated by the elementary symmetric polynomials, $\mathbb{Z}[\sigma_1, \cdots, \sigma_n]$. But it is well known that the elementary symmetric polynomials generate $Sym[x_1, \cdots, x_n]$ (see [**22**]). The complex case is proved similarly. □

This result gives another way of producing characteristic classes which is particularly useful in index theory.

Let $p(x)$ be a power series in one variable, which is assumed to have a grading equal to one. Say

$$p(x) = \sum_i a_i x^i.$$

Consider the corresponding symmetric power series in n -variables,

$$p(x_1, \cdots, x_n) = p(x_1) \cdots p(x_n).$$

Let $p_j(x_1, \cdots, x_n)$ be the homogeneous component of $p(x_1, \cdots, x_n)$ of grading j. So

$$p_j(x_1, \cdots, x_n) = \sum_{i_1 + \cdots + i_n = j} a_{i_1} \cdots a_{i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

Since p_j is symmetric, by the splitting principle we can think of

$$p_j \in H^j(BO(n); \mathbb{Z}_2)$$

and hence determines a characteristic class (i.e a polynomial in the Stiefel - Whitney classes).

Similarly if we give x grading 2, we can think of $p_j \in H^{2j}(BU(n))$ and so determines a polynomial in the Chern classes.

In particular, given a real valued smooth function $y = f(x)$, its Taylor series $p_f(x) = \sum_k \frac{f^{(k)}(0)}{k!} x^k$ determines characteristic classes $f_i \in H^i(BO(n); \mathbb{Z}_2)$ or $f_i \in H^{2i}(BU(n); \mathbb{Z}_2)$.

Exercise. Consider the examples $f(x) = e^x$, and $f(x) = tanh(x)$. Write the low dimensional characteristic classes f_i in $H^*(BU(n))$ for $i = 1, 2, 3$, as explicit polynomials in the Chern classes.

4. Applications

In this section all cohomology will be taken with $\mathbb{Z}_2$ - coefficients, even if not explicitly written.

4.1. Characteristic classes of manifolds. We have seen that the characteristic classes of trivial bundles are trivial. However the converse is not true, as we will now see, by examining the characteristic classes of manifolds.

DEFINITION 3.6. The characteristic classes of a manifold M, $w_j(M)$, $c_i(M)$, are defined to be the characteristic classes of the tangent bundle, τ_M.

THEOREM 3.21. $w_j(S^n) = 0$ *for all* $j, n > 0$.

PROOF. As we saw in chapter 1, the normal bundle of the standard embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a trivial line bundle. Thus

$$\tau_{S^n} \oplus \epsilon_1 \cong \epsilon_{n+1}$$

and so $\tau_{S^{n+1}}$ is stably trivial. The theorem follows. $\square$

Of course we know τ_{S^2} is nontrivial since it has no nowhere zero cross sections. Thus the Stiefel-Whitney classes do not form a complete invariant of the bundle. However they do constitute a very important class of invariants, as we will see below.

Write $a \in H^1(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ as the generator. Then the total Stiefel - Whitney class of the canonical line bundle γ_1 is

$$w(\gamma_1) = 1 + a \in H^*(\mathbb{RP}^n).$$

This allows us to compute the Stiefel - Whitney classes of $\mathbb{RP}^n$ (i.e of the tangent bundle $\tau_{\mathbb{RP}^n}$).

THEOREM 3.22. $w(\mathbb{RP}^n) = (1+a)^{n+1} \in H^*(\mathbb{RP}^n; \mathbb{Z}_2)$. *So* $w_j(\mathbb{RP}^n) = \binom{n+1}{j} a^j \in H^j(\mathbb{RP}^n)$.

Note: Even though the polynomial $(1+a)^{n+1}$ has highest degree term a^{n+1}, this class is zero in $H^*(\mathbb{RP}^n)$ since $H^{n+1}(\mathbb{RP}^n) = 0$.

PROOF. As seen in chapter 1,

$$\tau_{\mathbb{RP}^n} \oplus \epsilon_1 \cong \oplus_{n+1} \gamma_1.$$

Thus

$$\begin{aligned} w(\tau_{\mathbb{RP}^n}) &= w(\tau_{\mathbb{RP}^n} \oplus \epsilon_1) \\ &= w(\oplus_{n+1}\gamma_1) \\ &= w(\gamma_1)^{n+1}, \quad \text{by the Whitney sum formula} \\ &= (1+a)^{n+1}. \end{aligned}$$

□

Observation. The same argument shows that the total Chern class of $\mathbb{CP}^n$ is

$$c(\mathbb{CP}^n) = (1+a)^{n+1} \tag{4.1}$$

where $a \in H^2(\mathbb{CP}^n \mathbb{Z})$ is the generator.

This calculation of the Stiefel - Whitney classes of $\mathbb{RP}^n$ allows us to rule out the possibility that many of these projective spaces are parallelizable.

COROLLARY 3.23. *If* $\mathbb{RP}^n$ *is parallelizable, then* n *is of the form* $n = 2^k - 1$ *for some* k.

PROOF. We show that if $n \neq 2^k - 1$ then there is some $j > 0$ such that $w_j(\mathbb{RP}^n) \neq 0$. But $w_j(\mathbb{RP}^n) = \binom{n+1}{j} a^j$, so we are reduced to verifying that if m is not a power of 2, then there is a $j \in \{1, \cdots, m-1\}$ such that $\binom{m}{j} \equiv 1 \bmod 2$. This follows immediately from the following combinatorial lemma, whose proof we leave to the reader.

Lemma 3.24. *Let $j \in \{1, \cdots, m-1\}$. Write j and m in their binary representations,*

$$m = \sum_{i=0}^{k} a_i 2^i$$

$$j = \sum_{i=0}^{k} b_i 2^i$$

where the a_i's and b_i's are either 0 or 1. Then

$$\binom{m}{j} \equiv \prod_{i=0}^{k} \binom{a_i}{b_i} \, mod \, 2.$$

Note. Here we are adopting the usual conventions that $\binom{0}{0} = 1$ and $\binom{0}{1} = 0$.

□

Since we know that Lie groups are parallelizable, this result says that $\mathbb{RP}^n$ can only have a Lie group structure if n is of the form $2^k - 1$. However a famous theorem of Adams [**1**] says that the only $\mathbb{RP}^n$'s that are parallelizable are $\mathbb{RP}^1$, $\mathbb{RP}^3$, and $\mathbb{RP}^7$.

Now as seen in chapter 2 an n - dimensional vector bundle ζ^n has k - linearly independent cross sections if and only if

$$\zeta^n \cong \rho^{n-k} \oplus \epsilon_k$$

for some $n-k$ dimensional bundle ρ. Moreover, having this structure is equivalent to the classifying map

$$f_\zeta : X \to BO(n)$$

having a lift (up to homotopy) to a map $f_\rho : X \to BO(n-k)$.

Now the Stiefel - Whitney classes give natural obstructions to the existence of such a lift because the map $\iota : BO(n-k) \to BO(n)$ induces the map of rings

$$\iota^* : \mathbb{Z}_2[w_1, \cdots, w_n] \to \mathbb{Z}_2[w_1, \cdots w_{n-k}]$$

that maps w_j to w_j for $j \leq n-k$, and w_j to 0 for $n \geq j > n-k$. We therefore have the following result.

Theorem 3.25. *Let ζ be an n -dimensional bundle over X. Suppose $w_k(\zeta)$ is nonzero in $H^k(X;\mathbb{Z}_2)$. Then ζ has no more than $n-k$ linearly independent cross sections. In particular, if $w_n(\zeta) \neq 0$, then ζ does not have a nowhere zero cross section.*

This result has applications to the existence of linearly independent vector fields on a manifold. The following is an example.

Theorem 3.26. *If m is even, $\mathbb{RP}^m$ does not have a nowhere zero vector field.*

Proof. By 3.22

$$
\begin{aligned}
w_m(\mathbb{RP}^m) &= \binom{m+1}{m} a^m \\
&= (m+1)a^m \in H^m(\mathbb{RP}^m;\mathbb{Z}_2).
\end{aligned}
$$

For m even this is nonzero. Hence $w_m(\mathbb{RP}^m) \neq 0$. □

4.2. Normal bundles and immersions. Theorem 3.25 has important applications to the existence of immersions of a manifold M in Euclidean space, which we now discuss.

Let $e : M^n \looparrowright \mathbb{R}^{n+k}$ be an immersion. Recall that this means that the derivative at each point,

$$De(x) : T_x M^n \to T_{e(x)}\mathbb{R}^{n+k} = \mathbb{R}^{n+k}$$

is injective. Recall also that the Inverse Function Theorem implies that an immersion is a local embedding.

The immersion e defines a k - dimensional normal bundle ν_e^k whose fiber at $x \in M$ is the orthogonal complement of the image of $T_x M^n$ in $\mathbb{R}^{n+k}$ under $De(x)$. In particular we have

$$\tau_{M^n} \oplus \nu_e^k \cong e^* \tau_{\mathbb{R}^{n_k}} \cong \epsilon_{n+k}.$$

Thus we have the Whitney sum relation among the Stiefel - Whitney classes

$$w(M^n) \cdot w(\nu_e^k) = 1. \tag{4.2}$$

So we can compute the Stiefel - Whitney clases of the normal bundle formally as the power series

$$w(\nu_e^k) = 1/w(M) \in \bar{H}^*(M;\mathbb{Z}_2).$$

This proves the following:

Proposition 3.27. *The Stiefel - Whitney classes of the normal bundle to an immersion $e : M^n \looparrowright \mathbb{R}^{n+k}$ are independent of the immersion. They are called the normal Stiefel - Whitney classes, and are written $\bar{w}_i(M)$. These classes are determined by the formula*

$$w(M) \cdot \bar{w}(M) = 1.$$

Example. $\bar{w}(\mathbb{RP}^n) = 1/(1+a)^{n+1} \in \bar{H}^*(\mathbb{RP}^n;\mathbb{Z}_2)$.

So for example, when $n = 2^k$, $k > 0$, $w(\mathbb{RP}^{2^k}) = 1 + a + a^{2^k}$. This is true since by 3.24 $\binom{2^k+1}{r} \equiv 1 \bmod 2$ if and only if $r = 0, 1, 2^k$. Thus the total normal Stiefel - Whitney class is given by

$$\bar{w}(\mathbb{RP}^{2^k}) = 1/(1 + a + a^{2^k}) = 1 + a + a^2 + \cdots + a^{2^k - 1}.$$

Note. The reason this series is truncated a a^{2^k-1} is because

$$(1+a+a^{2^k})(1+a+a^2+\cdots+a^{2^k-1}) = 1 \in H^*(\mathbb{RP}^n;\mathbb{Z}_2)$$

since $H^q(\mathbb{RP}^n) = 0$ for $q > n$.

COROLLARY 3.28. *There is no immersion of* $\mathbb{RP}^{2^k}$ *in* $\mathbb{R}^N$ *for* $N \le 2^{k+1}-2$.

PROOF. The above calculation shows that $\bar{w}_{2^k-1}(\mathbb{RP}^{2^k}) \neq 0$. Thus it cannot have a normal bundle of dimension less than 2^k-1. The result follows. □

In the 1940's, Whitney proved the following seminal result in the theory of embeddings and immersions [**45**]

THEOREM 3.29. *Let* M^n *be a closed* n *- dimensional manifold. Then there is an embedding*

$$e : M^n \hookrightarrow \mathbb{R}^{2n}$$

and an immersion

$$\iota : M^n \looparrowright \mathbb{R}^{2n-1}.$$

Thus combining these results gives the following best immersion dimension for $\mathbb{RP}^{2^k}$.

COROLLARY 3.30. $\mathbb{RP}^{2k}$ *has an immersion in* $\mathbb{R}^{2^{k+1}-1}$ *but not in* $\mathbb{R}^{2^{k+1}-2}$.

A natural question raised by Whitney's theorem is to find the best possible immersion dimension for other manifolds, or for some class of manifolds. In general this is a very difficult problem. However by the following important result of Smale and Hirsch [**17**], this is purely a bundle theoretic question, and ultimately a homotopy theoretic question (via classifying maps).

THEOREM 3.31. *Let* M^n *be a closed* n *- manifold. Then* M^n *immerses in* $\mathbb{R}^{n+k}$ *if and only if there is a* k *- dimensional bundle* ν^k *over* M^n *with*

$$\tau_{M^n} \oplus \nu^k \cong \epsilon_{n+k}.$$

Thus questions of immersions boil down to bundle theoretic questions. By classifying space theory they can be viewed as homotopy theoretic questions. More specifically, let $\nu : M \to BO$ represent the element in $\tilde{K}O(X)$ given by

$$[\nu] = -[\tau_M] \in \tilde{K}O(X).$$

Notice that if viewed with values in $BO(N)$, for N large, ν classifies the normal bundle of an embedding of M^n in $\mathbb{R}^{n+N}$, and in particular

$$\nu^*(w_i) = \bar{w}_i(M) \in H^i(M; \mathbb{Z}_2).$$

$\nu : M \to BO$ is called the "stable normal bundle" map of M. The following is an interpretation of the above theorem of Smale and Hirsch using classifying space theory.

THEOREM 3.32. *M^n admits an immersion in $\mathbb{R}^{n+k}$ if and only if the stable normal bundle map $\nu : M \to BO$ has a homotopy lifting to a map*

$$\nu_k : M \to BO(k).$$

In the late 1950's, Wu, in China, computed a formula for how the Steenrod square cohomology operations are affected by Poincare duality in a manifold. W. Massey then used Wu's formulas to prove the following [**25**]:

Write an integer n in its binary expansion

$$n = \sum_{i=0}^{k} a_i \cdot 2^i$$

where each a_i is 0 or 1. Let

$$\alpha(n) = \sum_{i=0}^{k} a_i. \tag{4.3}$$

So $\alpha(n)$ is the number of ones in the base 2 representation of n.

THEOREM 3.33. *Let M^n be a closed n - dimensional manifold. Then*

$$\bar{w}_i(M^n) = 0$$

for $i > n - \alpha(n)$.

Thus Stiefel - Whitney classes give no obstruction to existence of immersions of n - manifolds in $\mathbb{R}^{2n-\alpha(n)}$. The conjecture that every n - manifold does indeed immerse in this dimension became known as the "Immersion Conjecture", and was proved in [**8**].

THEOREM 3.34. *Every closed manifold M^n immerses in $\mathbb{R}^{2n-\alpha(n)}$*

This theorem was proved homotopy theoretically. Namely it was shown that the stable normal bundle map $\nu : M^n \to BO$ always has a lift (up to homotopy) to a map $M^n \to BO(n - \alpha(n))$. The theorem then follows from the Hirsch - Smale theorem 3.31. The lifting to $BO(n - \alpha(n))$ was constructed in two steps. First, by work of Brown and Peterson [**6**] there is a "universal space for normal bundles" BO/I_n and a map $\rho : BO/I_n \to BO$ with the property that every stable normal bundle map from an n - manifold $\nu : M^n \to BO$ lifts to a map $\tilde{\nu} : M^n \to BO/I_n$. Then the main work in [**8**] was to develop an obstruction theory to analyze the homotopy types of BO/I_n and $BO(n-\alpha(n))$ to show that $\rho : BO/I_n \to BO$ lifts to a map $\tilde{\rho} : BO/I_n \to BO(n-\alpha(n))$. The composition

$$M^n \xrightarrow{\tilde{\nu}} BO/I_n \xrightarrow{\tilde{\rho}} BO(n-\alpha(n))$$

then classifies the normal bundle of an immersion $M^n \looparrowright \mathbb{R}^{2n-\alpha(n)}$.

This result, and indeed Massey's theorem 3.33 are best possible, as can be seen by the following example.

Let $e_j : \mathbb{RP}^{2^j} \looparrowright \mathbb{R}^{2^{j+1}-1}$ be an immersion which is guaranteed by Whitney's theorem. Now write n in its binary expansion

$$n = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_r}$$

where the $0 \leq j_1 < \cdots < j_r$ and $r = \alpha(n)$. Consider the n - dimensional manifold

$$M^n = \mathbb{RP}^{2^{j_1}} \times \cdots \times \mathbb{RP}^{2^{j_r}}.$$

Consider the product immersion

$$e : M^n = \mathbb{RP}^{2^{j_1}} \times \cdots \times \mathbb{RP}^{2^{j_r}} \xrightarrow{e_{j_1} \times \cdots \times e_{j_r}} \mathbb{R}^{2^{j_1+1}-1} \times \cdots \times \mathbb{R}^{2^{j_r+1}-1} = \mathbb{R}^{2n-\alpha(n)}.$$

Since $M^n = \mathbb{RP}^{2^{j_1}} \times \cdots \times \mathbb{RP}^{2^{j_r}}$, the Whitney sum formula will imply that

$$\bar{w}_{n-\alpha(n)}(M^n) = \bar{w}_{2^{j_1}-1}(\mathbb{RP}^{2^{j_1}}) \otimes \cdots \otimes \bar{w}_{2^{j_r}-1}(\mathbb{RP}^{2^{j_r}})$$

which, by the proof of 3.28 is nonzero. Hence M^n does not have an immersion in $\mathbb{R}^{2n-\alpha(n)-1}$.

Other results along these lines includes a fair amount known about the best immersion dimensions of projective spaces (see [**10**]). However the best immersion dimensions of all manifolds with structure, say an orientation or an almost complex structure, is unknown. Also the best embedding dimension for all n - manifolds is unknown.

5. Pontrjagin Classes

In this section we define and study Pontrjagin classes. These are integral characteristic classes for real vector bundles and are defined in terms of the Chern classes of the complexification of the bundle. We will then show that polynomials in Pontrjagin classes and the Euler class define all possible characteristic classes for oriented, real vector bundles when the values of the characteristic classes is cohomology with coefficients in an integral domain R which contains $1/2$. By the classification theorem,to deduce this we must compute $H^*(BSO(n); R)$. For this calculation we follow the treatment given in Milnor and Stasheff [**31**].

5.1. Orientations and Complex Conjugates. We begin with a reexamination of certain basic properties of complex vector bundles.

Let V be an n - dimensional $\mathbb{C}$ - vector space with basis $\{v_1, \cdots, v_n\}$. By multiplication of these basis vectors by the complex number i, we get a collection of $2n$ - vectors $\{v_1, iv_1, v_2, iv_2, \cdots, v_n, iv_n\}$ which forms a basis for V as a real $2n$ - dimensional vector space. This basis then determines an orientation of the underlying real vector space V.

Exercise. Show that the orientation of V that the basis $\{v_1, iv_1, v_2, iv_2, \cdots, v_n, iv_n\}$ determines is independent of the choice of the original basis $\{v_1, \cdots, v_n\}$

Thus every complex vector space V has a canonical orientation. By choosing this orientation for every fiber of a complex vector bundle ζ, we see that every complex vector bundle has a canonical orientation. By the results of section 2 this means that every n - dimensional complex vector bundle ζ over a space X has a canonical choice of Thom class $u \in H^{2n}(T(\zeta))$ and hence Euler class

$$\chi(\zeta) = c_n(\zeta) \in H^{2n}(X).$$

Now given a complex bundle ζ there exists a *conjugate bundle* $\bar{\zeta}$ which is equal to ζ as a real, $2n$ - dimensional bundle, but whose complex structure is conjugate. More specifically, recall that a complex structure on a $2n$ - dimensional real bundle ζ determines and is determined by a linear transformation

$$J_\zeta : \zeta \to \zeta$$

with the property that $J_\zeta^2 = J_\zeta \circ J_\zeta = -id$. If ζ has a complex structure then J_ζ is just scalar multiplication by the complex number i on each fiber. If we replace J_ζ by $-J_\zeta$ we define a new complex structure on ζ referred to as the *conjugate* complex structure. We write $\bar{\zeta}$ to denote ζ with this structure. That is,

$$J_{\bar{\zeta}} = -J_\eta.$$

Notice that the identity map

$$id : \zeta \to \bar{\zeta}$$

is anti-complex linear (or conjugate complex linear) in the sense that

$$id(J_\zeta \cdot v) = -J_{\bar{\zeta}} \cdot id(v).$$

We note that the conjugate bundle $\bar{\zeta}$ is often not isomorphic to ζ as complex vector bundles. For example, consider the two dimensional sphere as complex projective space

$$S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \infty.$$

The tangent bundle $\tau_{\mathbb{CP}^1}$ has the induced structure as a complex line bundle.

Proposition 3.35. *The complex line bundles τ_{S^2} and $\bar{\tau}_{S^2}$ are not isomorphic.*

PROOF. Suppose $\phi : \tau_{S^2} \to \bar{\tau}_{S^2}$ is a isomorphism as complex vector bundles. Then at every tangent space

$$\phi_x : T_x S^2 \to T_x S^2$$

is a an isomorphism that reverses the complex structure. Any such isomorphism is given by reflection through a line ℓ_x in the tangent plane $T_x S^2$. Therefore for every x we have picked a line $\ell_x \subset T_x S^2$. This defines a (real) one dimensional subbundle ℓ of τ_{S^2}, which, by the classification theorem is given by an element of

$$[S^2, BO(1)] \cong H^1(S^2, \mathbb{Z}_2) = 0.$$

Thus ℓ is a trivial subbundle of τ_{S^2}. Hence we can find a nowhere vanishing vector field on S^2, which gives us a contradiction. $\square$

Exercise. Let $\bar{\gamma}_n$ be the conjugate of the universal bundle γ_n over $BU(n)$. By the classification theorem, $\bar{\gamma}_n$ is classified by a map

$$q : BU(n) \to BU(n)$$

having the property that $q^*(\gamma_n) = \bar{\gamma}_n$. Using the Grassmannian model of $BU(n)$, find an explicit description of a map $q : BU(n) \to BU(n)$ with this property.

