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ON CONFORMALLY SYMMETRIC RICCI-RECURRENT MANIFOLDS WITH ABELIAN FUNDAMENTAL GROUPS.

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1. **Introduction.** An n-dimensional ($n \geqq 4$) Riemannian manifold M (not necessarily of definite metric) is called conformally symmetric [1][1)] if its Weyl's conformal curvature tensor $C^h{}_{ijk}$ is parallel, that is,

$$C^h{}_{ijk,l} = 0 \, . \tag{1}$$

A Riemannian manifold is said to be Ricci-recurrent provided that at each point x such that $R_{ij}(x) \neq 0$, there exists a tangent vector v satisfying the relation $R_{ij,k}(x) = v_k R_{ij}(x)$. Condition (1) holds clearly for any conformally flat as well as for any locally symmetric Riemannian manifold of dimension $n \geqq 4$. We shall restrict our consideration to manifolds which are essentially conformally symmetric, i.e., satisfy (1) but are neither conformally flat nor locally symmetric. In [7] W. Roter proved the existence and gave a complete local description of those essentially conformally symmetric manifolds which are at the same time Ricci-recurrent. Although the Ricci-recurrent ones do not exhaust the whole class of essentially conformally symmetric manifolds (see [2]), they form a remarkable subclass which contains, e.g., all essentially conformally symmetric Riemannian metrics of index one, i.e., of signature $(-, +, \cdots, +)$ (see [3], Corollary 1). The present paper deals with a global classification problem for complete, analytic, essentially confomally symmetric Ricci-recurrent manifolds. We observe first (Theorem 1) that in the simply connected case Roter's formulae remain valid (i.e., describe all possible isometry types) also in the large. Given a complete, analytic, essentially conformally symmetric Ricci-recurrent manifold M, we may express it as an orbit space $\bar{M}/G$ of the Riemannian universal covering $\bar{M}$ of M (determined in Theorem 1) by a discrete group G of isometries. Using this fact, we finally prove (Theorem 3) that there exists no compact, complete, analytic, four-dimensional essentially conformally symmetric Ricci-recurrent manifold with Abelian fundamental group. In particular, the four-torus admits no complete, analytic, essentially conformally symmetric Riemannian metric of index one.

2. **The general form of universal coverings.** The class of simply connected, complete, analytic, essentially conformally symmetric Ricci-recurrent manifolds can be described as follows:

Theorem 1. (i) *Let $\bar{M}$ denote the Euclidean n-space ($n \geqq 4$) endowed with the metric $\bar{g}$ given by*

$$\bar{g}_{ij} dx^i dx^j = \varphi (dx^1)^2 + k_{\lambda\mu} dx^\lambda dx^\mu + 2 dx^1 dx^n \, ,$$

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1) Numbers in brackets refer to the references at the end of the paper.

where Greek indices range over the set $\{2, \ldots, n-1\}$, *the function* φ *is defined by*

$$\varphi(x^1, \ldots, x^n) = (A(x^1)k_{\lambda\mu} + a_{\lambda\mu})x^\lambda x^\mu ,$$

A being a non-constant analytic function on $\boldsymbol{R}$ and $[k_{\lambda\mu}]$, $[a_{\lambda\mu}]$ non-zero symmetric matrices such that $[k_{\lambda\mu}]$ is non-singular and $k^{\lambda\mu}a_{\lambda\mu} = 0$, $[k^{\lambda\mu}]$ being the reciprocal of $[k_{\lambda\mu}]$. Then $\bar{M}$ is a simply connected, complete, analytic, n-dimensional essentially conformally symmetric Ricci-recurrent manifold. (ii) *Conversely, every simply connected, complete, analytic, n-dimensional essentially conformally symmetric Ricci-recurrent manifold M is isometric to a manifold $\bar{M}$ of the above type.*

Proof. (i) By an explicit computation (cf. [7], proof of Theorem 3) we verify that $\bar{M}$ is essentially conformally symmetric and Ricci-recurrent and that the geodesic equations for $\bar{M}$ reduce to a system of linear differential equations, so that $\bar{M}$ is complete.