The following describes the effect of conjugating a vector bundle on its Chern classes.

THEOREM 3.36. $c_k(\bar{\zeta}) = (-1)^k c_k(\zeta)$

PROOF. Suppose ζ is an n - dimensional bundle. By the classification theorem and the functorial property of Chern classes it suffices to prove this theorem when ζ is the universal bundle γ_n over $BU(n)$. Now in our calculations of the cohomology of these classifying spaces, we proved that the inclusion $\iota : BU(k) \to BU(n)$ induces an isomorphism in cohomology in dimension k,

$$\iota^* : H^{2k}(BU(n)) \xrightarrow{\cong} H^{2k}(BU(k)).$$

Hence it suffices to prove this theorem for the universal k - dimensional bundle γ_k over $BU(k)$.

Now $c_k(\gamma_k) = \chi(\gamma_k)$ and similarly, $c_k(\bar{\gamma}_k) = \chi(\bar{\gamma}_k)$. So it suffices to prove that

$$\chi(\gamma_k) = (-1)^{-k} \chi(\bar{\gamma}_k).$$

But by the observations above, this is equivalent to showing that the canonical orientation of the underlying real $2k$ - dimensional bundle from the complex structures of γ_k and $\bar{\gamma}_k$ are the same if k is even, and opposite if k is odd. To do this we only need to compare the orientations at a single point. Let $x \in BU(k)$ be given by $\mathbb{C}^k \subset \mathbb{C}^\infty$ as the first k - coordinates. If $\{e_1, \cdots, e_k\}$ forms the standard basis for $\mathbb{C}^k$, then the orientations of $\gamma_k(x)$ determined by the complex structures of γ_k and $\bar{\gamma}_k$ are respectively represented by the real bases

$$\{e_1, ie_1, \cdots, e_k, ie_k\} \quad \text{and} \quad \{e_1, -ie_1, \cdots, e_k, -ie_k\}.$$

The change of basis matrix between these two basis has determinant $(-1)^k$. The theorem follows. □

Now suppose η is a real n - dimensional vector bundle over a space X, we then let $\eta_{\mathbb{C}}$ be its complexification

$$\eta_{\mathbb{C}} = \eta \otimes_{\mathbb{R}} \mathbb{C}.$$

$\eta_{\mathbb{C}}$ has the obvious structure as an n - dimensional complex vector bundle.

PROPOSITION 3.37. *There is an isomorphism*

$$\phi : \eta_{\mathbb{C}} \xrightarrow{\cong} \bar{\eta}_{\mathbb{C}}.$$

PROOF. Define

$$\phi : \eta_{\mathbb{C}} \to \bar{\eta}_{\mathbb{C}}$$
$$\eta \times \mathbb{C} \to \eta \otimes \bar{\mathbb{C}}$$
$$v \otimes z \to v \otimes \bar{z}$$

for $v \in \eta$ and $z \in \mathbb{C}$. Clearly ϕ is an isomorphism of complex vector bundles. □

COROLLARY 3.38. *For a real n - dimensional bundle η, then for k odd,*

$$2c_k(\eta_{\mathbb{C}}) = 0.$$

PROOF. By 3.36 and 3.37

$$c_k(\eta_{\mathbb{C}}) = (-1)^k c_k(\eta_{\mathbb{C}}).$$

Hence for k odd $c_k(\eta_{\mathbb{C}})$ has order 2. □

5.2. Pontrjagin classes.

We now use these results to define Pontrjagin classes for real vector bundles.

DEFINITION 3.7. Let η be an n - dimensional real vector bundle over a space X. Then define the i^{th} - Pontrjagin class

$$p_i(\eta) \in H^{4i}(X;\mathbb{Z})$$

by the formula

$$p_i(\eta) = (-1)^i c_{2i}(\eta_{\mathbb{C}}).$$

Remark. The signs used in this definition are done to make calculations in the next section come out easily.

As we've done with Stiefel - Whitney and Chern classes, define the total Pontrjagin class

$$p(\eta) = 1 + p_1(\eta) + \cdots + p_i(\eta) + \cdots \in \bar{H}^*(X, \mathbb{Z}).$$

The following is the Whitney sum formula for Pontrjagin classes, and follows immediately for the Whitney sum formula for Chern classes and 3.38.

Theorem 3.39. *For real bundles η and ξ over X, we have*

$$2(p(\eta \oplus \xi) - p(\eta)p(\xi)) = 0 \in H^*(X; \mathbb{Z}).$$

In particular if R is a commutative integral domain containing $1/2$, then viewed as characteristic classes with values in $H^*(X; R)$, we have

$$p(\eta \oplus \xi) = p(\eta)p(\xi) \in \bar{H}^*(X : R).$$

Remark. Most often Pontryagin classes are viewed as having values in rational cohomology, and so the formula $p(\eta \oplus \xi) = p(\eta)p(\xi)$ applies.

We now study the Pontrjagin classes of a complex vector bundle. Let ζ be a complex n - dimensional bundle over a space X, and let $\zeta_{\mathbb{C}} = \zeta \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of its underlying real $2n$ - dimensional bundle. So $\zeta_{\mathbb{C}}$ is a complex $2n$ - dimensional bundle. We leave the proof of the following to the reader.

Proposition 3.40. *As complex $2n$ - dimensional bundles,*

$$\zeta_{\mathbb{C}} \cong \zeta \oplus \bar{\zeta}.$$

This result, together with 3.36 and the definition of Pontrjagin classes imply the following.

Corollary 3.41. *Let ζ be a complex n - dimensional bundle. Then its Pontryagin classes are determined by its Chern classes according to the formula*

$$\begin{aligned} 1 - p_1 + p_2 - \cdots \pm p_n &= (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + c_2 + \cdots + c_n) \\ &\in H^*(X, \mathbb{Z}). \end{aligned}$$

Example. We will compute the Pontrjagin classes of the tangent bundle of projective space, $\tau_{\mathbb{CP}^n}$. Recall that the total Chern class is given by

$$c(\tau_{\mathbb{CP}^n}) = (1 + a)^{n+1}$$

where $a \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$ is the generator. Notice that this implies that for the conjugate, $\bar{\tau}_{\mathbb{CP}^n}$ we have

$$c(\bar{\tau}_{\mathbb{CP}^n}) = (1 - a)^{n+1}$$

Thus by the above formula we have

$$\begin{aligned} 1 - p_1 + p_2 - \cdots \pm p_n &= (1+a)^{n+1}(1-a)^{n+1} \\ &= (1-a^2)^{n+1}. \end{aligned}$$

We therefore have the formula

$$p_k(\mathbb{CP}^n) = \binom{n+1}{k} a^{2k} \in H^{4k}(\mathbb{CP}^n).$$

Now let η be an oriented real n- dimensional vector bundle. Then the complexification $\eta_{\mathbb{C}} = \eta \otimes \mathbb{C} = \eta \oplus i\eta$ which is simply $\eta \oplus \eta$ as real vector bundles.

LEMMA 3.42. *The above isomorphism*

$$\eta_{\mathbb{C}} \cong \eta \oplus \eta$$

of real vector bundles takes the canonical orientation of $\eta_{\mathbb{C}}$ *to* $(-1)^{\frac{n(n-1)}{2}}$ *times the orientation of* $\eta \oplus \eta$ *induced from the given orientation of* η.

PROOF. Pick a particular fiber, η_x. Let $\{v_1, \cdots, v_n\}$ be a $\mathbb{C}$ - basis for V. Then the basis $\{v_1, iv_1, \cdots, v_n iv_n\}$ determines the orientation for $\eta_x \otimes \mathbb{C}$. However the basis $\{v_1, \cdots, v_n, iv_1, \cdots iv_n\}$ gives the natural basis for $(\eta \oplus i\eta)_x$. The change of basis matrix has determinant $(-1)^{\frac{n(n-1)}{2}}$. □

COROLLARY 3.43. *If* η *is an oriented* $2k$ *- dimensional real vector bundle, then*

$$p_k(\eta) = \chi(\eta)^2 \in H^{4k}(X).$$

PROOF.

$$\begin{aligned} p_k(\eta) &= (-1)^k c_{2k}(\eta \times \mathbb{C}) \\ &= (-1)^k \chi(\eta \otimes \mathbb{C}) \\ &= (-1)^k (-1)^{k(2k-1)} \chi(\eta \oplus \eta) \\ &= \chi(\eta \oplus \eta) \\ &= \chi(\eta)^2. \end{aligned}$$

□

5.3. Oriented characteristic classes. We now use the results above to show that Pontrjagin classes and the Euler class yield all possible characteristic classes for oriented vector bundles, if the coefficient ring contains 1/2. More specifically we prove the following.

THEOREM 3.44. *Let R be an integral domain containing* $1/2$. *Then*

$$H^*(BSO(2n+1);R) = R[p_1, \cdots, p_n]$$
$$H^*(BSO(2n);R) = R[p_1, \cdots, p_{n-1}, \chi(\gamma_{2n})]$$

Remark. This theorem can be restated by saying that $H^(BSO(n);R)$ is generated by $\{p_1, \cdots, p_{[n/2]}\}$ and χ, subject only to the relations

$$\chi = 0 \quad \text{if } n \text{ is odd}$$
$$\chi^2 = p_{[n/2]} \quad \text{if } n \text{ is even.}$$

PROOF. In this proof all cohomology will be taken with R coefficients. We first observe that since $SO(1)$ is the trivial group, $BSO(1)$ is contractible, and so $H^*(BSO(1)) = 0$. This will be the first step in an inductive proof. So we assume the theorem has been proved for $BSO(n-1)$, and we now compute $H^*(BSO(n))$ using the Gysin sequence:

(5.1)

$$\cdots \to H^{q-1}(BSO(n-1)) \xrightarrow{\delta} H^{q-n}(BSO(n)) \xrightarrow{\cup\chi} H^q(BSO(n)) \xrightarrow{\iota^*}$$
$$H^q(BSO(n-1)) \xrightarrow{\delta} H^{q-n+1}(BSO(n)) \xrightarrow{\cup\chi} H^{q+1}(BSO(n)) \to \cdots$$

Case 1. n is even.

Since the first $n/2-1$ Pontrjagin classes are defined in $H^*(BSO(n))$ as well as in $H^*(BSO(n-1))$, the inductive assumption implies that $\iota^* : H^*(BSO(n)) \to H^*(BSO(n-1))$ is surjective. Thus the Gysin sequence reduces to short exact sequences

$$0 \to H^q(BSO(n)) \xrightarrow{\cup\chi} H^{q+n}(BSO(n)) \xrightarrow{\iota^*} H^{q+n}(BSO(n-1)) \to 0.$$

The inductive step then follows.

Case 2. n is odd, say $n = 2m+1$.

By 3.13 in this case the Euler class χ has order two in integral cohomology. Thus since R contains $1/2$, in cohomology with R coefficients, the Euler class is zero. Thus the Gysin sequence reduces to short exact sequences:

$$0 \to H^j(BSO(2m+1)) \xrightarrow{\iota^*} H^*(BSO(2m)) \to H^{j-2m}(BSO(2m+1)) \to 0.$$

Thus the map ι^* makes $H^*(BSO(2m{+}1))$ a subalgebra of $H^*(BSO(2m))$. This subalgeabra contains the Pontrjagin classes and hence it contains the graded algebra $A^* = R[p_1, \cdots p_m]$. By computing ranks we will now show that this is the entire image of ι^*. This will complete the inductive step in this case.

So inductively assume that the rank of A^{j-1} is equal to the rank of $H^j(BSO(2m+1))$. Now we know that every element of $H^j(BSO(2m))$ can be written uniquely as a sum $a + \chi b$ where $a \in A^j$ and $b \in A^{j-2m}$. Thus

$$H^j(BSO(2m)) \cong A^j \oplus A^{j-2m}$$

which implies that

$$rk(H^j(BSO(2m)) = rk(A^j) + rk(A^{j-2m}).$$

But by the exactness of the above sequence,

$$rk(H^j(BSO(2m)) = rk(H^j(BSO(2m+1)) + rk(H^{j-2m}(BSO(2m+1))).$$

Comparing these two equations, and using our inductive assumption, we conclude that

$$rk(H^j(BSO(2m+1)) = rk(A^j).$$

Thus $A^j = \iota^*(H^j(BSO(2m+1)))$, which completes the inductive argument. $\square$

6. Connections, Curvature, and Characteristic Classes

In this section we describe how Chern and Pontrjagin classes can be defined using connections (i.e covariant derivatives) on vector bundles. What we will describe is an introduction to the theory of Chern and Weil that describe the cohomology of a classifying space of a compact Lie group in terms of invariant polynomials on its Lie algebra. The treatment we will follow is from Milnor and Stasheff [**31**].

Definition 3.8. Let $M_n(\mathbb{C})$ be the ring of $n \times n$ matrices over $\mathbb{C}$. Then an *invariant polynomial* on $M_n(\mathbb{C})$ is a function

$$P : M_n(\mathbb{C}) \to \mathbb{C}$$

which can be expressed as a complex polynomial in the entries of the matrix, and satisfies,

$$P(ABA^{-1}) = P(B)$$

for every $B \in M_n(\mathbb{C})$ and $A \in GL(n, \mathbb{C})$.

Examples. The trace function $(a_{i,j}) \to \sum_{j=1}^n a_{j,j}$ and the determinant function are examples of invariant polynomials on $M_n(\mathbb{C})$.

Now let $D_A : \Omega^0(M;\zeta) \to \Omega^1(M;\zeta)$ be a connection (or covariant derivative) on a complex n - dimensional vector bundle ζ. Its curvature is a a two- form with values in the endomorphism bundle

$$F_A \in \Omega^2(M; End(\zeta))$$

The endomorphism bundle can be described alternatively as follows. Let E_ζ be the principal $GL(n,\mathbb{C})$ bundle associated to ζ. Then of course $\zeta = E_\zeta \otimes_{GL(n,\mathbb{C})} \mathbb{C}^n$. The endomorphism bundle can then be described as follows. The proof is an easy exercise that we leave to the reader.

Proposition 3.45.

$$End(\zeta) \cong ad(\zeta) = E_\zeta \times_{GL(n,\mathbb{C})} M_n(\mathbb{C})$$

where $GL(n,\mathbb{C})$ acts on $M_n(bc)$ by conjugation,

$$A \cdot B = ABA^{-1}.$$

Let ω be a differential p - form on M with values in $End(\zeta)$,

$$\omega \in \Omega^p(M; End(\zeta)) \cong \Omega^p(M; ad(\zeta)) = \Omega^p(M; E_\zeta \times_{GL(n,\mathbb{C})} M_n(\mathbb{C})).$$

Then on a coordinate chart $U \subset M$ with local trivialization $\psi : \zeta_{|_U} \cong U \times \mathbb{C}^n$ for ζ, and hence the induced coordinate chart and local trivialization for $ad(\zeta)$, ω can be viewed as an $n \times n$ matrix of p -forms on M. We write

$$\omega = (\omega_{i,j}).$$

Of course this description depends on the coordinate chart and local trivialization chosen, but at any $x \in U$, then by the above proposition, two trivializations yield conjugate matrices. That is, if $(\omega_{i,j}(x))$ and $(\omega'_{i,j}(x))$ are two matrix descriptions of $\omega(x)$ defined by two different local trivializations of $\zeta_{|_U}$, then there exists an $A \in GL(n,\mathbb{C})$ with

$$A(\omega_{i,j}(x))A^{-1} = (\omega'_{i,j}(x)).$$

Now let P be an invariant polynomial on $M_n(\mathbb{C})$ of degree d. Then using the wedge bracket we can apply P to a matrix of p forms, and produce a differential form of top dimension pd on $U \subset M$: $P(\omega_{i,j}) \in \Omega^{pd}(U)$. Now since the polynomial P is invariant under conjugation the form $P(\omega_{i,j})$ is independent of the local trivialization of $\zeta_{|_U}$. These forms therefore fit together to give a well defined global form

$$P(\omega) \in \Omega^*(M). \tag{6.1}$$

If $P$is homogeneous of degree d, then

$$P(\omega) \in \Omega^{pd}(M) \tag{6.2}$$

An important example is when $\omega = F_A \in \Omega^2(M; End(\zeta))$ is the curvature form of a connection D_A on ζ. We have the following fundamental lemma, that will allow us to define characteristic classes in terms of these forms and invariant polynomials.

LEMMA 3.46. *For any connection D_A and invariant polynomial (or invariant power series) P, the differential form $P(F_A)$ is closed. That is,*

$$dP(F_A) = 0.$$

PROOF. (following Milnor and Stasheff [**31**]) Let P be an invariant polynomial or power series. We write $P(A) = P(a_{i,j})$ where the $a_{i,j}$'s are the entries of the matrix. We can then consider the matrix of partial derivatives $(\partial P/\partial(x_{i,j}))$ where the $x_{i,j}$'s are indeterminates. Let $F_A = (\omega_{i,j})$ be the curvature matrix of two - forms on an open set U with a given trivialization. Then the exterior derivative has the following local expression

$$dP(F_A) = \sum (\partial P/\partial \omega_{i,j}) d\omega_{i,j}. \tag{6.3}$$

In matrix notation this can be written as

$$dP(F_A) = trace(P'(F_A)dF_A)$$

Now as seen in chapter 1, on a trivial bundle, and hence on this local coordinate patch, a connection D_A can be viewed as a matrix valued one form,

$$D_A = (\alpha_{i,j})$$

and with respect to which the curvature F_A has the formula

$$\omega_{i,j} = d\alpha_{i,j} - \sum_k \omega_{i,k} \wedge \omega_{k,j}.$$

In matrix notation we write

$$F_A = d\alpha - \alpha \wedge \alpha.$$

Differentiating yields the following form of the Bianchi identity

$$dF_A = \alpha \wedge F_A - F_A \wedge \alpha. \tag{6.4}$$

We need the following observation.

Claim. The transpose of the matrix of first derivatives of an invariant polynomial (or power series) $P'(A)$ commutes with A.

Proof. Let $E_{j,i}$ be the matrix with entry 1 in the (j,i)-th place and zeros in all other coordinates. Now differentiate the equation

$$P((I+tE_{j,i})A) = P(A(I+tE_{j,i}))$$

with respect to t and then setting $t=0$ yields

$$\sum_k A_{i,k}(\partial P/\partial A_{j,k}) = \sum_k (\partial P/\partial A_{k,i})A_{k,i}.$$

Thus the matrix A commutes with the transpose of $(\partial P/\partial Ai, j)$ as claimed. □

We now complete the proof of the lemma. Substituting F_A for the matrix of indeterminates in the above claim means we have

$$F_A \wedge P'(F_A) = P'(F_A) \wedge F_A. \tag{6.5}$$

Now for notational convenience let $X = P'(F_A) \wedge \alpha$. Then substituting the Bianchi identity 6.4 into 6.3 and using 6.5 we obtain

$$\begin{aligned} dP(F_A) &= trace\,(X \wedge F_A - F_A \wedge X) \\ &= \sum (X_{i,j} \wedge \omega_{j,i} - \omega_{j,i} \wedge X_{i,j}). \end{aligned}$$

Since each $X_{i,j}$ commutes with the 2 - form $\omega_{j,i}$, this sum is zero, which proves the lemma. □

Thus for any connection D_A on the complex vector bundle ζ over M, and invariant polynomial P, the form $P(F_A)$ represents a deRham cohomology class with complex coefficients. That is,

$$[P(F_A)] \in H^*(M : \mathbb{C}).$$

Theorem 3.47. *The cohomology class $[P(F_A)] \in H^*(X, \mathbb{C})$ is independent of the connection D_A.*

Proof. Let D_{A_0} and D_{A_1} be two connections on ζ. Pull back the bundle ζ over $M \times \mathbb{R}$ via the projection map $M \times \mathbb{R} \to M$. Call this pull - back bundle $\bar{\zeta}$ over $M \times \mathbb{R}$. We get the induced pull back connections $\bar{D}_{A_i}$, $i = 0, 1$ as well. We can then form the linear combination of connections

$$D_A = t\bar{D}_{A_1} + (1-t)D_{A_0}.$$

Then $P(F_A)$ is a deRham cocycle on $M \times \mathbb{R}$. Now let $i = 0$ or 1 and consider the inclusions $j_i : M = M \times \{i\} \hookrightarrow M \times \mathbb{R}$. The induced connection $j_i^*(D_A) = D_{A_i}$ on ζ. But since there is an obvious homotopy between j_0 and j_1 and hence the cohomology classes

$$[j_0^*(P(F_A)) = P(F_{A_0})] = [j_1^*(P(F_A)) = P(F_{A_1})].$$

This proves the theorem. □

Thus the invariant polynomial P determines a cohomology class given any bundle ζ over a smooth manifold. It is immediate that these classes are preserved under pull - back, and are hence characteristic classes for $U(n)$ bundles, and hence are given by elements of

$$H^*(BU(n);\mathbb{C}) \cong \mathbb{C}[c_1, \cdots, c_n].$$

In order to see how an invariant polynomial corresponds to a polynomial in the Chern classes we need the following bit of algebra.

Recall the elementary symmetric polynomials $\sigma_1, \cdots \sigma_n$ in n -variables, discussed in section 3. If we view the n - variables as the eigenvalues of an $n \times n$ matrix, we can write

$$det(I + tA) = 1 + t\sigma_1(A) + \cdots + t^n\sigma_n(A). \tag{6.6}$$

Lemma 3.48. *Any invariant polynomial on $M_n(\mathbb{C})$ can be expressed as a polynomial of $\sigma_1, \cdots, \sigma_n$.*

Proof. Given $A \in M_n(\mathbb{C})$, chose a B such that BAB^{-1} is in Jordan canonical form. Replacing B with $diag(\epsilon, \epsilon^2, \cdots, \epsilon^n)B$, we can make the off diagonal entries arbitrarily close to zero. By continuity it follows that $P(A)$ depends only on the diagonal entries of BAB^{-1}, i,e the eigenvalues of A. Since $P(A)$ is invariant, it must be a symmetric polynomial of these eigenvalues. Hence it is a polynomial in the elementary symmetric polynomials. $\square$

So we now consider the elementary symmetric polynomials, viewed as invariant polynomials in $M_n(\mathbb{C})$. Hence by the above constructions they determine characteristic classes $[\sigma_r(F_A)] \in H^{2r}(M;\mathbb{C})$ where F_A is a connection on a vector bundle ζ over M.

Now we've seen the elementary symmetric functions before in the context of characteristic classes. Namely we've seen that $H^*(BU(n))$ can be viewed as the subalgebra of symmetric polynomials in $\mathbb{Z}[x_1, \cdots x_n] = H^*(BU(1) \times \cdots \times BU(1))$, with the Chern class C_r corresponding to the elementary symmetric polynomial σ_r. This was the phenomenon of the *splitting principle*.

We will now use a splitting principle argument to prove the following.

Theorem 3.49. *Let ζ be a complex n - dimensional vector bundle with connection D_A. Then the cohomology class $[\sigma_r(F_A)] \in H^{2r}(X;\mathbb{C})$ is equal to $(2\pi i)^r c_r(\zeta)$, for $r = 1, \cdots, n$.*

Proof. We first prove this theorem for complex line bundles. That is, $n = 1$. In this case $\sigma_1(F_A) = F_A$ which is a closed form in $\Omega^2(M; ad(\zeta)) = \Omega^2(M;\mathbb{C})$ because the adjoint action of $GL(1,\mathbb{C})$ is trivial since it is an abelian group. In particular F_A is closed in this case by 3.46. Thus F_A represents a cohomology class in $H^2(M;\mathbb{C})$. Moreover as seen above, this cohomology class $[F_A]$ is a characteristic class for line bundles and hence is an element of $H^2(BU(1);\mathbb{C}) \cong \mathbb{C}$ generated by the first Chern class $c_1 \in H^2(BU(1))$. So for this case we need to prove the following generalization of the Gauss - Bonnet theorem.

LEMMA 3.50. *Let ζ be a complex line bundle over a manifold M with connection D_A. Then the curvature form F_A is a closed two - form representing the cohomology class*

$$[F_A] = 2\pi i c_1(\zeta) = 2\pi i \chi(\zeta).$$

Before we prove this lemma we show how this lemma can in fact be interpreted as a generalization of the classical Gauss - Bonnet theorem. So let D_A be a unitary connection on ζ. (That is, D_A is induced by a connection on an associated principal $U(1)$ - bundle.) If we view ζ as a two dimensional, oriented vector bundle which, to keep notation straight we refer to as $\zeta_{\mathbb{R}}$, then D_A induces (and is induced by) a connection $D_{A_{\mathbb{R}}}$ on the real bundle $\zeta_{\mathbb{R}}$. Notice that since $SO(2) \cong U(1)$ then orthogonal connections on oriented real two dimensional bundes are equivalent to unitary connections on complex line bundles.

Since $SO(2)$ is abelian, the real adjoint bundle

$$ad(\zeta_{\mathbb{R}}) = E_{\zeta_{\mathbb{R}}} \times_{SO(2)} M_2(\mathbb{R})$$

is trivial. Hence the curvature $F_{A_{\mathbb{R}}}$ is then a 2×2 matrix valued two - form.

$$F_{A_{\mathbb{R}}} \in \Omega^2(M; M_2(\mathbb{R})).$$

Moreover, since the Lie algebra of $SO(2)$ consists of skew symmetric 2×2 real matrices, then it is straightforward to check the following relation between the original complex valued connection $F_A \in \Omega^2(M; \mathbb{C})$ and the real curvature form $F_{A_{\mathbb{R}}} \in \Omega^1(M; M_2(\mathbb{R}))$.

Claim. If $F_{A_{\mathbb{R}}}$ is written as the skew symmetric matrix of $2 - forms$

$$F_{A_{\mathbb{R}}} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \in \Omega^2(M; M_2(\mathbb{R}))$$

then

$$F_A = i\omega \in \Omega^2(M; \mathbb{C}).$$

When the original connection $D_{A_{\mathbb{R}}}$ is the *Levi - Civita* connection associated to a Riemannian metric on the tangent bundle of a Riemann surface, the curvature form

$$\omega \in \Omega^2(M, \mathbb{R})$$

is referred to as the "Gauss - Bonnet"" connection. If dA denotes the area form with respect to the metric, then we can write

$$\omega = \kappa \, dA$$

then κ is a scalar valued function called the "Gaussian curvature" of the Riemann surface M. In this case, by the claim we have $[F_A] = 2\pi i\chi(\tau(M))$, and since

$$\langle \chi(\tau(M)), [M]\rangle = \chi_M,$$

Where χ_M the Euler characteristic of M, we have

$$\langle [F_A], [M]\rangle = \int_M F_A = i\int_M \omega = i\int_M \kappa\, dA.$$

Thus the above lemma applied to this case, which states that

$$\langle [F_A], [M]\rangle = 2\pi i\chi_M$$

is equivalent to the classical Gauss - Bonnet theorem which states that

$$\int_M \kappa\, dA = 2\pi\chi_M = 2\pi(2-2g) \tag{6.7}$$

where g is the genus of the Riemann surface M.

We now prove the above lemma.

PROOF. As mentioned above, since $[F_A]$ is a characteristic class for line bundles, and so it is some multiple of the first Chern class, say $[F_A] = qc_1(\zeta)$. By the naturality, the coefficient q is independent of the bundle. So to evaluate q it is enough to compute it on a specific bundle. We choose the tangent bundle of the unit sphere τ_{S^2}, equipped with the Levi - Civita connection D_A corresponding to the usual round metric (or equivalently the metric coming from the complex strucure $S^2 = \mathbb{CP}^1$). In this case the Gaussian curvature is constant at one,

$$\kappa = 1.$$

Moreover since $\tau_{S^2} \oplus \epsilon_1 \cong \gamma_1 \oplus \gamma_1$, the Whitney sum formula yields

$$\langle c_1(S^2), [S^2]\rangle = 2\langle c_1(\gamma_1), [S^2]\rangle = 2.$$

Thus we have

$$\begin{aligned}\langle [F_A], [S^2]\rangle &= q\langle c_1(S^2), [S^2]\rangle \\ &= 2q.\end{aligned}$$

Putting these facts together yields that

$$\begin{aligned}
2q &= \langle [F_A], [S^2] \rangle \\
&= \int_{S^2} F_A \\
&= i \int_{S^2} \kappa \, dA \\
&= i \int_{S^2} dA \qquad = i \cdot \text{surface area of } S^2 \\
&= i \cdot 4\pi.
\end{aligned}$$

Hence $q = 2\pi i$, as claimed. □

We now proceed with the proof of theorem 3.49 in the case when the bundle is a sum of line bundles. By the splitting principal we will then be able to conclude the theorem is true for all bundles.

So let $\zeta = L_1 \oplus \cdots \oplus L_n$ where $L_1, \cdots, L_n$ are complex line bundles over M. Let $D_1, \cdots, D_n$ be connections on $L_!, \cdots, L_n$ respectively. Now let D_A be the connection on ζ given by the sum of these connections

$$D_A = D_1 \oplus \cdots \oplus D_n.$$

Notice that with respect to any local trivialization, the curvature matrix F_A is the diagonal $n \times n$ matrix with diagonal entries, the curvatures $F_1, \cdots, F_n$ of the connections $D_1, \cdots D_n$ respectively. Thus the invariant polynomial applied to the curvature form $\sigma_r(F_A)$ is given by the symmetric polynomial in the diagonal entries,

$$\sigma_r(F_A) = \sigma_r(F_1, \cdots, F_r).$$

Now since the curvatures F_i are closed 2 - forms on M, we have an equation of cohomology classes

$$[\sigma_r(F_A)] = \sigma_r([F_1], \cdots, [F_r]).$$

By the above lemma we therefore have

$$\begin{aligned}
[\sigma_r(F_A)] &= \sigma_r([F_1], \cdots, [F_n]) \\
&= \sigma_r((2\pi i)c_1(L_1), \cdots, (2\pi i)c_1(L_n)) \\
&= (2\pi i)^r \sigma_r(c_1(L), \cdots, c_1(L_n)) \quad \text{since } \sigma_r \text{ is symmetric} \\
&= (2\pi i)^r c_r(L_1 \oplus \cdots \oplus L_n) \quad \text{by the splitting principal 3.20} \\
&= (2\pi i)^r c_r(\zeta)
\end{aligned}$$

as claimed.

This proves the theorem when ζ is a sum of line bundles. As observed above, the splitting principal implies that the theorem then must be true for all bundles. □

We end this section by describing two corollaries of this important theorem.

COROLLARY 3.51. *For any real vector bundle η, the deRham cocycle $\sigma_{2k}(F_A)$ represent the cohomology class $(2\pi)^{2k}p_k(\eta) \in H^{4k}(M;\mathbb{R})$, while $[\sigma_{2k+1}(F_A)]$ is zero in $H^{4k+2}(M;\mathbb{R})$.*

PROOF. This just follows from the definition of the Pontrjagin classes in terms of the even Chern classes of the complexification, and the fact that the odd Chern classes of the complexification have order two and therefore represent the zero class in $H^*(M;\mathbb{R})$. □

Recall that a flat connection is one whose curvature is zero. The following is immediate form the above theorem.

COROLLARY 3.52. *If a real (or complex) vector bundle has a flat connection, then all its Pontrjagin (or Chern) classes with rational coefficients are zero.*

We recall that a bundle has a flat connection if and only if its structure group can be reduced to a discrete group. Thus a complex vector bundle with a discrete structure group has zero Chern classes with rational coefficients. This can be interpreted as saying that if $\iota : G \subset GL(n,\mathbb{C})$ is the inclusion of a discrete subgroup, then the map in cohomology,

$$\mathbb{Q}[c_1, \cdots, c_n] = H^*(BU(n);\mathbb{Q}) = H^*(BGL_n(\mathbb{C});\mathbb{Q}) \xrightarrow{\iota^*} H^*(BG;\mathbb{Q})$$

is zero.

CHAPTER 4

Homotopy Theory of Fibrations

In this chapter we study the basic algebraic topological properties of fiber bundles, and their generalizations, "Serre fibrations". We begin with a discussion of homotopy groups and their basic properties. We then show that fibrations yield long exact sequences in homotopy groups and use it to show that the loop space of the classifying space of a group is homotopy equivalent to the group. We then develop basic obstruction theory for liftings in fibrations, use it to interpret characteristic classes as obstructions, and apply them in several geometric contexts, including vector fields, Spin structures, and classification of $SU(2)$ - bundles over four dimensional manifolds. We also use obstruction theory to prove the existence of Eilenberg - MacLane spaces, and to prove their basic property of classifying cohomology. We then develop the theory of spectral sequences and then discuss the famous Leray - Serre spectral sequence of a fibration. We use it in several applications, including a proof of the theorem relating homotopy groups and homology groups, a calculation of the homology of the loop space ΩS^n, and a calculation of the homology of the Lie groups $U(n)$ and $O(n)$.

1. Homotopy Groups

We begin by adopting some conventions and notation. In this chapter, unless otherwise specified, we will assume that all spaces are connected and come equipped with a basepoint. When we write $[X, Y]$ we mean homotopy classes of basepoint preserving maps $X \to Y$. Suppose $x_0 \in X$ and $y_0 \in Y$ are the basepoints. Then a basepoint preserving homotopy between basepoint preserving maps f_0 and $f_1 : X \to Y$ is a map

$$F : X \times I \to Y$$

such that each $F_t : X \times \{t\} \to Y$ is a basepoint preserving map and $F_0 = f_0$ and $F_1 = f_1$. If $A \subset X$ and $B \subset Y$, are subspaces that contain the basepoints, ($x_0 \in A$, and $y_0 \in B$), we write $[X, A; Y, B]$ to mean homotopy classes of maps $f : X \to Y$ so that the restriction $f_{|A}$ maps A to B. Moreover homotopies are assumed to preserve these subsets as well. That is, a homotopy defining this equivalence relation is a map $F : X \times I \to Y$ that restricts to a basepoint preserving homotopy $F : A \times I \to B$. We can now give a strict definition of homotopy groups.

Definition 4.1. The n^{th} homotopy group of a space X with basepoint $x_0 \in X$ is defined to be the set

$$\pi_n(X) = \pi_n(X, x_0) = [S^n, X].$$

Equivalently, this is the set

$$\pi_n(X) = [D^n, S^{n-1}; X, x_0]$$

where $S^{n-1} = \partial D^n$ is the boundary sphere.

Exercise. Prove that these two definitions are in fact equivalent.

Remarks. 1. It will often helpful to us to use as our model of the disk D^n the n - cube $I^n = [0,1]^n$. Notice that in this model the boundary ∂I^n consists of n - tuples $(t_1, \cdots, t_n)$ with $t_i \in [0,1]$ where at least one of the coordinates is either 0 or 1.

2. Notice that for $n = 1$, this definition of the first homotopy group is the usual definition of the fundamental group.

So far the homotopy "groups" have only been defined as sets. We now examine the group structure. To do this, we will define our homotopy groups via the cube I^n, which we give the basepoint $(0, \cdots, 0)$. Let

$$f \quad \text{and} \quad g : (I^n, \partial I^n) \longrightarrow (X, x_0)$$

be two maps representing elements $[f]$ and $[g] \in \pi_n(X, x_0)$. Define

$$f \cdot g : I^n \longrightarrow X$$

by

$$f \cdot g(t_1, t_2, \cdots, t_n) = \begin{cases} f(2t_1, t_2, \cdots, t_n) & \text{for } t_1 \in [0, 1/2] \\ g(2t-1, t_2, \cdots, t_n) & \text{for } t_1 \in [1/2, 1] \end{cases}$$

The map $f \cdot g : (I^n, \partial I^n) \to (X, x_0)$ represents the product of the classes

$$[f \cdot g] = [f] \cdot [g] \in \pi_n(X, x_0).$$

Notice that in the case $n = 1$ this is precisely the definition of the product structure on the fundamental group $\pi_1(X, x_0)$. The same proof that this product structure is well defined and gives the fundamental group the structure of an associative group extends to prove that all of the homotopy groups are in fact groups under this product structure. We leave the details of checking this to the reader. We refer the reader to any introductory textbook on algebraic topology for the details.

As we know the fundamental group of a space can be quite complicated. Indeed any group can be the fundamental group of a space. In particular fundamental groups can be very much noncommutative. However we recall the relation of the fundamental group to the first homology group, for which we again refer the reader to any introductory textbook:

THEOREM 4.1. *Let X be a connected space. Then the abelianization of the fundamental group is isomorphic to the first homology group,*

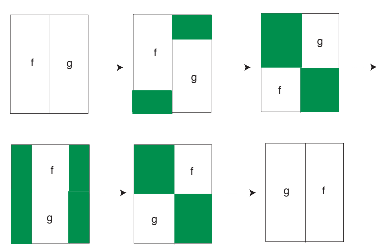
$$\pi_1(X)/[\pi_1,\pi_1] \cong H_1(X)$$

where $[\pi_1,\pi_1]$ is the commutator subgroup of $\pi_1(X)$.

We also have the following basic result about higher homotopy groups.

PROPOSITION 4.2. *For $n \geq 2$, the homotopy group $\pi_n(X)$ is abelian.*

PROOF. Let $[f]$ and $[g]$ be elements of $\pi_n(X)$ represented by basepoint preserving maps $f : (I^n, \partial I^n) \to (X, x_0)$ and $g : (I^n, \partial I^n) \to (X, x_0)$, respectively. We need to find a homotopy between the product maps $f \cdot g$ and $g \cdot f$ defined above. The following schematic diagram suggests such a homotopy. We leave it to the reader to make this into a well defined homotopy.



□

Now assume $A \subset X$ is a subspace containing the basepoint $x_0 \in A$.

DEFINITION 4.2. For $n \geq 1$ we define the *relative homotopy group* $\pi_n(X,A) = \pi_n(X,A,x_0)$ *to be homotopy classes of maps of pairs*

$$\pi_n(X,A) = [(D^n, \partial D^n, t_0); (X, A, x_0)].$$

where $t_0 \in \partial D^n = S^{n-1}$ and $x_0 \in A$ are the basepoints.

Exercise. Show that for $n > 1$ the relative homotopy group $\pi_n(X, A)$ is in fact a group. Notice here that the zero element is represented by any basepoint preserving map of pairs$f : (D^n, \partial I^n) \to (X, A)$ that is homotopic (through maps of pairs) to one whose image lies entirely in $A \subset X$.

Again, let $A \in X$ be a subset containing the basepoint $x_0 \in A$, and let $i : A \hookrightarrow X$ be the inclusion. This induces a homomorphism of homotopy groups

$$i_* : \pi_n(A, x_0) \to \pi_n(X, x_0).$$

Also, by ignoring the subsets, a basepoint preserving map $f : (D^n, \partial D^n) \to (X, x_0)$ defines a map of pairs $f : (D^n, \partial D^n, t_0) \to (X, A, x_0)$ which defines a homomorphism

$$j_* : \pi_n(X, x_0) \to \pi_n(X, A, x_0).$$

Notice furthermore, that by construction, the composition

$$j_* \circ i_* : \pi_n(A) \to \pi_n(X) \to \pi_n(X, A)$$

is zero. Finally, if given a map of pairs $g : (D^n, S^{n-1}, t_0) \to (X, A, x_0)$, then we can restrict g to the boundary sphere S^{n-1} to produce a basepoint preserving map

$$\partial g : (S^{n-1}, t_0) \to (A, x_0).$$

This defines a homomorphism

$$\partial_* : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0).$$

Notice here that the composition

$$\partial_* \circ j_* : \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A)$$

is also zero, since the application of this composition to any representing map $f : (D^n, S^{n-1}) \to (X, x_0)$ yields the constant map $S^{n-1} \to x_0 \in A$. We now have the following fundamental property of homotopy groups. Compare with the analogous theorem in homology.

THEOREM 4.3. *Let $A \subset X$ be a subspace containing the basepoint $x_0 \in A$. Then we have a long exact sequence in homotopy groups*

$$\cdots \xrightarrow{\partial_*} \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial_*} \pi_{n-1}(A) \to \cdots \to \pi_1(A) \xrightarrow{i_*} \pi_1(X)$$

PROOF. We've already observed that $j_* \circ i_*$ and $\partial_* \circ j_*$ are zero. Similarly, $i_* \circ \partial_*$ is zero because an element in the image of ∂_* is represented by a basepoint preserving map $S^{n-1} \to A$ that extends to a map $D^n \to X$. Thus the image under i_*, namely the composition $S^{n-1} \to A \hookrightarrow X$ has an extension to D^n and is therefore null homotopic. We therefore have

$$image(\partial_*) \subset kernel(i_*)$$
$$image(i_*) \subset kernel(j_*)$$
$$image(j_*) \subset kernel(\partial_*).$$

To finish the proof we need to show that all of these inclusions are actually equalities. Consider the kernel of (i_*). An element $[f] \in \pi_n(A)$ is in $ker(i_*)$ if and only if the basepoint preserving composition $f : S^n \to A \subset X$ is null homotopic. Such a null - homotopy gives an extension of this map to the disk $F : D^{n+1} \to X$. The induced map of pairs $F : (D^{n+1}, S^n) \to (X, A)$ represents an element in $\pi_{n+1}(X, A)$ whose image under ∂_* is $[f]$. This proves that $image(\partial_*) = kernel(i_*)$. The other equalities are proved similarly, and we leave their verification to the reader. $\square$

Remark. Even though this theorem is analogous to the existence of exact sequences for pairs in homology, notice that its proof is much easier.

Notice that $\pi_0(X)$ is the set of path components of X. So a space is (path) - connected if and only if $\pi_0(X) = 0$ (i.e the set with one element). We generalize this notion as follows.

Definition 4.3. A space X is said to be m - connected if $\pi_q(X) = 0$ for $0 \leq q \leq m$.

We now do our first calculation.

Proposition 4.4. *An n - sphere is $n-1$ connected.*

Proof. We need to show that any map $S^k \to S^n$, where $k < n$ is null homotopic. Now since spheres can be given the structure of simplicial complexes, the simplicial approximation theorem says that any map $f : S^k \to S^n$ is homotopic to a simplicial map (after suitable subdivisions). So we assume without loss of generality that f is simplicial. But since $k < n$, the image of of f lies in the k - skeleton of the n - dimensional simplicial complex S^n. In particular this means that $f : S^k \to S^n$ is not surjective. Let $y_0 \in S^n$ be a point that is not in the image of f. Then f has image in $S^n - y_0$ which is homeomorphic to the open disk D^n, and is therefore contractible. This implies that f is null homotopic. $\square$

2. Fibrations

Recall that in chapter 2 we proved that locally trivial fiber bundles satisfy the Covering Homotopy Theorem 2.2. A generalization of the notion of a fiber bundle, due to Serre, is simply a map that satisfies this type of lifting property.

Definition 4.4. A Serre fibration is a surjective, continuous map $p : E \to B$ that satisfies the *Homotopy Lifting Property* for CW - complexes. That is, if X is any CW - complex and $F : X \times I \to B$ is any continuous homotopy so that $F_0 : X \times \{0\} \to B$ factors through a map $f_0 : X \to E$, then there exists a lifting $\bar{F} : X \times I \to E$ that extends f_0 on $X \times \{0\}$, and makes the following diagram commute:

$$\begin{array}{ccc} X \times I & \xrightarrow{\bar{F}} & E \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & B. \end{array}$$

A Hurewicz fibration is a surjective, continuous map $p : E \to B$ that satisfies the homotopy lifting property for all spaces.