(ii) By Theorem 3 of [7] we may find a point $x \in M$ and a chart at x, i.e., a connected neighbourhood U of x together with a diffeomorphism $f = (f^1, \ldots, f^n): U \to f(U) \subset \boldsymbol{R}^n$ such that the metric g of M is expressed in U as

$$g_{ij}df^i df^j = \varphi(df^1)^2 + k_{\lambda\mu}df^\lambda df^\mu + 2df^1 df^n$$

with $\varphi(y) = (A(f^1(y))k_{\lambda\mu} + a_{\lambda\mu})f^\lambda(y)f^\mu(y)$ for any $y \in U$, where A is a non-constant analytic function defined on some interval of $\boldsymbol{R}$ and $[k_{\lambda\mu}]$, $[a_{\lambda\mu}]$ are non-zero symmetric matrices satisfying $k^{\lambda\mu}a_{\lambda\mu} = 0$, $[k_{\lambda\mu}]$ being non-singular and $[k^{\lambda\mu}]$ its reciprocal. Using the fact that the only Christoffel symbols with respect to this chart which may not vanish are Γ^n_{11}, Γ^λ_{11} and $\Gamma^n_{1\lambda} = \Gamma^n_{\lambda 1}$, $\lambda = 2, \ldots, n-1$ (see [7], p. 93), it is easy to see that

$$f^1{}_{,ij} = 0 , \tag{2}$$

which means that f^1 is an affine function on U. By Theorem 6.1 of ([5], p. 252), f^1 can be extended to an analytic function on M, denoted again by f^1 and satisfying (2) in view of analyticity. Computing (in our chart) the components of the Ricci tensor (see [7], p. 93), we verify that they all vanish except for

$$R_{11} = (n-2)A \circ f^1 . \tag{3}$$

Therefore the equality

$$R_{ij}f^1{}_{,k}f^1{}_{,l} = R_{kl}f^1{}_{,i}f^1{}_{,j} \tag{4}$$

holds on U and, by analyticity, it extends to the whole of M. Transvecting (4) with $u^l u^k$ for a suitably chosen vector u, we conclude that

$$R_{ij} = Sf^1{}_{,i}f^1{}_{,j} \tag{5}$$

for some analytic function S on M. Now choose $v \in T_xM$ such that $df^1(v) = 1$. By (2), df^1 is parallel, so $d[f^1(\exp(t - f^1(x))v)]/dt = 1$, which easily implies

$$f^1(\exp(t - f^1(x))v) = t \tag{6}$$

for each $t \in \boldsymbol{R}$. The components of df^1 in our chart are clearly $(1, 0, \ldots, 0)$, so that formulae (3), (5) and (6) yield

$$A(t) = S(\exp(t - f^1(x))v)/(n-2) ,$$

wherever $A(t)$ is defined. By completeness of M, this implies that A can be extended to an analytic function on the whole real line, denoted also by A. The analytic Riemannian manifold $\bar{M}$, defined to be the R^n together with the metric $\bar{g}$ given by $\bar{g}_{ij}dx^i dx^j = \bar{\varphi}(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1 dx^n$, where $\bar{\varphi}(x^1, \ldots, x^n) = (A(x^1)k_{\lambda\mu} + a_{\lambda\mu})x^\lambda x^\mu$, is of the type determined in (i). Clearly, $f: U \to f(U) \subset \bar{M}$ is now an isometry, and by Corollary 6.2 of ([5], p. 255), it extends to an affine diffeomorphism of M onto $\bar{M}$, denoted again by f. The equality $f^*\bar{g} = g$ on U remains valid on M in view of analyticity. Hence f is an isometry, which completes the proof.