Remarks. 1. Obviously every Hurewicz fibration is a Serre fibration. The converse is false. In these notes, unless otherwise stated, we will deal with Serre fibrations, which we will simply refer to as fibrations.

2. The Covering Homotopy Theorem implies that a fiber bundle is a fibration in this sense.

The following is an important example of a fibration.

Proposition 4.5. *Let X be any connected space with basepoint $x_0 \in X$. Let PX denote the space of based paths in X. That is,*

$$PX = \{\alpha : I \to X : \alpha(0) = x_0\}.$$

The path space PX is topologized using the compact - open function space topology. Define

$$p : PX \to X$$

by $p(\alpha) = \alpha(1)$. Then PX is a contractible space, and the map $p : PX \to X$ is a fibration, whose fiber at x_0, $p^{-1}(x_0)$ is the loop space ΩX.

Proof. The fact that PX is contractible is straightforward. For a null homotopy of the identity map one can take the map $H : PX \times I \to PX$, defined by $H(\alpha, s)(t) = \alpha((1 - s)t)$.

To prove that $p : PX \to X$ is a fibration, we need to show it satisfies the Homotopy Lifting Property. So let $F : Y \times I \to X$ and $f_0 : X \to PX$ be maps making the following diagram commute:

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{f_0} & PX \\ \cap \big\downarrow & & \big\downarrow p \\ Y \times I & \xrightarrow[F]{} & X \end{array}$$

Then we can define a homotopy lifting, $\bar{F} : Y \times I \to PX$ by defining for $(y, s) \in Y \times I$, the path

$$\bar{F}(y, s) : I \to X$$
$$\bar{F}(y, s)(t) = \begin{cases} f_0(y)(\frac{2t}{2-s}) & \text{for } t \in [0, \frac{2-s}{2}] \\ F(y, 2t - 2 + s) & \text{for } t \in [\frac{2-s}{2}, 1] \end{cases}$$

One needs to check that this definition makes $\bar{F}(y, s)(t)$ a well defined continous map and satisfies the boundary conditions

$$\bar{F}(y, 0)(t) = f_0(y, t)$$
$$\bar{F}(y, s)(0) = x_0$$
$$\bar{F}(y, s)(1) = F(y, s)$$

These verifications are all straightforward. □

The following is just the observation that one can pull back the Homotopy Lifting Property.

Proposition 4.6. *Let $p : E \to B$ be a fibration, and $f : X \to B$ a continuous map. Then the pull back, $p_f : f^*(E) \to X$ is a fibration, where*

$$f^*(E) = \{(x, e) \in X \times E \, \textit{such that} \, f(x) = p(e)\}$$

and $p_f(x, e) = x$.

The following shows that in the setting of homotopy theory, every map can be viewed as a fibration in this sense.

Theorem 4.7. *Every continuous map $f : X \to Y$ is homotopic to a fibration in the sense that there exists a fibration*

$$\tilde{f} : \tilde{X} \to Y$$

and a homotopy equivalence

$$h : X \xrightarrow{\simeq} \tilde{X}$$

making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow[\simeq]{h} & \tilde{X} \\ f\big\downarrow & & \big\downarrow\tilde{f} \\ Y & = & Y. \end{array}$$

PROOF. Define $\tilde{X}$ to be the space

$$\tilde{X} = \{(x, \alpha) \in X \times Y^I \text{ such that } \alpha(0) = x.\}$$

where here Y^I denotes the space of continuous maps $\alpha : [0,1] \to Y$ given the compact open topology. The map $\tilde{f} : \tilde{X} \to Y$ is defined by $\tilde{f}(x, \alpha) = \alpha(1)$. The fact that $\tilde{f} : \tilde{X} \to Y$ is a fibration is proved in the same manner as theorem 4.5, and so we leave it to the reader.

Define the map $h : X \to \tilde{X}$ by $h(x) = (x, \epsilon_x) \in \tilde{X}$, where $\epsilon_x(t) = x$ is the constant path at $x \in X$. Clearly $\tilde{f} \circ h = f$ so the diagram in the statement of the theorem commutes. Now define $g : \tilde{X} \to X$ by $g(x, \alpha) = x$. Clearly $g \circ h$ is the identity map on X. To see that $h \circ g$ is homotopic to the identity on $\tilde{X}$, consider the homotopy $F : \tilde{X} \times I \to \tilde{X}$, defined by $F((x, \alpha), s) = (x, \alpha_s)$, where $\alpha_s : I \to X$ is the path $\alpha_s(t) = \alpha(st)$. So in particular $\alpha_0 = \epsilon_x$ and $\alpha_1 = \alpha$. Thus F is a homotopy between $h \circ g$ and the identity map on $\tilde{X}$. Thus h is a homotopy equivalence, which completes the proof of the theorem. $\square$

The homotopy fiber of a map $f : X \to Y$, F_f, is defined to be the fiber of the fibration $\tilde{f} : \tilde{X} \to Y$ defined in the proof of this theorem. That is,

DEFINITION 4.5. The homotopy fiber F_f of a basepoint preserving map $f : X \to Y$ is defined to be

$$F_f = \{(x, \alpha) \in X \times Y^I \text{ such that } \alpha(0) = f(x) \text{ and } \alpha(1) = y_0.\}$$

where $y_0 \in Y$ is the basepoint.

So for example, the homotopy fiber of the inclusion of the basepoint $y_0 \hookrightarrow Y$ is the loop space ΩY. The homotopy fiber of the identity map $id : Y \to Y$ is the path space PY. The homotopy fibers are important invariants of the map $f : X \to Y$.

The following is the basic homotopy theoretic property of fibrations.

THEOREM 4.8. *Let $p : E \to B$ be a fibration over a connected space B with fiber F. So we are assuming the basepoint of E, is contained in F, $e_0 \in F$, and that $p(e_0) = b_0$ is the basepoint in B. Let $i : F \hookrightarrow E$ be the inclusion of the fiber. Then there is a long exact sequence of homotopy groups:*

$$\cdots \xrightarrow{\partial_*} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial_*} \pi_{n-1}(F) \to$$
$$\cdots \to \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B).$$

PROOF. Notice that the projection map $p : E \to B$ induces a map of pairs

$$p : (E, F) \to (B, b_0).$$

By the exact sequence for the homotopy groups of the pair (E, F), 4.3 it is sufficient to prove that the induced map in homotopy groups

$$p_* : \pi_n(E, F) \to \pi_n(B, b_0)$$

is an isomorphism for all $n \geq 1$. We first show that p_* is surjective. So let $f : (I^n, \partial I^n) \to (B, b_0)$ represent an element of $\pi_n(B)$. We can think of a map from a cube as a homotopy of maps of cubes of one lower dimension. Therefore by induction on n, the homotopy lifting property says that that $f : I^n \to B$ has a basepoint preserving lifting $\bar{f} : I^n \to E$. Since $p \circ \bar{f} = f$, and since the restriction of f to the boundary ∂I^n is constant at b_0, then the image of the restriction of $\bar{f}$ to the boundary ∂I^n has image in the fiber F. That is, $\bar{f}$ induces a map of pairs

$$\bar{f} : (I^n, \partial I^n) \to (E, F)$$

which in turn represents an element $[\bar{f}] \in \pi_n(E, F)$ whose image under p_* is $[f] \in \pi_n(B, b_0)$. This proves that p_* is surjective.

We now prove that $p_* : \pi_n(E, F) \to \pi_n(B, b_0)$ is injective. So let $f : (D^n, \partial D^n) \to (E, F)$ be a map of pairs that represents an element in the kernel of p_*. That means $p \circ f : (D^n, \partial D^n) \to (B, b_0)$ is null homotopic. Let $F : (D^n, \partial D^n) \times I \to (B, b_0)$ be a null homotopy between $F_0 = f$ and the constant map $\epsilon : D^n \to b_0$. By the Homotopy Lifting Property there exists a basepoint preserving lifting

$$\bar{F} : D^n \times I \to E$$

having the properties that $p \circ \bar{F} = F$ and $\bar{F} : D^n \times \{0\} \to E$ is equal to $f : (D^n, \partial D^n) \to (E, F)$. Since $p \circ \bar{F} = F$ maps $\partial D^n \times I$ to the basepoint b_0, we must have that $\bar{F}$ maps $\partial D^n \times I$ to $p^{-1}(b_0) = F$. Thus $\bar{F}$ determines a homotopy of pairs,

$$\bar{F} : (D^n, \partial D^n) \times I \to (E, F)$$

with $\bar{F}_0 = f$. Now consider $\bar{F}_1 : (D^n, \partial D^n) \times \{1\} \to E$. Now $p \circ \bar{F}_1 = F_1 = \epsilon : D^n \to b_0$. Thus the image of $\bar{F}_1$ lies in $p^{-1}(b_0) = F$. Thus $\bar{F}$ gives a homotopy of the map of pairs $f : (D^n, \partial D^n) \to (E, F)$ to a map of pairs whose image lies entirely in F. Such a map represents the zero element of $\pi_n(E, F)$. This completes the proof that p_* is injective, and hence is an isomorphism. As observed earlier, this is what was needed to prove the theorem. □

We now use this theorem to make several important calculations of homotopy groups. In particular, we prove the following seminal result of Hopf.

THEOREM 4.9.

$$\pi_2(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}.$$
$$\pi_k(S^3) \cong \pi_k(S^2) \ \textit{for all } k \geq 3. \textit{ In particular,}$$
$$\pi_3(S^2) \cong \mathbb{Z}, \textit{ generated by the Hopf map } \eta : S^3 \to S^2.$$

PROOF. Consider the Hopf fibration $\eta : S^3 \to S^2 = \mathbb{CP}^1$ with fiber S^1. Recall that S^1 is an Eilenberg - MacLane space $K(\mathbb{Z}, 1)$ since it is the classifying space of bz. Thus

$$\pi_q(S^1) = \begin{cases} \mathbb{Z} & \text{for q} = 1 \\ 0 & \text{for all other } q. \end{cases}$$

(**Remark.** The fact that the classifying space $B\pi$ of a *discrete* group π is an Eilenberg - MacLane space $K(\pi, 1)$ can now be given a simpler proof, using the exact sequence in homotopy groups of the universal bundle $E\pi \to B\pi$.)

Using this fact in the exact sequence in homotopy groups for the Hopf fibration $\eta : S^3 \to S^2$, together with the fact that $\pi_q(S^3) = 0$ for $q \leq 2$, one is led to the facts that $\pi_2(S^2) \cong \pi_1(S^1) = \mathbb{Z}$, and that $\eta_* : \pi_k(S^3) \to \pi_k(S^2)$ is an isomorphism for $k \geq 3$. To examine the case $k = 3$, consider the homomorphism (called the *Hurewicz homomorphism*)

$$h : \pi_3(S^3) \to H_3(S^3) = \mathbb{Z}$$

defined by sending a class represented by a self map $f : S^3 \to S^3$, to the image of the fundamental class in homology, $f_*([S^3]) \in H^3(S^3) \cong \mathbb{Z}$. Clearly this is a homomorphism (check this!). Moreover it is surjective since the image of the identity map is the fundamental class, and thus generates, $H_3(S^3)$, $H([id]) = [S^3] \in H_3(S^3)$. Thus $\pi_3(S^3)$ contains an integral summand generated by the identity. In particular, since $\eta_* : \pi_3(S^3) \to \pi_3(S^2)$ is an isomorphism, this implies that $\pi_3(S^2)$ contains an integral summand generated by the Hopf map $[\eta] \in \pi_3(S^2)$. The fact that these integral summands generate the entire groups $\pi_3(S^3) \cong \pi_3(S^2)$ will follow once we know that the Hurewicz homomorphism is an isomorphism in this case. Later in this chapter we will prove the more general "Hurewicz theorem" that says that for any $k > 1$, and any $(k-1)$ - connected space X, the Hurewicz homomorphism is an isomorphism in dimension k: $h : \pi_k(X) \cong H_k(X)$. $\square$

Remark. As we remarked earlier in these notes. these were the first nontrivial elements found in the higher homotopy groups of spheres, $\pi_{n+k}(S^n)$, and Hopf's proof of their nontriviality is commonly viewed as the beginning of modern Homotopy Theory [**43**]

We end this section with an application to the "homotopy stability" of the orthogonal and unitary groups, as well as their classifying spaces.

THEOREM 4.10. *The inclusion maps*

$$\iota : O(n) \hookrightarrow O(n+1) \quad \textit{and}$$
$$U(n) \hookrightarrow U(n+1)$$

induce isomorphisms in homotopy groups through dimensions $n-2$ and $2n-1$ respectively. Also, the induced maps on classifying spaces,

$$B\iota : BO(n) \to BO(n+1) \quad \textit{and}$$
$$BU(n) \to BU(n+1)$$

induce isomorphisms in homotopy groups through dimensions $n-1$ and $2n$ respectively.

PROOF. The first two statements follow from the existence of fiber bundles

$$O(n) \hookrightarrow O(n+1) \to S^n$$

and

$$U(n) \hookrightarrow U(n+1) \to S^{2n+1},$$

the connectivity of spheres 4.4, and by applying the exact sequence in homotopy groups to these fiber bundles. The second statement follows from the same considerations, after recalling from 2.28 the sphere bundles

$$S^n \to BO(n) \to BO(n+1)$$

and

$$S^{2n+1} \to BU(n) \to BU(n+1).$$

□

3. Obstruction Theory

In this section we discuss the obstructions to obtaining a lifting to the total space of a fibration of a map to the basespace. As an application we prove the important "Whitehead theorem" in homotopy theory, and we prove general results about the existence of cross sections of principal $O(n)$ or $U(n)$ - bundles. We do not develop a formal theory here - we just develop what we will need for our applications to fibrations. For a full development of obstruction theory we refer the reader to [**42**].

Let X be a CW - complex. Recall that its cellular k - chains, $C_k(X)$ is the free abelian group generated by the k - dimensional cells in X. The co-chains with coefficients in a group G are defined by

$$C^k(X, G) = Hom(C_k(X), G).$$

THEOREM 4.11. *Let $p : E \to B$ be a fibration with fiber F. Let $f : X \to B$ be a continuous map, where X is a CW - complex. Suppose there is a lifting of the $(k-1)$ - skeleton $\tilde{f}_{k-1} : X^{(k-1)} \to E$. That is, the following diagram commutes:*

$$\begin{array}{ccc} X^{(k-1)} & \xrightarrow{\tilde{f}_{k-1}} & E \\ \cap\downarrow & & \downarrow p \\ X & \xrightarrow[f]{} & B. \end{array}$$

Then the obstruction to the existence of a lifting to the k -skeleton, $\tilde{f}_k : X^{(k)} \to E$ that extends $\tilde{f}_{k-1}$, is a cochain $\gamma \in C^k(X; \pi_{k-1}(F))$. That is, $\gamma = 0$ if and only if such a lifting $\tilde{f}_k$ exists.

PROOF. We will first consider the special case where $X^{(k)}$ is obtained from $X^{k-1)}$ by adjoining a single k -dimensional cell. So assume

$$X^{(k)} = X^{(k-1)} \cup_\alpha D^k$$

where $\alpha : \partial D^k = S^{(k-1)} \to X^{k-1)}$ is the attaching map. We therefore have the following commutative diagram:

$$\begin{array}{ccccc} S^{k-1} & \xrightarrow{\alpha} & X^{(k-1)} & \xrightarrow{\tilde{f}_{k-1}} & E \\ \cap\downarrow & & \cap\downarrow & & \downarrow p \\ D^k & \xrightarrow[\subset]{} & X^{(k-1)} \cup_\alpha D^k & \xrightarrow[f]{} & B \end{array}$$

Notice that $\bar{f}_{k-1}$ has an extension to $X^{(k-1)} \cup_\alpha D^k = X^{(k)}$ that lifts f, if and only if the composition $D^k \subset X^{(k-1)} \cup_\alpha D^k \xrightarrow{f} B$ lifts to E in such a way that it extends $\bar{f}_{k-1} \circ \alpha$.

Now view the composition $D^k \subset X^{(k-1)} \cup_\alpha D^k \xrightarrow{f} B$ as a a map from the cone on S^{k-1} to B, or in other words, as a null homotopy$F : S^{k-1} \times I \to B$ from $F_0 = p \circ \bar{f}_{k-1} \circ \alpha : S^{k-1} \to X^{(k-1)} \to E \to B$ to the constant map $F_1 = \epsilon : S^{(k-1)} \to b_0 \in B$. By the Homotopy Lifting Property, F lifts to a homotopy

$$\bar{F} : S^{(k-1)} \times I \to E$$

with $\bar{F}_0 = \bar{f}_{k-1} \circ \alpha$. Thus the extension f_k exists on $X^{(k-1)} \cup_\alpha D^k$ if and only if this lifting $\bar{F}$ can be chosen to be a null homotopy of $\bar{f}_{k-1} \circ \alpha$. But we know $\bar{F}_1 : S^{k-1} \times \{1\} \to E$ lifts F_1 which is the constant map $\epsilon : S^{k-1} \to b_0 \in B$. Thus the image of $\bar{F}_1$ lies in the fiber F, and therefore determines an element $\gamma \in \pi_{k-1}(F)$. The homotopy $\bar{F}_1$ can be chosen to be a null homotopy if and only if $\bar{F}_1 : S^{k-1} \to F$ is null homotopic. (Because combining $\bar{F}$ with a null homotopy of $\bar{F}_1$, i.e an extension of $\bar{F}_1$ to a map $D^k \to F$, is still a lifting of F, since the extension lives in a fiber over a point.) But this is only true if the homotopy class $\gamma = 0 \in \pi_{k-1}(F)$.

This proves the theorem in the case when $X^{(k)} = X^{(k-1)} \cup_\alpha D^k$. In the general case, suppose that $X^{(k)}$ is obtained from $X^{(k-1)}$ by attaching a collection of k - dimensional disks, indexed on a set, say J. That is,

$$X^{(k)} = X^{(k-1)} \bigcup_{j\in J} \cup_{\alpha_j} D^k.$$

The above procedure assigns to every $j \in J$ an "obstruction" $\gamma_j \in \pi_{k-1}(F)$. An extension $\bar{f}_k$ exists if and only if all these obstructions are zero. This assignment from the indexing set of the k - cells to the homotopy group can be extended linear to give a homomorphism γ from the free abelian group generated by the k - cells to the homotopy group $\pi_{k-1}(F)$, which is zero if and only if the extension $\bar{f}_k$ exists. Such a homomorphism γ is a cochain, $\gamma \in C^k(X; \pi_{k-1}(F))$. This completes the proof of the theorem. □

We now discuss several applications of this obstruction theory.

Corollary 4.12. *Any fibration $p : E \to B$ over a CW - complex with an aspherical fiber F admits a cross section.*

Proof. Since $\pi_q(F) = 0$ for all q, by the theorem, there are no obstructions to constructing a cross section inductively on the skeleta of B. □

Proposition 4.13. *Let X be an n - dimensional CW - complex, and let ζ be an m - dimensional vector bundle over X, with $m \geq n$. Then ζ has $m - n$ linearly independent cross sections. If ξ is a d - dimensional complex bundle over X, then ξ admits $d - [n/2]$ linearly independent cross sections, where $[n/2]$ is the integral part of $n/2$.*

.

Proof. Let ζ be classified by a map $f_m : X \to BO(m)$. To prove the theorem we need to prove that f_m lifts (up to homotopy) to a map $f_n X \to BO(n)$. We would then have that

$$\zeta \cong f_m^*(\gamma_m) \cong f_n^*(\gamma_n) \oplus \epsilon_{m-n}$$

where γ_k is the universal k - dimensional vector bundle over $BO(k)$, and ϵ_j represents the j - dimensional trivial bundle. These isomorphisms would then produce the $m-n$ linearly independent cross sections of ζ. over X. Now recall there is a fibration

$$O(m)/O(n) \to BO(n) \to BO(m).$$

That is, the fiber of $p : BO(n) \to BO(m)$ is the quotient space $O(m)/O(n)$. Now by a simple induction argument using 4.10 shows that the fiber $O(m)/O(n)$ is $n-1$ connected. That is, $pi_q(O(m)/O(n)) = 0$ for $q \leq n-1$. By 4.11 all obstructions vanish for lifting the n - skeleton of X

to the total space $BO(n)$. Since we are assuming X is n - dimensional, this completes the proof. The complex case is proved similarly. $\square$

Corollary 4.14. *Let X be a compact, n - dimensional CW complex. Then every element of the reduced real K - theory, $\tilde{K}O(X)$ can be represented by a n - dimensional vector bundle. Every element of the complex K - theory, $\tilde{K}(X)$ can be represented by an $[n/2]$ - dimensional complex vector bundle.*

Proof. By 2.32 we know

$$\tilde{K}O(X) \cong [X, BO] \quad \text{and}$$
$$\tilde{K}(X) \cong [X, BU].$$

But by the above proposition, any element $\alpha \in [X, BO]$ lifts to an element $\alpha_n \in [X, BO(n)]$ which in turn classifies an n - dimensional real vector bundle representing the $\tilde{K}O$ - class α.

Similarly, any element $\beta \in [X, BU]$ lifts to an element $\alpha_n \in [X, BU([n/2])]$ which in turn classifies an $[n/2]$ - dimensional complex vector bundle representing the $\tilde{K}$ - class β. $\square$

We now use this obstruction theory to prove the well known "Whitehead Theorem", one of the most important foundational theorems in homotopy theory.

Theorem 4.15. *Suppose X and Y are CW - complexes and $f : X \to Y$ a continuous map that induces an isomorphism in homotopy groups,*

$$f_* : \pi_k(X) \xrightarrow{\cong} \pi_k(Y) \quad \text{for all } k \geq 0$$

Then $f : X \to Y$ is a homotopy equivalence.

Proof. By 4.7 we can replace $f : X \to Y$ by a homotopy equivalent fibration

$$\tilde{f} : \tilde{X} \to Y.$$

That is, there is a homotopy equivalence $h : X \to \tilde{X}$ so that $\tilde{f} \circ h = f$. Since f induces an isomorphism in homotopy groups, so does $\tilde{f}$. By the exact sequence in homotopy groups for this fibration, this means that the fiber of the fibration $\tilde{f} : \tilde{X} \to Y$, i.e the homotopy fiber of f, is aspherical. thus by 4.11 there are no obstructions to finding a lifting $\tilde{g} : Y \to \tilde{X}$ of the identity map of Y. Thus $\tilde{g}$ is a section of the fibration, so that $\tilde{f} \circ \tilde{g} = id : Y \to Y$. Now let $h^{-1} : \tilde{X} \to X$ denote a homotopy inverse to the homotopy equivalence h. Then if we define

$$g = h^{-1} \circ \tilde{g} : Y \to X$$

we then have $f \circ g : Y \to Y$ is given by

$$\begin{aligned} f \circ g &= f \circ h^{-1} \circ \tilde{g} \\ &= \tilde{f} \circ h \circ h^{-1} \circ \tilde{g} \\ &\sim \tilde{f} \circ \tilde{g} \\ &= id : Y \to Y. \end{aligned}$$

Thus $f \circ g$ is homotopic to the identity of Y. To show that $g \circ f$ is homotopic to the identity of X, we need to construct a homotopy $X \times I \to X$ that lifts a homotopy $X \times I \to Y$ from $f \circ g \circ f$ to f. This homotopy is constructed inductively on the skeleta of X, and like in the argument proving 4.11, one finds that there are no obstructions in doing so because the homotopy fiber of f is aspherical. We leave the details of this obstruction theory argument to the reader. Thus f and g are homotopy inverse to each other, which proves the theorem. $\square$

The following is an immediate corollary.

Corollary 4.16. *An aspherical CW - complex is contractible.*

Proof. If X is an aspherical CW - complex, then the constant map to a point, $\epsilon : X \to pt$ induces an isomorphism on homotopy groups, and is therefore, by the above theorem, a homotopy equivalence. $\square$

The Whitehead theorem will now allow us to prove the following important relationship between the homotopy type of a topological group and its classifying space.

Theorem 4.17. *Let G be a topological group with the homotopy type of a CW complex., and BG its classiftying space. Then there is a homotopy equivalence between G and the loop space,*

$$G \simeq \Omega BG.$$

Proof. It was shown in chapter 2 that there is a model for a universal G - bundle, $p : EG \to BG$ with EG a G - equivariant CW - complex. In particular, EG is aspherical, and hence by the Whitehead theorem, it is contractible. Let

$$H : EG \times I \to EG$$

be a contraction. That is, $H_0 : EG \times \{0\} \to EG$ is the constant map at the basepoint $e_0 \in EG$, , and $H_1 : EG \times \{1\} \to EG$ is the identity. Composing with the projection map,

$$\Phi = p \circ H : EG \times I \to BG$$

is a homotopy between the constant map to the basepoint $\Phi_0 : EG \times \{0\} \to b_0 \in BG$ and the projection map $\Phi_1 = p : EG \times \{1\} \to BG$. Consider the adjoint of Φ,

$$\bar{\Phi} : EG \to P(BG) = \{\alpha : I \to BG \text{ such that } \alpha(0) = b_0.\}$$

defined by $\bar{\Phi}(e)(t) = \Phi(e,t) \in BG$. Then by definition, the following diagram commutes:

$$\begin{array}{ccc} EG & \xrightarrow{\bar{\Phi}} & (BG) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ BG & = & BG \end{array}$$

where $q(\alpha) = \alpha(1)$, for $\alpha \in P(BG)$. Thus Φ is a map of fibrations that induces a map on fibers

$$\phi : G \to \Omega BG.$$

Comparing the exact sequences in homotopy groups of these two fibrations, we see that ϕ induces an isomorphism in homotopy groups. A result of Milnor [**29**] that we will not prove says that if X is a CW complex, then the loop space ΩX has the homotopy type of a CW - complex. Then the Whitehead theorem implies that $\phi : G \to \Omega BG$ is a homotopy equivalence. □

4. Eilenberg - MacLane Spaces

In this section we prove a classification theorem for cohomology. Recall that in chapter 2 we proved that there are spaces BG that classify principal G - bundles over a space X, in the sense that homotopy classes of basepoint preserving maps, $[X, BG]$ are in bijective correspondence with isomorphism classes of principal G - bundles. Similarly $BO(n)$ and $BU(n)$ classify real and complex n - dimensional vector vector bundles in this same sense, and BO and BU classify K -theory. In this section we show that there are classifying spaces $K(G,n)$ that classify n - dimensional cohomology with coefficients in G in this same sense. These are Eilenberg - MacLane spaces. We have discussed these spaces earlier in these notes, but in this section we prove their existence and their classification properties.

4.1. Obstruction theory and the existence of Eilenberg - MacLane spaces. In chapter 2 we proved that for any topological group G there is a space BG classifying G bundles. For G discrete, we saw that $BG = K(G,1)$, an Eilenberg - MacLane space whose fundamental group is G, and whose higher homotopy groups are all zero. In this section we generalize this existence theorem as follows.

THEOREM 4.18. *Let G be any abelian group and n an integer with $n \geq 2$. Then there exists a space $K(G, n)$ with*

$$\pi_k(K(G,n)) = \begin{cases} G, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

This theorem will basically be proven using obstruction theory. For this we will assume the following famous theorem of Hurewicz, which we will prove later in this chapter. We first recall the Hurewicz homomorphism from homotopy to homology.

Let $f : (D^n, S^{n-1}) \to (X, A)$ represent an element $[f] \in \pi_n(X, A)$. Let $\sigma_n \in H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ be a preferred, fixed generator. Define $h([f]) = f_*(\sigma_n) \in H_n(X, A)$. The following is straightforward, and we leave its verification to the reader.

LEMMA 4.19. *The above construction gives a well defined homomorphism*

$$h_* : \pi_n(X, A) \to H_n(X, A)$$

called the "Hurewicz homomorphism".

The following is the "Hurewicz theorem".

THEOREM 4.20. *Let X be simply connected, and let $A \subset X$ be a simply connected subspace. Suppose that the pair (X, A) is $(n-1)$ - connected, for $n > 2$. That is,*

$$\pi_k(X, A) = 0 \quad \text{if} \quad k \leq n - 1.$$

Then the Hurewicz homomorphism $h_* : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.

We now prove the following basic building block type result concerning how the homotopy groups change as we build a CW - complex cell by cell.

THEOREM 4.21. *Let X be a simply connected, CW - complex and let*

$$f : S^k \to X$$

be a map. Let X' be the mapping cone of f. That is,

$$X' = X \cup_f D^{n+1}$$

which denotes the union of X with a disk D^{n+1} glued along the boundary sphere $S^k = \partial D^{k+1}$ via f. That is we identify $t \in S^k$ with $f(t) \in X$. Let

$$\iota : X \hookrightarrow X'$$

be the inclusion. Then

$$\iota_* : \pi_k(X) \to \pi_k(X')$$

is surjective, with kernel equal to the cyclic subgroup generated by $[f] \in \pi_k(X)$.

PROOF. Let $g : S^q \to X'$ represent an element in $\pi_q(X')$ with $q \leq k$. By the cellular approximation theorem, g is homotopic to a cellular map, and therefore one whose image lies in the q - skeleton of X'. But for $q \leq k$, the q - skeleton of X' is the q - skeleton of X. This implies that

$$\iota_* : \pi_q(X) \to \pi_q(X')$$

is surjective for $q \leq k$. Now assume $q \leq k-1$, then if $g : S^q \to X \subset X'$ is null homotopic, any null homotopy, i.e extension to the disk $G : D^{q+1} \to X'$ can be assumed to be cellular, and hence has image in X. This implies that for $q \leq k-1$, $\iota_* : \pi_q(X) \to \pi_q(X')$ is an isomorphism. By the exact sequence in homotopy groups of the pair (X', X), this implies that the pair (X', X) is k - connected. By the Hurewicz theorem that says that

$$\pi_{k+1}(X', X) \cong H_{k+1}(X', X) = H_{k+1}(X \cup_f D^{k+1}, X)$$

which, by analyzing the cellular chain complex for computing $H_*(X')$ is $\mathbb{Z}$ if and only if $f : S^k \to X$ is zero in homology, and zero otherwise. In particular, the generator $\gamma \in \pi_{k+1}(X', X)$ is represented by the map of pairs given by the inclusion

$$\gamma : (D^{k+1}, S^k) \hookrightarrow (X \cup_f D^{k+1}, X)$$

and hence in the long exact sequence in homotopy groups of the pair (X', X),

$$\cdots \to \pi_{k+1}(X', X) \xrightarrow{\partial_*} \pi_k(X) \xrightarrow{\iota_*} \pi_k(X') \to \cdots$$

we have $\partial_*(\gamma) = [f] \in \pi_k(X)$. Thus $\iota_* : \pi_k(X) \to \pi_k(X')$ is surjective with kernel generated by $[f]$. This proves the theorem. $\square$

We will now use this basic homotopy theory result to establish the existence of Eilenberg - MacLane spaces.

PROOF. of 4.18 Fix the group G and the integer $n \geq 2$. Let $\{\gamma_\alpha : \alpha \in \mathcal{A}\}$ be a set of generators of G, where $\mathcal{A}$ denotes the indexing set for these generators. Let $\{\theta_\beta : \beta \in \mathcal{B}\}$ be a corresponding set of relations. In other words G is isomorphic to the free abelian group $F_{\mathcal{A}}$ generated by $\mathcal{A}$, modulo the subgroup $R_{\mathcal{B}}$ generated by $\{\theta_\beta : \beta \in \mathcal{B}\}$.

Consider the wedge of spheres $\bigvee_{\mathcal{A}} S^n$ indexed on the set $\mathcal{A}$. Then by the Hurewicz theorem,

$$\pi_n(\bigvee_{\mathcal{A}} S^n) \cong H_n(\bigvee_{\mathcal{A}} S^n) \cong F_{\mathcal{A}}.$$

Now the group $R_{\mathcal{B}}$ is a subgroup of a free abelian group, and hence is itself free abelian. Let $\bigvee_{\mathcal{B}} S^n$ be a wedge of spheres whose n^{th} - homotopy group (which by the Hurewicz theorem is isomorphic to its homology, which is free abelian) is $R_{\mathcal{B}}$. Moreover there is a natural map

$$j : \bigvee_{\mathcal{B}} S^n \to \bigvee_{\mathcal{A}} S^n$$

which, on the level of the homotopy group π_n is the inclusion $R_{\mathcal{B}} \subset F_{\mathcal{A}}$. Let X_{n+1} be the mapping cone of j:

$$X_{n+1} = \bigvee_{\mathcal{A}} S^n \cup_j \bigcup_{\mathcal{B}} D^{n+1}$$

where the disk D^{n+1} corresponding to a generator in $R_{\mathcal{B}}$ is attached via the map $S^n \to \bigvee_{\mathcal{A}} S^n$ giving the corresponding element in $\pi_n(\bigvee_{\mathcal{A}} S^n) = F_{\mathcal{A}}$. Then by using 4.21 one cell at a time, we see that X_{n+1} is an $n-1$ - connected space and $\pi_n(X_n)$ is generated by $F_{\mathcal{A}}$ modulo the subgroup $R_{\mathcal{B}}$. In other words,

$$\pi_n(X_{n+1}) \cong G.$$

Now inductively assume we have constructed an space X_{n+k} with

$$\pi_q(X_{n+k}) = \begin{cases} 0 & \text{if } q < n, \\ G & \text{if } q = n \quad \text{and} \\ 0 & \text{if } n < q \leq n+k-1 \end{cases}$$

Notice that we have begun the inductive argument with $k = 1$, by the construction of the space X_{n+1} above. So again, assume we have constructed X_{n+k}, and we need to show how to construct X_{n+k+1} with these properties. Once we have done this, by induction we let $k \to \infty$, and clearly X_∞ will be a model for $K(G, n)$.

Now suppose $\pi = \pi_{n+k}(X_{n+k})$ is has a generating set $\{\gamma_u : u \in \mathcal{C}\}$, where $\mathcal{C}$ is the indexing set. Let $F_{\mathcal{C}}$ be the free abelian group generated by the elements in this generating set. Let $\bigvee_{u \in \mathcal{C}} S^{n+k}_u$ denote a wedge of spheres indexed by this indexing set. Then, like above, by applying the Hurewicz theorem we see that

$$\pi_{n+k}(\bigvee_{u \in \mathcal{C}} S^{n+k}_u) \cong H_{n+k}(\bigvee_{\mathcal{C}} S^{n+k}) \cong F_{\mathcal{C}}.$$

Let

$$f : \bigvee_{\mathcal{C}} S^{n+k} \to X_{n+k}$$

be a map which, when restricted to the sphere S^{n+k}_u represents the generator $\gamma_u \in \pi = \pi_{n+k}(X_{n+k})$. We define X_{n+k+1} to be the mapping cone of f:

$$X_{n+k+1} = X_{n+k} \cup_f \bigcup_{u \in \mathcal{C}} D^{n+k+1}.$$

Then by 4.21 we have that $\pi_q(X_{n+k}) \to \pi_q(X_{n+k+1})$ is an isomorphism for $q < n+k$, and

$$\pi_{n+k}(X_{n+k}) \to \pi_{n+k}(X_{n+k+1})$$

is surjective, with kernel the subgroup generated by $\{\gamma_u : u \in \mathcal{C}\}$. But since this subgroup generates $\pi = \pi_{n+k}(X_{n+k})$ we see that this homomorphism is zero. Since it is surjective, that implies $\pi_{n+k}(X_{n+k+1}) = 0$. Hence X_{n+k+1} has the required properties on its homotopy groups, and so we have completed our inductive argument. $\square$

4.2. The Hopf - Whitney theorem and the classification theorem for Eilenberg - MacLane spaces. We now know that the Eilenberg - MacLane spaces $K(G,n)$ exist for every n and every abelian group G, and when $n = 1$ for every group G. Furthermore, by their construction in the proof of 4.18 they can be chosen to be CW - complexes. In this section we prove their main property, i.e they classify cohomology.

In order to state the classification theorem properly, we need to recall the universal coefficient theorem, which says the following.

THEOREM 4.22. *(Universal Coefficient Theorem) Let G be an abelian group. Then there is a split short exact sequence*

$$0 \to Ext(H_{n-1}(X);G) \to H^n(X;G) \to Hom(H^n(X),G) \to 0.$$

COROLLARY 4.23. *If Y is $(n-1)$ - connected, and $\pi = \pi_n(Y)$, then*

$$H^n(Y;\pi) \cong Hom(\pi,\pi).$$

PROOF. Since Y is $(n-1)$ connected, $H_{n-1}(Y) = 0$, so the universal coefficient theorem says that $H^n(Y;\pi) \cong Hom(H_n(Y),\pi)$. But the Hurewicz theorem says that the Hurewicz homomorphism $h_* : \pi = \pi_n(Y) \to H_n(Y)$ is an isomorphism. The corollary follows by combining these two isomorphisms. □

For an $(n-1)$ - connected space Y as above, let $\iota \in H^n(Y;\pi)$ be the class corresponding to the identity map $id \in Hom(\pi,\pi)$ under the isomorphism in this corollary. This is called the fundamental class. Given any other space X, we therefore have a set map

$$\phi : [X,Y] \to H^n(X,\pi)$$

defined by $\phi([f]) = f^*(\iota) \in H^n(X;\pi)$. The classification theorem for Eilenberg - MacLane spaces is the following.

THEOREM 4.24. *For $n \geq 2$ and $G\pi$ any abelian group, let $K(\pi,n)$ denote an Eilenberg - MacLane space with $\pi_n(K(\pi,n)) = \pi$, and all other homotopy groups zero. Let $\iota \in H^n(K(\pi,n);\pi)$ be the fundamental class. Then for any CW - complex X, the map*

$$\phi : [X, K(\pi,n)] \to H^n(X;\pi)$$
$$[f] \to f^*(\iota)$$

is a bijective correspondence.

We have the following immediate corollary, giving a uniqueness theorem regarding Eilenberg - MacLane spaces.

Corollary 4.25. *Let $K(\pi,n)_1$ and $K(\pi,n)_2$ be CW - complexes that are both Eilenberg - MacLane spaces with the same homotopy groups. Then there is a natural homotopy equivalence between $K(\pi,n)_1$ and $K(\pi,n)_2$.*

Proof. Let $f : K(\pi,n)_1 \to K(\pi,n)_2$ be a map whose homotopy class is the inverse image of the fundamental class under the bijection

$$\phi : [K(\pi,n)_1, K(\pi,n)_2] \xrightarrow{\cong} H^n(K(\pi,n)_1;\pi) \cong Hom(\pi,\pi).$$

This means that $f : K(\pi,n)_1 \to K(\pi,n)2$ induces the identity map in $Hom(\pi,\pi)$, and in particular induces an isomorphism on π_n. Since all other homotopy groups are zero in both of these complexes, f induces an isomorphism in homotopy groups in all dimensions. Therefore by the Whitehead theorem 4.15, f is a homotopy equivalence. □

We begin our proof of this classification theorem by proving a special case, known as the Hopf - Whitney theorem. This predates knowledge of the existence of Eilenberg - MacLane spaces.

Theorem 4.26. *(Hopf-Whitney theorem) Let Y be any $(n-1)$ - connected space with $\pi = \pi_n(Y)$. Let X be any n - dimensional CW complex. Then the map*

$$\phi : [X,Y] \to H^n(X;\pi)$$
$$[f] \to f^*(\iota)$$

is a bijective correspondence.

Remark. This theorem is most often used in the context of manifolds, where it implies that if M^n is any closed, orientable manifold the correspondence

$$[M^n, S^n] \to H^n(M^n;\mathbb{Z}) \cong \mathbb{Z}$$

is a bijection.

Exercise. Show that this correspondence can alternatively be described as assigning to a smooth map $f : M^n \to S^n$ its degree, $deg(f) \in \mathbb{Z}$.

Proof. (Hopf - Whitney theorem) We first set some notation. Let Y be $(n-1)$ - connected, and have basepoint $y_0 \in Y$. Let $X^{(m)}$ denote the m - skeleton of the n - dimensional complex X. Let $C_k(X) = H_k(X^{(k)}, X^{(k-1)})$ be the cellular k - chains in X. Alternatively, $C_k(X)$ can be thought of as the free abelian group on the k - dimensional cells in the CW - decomposition of X. Let $Z^k(X)$

and $B^k(X)$ denote the subgroups of cocycles and coboundaries respectively. Let J_k be the indexing set for the set of k - cells in this CW - structure. So that there are attaching maps

$$\alpha_k : \bigvee_{j \in J_k} S^k_j \to X^{(k)}$$

so that the $(k+1)$ - skeleton $X^{(k+1)}$ is the mapping cone

$$X^{(k+1)} = X^{(k)} \cup_{\alpha_k} \bigcup_{j \in J_k} D^{k+1}_j.$$

We prove this theorem in several steps, each translating between cellular cochain complexes or cohomology on the one hand, and homotopy classes of maps on the other hand. The following is the first step.

Step 1. There is a bijective correspondence between the following set of homotopy classes of maps of pairs, and the cochain complex with values in π:

$$\phi : [(X^{(n)}, X^{(n-1)}), (Y, y_0)] \to C^n(X; \pi).$$

Proof. A map of pairs $f : (X^{(n)}, X^{(n-1)}) \to (Y, y_0)$ is the same thing as a basepoint preserving map from the quotient,

$$f : X^{(n)}/X^{(n-1)} = \bigvee_{j \in J_n} S^n_j \to Y.$$

So the homotopy class of f defines and is defined by an assignment to every $j \in J_n$, an element $[f_j] \in \pi_n(Y) = \pi$. But by extending linearly, this is the same as a homomorphism from the free abelian group generated by J_n, i.e the chain group $C_n(X)$, to π. That is, this is the same thing as a cochain $[f] \in C^n(X; \pi)$. □

Step 2. The map $\phi : [X, Y] \to H^n(X; \pi)$ is surjective.