By Theorem 1, the universal covering of any complete, analytic, essentially conformally symmetric Ricci-recurrent manifold is Euclidean, so that we obtain

Corollary 1. *Let M be a complete, analytic, essentially conformally symmetric Ricci-recurrent manifold. Then $\pi_k M = 0$ for any integer $k \geqq 2$.*

Theorem 2. *Let $\bar{M}$ be the Riemannian manifold defined as in* (i) *of Theorem* 1. *Then*

(i) *Any isometry $F = (F^1, \ldots, F^n)$ of $\bar{M}$ onto itself is of the form*

$$\begin{aligned} &F^1(x^1, \ldots, x^n) = \varepsilon x^1 + T\,, \\ (7)\qquad &F^\lambda(x^1, \ldots, x^n) = H^\lambda_\mu x^\mu + C^\lambda(x^1)\,, \quad \lambda = 2, \ldots, n-1\,, \\ &F^n(x^1, \ldots, x^n) = -\varepsilon k_{\lambda\mu}\dot{C}^\lambda(x^1)(H^\mu_\nu x^\nu + \tfrac{1}{2}C^\mu(x^1)) + \varepsilon x^n + r\,, \end{aligned}$$

where r, ε, T are real numbers satisfying the conditions

$$(8)\qquad |\varepsilon| = 1\,, \qquad A(t) = A(\varepsilon t + T) \ \textit{for any real}\ \ t\,,$$

and $[H^\lambda_\mu]$ is an $(n-2) \times (n-2)$ matrix such that

$$(9)\qquad \text{a)}\ \ k_{\nu\tau}H^\nu_\lambda H^\tau_\mu = k_{\lambda\mu} \qquad \textit{and} \qquad \text{b)}\ \ a_{\nu\tau}H^\nu_\lambda H^\tau_\mu = a_{\lambda\mu}$$

and the functions C^λ, $\lambda = 2, \ldots, n-1$, form a solution of the following system of ordinary differential equations:

$$(10)\qquad \ddot{C}^\lambda(t) = A(t)C^\lambda(t) + k^{\lambda\nu}a_{\nu\mu}C^\mu(t)\,.$$

(ii) *Conversely, for any r, ε, T, H^λ_μ and C^λ satisfying* (8)-(10), *formulae* (7) *define an isometry of $\bar{M}$ onto itself.*

Proof. An easy computation shows that $v^1 = v^\lambda = 0$, $v^n = 1$ are the components of the unique (up to a constant factor) parallel vector field v on $\bar{M}$. For any isometry $F = (F^1, \ldots, F^n)$ we have thus

$$(11)\qquad F_* v = \varepsilon^{-1} v\,,$$

ε being a non-zero constant. Moreover, F leaves invariant the orthogonal complement of v, which is a parallel $(n-1)$-plane field determining a foliation whose leaves are totally geodesic submanifolds of $\bar{M}$, defined by $x^1 =$ constant. It is easy to see that the restrictions of $dx^2, \ldots, dx^n$ to any leaf are parallel in the symmetric connection the leaf inherits from $\bar{M}$ (see [6], pp. 56-59). Therefore any leaf is affinely equivalent to R^{n-1}, the functions $x^2, \ldots, x^n$ refering to an affine coordinate system and F restricted to any leaf is an affine equivalence onto another one. This, together with (11), implies that F is of the form

$$
(12) \qquad \begin{aligned}
F^1(x^1, \ldots, x^n) &= f(x^1) , \\
F^\lambda(x^1, \ldots, x^n) &= F^\lambda_\mu(x^1)x^\mu + C^\lambda(x^1) , \\
F^n(x^1, \ldots, x^n) &= F^n_\mu(x^1)x^\mu + \varepsilon^{-1}x^n + C^n(x^1)
\end{aligned}
$$

with

$$
(13) \qquad \det [F^\lambda_\mu(x^1)] \neq 0 .
$$

Comparing now the components of $\bar{g}$ with those transformed by F, we obtain $1 = \bar{g}_{1n} = (F^*\bar{g})_{1n} = \varepsilon^{-1}\dot{f}(x^1)$, which gives

$$
(14) \qquad F^1(x^1, \ldots, x^n) = \varepsilon x^1 + T
$$

for some real T. Next we have

$$
0 = \bar{g}_{1\lambda} = (F^*\bar{g})_{1\lambda} = F^n_\lambda(x^1) + k_{\mu\nu}\dot{C}^\mu(x^1)F^\nu_\lambda(x^1) + k_{\nu\tau}\dot{F}^\nu_\mu(x^1)F^\tau_\lambda(x^1)x^\mu .
$$