Proof. . Notice that since X is an n - dimensional CW - complex, all n - dimensional cochains are cocycles, $C^n(X; \pi) = Z^n(X; \pi)$. So in particular there is a surjective homomorphism $\mu : C^n(X; \pi) = Z^n(X; \pi) \to Z^n(X; \pi)/B^n(X; \pi) = H^n(X; \pi)$. A check of the definitions of the maps defined so far yields that the following diagram commutes:

$$\begin{tikzcd}
{[(X^{(n)}, X^{(n-1)}), (Y, y_0)]} \arrow[r, "\phi", "\cong"'] \arrow[d, "\rho"'] & C^n(X; \pi) \arrow[d, "\mu"] \\
{[X, Y]} \arrow[r, "\phi"'] & H^n(X; \pi)
\end{tikzcd}$$

where ρ is the obvious restriction map. By the commutativity of this diagram, since μ is surjective and $\phi : [(X^{(n)}, X^{(n-1)}), (Y, y_0)] \to C^n(X; \pi)$ is bijective, then we must have that $\phi : [X, Y] \to H^n(X; \pi)$ is surjective, as claimed. □

In order to show that ϕ is injective, we will need to examine the coboundary map

$$\delta : C^{n-1}(X;\pi) \to C^n(X;\pi)$$

from a homotopy point of view. To do this, recall that the boundary map on the chain level, $\partial : C_k(X) \to C_{k-1}(X)$ is given by the connecting homomorphism $H_n(X^{(k)}, X^{(k-1)}) \to H_{k-1}(X^{(k-1)}, X^{(k-2)})$ from the long exact sequence in homology of the triple, $(X^{(k)}, X^{(k-1)}, X^{(k-2)})$. This boundary map can be realized homotopically as follows. Let $c(X^{(k-1)})$ be the cone on the subcomplex $X^{(k-1)}$,

$$c(X^{(k-1)}) = X^{(k-1)} \times I/(X^{(k-1)} \times \{1\} \cup \{x_0\} \times I),$$

which is obviously a contractible space. Consider the mapping cone of the inclusion $X^{(k-1)} \hookrightarrow X^{(k)}$, $X^{(k)} \cup c(X^{(k-1)}$. By projecting the cone to a point, there is a projection map

$$p_k : X^{(k)} \cup c(X^{(k-1)}) \to X^{(k)}/X^{(k-1)} = \bigvee_{j \in J_k} S^k_j$$

which is a homotopy equivalence. (**Note.** The fact that this map induces an isomorphism in homology is straight forward by computing the homology exact sequence of the pair $(X^{(k)} \cup c(X^{(k-1)}), X^{(k)})$. The fact that this map is a homotopy equivalence is a basic point set topological property of CW - complexes coming from the so - called "Homotopy Extension Property". However it can be proved directly, by hand, in this case. We leave its verification to the reader.) Let

$$u_k : X^{(k)} \to \bigvee_{j \in J_k} S^k_j$$

be the composition

$$X^{(k)} \hookrightarrow X^{(k)} \cup c(X^{(k-1)}) \xrightarrow{p_k} X^{(k)}/X^{(k-1)} = \bigvee_{j \in J_k} S^k_j.$$

Then the composition of u_k with the attaching map

$$\alpha_{k+1} : \bigvee_{j \in J_{k+1}} S^k_j \to X^{(k)}$$

(whose mapping cone defines the $(k+1)$ - skeleton $X^{(k+1)}$), is a map between wedges of k - spheres,

$$d_{k+1} : \bigvee_{j \in J_{k+1}} S^k_j \xrightarrow{\alpha_{k+1}} X^{(k)} \xrightarrow{u_k} \bigvee_{j \in J_k} S^k_j.$$

The following is immediate from the definitions.

Step 3. The induced map in homology,

$$(d_{k+1})_* : H_k(\bigvee_{j \in J_{k+1}} S^k_j) \to H_k(\bigvee_{j \in J_k} S^k_j)$$
$$C_{k+1}(X) \to C_k(X)$$

is the boundary homomorphism in the chain complex $\partial_{k+1} : C_{k+1}(X) \to C_k(X)$.

Now consider the map

$$[(X^{(n)}, X^{(n-1)}), (Y, y_0)] \xrightarrow[\cong]{\phi} C^n(X;\pi) = Z^n(X;\pi) \xrightarrow{\mu} H^n(X;\pi).$$

We then have the following corollary.

Step 4. A map $f : X^{(n)}/X^{(n-1)} = \bigvee_{j \in J_n} S^n_j \to Y$ has the property that

$$\mu \circ \phi([f]) = 0 \in H^n(X; \pi)$$

if and only if there is a map

$$f_{n-1} : \bigvee_{j \in J_{(n-1)}} S^n_j \to Y$$

so that f is homotopic to the composition

$$\bigvee_{j \in J_n} S^n_j \xrightarrow{d_n} \bigvee_{j \in J_{n-1}} S^n_j \xrightarrow{f_{n-1}} Y.$$

Proof. Since $\phi : [(X^{(n)}, X^{(n-1)}), (Y, y_0)] \to C^n(X; \pi) = Z^n(X; \pi)$ is a bijection, $\mu \circ \phi([f]) = 0$ if and only if $\phi([f])$ is in the image of the coboundary map. The result then follows from step 3. $\square$

Step 5. The composition

$$X^{(n)} \xrightarrow{u_n} \bigvee_{j \in J_n} S^n_j \xrightarrow{d_n} \bigvee_{j \in J_{n-1}} S^n_j$$

is null homotopic.

Proof. The map u_n was defined by the composition

$$X^{(n)} \hookrightarrow X^{(n)} \cup c(X^{(n-1)}) \xrightarrow[\simeq]{p_n} \bigvee_{j \in J_n} S^n_j.$$

But notice that if we take the quotient $X^{(n)} \cup c(X^{(n-1)})/X^{(n)}$ we get the suspension

$$X^{(n)} \cup c(X^{(n-1)})/X^{(n)} = \Sigma X^{(n-1)}.$$

Furthermore, the map between the wedges of the spheres, $d_n : \bigvee_{j \in J_n} S^n_j \to \bigvee_{j \in J_{n-1}} S^n_j$ is directly seen to be the composition

$$d_n : \bigvee_{j \in J_n} S^n_j \simeq X^{(n)} \cup c(X^{(n-1)}) \xrightarrow{proj.} X^{(n)} \cup c(X^{(n-1)})/X^{(n)} = \Sigma X^{(n-1)} \xrightarrow{\Sigma u_{n-1}} \bigvee_{j \in J_{n-1}} S^n_j.$$

Thus the composition $d_n \circ u_n : X^{(n)} \to \bigvee_{j \in J_n} S^n_j \to \bigvee_{j \in J_{n-1}} S^n_j$ factors as the composition

$$X^{(n)} \hookrightarrow X^{(n)} \cup c(X^{(n-1)}) \xrightarrow{proj.} X^{(n)} \cup c(X^{(n-1)})/X^{(n)} = \Sigma X^{(n-1)} \xrightarrow{\Sigma u_{n-1}} \bigvee_{j \in J_{n-1}} S^n_j.$$

But the composite of the first two terms in this composition,

$$X^{(n)} \hookrightarrow X^{(n)} \cup c(X^{(n-1)}) \xrightarrow{proj.} X^{(n)} \cup c(X^{(n-1)})/X^{(n)}$$

is clearly null homotopic, and hence so is $d_n \circ u_n$. $\square$

We now complete the proof of the theorem by doing the following step.

Step. 6. The correspondence $\phi : [X, Y] \to H^n(X; \pi)$ is injective.

PROOF. Let $f,\, g : X \to Y$ be maps with $\phi([f]) = \phi([g]) \in H^n(X;\pi)$. Since Y - is $(n-1)$ - connected, given any map $h : X \to Y$, the restriction to its $(n-1)$ - skeleton is null homotopic. (**Exercise.** Check this!) Null homotopies define maps

$$\tilde{f},\, \tilde{g} : X \cup c(X^{(n-1)}) \to Y$$

given by f and g respectively on X, and by their respective null homotopies on the cones, $c(X^{(n-1)})$. Using the homotopy equivalence $p_n : X^{(n)} \cup c(X^{(n-1)}) \simeq X^{(n)}/X^{(n-1)} = \bigvee_{j \in J_n} S^n_j$, we then have maps

$$\bar{f},\, \bar{g} : X^{(n)}/X^{(n-1)} \to Y$$

which, when composed with the projection $X = X^{(n)} \to X^{(n)}/X^{(n-1)}$ are homotopic to f and g respectively. Now by the commutativity of the diagram in step 2, since $\phi([f]) = \phi([g])$, then $\mu \circ \phi([\bar{f}]) = \mu \circ \phi([\bar{g}])$. Or equivalently,

$$\mu \circ \phi([\bar{f}] - [\bar{g}]) = 0$$

where we are using the fact that

$$[(X^{(n)}, X^{(n-1)}), Y] = [\bigvee_{j \in J_n} S^n_j, Y] = \oplus_{j \in J_n} \pi_n(Y)$$

is a group, and maps to $C^n(X;\pi)$ is a group isomorphism.

Let $\psi : X^{(n)}/X^{(n-1)} \to Y$ represent $[\bar{f}] - [\bar{g}] \in [\bigvee_{j \in J_n} S^n_j, Y]$. Then $\mu \circ \phi(\psi) = 0$. Then by step 4, there is a map $\psi_{n-1} : \bigvee_{j \in J_{n-1}} S^n_j \to Y$ so that $\psi_{n-1} \circ d_n$ is homotopic to ψ. Thus the composition

$$X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\psi} Y$$

is homotopic to the composition

$$X \to X^{(n)}/X^{n-1} = \bigvee_{j \in J_n} S^n_j \xrightarrow{d_n} \bigvee_{j \in J_{n-1}} S^n_j \xrightarrow{\psi_{n-1}} Y.$$

But by step 5, this compostion is null homotopic. Now since ψ represents $[\bar{f}] - [\bar{g}]$, a null homotopy of the composition

$$X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\psi} Y$$

defines a homotopy between the compositions

$$X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\bar{f}} Y \quad \text{and} \quad X \xrightarrow{proj.} X/X^{(n-1)} \xrightarrow{\bar{g}} Y.$$

The first of these maps is homotopic to $f : X \to Y$, and the second is homotopic to $g : X \to Y$. Hence $f \simeq g$, which proves that ϕ is injective. □

We now know that the correspondence $\phi : [X, Y] \to H^n(X;\pi)$ is surjective (step 2) and injective (step 6). This completes the proof of this theorem. □

We now proceed with the proof of the main classification theorem for cohomology, using Eilenberg - MacLane spaces (4.24).

Proof. The Hopf Whitney theorem proves this theorem when X is an n - dimensional CW - complex. We split the proof for general CW - complexes into two cases.

Case 1. X is $n+1$ - dimensional.

Consider the following commutative diagram

$$\begin{array}{ccc} [X, K(\pi, n)] & \xrightarrow{\phi} & H^n(X;\pi) \\ \rho\downarrow & & \downarrow\rho \\ [X^{(n)}, K(\pi, n)] & \xrightarrow[\cong]{\phi_n} & H^n(X^{(n)};\pi) \end{array} \tag{4.1}$$

where the vertical maps ρ denote the obvious restriction maps, and ϕ_n denotes the restriction of the correspondence ϕ to the n - skeleton, which is an isomorphism by the Hopf - Whitney theorem.

Now by considering the exact sequence for cohomology of the pair $(X, X^{(n)}) = (X^{(n+1)}, X^{(n)})$, one sees that the restriction map $\rho : H^n(X, \pi) \to H^n(X^{(n)}, \pi)$ is injective. Using this together with the fact that ϕ_n is an isomorphism and the commutativity of this diagram, one sees that to show that $\phi : [X, K(\pi, n)] \to H^n(X;\pi)$ is surjective, it suffices to show that for $\gamma \in H^n(X, \pi)$ with $\rho(\gamma) = \phi_n([f_n])$, where $f_n : X^{(n)} \to K(\pi, n)$, then f_n can be extended to a map $f : X \to K(\pi, n)$.

Using the same notation as was used in the proof of the Hopf - Whitney theorem, since $X = X^{(n+1)}$, we can write

$$X = X^{(n)} \cup_{\alpha_{n+1}} \bigcup_{j \in J_{n+1}} D^{(n+1)}$$

where $\alpha_{n+1} : \bigvee_{j\in J_{n+1}} S^n_j \to X^{(n)}$ is the attaching map. Thus the obstruction to finding an extension $f : X \to K(\pi, n)$ of the map $f_n : X^{(n)} \to K(\pi, n)$, is the compostion

$$\bigvee_{j\in J_{n+1}} S^n_j \xrightarrow{\alpha_{n+1}} X^{(n)} \xrightarrow{f_n} K(\pi, n).$$

Now since $\bigvee_{j\in J_{n+1}} S^n_j$ is n - dimensional, the Hopf - Whitney theorem says that this map is determined by its image under ϕ,

$$\phi([f_n \circ \alpha_{n+1}]) \in H^n(\bigvee_{j\in J_{n+1}} S^n_j;\pi).$$

But this class is $\alpha^*_{n+1}(\phi([f_n]))$, which by assumption is $\alpha^*_{n+1}(\rho(\gamma))$. But the composition

$$H^n(X;\pi) \xrightarrow{\rho} H^n(X^{(n)}, \pi) \xrightarrow{\alpha^*_{n+1}} H^n(\bigvee_{j\in J_{n+1}} S^n_j;\pi)$$

are two successive terms in the long exact sequence in cohomology of the pair $(X^{(n+1)}, X^{(n)})$ and is therefore zero. Thus the obstruction to finding the extension $f : X \to K(\pi, n)$ is zero. As observed above this proves that $\phi : [X, K(\pi, n)] \to H^n(X;\pi)$ is surjective.

We now show that ϕ is injective. So suppose $\phi([f]) = \phi([g])$ for f, $g : X \to K(\pi, n)$. To prove that ϕ is injective we need to show that this implies that f is homotopic to g. Let f_n and g_n be the restrictions of f and g to $X^{(n)}$. That is,

$$f_n = \rho([f]) : X^{(n)} \to K(\pi, n) \quad \text{and} \quad g_n = \rho([g]) : X^{(n)} \to K(\pi, n)$$

Now by the commutativity of diagram 4.1 and the fact that ϕ_n is an isomorphism, we have that f_n and g_n are homotopic maps. Let

$$F_n : X^{(n)} \times I \to K(\pi, n)$$

be a homotopy between them. That is, $F_0 = f_n : X^{(n)} \times \{0\} \to K(\pi, n)$ and $F_1 = g_n : X^{(n)} \times \{1\} \to K(\pi, n)$. This homotopy defines a map on the $(n+1)$ -subcomplex of $X \times I$ defined to be

$$\tilde{F} : (X \times \{0\}) \cup (X \times \{1\}) \cup X^{(n)} \times I \to K(\pi, n)$$

where $\tilde{F}$ is defined to be f and g on $X \times \{0\}$ and $X \times \{1\}$ respectively, and F on $X^{(n)} \times I$. But since X is $(n+1)$ - dimensional, $X \times I$ is $(n+2)$ - dimensional, and this subcomplex is its $(n+1)$ - skeleton. So $X \times I$ is the union of this complex with $(n+2)$ - dimensional disks, attached via maps from a wedge of $(n+1)$ - dimensional spheres. Hence the obstruction to extending $\tilde{F}$ to a map $F : X \times I \to K(\pi, n)$ is a cochain in $C^{n+2}(X \times I; \pi_{n+1}(K(\pi, n))$. But this group is zero since $\pi_{n+1}(K(\pi, n)) = 0$. Thus there is no obstruction to extending $\tilde{F}$ to a map $F : X \times I \to K(\pi, n)$, which is a homotopy between f and g. As observed before this proves that ϕ is injective. This completes the proof of the theorem in this case.

General Case. Since, by case 1, we know the theorem for $(n+1)$ - dimensional CW - complexes, we assume that the dimension of X is $\geq n+2$. Now consider the following commutative diagram:

$$\begin{array}{ccc}
[X, K(\pi, n)] & \xrightarrow{\phi} & H^n(X; \pi) \\
\rho \downarrow & & \downarrow \rho \\
[X^{(n+1)}, K(\pi, n)] & \xrightarrow[\cong]{\phi_{n+1}} & H^n(X^{(n+1)}; \pi)
\end{array}$$

where, as earlier, the maps ρ denote the obvious restriction maps, and ϕ_{n+1} denotes the restriction of ϕ to the $(n+1)$ skeleton, which we know is an isomorphism, by the result of case 1.

Now in this case the exact sequence for the cohomology of the pair $(X, X^{(n+1)})$ yields that the restriction map $\rho : H^n(X; \pi) \to H^n(X^{(n+1)}, \pi)$ is an isomorphism. Therefore by the commutativity of this diagram, to prove that $\phi : [X, K(\pi, n)] \to H^n(X; \pi)$ is an isomorphism, it suffices to show that the restriction map

$$\rho : [X, K(\pi, n)] \to [X^{(n+1)}, K(\pi, n)]$$

is a bijection. This is done by induction on the skeleta $X^{(K)}$ of X, with $K \geq n+1$. To complete the inductive step, one needs to analyze the obstructions to extending maps $X^{(K)} \to K(\pi, n)$ to $X^{(K+1)}$ or homotopies $X^{(K)} \times I \to K(\pi, n)$ to $X^{(K+1)} \times I$, like what was done in the proof of case 1. However in these cases the obstructions will always lie is spaces of cochains with coefficients in $\pi_q(K(\pi, n))$ with $q = K$ or $K+1$, and so $q \geq n+1$. But then $\pi_q(K(\pi, n)) = 0$ and so these obstructions will always vanish. We leave the details of carrying out this argument to the reader. □

5. Spectral Sequences

One of the great achievements of Algebraic Topology was the development of spectral sequences. They were originally invented by Leray in the late 1940's and since that time have become fundamental calculational tools in many areas of Geometry, Topology, and Algebra. One of the earliest and most important applications of spectral sequences was the work of Serre [**36**] for the calculation of the homology of a fibration. We divide our discussion of spectral sequences in these notes into three parts. In the first section we develop the notion of a spectral sequence of a filtration. In the next section we discuss the Leray - Serre spectral sequence for a fibration. In the final two sections we discuss applications: we prove the Hurewicz theorem, calculate the cohomology of the Lie groups $U(n)$, and $O(n)$, and of the loop spaces ΩS^n, and we discuss *Spin* and $Spin_{\mathbb{C}}$ - structures on manifolds.

5.1. The spectral sequence of a filtration. A spectral sequence is the algebraic machinery for studying sequences of long exact sequences that are interelated in a particular way. We begin by illustrating this with the example of a filtered complex.

Let C_* be a chain complex, and let $A_* \subset C_*$ be a subcomplex. The short exact sequence of chain complexes

$$0 \longrightarrow A_* \hookrightarrow C_* \longrightarrow C_*/A_* \longrightarrow 0$$

leads to a long exact sequence in homology:

$$\longrightarrow \cdots \longrightarrow H_{q+1}(C_*, A_*) \longrightarrow H_q(A_*) \longrightarrow H_q(C_*) \longrightarrow H_q(C_*, A_*) \longrightarrow H_{q-1}(A_*) \longrightarrow \cdots$$

This is useful in computing the homology of the big chain complex, $H_*(C_*)$ in terms of the homology of the subcomplex $H_*(A_*)$ and the homology of the quotient complex $H_*(C_*, A_*)$. A spectral sequence is the machinery used to study the more general situation when one has a *filtration* of a chain complex C_* by subcomplexes

$$0 = F_0(C_*) \hookrightarrow F_1(C_*) \hookrightarrow \cdots \hookrightarrow F_k(C_*) \hookrightarrow F_{k+1}(C_*) \hookrightarrow \cdots \hookrightarrow C_* = \bigcup_k F_k(C_*).$$

Let D_*^k be the subquotient complex $D_*^k = F_k(C_*)/F_{k-1}(C_*)$ and so for each k there is a long exact sequence in homology

$$\longrightarrow H_{q+1}(D_*^k) \longrightarrow H_q(F_{k-1}(C_*)) \longrightarrow H_q(F_k(C_*)) \longrightarrow H_q(D_*^k) \longrightarrow \cdots$$

By putting these long exact sequences together, in principle one should be able to use information about $\oplus_k H_*(D_*^k)$ in order to obtain information about

$$H_*(C_*) = \varinjlim_k H_*(F_k(C_*)).$$

A spectral sequence is the bookkeeping device that allows one to do this. To be more specific, consider the following diagram.

$$
\begin{tikzcd}
0 \arrow[d, "i"] & & 0 \arrow[d, "i"] & \\
H_q(F_1(C_*)) \arrow[d, "i"] & & H_{q-1}(F_1(C_*)) \arrow[r, "="] \arrow[d, "i"] & H_{q-1}(D^1_*) \\
\vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
H_q(F_{k-p}(C_*)) \arrow[r, "j"] \arrow[d, "i"] & H_q(D^{k-p}_*) \arrow[r, "\partial"] & H_{q-1}(F_{k-p-1}(C_*)) \arrow[r, "j"] \arrow[d, "i"] & H_{q-1}(D^{k-p-1}_*) \\
\vdots \arrow[d, "i"] & & H_{q-1}(F_{k-p}(C_*)) \arrow[r, "j"] \arrow[d, "i"] & H_{q-1}(D^{k-p}_*) \\
\vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
H_q(F_{k-2}(C_*)) \arrow[d, "i"] & & H_{q-1}(F_{k-3}(C_*)) \arrow[d, "i"] & \\
H_q(F_{k-1}(C_*)) \arrow[r, "j"] \arrow[d, "i"] & H_q(D^{k-1}_*) \arrow[r, "\partial"] & H_{q-1}(F_{k-2}(C_*)) \arrow[r, "j"] \arrow[d, "i"] & H_{q-1}(D^{k-2}_*)) \\
H_q(F_k(C_*)) \arrow[r, "j"] \arrow[d, "i"] & H_q(D^k_*) \arrow[r, "\partial"] & H_{q-1}(F_{k-1}(C_*)) \arrow[r, "j"] \arrow[d, "i"] & H_{q-1}(D^{k-1}_*)) \\
\vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
H_q(C_*) & & H_{q-1}(C_*) &
\end{tikzcd}
\tag{5.1}
$$

The columns represent the homology filtration of $H_*(C_*)$ and the three maps ∂, j, and i combine to give long exact seqences at every level.