The right-hand side is a polynomial in variables x^λ, $\lambda = 2, \ldots, n-1$, so that

$$
(15) \qquad F^n_\lambda(x^1) = -\varepsilon^{-1}k_{\mu\nu}\dot{C}^\mu(x^1)F^\nu_\lambda(x^1)
$$

and $k_{\nu\tau}\dot{F}^\nu_\mu(x^1)F^\tau_\lambda(x^1) = 0$, which, by (13), implies that $\dot{F}^\lambda_\mu = 0$, i.e., $F^\lambda_\mu =$ constant, say

$$
(16) \qquad F^\lambda_\mu(x^1) = H^\lambda_\mu .
$$

It is also easy to see that $k_{\lambda\mu} = \bar{g}_{\lambda\mu} = (F^*\bar{g})_{\lambda\mu} = k_{\nu\tau}H^\nu_\lambda H^\tau_\mu$, which implies (9) a). Moreover, we have

$$
\begin{aligned}
&(A(x^1)k_{\lambda\mu} + a_{\lambda\mu})x^\lambda x^\mu \\
&\quad = \bar{g}_{11} = (F^*\bar{g})_{11} \\
&\quad = \varepsilon^2(A(\varepsilon x^1 + T)k_{\nu\tau} + a_{\nu\tau})(H^\nu_\lambda H^\tau_\mu x^\lambda x^\mu + 2H^\nu_\mu C^\tau(x^1)x^\mu + C^\nu(x^1)C^\tau(x^1)) \\
&\qquad + k_{\lambda\mu}\dot{C}^\lambda(x^1)\dot{C}^\mu(x^1) + 2\varepsilon\dot{F}^n_\mu(x^1)x^\mu + 2\varepsilon\dot{C}^n(x^1) .
\end{aligned}
$$

Both sides of this equality are polynomials in variables x^λ, so, by (9) a),

$$
(17) \qquad A(x^1)k_{\lambda\mu} + a_{\lambda\mu} = \varepsilon^2(A(\varepsilon x^1 + T)k_{\lambda\mu} + a_{\nu\tau}H^\nu_\lambda H^\tau_\mu) ,
$$

$$
(18) \qquad 2\varepsilon\dot{F}^n_\mu(x^1) + 2\varepsilon^2(A(\varepsilon x^1 + T)k_{\nu\tau} + a_{\nu\tau})H^\nu_\mu C^\tau(x^1) = 0 ,
$$

$$
(19) \qquad 2\varepsilon\dot{C}^n(x^1) + k_{\lambda\mu}\dot{C}^\lambda(x^1)\dot{C}^\mu(x^1) + \varepsilon^2(A(\varepsilon x^1 + T)k_{\nu\tau} + a_{\nu\tau})C^\nu(x^1)C^\tau(x^1) = 0 .
$$

Transvecting (17) with $k^{\lambda\mu}$ and using the equalities $k^{\lambda\mu}a_{\lambda\mu} = 0$ and $k^{\lambda\mu}H^\nu_\lambda H^\tau_\mu = k^{\nu\tau}$, we obtain

$$
(20) \qquad A(t) = \varepsilon^2 A(\varepsilon t + T)
$$

for any real t. By induction on k, we obtain for the k-th derivative $A^{(k)}$ of A

$$
A^{(k)}(t) = \varepsilon^{k+2}A^{(k)}(\varepsilon t + T) .
$$

If we had $|\varepsilon| \neq 1$, then this would yield, for $t_0 = T/(1 - \varepsilon)$,

$$
A^{(k)}(t_0) = \varepsilon^{k+2}A^{(k)}(t_0) ,
$$

i.e., $A^{(k)}(t_0) = 0$ for $k \geqq 0$. By analyticity, A would vanish identically, a contradiction. Therefore $|\varepsilon| = 1$, and (20) implies (8). Formula (9) b) follows now from (17).

By (18) we have $\dot{F}^n_\mu(x^1) = -\varepsilon(A(x^1)k_{\nu\tau} + a_{\nu\tau})H^\nu_\mu C^\tau(x^1)$, which, together with (15), (16) and (8), implies (10).

Using (19) and taking into account (8) and (10) we see that the derivative of $C^n(t)$ coincides with that of $-\frac{1}{2}\varepsilon k_{\lambda\mu}\dot{C}^\lambda(t)C^\mu(t)$, so that

(21) $$C^n(t) = -\tfrac{1}{2}\varepsilon k_{\lambda\mu}\dot{C}^\lambda(t)C^\mu(t) + r$$

for some real constant r. Assertion (i) follows now from (12) together with (14), (15), (16) and (21). Assertion (ii) can be verified by an explicit computation, which completes the proof.

Remark 1. For our purposes it will be convenient to adopt the following notations and conventions. Elements of the isometry group $I(\bar{M})$ determined in Theorem 2 will be identified with quintuples $a = (\varepsilon, T, H, C(t), r)$, where ε belongs to the multiplicative group $Z_2 = \{-1, 1\}$, T satisfies (8) and so its range is a discrete subset of R, $H = [H^\lambda_\mu]$ lies in the group D of all $(n-2)\times(n-2)$ matrices satisfying (9), the curve $t \mapsto C(t) = (C^2(t), \ldots, C^{n-1}(t))$ in R^{n-2} is an element of the vector space V of all solutions of (10) and r is an arbitrary real number.

Define the exterior 2-form ω on V by

$$\omega(C_1, C_2) = \tfrac{1}{2}k_{\lambda\mu}(\dot{C}^\lambda_1(t)C^\mu_2(t) - C^\lambda_1(t)\dot{C}^\mu_2(t))$$

(this is a constant independent of t, which can be verified easily by differentiating and taking into account (10)). The group operation of $I(\bar{M})$ can now be written as

(22) $$\begin{aligned}(\varepsilon_1, T_1, H_1, C_1(t), r_1)(\varepsilon_2, T_2, H_2, C_2(t), r_2) \\ = (\varepsilon_1\varepsilon_2, \varepsilon_1 T_2 + T_1, H_1H_2, H_1C_2(t) + C_1(\varepsilon_2 t + T_2), \\ \varepsilon_1\varepsilon_2\omega(H_1C_2(t), C_1(\varepsilon_2 t + T_2)) + \varepsilon_1 r_2 + r_1)\,,\end{aligned}$$

where the curves $t \mapsto H_1C_2(t)$ and $t \mapsto C_1(\varepsilon_2 t + T_2)$ are easily seen to lie in V again.

Points of our manifold $\bar{M}$, whose underlying set is just R^n, will be described as triples (x, u, w), $x, w \in R$, $u \in R^{n-2}$, so that for an isometry $a = (\varepsilon, T, H, C(t), r) \in I(\bar{M})$ we have

(23) $$a(x, u, w) = (\varepsilon x + T, Hu + C(x), -\varepsilon\langle\dot{C}(x), Hu + \tfrac{1}{2}C(x)\rangle + \varepsilon w + r)\,,$$

$\langle\ldots,\ldots\rangle$ being the (possibly indefinite) inner product in R^{n-2} determined by $k_{\lambda\mu}$.

Finally, we can define two natural homomorphisms $h_{13}\colon I(\bar{M}) \to Z_2 \oplus D$ and $h_2\colon \ker h_{13} \to R$ by $h_{13}(\varepsilon, T, H, C(t), r) = (\varepsilon, H)$ and $h_2(1, T, I, C(t), r) = T$, I being the identity matrix. Note that $\operatorname{im} h_2$ is either trivial or infinite cyclic in view of (8). Moreover, in the case $n = 4$ D is finite (a simple computation shows that it has 4 or 6 elements).

3. **The four-dimensional case.** We start with some preliminary remarks and lemmas.