Let $\alpha \in H_q(C_*)$. We say that α has *algebraic filtration* k, if α is in the image of a class $\alpha_k \in H_q(F_k(C_*))$ but is not in the image of $H_q(F_{k-1}(C_*))$. In such a case we say that the image

$j(\alpha_k) \in H_q(D^k_*)$ is a *representative* of α. Notice that this representative is not unique. In particular we can add any class in the image of

$$d_1 = j \circ \partial : H_{q+1}(D^{k+1}_*) \longrightarrow H_q(D^k_*)$$

to $j(\alpha_k)$ and we would still have a representative of $\alpha \in H_q(C_*)$ under the above definition.

Conversely, let us consider when an arbitrary class $\beta \in H_q(D^k_*)$ represents a class in $H_q(C_*)$. By the exact sequence this occurs if and only if the image $\partial(\beta) = 0$, for this is the obstruction to β being in the image of $j : H_q(F_k(C_*)) \to H_q(D^k_*)$ and if $j(\tilde{\beta}) = \beta$ then β represents the image

$$i \circ \cdots \circ i(\tilde{\beta}) \in H_q(C_*).$$

Now $\partial(\beta) = 0$ if and only if it lifts all the way up the second vertical tower in diagram 5.1 The first obstruction to this lifting, (i.e the obstruction to lifting $\partial(\beta)$ to $H_{q-1}(F_{k-2}(C_*))$ is that the composition

$$d_1 = j \circ \partial : H_q(D^k) \longrightarrow H_{q-1}D^{k-1}_*)$$

maps β to zero. That is elements of $H_q(C_*)$ are represented by elements in the subquotient

$$ker(d_1)/Im(d_1)$$

of $H_q(D^k_*)$. We use the following notation to express this. We define

$$E^{r,s}_1 = H_{r+s}(D^r_*)$$

and define

$$d_1 = j \circ \partial : E^{r,s}_1 \longrightarrow E^{r-1,s}_1.$$

r is said to be the *algebraic filtration* of elements in $E^{r,s}_1$ and $r+s$ is the *total degree* of elements in $E^{r,s}_1$. Since $\partial \circ j = 0$, we have that

$$d_1 \circ d_1 = 0$$

and we let

$$E^{r,s}_2 = Ker(d_1 : E^{r,s}_1 \to E^{r-1,s}_1)/Im(d_1 : E^{r+1,s}_1 \to E^{r,s}_1)$$

be the resulting homology group. We can then say that the class $\alpha \in H_q(C_*)$ has as its representative, the class $\alpha_k \in E^{k,q-k}_2$.

Now let us go back and consider further obstructions to an arbitrary class $\beta \in E^{k,q-k}_2$ representing a class in $H_q(C_*)$. Represent β as a cycle in E_1: $\beta \in Ker(d_1 = j \circ \partial \in H_q(D^k_*))$. Again, β represents a class in $H_q(C_*)$ if and only if $\partial(\beta) = 0$. Now since $j \circ \partial(\beta) = 0$, $\partial(\beta) \in H_{q-1}(F_{k-1}(C_*))$ lifts to a class, say $\tilde{\beta} \in H_{q-1}F_{k-2}(C_*)$. Remember that the goal was to lift $\partial(\beta)$ all the way up the vertical tower (so that it is zero). The obstruction to lifting it the next stage, i.e to $H_{q-1}(F_{k-3}(C_*))$ is that $j(\tilde{\beta}) \in H_{q-1}(D^{k-2}_*)$ is zero. Now the fact that a d_1 cycle β has the property that $\partial(\beta)$ lifts to $H_{q-1}F_{k-2}(C_*))$ allows to define a map

$$d_2 : E^{k,q-k}_2 \longrightarrow E^{k-2,q-k+1}_2$$

and more generally,

$$d_2 : E_2^{r,s} \longrightarrow E_2^{r-2,s+1}$$

by composing this lifting with

$$j : H_{s+r-1}(F_{r-2}(C_*)) \longrightarrow H_{s+r-1}(D_*^{r-2}).$$

That is, $d_2 = j \circ i^{-1} \circ \partial$. It is straightforward to check that $d_2 : E_2^{r,s} \longrightarrow E_2^{r-2,s+1}$ is well defined, and that elements of $H_q(C_*)$ are actually represented by elements in the subquotient homology groups of $E_2^{*,*}$:

$$E_3^{r,s} = Ker(d_2 : E_2^{r,s} \to E_2^{r-2,s+1})/Im(d_2 : E_2^{r+2,s-1} \to E_1^{r,s})$$

Inductively, assume the subquotient homology groups $E_j^{r,s}$ have been defined for $j \leq p-1$ and differentials

$$d_j : E_j^{r,s} \longrightarrow E_j^{r-j,s+j-1}$$

defined on representative classes in $H_{r+s}(D_*^r)$ to be the composition

$$d_j = j \circ (i^{j-1} = i \circ \cdots \circ i)^{-1} \circ \partial$$

so that $E_{j+1}^{*,*}$ is the homology $Ker(d_j)/Im(d_j)$. We then define

$$E_p^{r,s} = Ker(d_{p-1} : E_{p-1}^{r,s} \to E_{p-1}^{r-p+1,s+p-2})/Im(d_{p-1} : E_{p-1}^{r+p-1,s-p+2} \to E_{p-1}^{r,s}).$$

Thus $E_p^{k,q-k}$ is a subquotient of $H_q(D_*^k)$, represented by elements β so that $\partial(\beta)$ lifts to $H_q(F_{k-p}(C_*))$. That is, there is an element $\tilde{\beta} \in H_q(F_{k-p}(C_*))$ so that

$$i^{p-1}(\tilde{\beta}) = \partial(\beta) \in H_{q-1}(F_{k-1}(C_*)).$$

The obstruction to $\tilde{\beta}$ lifting to $H_{q-1}(F_{k-p-1}(C_*))$ is $j(\beta) \in H_q(D_*^{k-p})$. This procedure yields a well defined map

$$d_p : E_p^{r,s} \longrightarrow E_p^{r-p,s+p-1}$$

given by $j \circ (i^{p-1})^{-1} \circ \partial$ on representative classes in $H_q(D_*^k)$. This completes the inductive step. Notice that if we let

$$E_\infty^{r,s} = \varinjlim_p E_p^{r,s}$$

then $E_\infty^{k,q-k}$ is a subquotient of $H_q(D_*^k)$ consisting of precisely those classes represented by elements $\beta \in H_q(D_*^k)$ so that $\partial(\beta)$ lifts all the way up the vertical tower i.e $\partial(\beta)$ is in the image of i^p for all p. This is equivalent to the condition that $\partial(\beta) = 0$ which as observed above is precisely the condition necessary for β to represent a class in $H_q(C_*)$. These observations can be made more precise as follows.

THEOREM 4.27. *Let $I^{r,s} = Image(H_{r+s}(F_r(C_*)) \longrightarrow H_{r+s}(C_*))$. Then $E_\infty^{r,s}$ is isomorphic to the quotient group*

$$E_\infty^{r,s} \cong I^{r,s}/I^{r-1,s+1}.$$

Thus the $E_\infty^{*,*}$ determines $H_*(C_*)$ up to extensions. In particular, if all homology groups are taken with field coefficients we have

$$H_q(C_*) \cong \bigoplus_{r+s=q} E_\infty^{r,s}.$$

In this case we say that $\{E_p^{r,s}, d_p\}$ is a *spectral sequence* starting at $E_1^{r,s} = H_{r+s}(D_*^r)$, and converging to $H_{r+s}(C_*)$.

Often times a filtration of this type occurs when one has a topological space X filtered by subspaces,

$$* = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow X_{k+1} \hookrightarrow \cdots \hookrightarrow X.$$

An important example is the filtration of a CW - complex X by its skeleta, $X_k = X^{(k)}$. We get a spectral sequence as above by applying the homology of the chain complexes to this topological filtration. This spectral sequence converges to $H_*(X)$ with E_1 term $E_1^{r,s} = H_{r+s}(X_r, X_{r-1})$. From the construction of this spectral sequence one notices that chain complexes are irrelevant in this case; indeed all one needs is the fact that each inclusion $X_{k-1} \hookrightarrow X_k$ induces a long exact sequence in homology.

Exercise. Show that in the case of the filtration of a CW - complex X by its skeleta, that the E_1 -term of the corresponding spectral sequence is the cellular chain complex, and the E_2 - term is the homology of X,

$$E_2^{r,s} = \begin{cases} H_r(X), & \text{if } s = 0 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, show that this spectral sequence "collapses" at the E_2 level, in the sense that

$$E_p^{r,s} = E_2^{r,s} \quad \text{for all } p \geq 2$$

and hence

$$E_\infty^{r,s} = E_2^{r,s}.$$

Now if $h_*(-)$ is any *generalized* homology theory (that is, a functor that obeys all the Eilenberg - Steenrod axioms but dimension) then the inclusions of a filtration as above $X_{k-1} \hookrightarrow X_k$ induce long exact sequences in $h_*(-)$, and one gets, by a procedure completely analogous to the above, a spectral sequence converging to $h_*(X)$ with E_1 term

$$E_1^{r,s} = h_{r+s}(X_r, X_{r-1}).$$

Again, for the skeletal filtration of a CW complex, this spectral sequence is called the Atiyah - Hirzebruch spectral sequence for the generalized homology h_*.

Exercise. Show that the E_2 -term of the Atiyah - Hirzebruch spectal sequence for the generalized homology theory h_* is

$$E_2^{r,s} = h_{r+s}(S^r) \otimes H_r(X).$$

Particularly important examples of such generalized homology theories include stable homotopy ($\cong$ framed bordism), other bordism theories, and K - homology theory. Similar spectral sequences also exist for cohomology theories. The reader is referred to [**27**] for a good general reference on spectral sequences with many examples of those most relevant in Algebraic Topology.

5.2. The Leray - Serre spectral sequence for a fibration. The most important example of a spectral sequence from the point of view of these notes is the Leray - Serre spectral sequence of a fibration. Given a fibration $F \to E \to B$, the goal is to understand how the homology of the three spaces (fiber, total space, base space) are related. In the case of a trivial fibration, $E = B \times F \to B$, the answer to this question is given by the Kunneth formula, which says, that when taken with field coefficients,

$$H_*(B \times F; k) \cong H_*(B; k) \otimes_k H_*(F : k),$$

where k is the field.

When $p : E \to B$ is a nontrivial fibration, one needs a spectral sequence to study the homology. The idea is to construct a filtration on a chain complex $C_*(E)$ for computing the homology of the total space E, in terms of the skeletal filtration of a CW - decomposition of the base space B.

Assume for the moment that $p : E \to B$ is a fiber bundle with fiber F. For the purposes of our discussion we will assume that the base space B is simply connected. Let $B^{(k)}$ be the k - skeleton of B, and define

$$E(k) = p^{-1}(B^{(k)}) \subset E.$$

We then have a filtration of the total space E by subspaces

$$* \hookrightarrow E(0) \hookrightarrow E(1) \hookrightarrow \cdots \hookrightarrow E(k) \hookrightarrow E(k+1) \hookrightarrow \cdots \hookrightarrow E.$$

To analyze the E_1 - term of the associated homology spectral spectral sequence we need to compute the E_1 - term, $E_1^{r,s} = H_{r+s}(E(r), E(r-1))$. To do this, write the skeleta of B in the form

$$B^{(r)} = B^{(r-1)} \cup \bigcup_{j \in J_r} D_j^r.$$

Now since each cell D_r is contractible, the restriction of the fibration E to the cells is trivial, and so

$$E(r) - E(r-1) \cong \bigcup_{j \in J_r} D^r \times F.$$

Moreover the attaching maps are via the maps

$$\tilde{\alpha}_r : \bigvee_{j \in J_r} S_j^{r-1} \times F \to E(r-1)$$

induced by the cellular attaching maps $\alpha_k : \bigvee_{j \in J_k} S_j^{k-1} \to B^{(k-1)}$. Using the Mayer - Vietoris sequence, one then computes that

$$\begin{aligned} E_1^{r,s} = H_{r+s}(E(r), E(r-1)) &= H_{r+s}(\bigcup_{j \in J_r} D^r \times F, \bigcup_{j \in J_r} S^{r-1} \times F) \\ &= H_{r+s}(\bigvee_{j \in J_r} S^r \times F, F) \\ &= H_r(\bigvee_{j \in J_r} S^r) \otimes H_s(F) \\ &= C_r(B; H_s(F)). \end{aligned}$$

These calculations indicate the following result, due to Serre in his thesis [**36**]. We refer the reader to that paper for details. It is one of the great pieces of mathematics literature in the last 50 years.

THEOREM 4.28. *Let $p : E \to B$ be a fibration with fiber F. Assume that F is connected and B is simply connected. Then there are chain complexes $C_*(E)$ and $C_*(B)$ computing the homology of E and B respectively, and a filtation of $C_*(E)$ leading to a spectral sequence converging to $H_*(E)$ with the following properties:*

(1) $E_1^{r,s} = C_r(B) \otimes H_s(F)$

(2) $E_2^{r,s} = H_r(B; H_s(F))$

(3) *The differential d_j has bidegree $(-j, j-1)$:*

$$d_j : E_j^{r,s} \to E_j^{r-j, s+j-1}.$$

(4) *The inclusion of the fiber into the total space induces a homomorphism*

$$i_* : H_n(F) \to H_n(E)$$

which can be computed as follows:

$$i_* : H_n(F) = E_2^{0,n} \to E_\infty^{0,n} \subset H_n(E)$$

where $E_2^{0,n} \to E_\infty^{0,n}$ is the projection map which exists because all the differentials d_j are zero on $E_j^{0,n}$.

(5) *The projection map induces a homomorphism*

$$p_* : H_n(E) \to H_n(B)$$

which can be computed as follows:

$$H_n(E) \to E_\infty^{n,0} \subset E_2^{n,0} = H_n(B)$$

where $E_\infty^{n,0}$ *includes into* $E_2^{n,0}$ *as the subspace consisting of those classes on which all differentials are zero. This is well defined because no class in* $E_j^{n,0}$ *can be a boundary for any* j.

Remark. The theorem holds when the base space is not simply connected also. However in that case the E_2 -term is homology with "twisted coefficients". This has important applications in many situations, however we will not consider this issue in these notes. Again, we refer the reader to Serre's thesis [**36**] for details.

We will finish this chapter by describing several applications of this important spectal sequence. The first, due to Serre himself [**36**], is the use of this spectral sequence to prove that even though fibrations do not, in general, admit long exact sequences in homology, they do admit exact sequences in homology through a range of dimensions depending on the connectivity of the base space and fiber.

THEOREM 4.29. *Let* $p : E \to B$ *be a fibration with connected fiber* F*, where* B *is simply connected and* $H_i(B) = 0$ *for* $0 < i < n$*, and* $H_i(F) = 0$ *for* $i < i < m$*. Then there is an exact sequence*

$$H_{n+m-1}(F) \xrightarrow{i_*} H_{n+m-1}(E) \xrightarrow{p_*} H_{n+m-1}(B) \xrightarrow{\tau} H_{n+m-2}(F) \to \cdots \to H_1(E) \to 0$$

PROOF. The E_2 -term of the Serre spectral sequence is given by

$$E_2^{r,s} = H_r(B; H_s(F))$$

which, by hypothesis is zero for $0 < r < n$ or $0 < j < m$. Let $q < n + m$. Then this implies that the composition series for $H_q(E)$, given by the filtration defining the spectral sequence, reduces to the short exact sequence

$$0 \to E_\infty^{0,q} \to H_q(E) \to E_\infty^{q,0} \to 0.$$

Now in general, for these "edge terms", we have

$$E_\infty^{q,0} = kernel\{d_q : E_q^{q,0} \to E_q^{0,q-1}\} \quad \text{and}$$
$$E_\infty^{0,q} = coker\{d_q : E_q^{q,0} \to E_q^{0,q-1}\}.$$

But when $q < n + m$, we have $E_q^{q,0} = E_2^{q,0} = H_q(B)$ and $E_q^{0,q-1} = E_2^{0,q-1} = H_{q-1}(F)$ because there can be no other differentials in this range. Thus if we define

$$\tau : H_q(B) \to H_{q-1}(F)$$

to be $d_q : E_q^{q,0} \to E_q^{0,q-1}$, for $q < n+m$, we then have that $p_* : H_q(E) \to H_q(B)$ maps surjectively onto the kernel of τ, and if $q < n+m-1$, then the kernel of p_* is the cokernel of $\tau : H_{q+1}(B) \to H_q(F)$. This establishes the existence of the long exact sequence in homology in this range. □

Remark. The homomorphism $\tau : H_q(B) \to H_{q-1}(F)$ for $q < n+m$ in the proof of this theorem is called the "transgression" homomorphism.

5.3. Applications I: The Hurewicz theorem. As promised earlier in this chapter, we now use the Serre spectral sequence to prove the Hurewicz theorem. The general theorem is a theorem comparing relative homotopy groups with relative homology groups. We begin by proving the theorem comparing homotopy groups and homology of a single space.

THEOREM 4.30. *Let X be an $n-1$ - connected space, $n \geq 2$. That is, we assume $\pi_q(X) = 0$ for $q \leq n-1$. Then $H_q(X) = 0$ for $q \leq n-1$ and the previously defined "Hurewicz homomorphism"*

$$h : \pi_n(X) \to H_n(X)$$

is an isomorphism.

PROOF. We assume the reader is familiar with the analogue of the theorem when $n = 1$, which says that for X connected, the first homology group $H_1(X)$ is given by the abelianization of the fundamental group

$$h : \pi_1(X)/[\pi_1, \pi_1] \cong H_1(X)$$

where $[\pi_1, \pi_1] \subset \pi_1(X)$ is the commutator subgroup. We use this preliminary result to begin an induction argument to prove this theorem. Namely we assume that the theorem is true for $n-1$ replacing n in the statement of the theorem. We now complete the inductive step. By our inductive hypotheses, $H_i(X) = 0$ for $i \leq n-2$ and $\pi_{n-1}(X) \cong H_{n-1}(X)$. But we are assuming that $\pi_{n-1}(X) = 0$. Thus we need only show that $h : \pi_n(X) \to H_n(X)$ is an isomorphism.

Consider the path fibration $p : PX \to X$ with fiber the loop space ΩX. Now $\pi_i(\Omega X) \cong \pi_{i+1}(X)$, and so $\pi_i(\Omega X) = 0$ for $i \leq n-2$. So our inductive assumption applied to the loop space says that

$$h : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$$

is an isomorphism. But $\pi_{n-1}(\Omega X) = \pi_n(X)$. Also, by the Serre exact sequence applied to this fibration, using the facts that

(1) the total space PX is contractible, and

(2) the fiber ΩX is $n-2$ - connected and the base space X is $(n-1)$ - connected

we then conclude that the transgression,

$$\tau : H_n(X) \to H_{n-1}(\Omega X)$$

is an isomorphism. Hence the Hurewicz map $h : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$ is the same as the Hurewicz map $h : \pi_n(X) \to H_n(X)$, which is therefore an isomorphism. $\square$

We are now ready to prove the more general relative version of this theorem 4.20

Theorem 4.31. *Let X be simply connected, and let $A \subset X$ be a simply connected subspace. Suppose that the pair (X, A) is $(n-1)$ - connected, for $n > 2$. That is,*

$$\pi_k(X, A) = 0 \quad \textit{if} \quad k \leq n-1.$$

Then the Hurewicz homomorphism $h_* : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.

Proof. . Replace the inclusion

$$\iota : A \hookrightarrow X.$$

by a homotopy equivalent fibration $\tilde{\iota} : \tilde{A} \to X$ as in 4.7. Let F_ι be the fiber. Then $\pi_i(F_\iota) \cong \pi_{i+1}(X, A)$, by comparing the long exact sequences of the pair (X, A) to the long exact sequence in homotopy groups for the fibration $\tilde{A} \to X$. So by the Hurewicz theorem 4.30 we know that $\pi_i(F) = H_i(F) = 0$ for $i \leq n-2$ and

$$h : \pi_{n-1}(F) \to H_{n-1}(F)$$

is an isomorphism. But as mentioned, $\pi_{n-1}(F) \cong \pi_n(X, A)$ and by comparing the homology long exact sequence of the pair (X, A) to the Serre exact sequence for the fibration $F \to \tilde{A} \to B$, one has that $H_{n-1}(F) \cong H_n(X, A)$. The theorem follows. $\square$

As a corollary, we obtain the following strengthening of the Whitehead theorem 4.15 which is quite useful in calculations.

Corollary 4.32. *Suppose X and Y are simply connected CW - complexes and $f : X \to Y$ a continuous map that induces an isomorphism in homology groups,*

$$f_* : H_k(X) \xrightarrow{\cong} H_k(Y) \quad \textit{for all } k \geq 0$$

Then $f : X \to Y$ is a homotopy equivalence.

PROOF. Replace $f : X \to Y$ by the inclusion into the mapping cylinder

$$\bar{f} : X \hookrightarrow \bar{Y}$$

where $\bar{Y} = Y \cup_f X \times I$ which is homotopy equivalent to Y, and $\bar{f}$ includes X into $\bar{Y}$ as $X \times \{1\}$.

Since X and Y are simply connected, we have that $\pi_2(X) \cong H_2(X)$ and $\pi_2(Y) \cong H_2(Y)$. Thus $f_* : \pi_2(X) \to \pi_2(Y)$ is an isomorphism. Again, since X and Y are simply connected, this implies that $\pi_q(\bar{Y}, X) = 0$ for $q = 1, 2$. Thus we can apply the relative Hurewicz theorem. However since $f_* : H_k(X) \cong H_k(Y) \quad$ for all $k \geq 0$, we have that $H_k(\bar{Y}, X) = 0$ for all $k \geq 0$. But then the Hurewicz theorem implies that $\pi_k(\bar{Y}, X) = 0$ for all k, which in turn implies that $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for all k. The theorem follows from the Whitehead theorem 4.15. □

5.4. Applications II: $H_*(\Omega S^n)$ and $H^*(U(n))$. In this section we will use the Serre spectral sequence to compute the homology of the loop space ΩS^n and the cohomology ring of the Lie groups, $H^*(U(n))$.

THEOREM 4.33.

$$H_q(\Omega S^n) = \begin{cases} \mathbb{Z} & \textit{if } q \textit{ is a multiple of } n-1, \textit{ i.e } q = k(n-1) \\ 0 & \textit{otherwise} \end{cases}$$

PROOF. ΩS^n is the fiber of the path fibration $p : PS^n \to S^n$. Since the total space of this fibration is contractible, the Serre spectral sequence converges to 0 in positive dimensions. That is,

$$E_\infty^{r,s} = 0$$

for all r, s, except that $E_\infty^{0,0} = \mathbb{Z}$. Now since the base space, S^n has nonzero homology only in dimensions 0 and n (when it is $\mathbb{Z}$), then

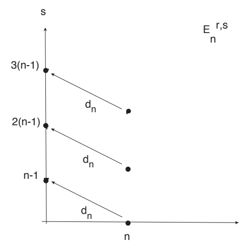
$$E_2^{r,s} = H_r(S^n; H_s(\Omega S^n))$$

is zero unless $r = 0$ or n. In particular, since $d_q : E_q^{r,s} \to E_q^{r-q, s+q-1}$, we must have that for $q < n$, $d_q = 0$. Thus $E_2^{r,s} = E_n^{r,s}$ and the only possible nonzero differential d_n occurs in dimensions

$$d_n : E_n^{n,s} \to E_n^{0, s+n-1}.$$

It is helpful to picture this spectral sequence as in the following diagram, where a dot in the (r, s) - entry denotes a copy of the integers in $E_n^{r,s} = H_r(S^n; H_s(\Omega S^n))$.

Notice that if the generator $\sigma_{n,0} \in E_n^{n,0}$ is in the kernel of d_n, then it would represent a nonzero class in $E_{n+1}^{n,0}$. But d_{n+1} and all higher differentials on $E_{n+1}^{n,0}$ must be zero, for dimensional reasons. That is, $E_{n+1}^{n,0} = E_\infty^{n,0}$. But we saw that $E_\infty^{n,0} = 0$. Thus we must conclude that $d_n(\sigma_{n,0}) \neq 0$. For



the same reasoning, (i.e the fact that $E_{n+1}^{n,0} = 0$) we must have that $d_n(k\sigma_{n,0}) \neq 0$ for all integers k. This means that the image of

$$d_n : E_n^{n,0} \to E_n^{0,n-1}$$

is $\mathbb{Z} \subset E_n^{0,n-1} = H_{n-1}(\Omega S^n)$. On the other hand, we claim that $d_n : E_n^{n,0} \to E_n^{0,n-1}$ must be surjective. For if $\alpha \in E_n^{0,n-1}$ is not in the image of d_n, then it represents a nonzero class in $E_{n+1}^{0,n-1} = E_\infty^{0,n-1}$. But as mentioned earlier $E_\infty^{0,n-1} = 0$. So d_n is surjective as well. In fact we have proven that

$$d_n : \mathbb{Z} = H_n(S^n) = E_n^{n,0} \to E_n^{0,n-1} = E_2^{0,n-1} = H_{n-1}(\Omega S^n)$$

is an isomorphism. Hence $H_{n-1}(\Omega S^n) \cong \mathbb{Z}$, as claimed. Now notice this calculation implies a calculation of $E_2^{n,n-1}$, namely,

$$E_2^{n,n-1} = H_n(S^n; H_{n-1}(\Omega S^n) = bz.$$

Repeating the above argument shows that $E_2^{n,n-1} = E_n n, n-1$ and that

$$d_n : E_n^{n,n-1} \to E_n^{0,2(n-1)}$$

must be an isomorphism. This yields that

$$\mathbb{Z} = E_2^{0,2(n-1)} = H_{2(n-1)}(\Omega S^n).$$

Repeating this argument shows that for every q, $\mathbb{Z} = E_2^{n,q(n-1)} = E_n^{n,q(n-1)}$ and that

$$d_n : E_n^{n,q(n-1)} \to E_n 0, (q+1)(n-1) \cong H_{(q+1)(n-1)}(\Omega S^n)$$

is an isomorphism. And so $H_{k(n-1)}(\Omega S^n) = \mathbb{Z}$ for all k.

We can also conclude that in dimensions j not a multiple of $n-1$, then $H_j(\Omega S^n)$ must be zero. This is true by the following argument. Assume the contrary, so that there is a smallest $j>0$ not a multiple of $n-1$ with $H_j(\Omega S^n) = E_2^{0,j} \neq 0$. But for dimensional reasons, this group cannot be in the image of any differential, because the only $E_q^{r,s}$ that can be nonzero with $r>0$ is when $r=n$. So the only possibility for a class $\alpha \in E_2^{0,j}$ to represent a class which is in the image of a differential is $d_n : E_n^{n,s} \to E_n^{0,s+n-1}$. So $j = s+n-1$. But since j is the smallest positive integer not of the form a multiple of $n-1$ with $H_j(\Omega S^n)$ nonzero, then for $s<j$, $E_n^{n,s} = H_n(S^n, H_s(\Omega S^n)) = H_s(\Omega S^n)$ can only be nonzero if s is a multiple of $(n-1)$, and therefore so is $s+n-1=j$. This contradiction implies that if j is not a multiple of $n-1$, then $H_j((\Omega S^n))$ is zero. This completes our calculation of$H_*(\Omega S^n)$. $\square$

We now use the cohomology version of the Serre spectral sequence to compute the cohomology of the unitary groups. We first give the cohomological analogue of 4.28. Again, the reader should consult [**36**] for details.

THEOREM 4.34. *Let $p : E \to B$ be a fibration with fiber F. Assume that F is connected and B is simply connected. Then there is a cohomology spectral sequence converging to $H^*(E)$, with $E_2^{r,s} = H^r(B; H^s(F))$, having the following properties.*

(1) *The differential d_j has bidegree $(j, -j+1)$:*

$$d_j : E_j^{r,s} \to E_j^{r+j,s-j+1}.$$

(2) *For each j, $E_j^{*,*}$ is a bigraded ring. The ring multiplication maps*

$$E_j^{p,q} \otimes E_j^{i,j} \to E_j^{p+i,q+j}.$$

(3) *The differential $d_j : E_j^{r,s} \to E_j^{r+j,s-j+1}$. is an antiderivation in the sense that it satisfies the product rule:*

$$d_j(ab) = d_j(a) \cdot b + (-1)^{u+v} a \cdot d_j(b)$$

where $a \in E_j^{u,v}$.

(4) *The product in the ring E_{j+1} is induced by the product in the ring E_j, and the product in E_∞ is induced by the cup product in $H^*(E)$.*

We apply this to the following calculation.

THEOREM 4.35. *There is an isomorphism of graded rings,*

$$H^*(U(n)) \cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}],$$

the graded exterior algebra on one generator σ_{2k-1} in every odd dimension $2k-1$ for $1 \leq k \leq n$.

PROOF. We prove this by induction on n. For $n = 1$, $U(1) = S^1$ and we know the assertion is correct. Now assume that $H^*(U(n-1)) \cong \Lambda[\sigma_1, \cdots \sigma_{2n-3}]$. Consider the Serre cohomology spectral sequence for the fibration

$$U(n-1) \subset U(n) \to U(n)/U(n-1) \cong S^{2n-1}.$$

Then the E_2 - term is given by

$$E_2^{*,*} \cong H^*(S^{2n-1}; H^*(U(n-1)) = H^*(S^{2n-1}) \otimes H^*(U(n-1))$$

and this isomorphism is an isomorphism of graded rings. But by our inductive assumption we have that

$$\begin{aligned} H^*(S^{2n-1}) \otimes H^*(U(n-1)) &\cong \Lambda[\sigma_{2n-1}] \otimes \Lambda[\sigma_1, \cdots \sigma_{2n-3}] \\ &\cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}]. \end{aligned}$$

Thus

$$E_2^{**} \cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}]$$

as graded algebras. Now since all the nonzero classes in $E_2^{*,*}$ have odd total degree (where the total degree of a class $\alpha \in E_2^{r,s}$ is $r + s$), and all differentials increase the total degree by one, we must have that all differentials in this spectral sequence are zero. Thus

$$E_\infty^{*,*} = E_2^{*,*} \cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}].$$

We then conclude that $H^*(U(n)) =\cong \Lambda[\sigma_1, \sigma_3, \cdots, \sigma_{2n-1}]$ which completes the inductive step in our proof. □

5.5. Applications III: ***Spin*** **and** ***Spin***$_{\mathbb{C}}$ **structures.** In this section we describe the notions of *Spin* and *Spin*$_{\mathbb{C}}$ structures on vector bundles. We then use the Serre spectral sequence to identify characteristic class conditions for the existence of these structures. These structures are particularly important in geometry, geometric analysis, and geometric topology.

Recall from chapter 2 that an n - dimensional vector bundle ζ over a space X is orientable if and only if it has a $SO(n)$ - structure, which exists if and only if the classifying map $f_\zeta : X \to BO(n)$ has a homotopy lifting to $BSO(n)$. In chapter 3 we proved the following property as well.

PROPOSITION 4.36. *The n - dimensional bundle ζ is orientable if and only if its first Stiefel - Whitney class is zero,*

$$w_1(\zeta) = 0 \in H^1(X; \mathbb{Z}_2).$$

A *Spin* structure on ζ is a refinement of an orientation. To define it we need to further study the topology of $SO(n)$.

The group $O(n)$ has two path components, i.e $\pi_0(O(n)) \cong \mathbb{Z}_2$ and $SO(n)$ is the path component of the identity map. In particular $SO(n)$ is connected, so $\pi_0(SO(n)) = 0$. We have the following information about $\pi_1(SO(n))$.

PROPOSITION 4.37. $\pi_1(SO(2)) = \mathbb{Z}$. *For* $n \geq 3$, *we have*

$$\pi_1(SO(n)) = \mathbb{Z}_2.$$

PROOF. $SO(2)$ is topologically a circle, so the first part of the theorem follows. $SO(3)$ is topologically the projective space

$$SO(3) \cong \mathbb{RP}^3$$

which has a double cover $\mathbb{Z}_2 \to S^3 \to \mathbb{RP}^3$. Since S^3 is simply connected, this is the universal cover of $\mathbb{RP}^3$ and hence $\mathbb{Z}_2 = \pi_1(\mathbb{RP}^3) = \pi_1(SO(3))$.

Now for $n \geq 3$, consider the fiber bundle $SO(n) \to SO(n+1) \to SO(n+1)/SO(n) = S^n$. By the long exact sequence in homotopy groups for this fibration we see that $\pi_1(SO(n)) \to \pi_1(SO(n+1))$ is an isomorphism for $n \geq 3$. The result follows by induction on n. □

Since $\pi_1(SO(n)) = \mathbb{Z}_2$, the universal cover of $SO(n)$ is a double covering. The group $Spin(n)$ is defined to be this universal double cover:

$$\mathbb{Z}_2 \to Spin(n) \to SO(n).$$

Exercise. Show that $Spin(n)$ is a group and that the projection map $p : Spin(n) \to SO(n)$ is a group homomorphism with kernel $\mathbb{Z}_2$.

Now the group $Spin(n)$ has a natural $\mathbb{Z}_2$ action, since it is the double cover of $SO(n)$. Define the group $Spin_{\mathbb{C}}(n)$ using this $\mathbb{Z}_2$ - action in the following way.

DEFINITION 4.6. The group $Spin_{\mathbb{C}}(n)$ is defined to be

$$Spin_{\mathbb{C}}(n) = Spin(n) \times_{\mathbb{Z}_2} U(1).$$

where $\mathbb{Z}_2$ acts on $U(1)$ by $z \to -z$ for $z \in U(1) \subset \mathbb{C}$.

Notice that there is a principal $U(1)$ - bundle,

$$U(1) \to Spin_{\mathbb{C}}(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) \to Spin(n)/\mathbb{Z}_2 = SO(n).$$

$Spin_{\mathbb{C}}$ - structures have been recently shown to be quite important in the Seiberg - Witten theory approach to the study of smooth structures on four dimensional manifolds [**21**].

The main theorem of this section is the following:

THEOREM 4.38. *Let ζ be an oriented n - dimensional vector bundle over a CW - complex X. Let $w_2(\zeta) \in H^2(X;\mathbb{Z}_2)$ be the second Stiefel - Whitney class of ζ. Then*

(1) *ζ has a $Spin(n)$ structure if and only if $w_2(\zeta) = 0$.*

(2) *ζ has a $Spin_{\mathbb{C}}(n)$ - structure if and only if $w_2(\zeta) \in H^2(X;\mathbb{Z}_2)$ comes from an integral cohomology class. That is, if and only if there is a class $c \in H^2(X;\mathbb{Z})$ which maps to $w_2(\zeta)$ under the projection map*

$$H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_2).$$

PROOF. The question of the existence of a $Spin$ or $Spin_{\mathbb{C}}$ structure is equivalent to the existence of a homotopy lifting of the classifying map $f_\zeta : X \to BSO(n)$ to $BSpin(n)$ or $BSpin_{\mathbb{C}}(n)$. To examine the obstructions to obtaining such liftings we first make some observations about the homotopy type of $BSO(n)$.

We know that $BSO(n) \to BO(n)$ is a double covering (the orientation double cover of the universal bundle). Furthermore $\pi_1(BO(n) = \pi_0(O(n)) = \mathbb{Z}_2$, so this is the universal cover of $BO(n)$. In particular this says that $BSO(n)$ is simply connected and

$$\pi_i(BSO(n)) \to \pi_i(BO(n))$$

is an isomorphism for $i \geq 2$.

Recall that for n odd, say $n = 2m+1$, then there is an isomorphism of groups

$$SO(2m+1) \times \mathbb{Z}_2 \cong O(2m+1).$$

Exercise. Prove this!

This establishes a homotopy equivalence

$$BSO(2m+1) \times B\mathbb{Z}_2 \cong BO(2m+1).$$

The following is then immediate from our knowledge of $H^*(BO(2m+1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_{2m+1}]$ and $H^*(B\mathbb{Z}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1]$.

LEMMA 4.39.

$$H^*(BSO(2m+1);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_2, \cdots, w_{2m+1}]$$

where $w_i \in H^i(BSO(2m+1);\mathbb{Z}_2)$ is the i^{th} Stiefel - Whitney class of the universal oriented $(2m+1)$ - dimensional bundle classified by the natural map $BSO(2m+1) \to BO(2m+1)$.

COROLLARY 4.40. *For $n \geq 3$, $H^2(BSO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2$, with nonzero class w_2.*

Proof. This follows from the lemma and the fact that for $n \geq 3$ the inclusion $BSO(n) \to BSO(n+1)$ induces an isomorphism in H^2, which can be seen by looking at the Serre exact sequence for the fibration $S^n \to BSO(n) \to BSO(n+1)$. $\square$

This allows us to prove the following.

Lemma 4.41. *The classifying space $BSpin(n)$ is homotopy equivalent to the homotopy fiber F_{w_2} of the map*

$$w_2 : BSO(n) \to K(\mathbb{Z}_2, 2)$$

classifying the second Stiefel - Whitney class $w_2 \in H^2(BSO(n); \mathbb{Z}_2)$.

Proof. The group $Spin(n)$ is the universal cover of $SO(n)$, and hence is simply connected. This means that $BSpin(n)$ is 2 - connected. By the Hurewicz theorem this implies that $H^2(BSpin(n); \mathbb{Z}_2) = 0$. Thus the composition

$$BSpin(n) \xrightarrow{p} BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$$

is null homotopic. Convert the map w_2 to a homotopy equivalent fibration, $\tilde{w}_2 : \tilde{B}SO(n) \to K(\mathbb{Z}_2, 2)$. The map p defines a map (up to homotopy) $\tilde{p} : BSpin(n) \to \tilde{B}SO(n)$, and the composition $\tilde{p} \circ w_2$ is still null homotopic. A null homotopy $\Phi : BSpin(n) \times I \to K(\mathbb{Z}_2, 2)$ between $\tilde{p} \circ w_2$ and the constant map at the basepoint, lifts, due to the homotopy lifting property, to a homotopy $\tilde{\Phi} : BSpin(n) \times I \to \tilde{B}SO(n)$ between $\tilde{p}$ and a map $\bar{p}$ whose image lies entirely in the fiber over the basepont, F_{w_2},

$$\bar{p} : BSpin(n) \to F_{w_2}.$$

We claim that $\bar{p}$ induces an isomorphism in homotopy groups. To see this, observe that the homomorphism $p_q : \pi_q(BSpin(n)) \to \pi_q(BSO(n))$ is equal to the homomorphism $\pi_{q-1}(Spin(n)) \to \pi_{q-1}(SO(n))$ which is an isomorphism for $q \geq 3$ because $Spin(n) \to SO(n)$ is the universal cover. But similarly $\pi_q(F_{w_2}) \to \pi_q(BSO(n))$ is also an isomorphism for $q \geq 3$ by the exact sequence in homotopy groups of the fibration $F_{w_2} \to BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$, since w_2 induces an isomorphism on π_2. $BSpin(n)$ and F_{w_2} are also both 2 - connected. Thus they have the same homotopy groups, and we have a commutative square for $q \geq 3$,

$$\begin{tikzcd}
\pi_q(BSpin(n)) \arrow[r, "\bar{p}_*"] \arrow[d, "p"', "\cong"] & \pi_q(F_{w_2}) \arrow[d, "\cong"] \\
\pi_q(BSO(n)) \arrow[r, "="'] & \pi_q(BSO(n)).
\end{tikzcd}$$

Thus $\bar{p} : BSpin(n) \to F_{w_2}$ induces an isomorphism in homotopy groups, and by the Whitehead theorem is a homotopy equivalence. $\square$

Notice that we are now able to complete the proof of the first part of the theorem. If ζ is any oriented, n - dimensional bundle with $Spin(n)$ structure, its classifying map$f_\zeta : X \to BSO(n)$ lifts to a map $\tilde{f}_\zeta : X \to BSpin(n)$, and hence by this lemma, $w_2(\zeta) = f_\zeta^*(w_2) = f_\zeta^* \circ p^*(w_2) = 0$. Conversely, if $w_2(\zeta) = 0$, then the classifying map $f_\zeta : X \to BSO(n)$ has the property that $f_\zeta^*(w_2) = 0$. This implies that the composition

$$X \xrightarrow{f_\zeta} BSO(n) \xrightarrow{w_2} K(\mathbb{Z}_2, 2)$$

is null homotopic. A null homotopy lifts to give a homotopy between f_ζ and a map whose image lies in the homotopy fiber F_{w_2}, which, by the above lemma is homotopy equivalent to $BSpin(n)$. Thus $f_\zeta : X \to BSO(n)$ has a homotopy lift $\tilde{f}_\zeta : X \to BSpin(n)$, which implies that ζ has a $Spin(n)$ - structure.

We now turn our attentiona to $Spin_{\mathbb{C}}$ - structures.

Consider the projection map

$$p : Spin_{\mathbb{C}}(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) \to U(1)/\mathbb{Z}_2 = U(1).$$

p is a group homomorpism with kernel $Spin(n)$. p therefore induces a map on classifying spaces, which we call c,

$$c : BSpin_{\mathbb{C}}(n) \to BU(1) = K(\mathbb{Z}, 2)$$

which has homotopy fiber $BSpin(n)$. But clearly we have the following commutative diagram

$$\begin{array}{ccccc} BSpin(n) & \xrightarrow{\subset} & B(Spin(n) \times_{bz_2} U(1)) & \xrightarrow{=} & BSpin_{\mathbb{C}}(n) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ BSpin(n) & \longrightarrow & B(Spin(n)/\mathbb{Z}_2) & \xrightarrow[=]{} & BSO(n) \end{array}$$

Therefore we have the following diagram between homotopy fibrations

$$\begin{array}{ccccc} BSpin(n) & \longrightarrow & BSpin_{\mathbb{C}}(n) & \xrightarrow{c} & K(\mathbb{Z}, 2) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ BSpin(n) & \longrightarrow & BSO(n) & \xrightarrow{w_2} & K(\mathbb{Z}_2, 2) \end{array}$$

where $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$ is induced by the projection $\mathbb{Z} \to \mathbb{Z}_2$. As we've done before we can assume that $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$ and $w_2 : BSO(n) \to K(\mathbb{Z}_2, 2)$ have been modified to be fibrations. Then this means that $BSpin_{\mathbb{C}}(n)$ is homotopy equivalent to the pull - back along w_2 of the fibration $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}_2, 2)$:

$$BSpin_{\mathbb{C}}(n) \simeq w_2^*(K(\mathbb{Z}, 2)).$$

But this implies that the map $f_\zeta : X \to BSO(n)$ homotopy lifts to $BSpin_{\mathbb{C}}(n)$ if and only if there is a map $u : X \to K(\mathbb{Z}, 2)$ such that $p \circ u : X \to K(\mathbb{Z}_2, 2)$ is homotopic to $w_2 \circ f_\zeta : X \to K(\mathbb{Z}_2, 2)$. Interpreting these as cohomology classes, this says that f_ζ lifts to $BSpin_{\mathbb{C}}(n)$ (i.e ζ has a $Spin_{\mathbb{C}}(n)$

- structure) if and only if there is a class $u \in H^2(X;\mathbb{Z})$ so that the $\mathbb{Z}_2$ reduction of u, $p_*(u)$ is equal to $w_2(\zeta) \in H^2(X;\mathbb{Z}_2)$. This is the statement of the theorem. $\square$

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