Given a (connected) manifold M, we can express it as the orbit space $\bar{M}/\Gamma$, where $\bar{M}$ is the universal covering of M and Γ is a group isomorphic to $\pi_1 M$ and acting on $\bar{M}$ freely and properly discontinuously, i.e., as the group of deck transformations (see [10], pp. 40–41). In the Riemannian case Γ acts as a group of isometries. Proper discontinuity of Γ implies in particular that for any sequences $x_k \in \bar{M}$, $a_k \in \Gamma$ such that $x_k \to x$ and $a_k x_k \to x$, a_k must equal the identity for all but finitely many k.

Lemma 1. *Suppose G is an Abelian group acting freely and properly discontinuously on R^n so that the orbit space R^n/G is compact. Then G is isomorphic to Z^n.*

Proof. We have $G \cong \pi_1(R^n/G)$, hence G is finitely generated. By Smith's theorem ([4], p. 287) G is torsionfree. Therefore G is free, say $G \cong Z^k$. Thus R^n/G is an Eilenberg–MacLane space of type $(Z^k, 1)$, which implies ([9], p. 93–95) that it is homotopy equivalent to the k-torus T^k. We have ([8], p. 303 and p. 294) $H_n(R^n/G, Z_2) = Z_2 = H_n(T^k, Z_2)$. It follows now (e.g., from Künneth formula) that $k = n$, as desired.

Lemma 2. *Given vectors $x_1, \ldots, x_s$ of R^n, $s > n$, there exist s sequences of integers $k^1_m, \ldots, k^s_m$, $m = 1, 2, \ldots$ with the following two properties:*
(i) *there exists $i \leq s$ such that for all m, $k^i_m \neq 0$.*
(ii) *$k^1_m x_1 + \cdots + k^s_m x_s \to 0$ as $m \to \infty$.*

Proof. Restricting ourselves to the span of the vectors $x_1, \ldots, x_s$ and changing their order, if necessary, we may assume that they span the whole R^n and $x_1, \ldots, x_n$ is a basis. Let G be the group of translations generated by this basis and define an action of Z^s on R^n by $(k^1, \ldots, k^s)x = x + k^1 x_1 + \cdots + k^s x_s$, $x \in R^n$, $k^i \in Z$. Suppose now that our assertion fails. Then the action of Z^s is clearly free and properly discontinuous. Thus R^n/Z^s is a manifold and the obvious surjective map $R^n/G \to R^n/Z^s$ shows that it is compact, for R^n/G is a torus. By Lemma 1 we have $s = n$, a contradiction. This completes the proof.

Lemma 3. *Let $\langle \ldots, \ldots \rangle$ be a non-degenerate symmetric bilinear form in R^n. Given sequences $y_m \in R^n$, $y_m \neq 0$, and $t_m \in R$, $m = 1, 2, \ldots$, the following two conditions are equivalent:* (i) *there exists a sequence $u_m \in R^n$ which is bounded* (*in a norm*) *and satisfies*

$$(24) \qquad \langle y_m, u_m \rangle = t_m \quad \text{for all} \quad m\,.$$

(ii) *there exists $S \geq 0$ such that $|t_m| \leq S\|y_m\|$, $\|\ \|$ being a norm in R^n.*

Proof. Choose covariant vectors $e^1, \ldots, e^n$ such that $\langle \ldots, \ldots \rangle = \sum_i \varepsilon_i e^i \otimes e^i$, $|\varepsilon_i| = 1$, and define an inner product by $(\ldots, \ldots) = \sum_i e^i \otimes e^i$. Note that (24) is equivalent to

$$(25) \qquad (y_m, u'_m) = t_m\,,$$

where u'_m is determined uniquely by $e^i(u'_m) = \varepsilon_i e^i(u_m)$ (no summing), so that u'_m is bounded if and only if u_m is. If (i) holds, then (25) yields $|t_m| \leq (\sup_m \|u'_m\|)\|y_m\|$, $\|\ \|$ being the norm induced by $(\ldots, \ldots)$. Assume now (ii). Setting $u'_m = t_m\|y_m\|^{-2} y_m$ we obtain (25) and $\|u_m\| \leq S$, as desired.

Lemma 4. *Let the indices λ, μ, ν assume values 2, 3. Suppose A is a C^∞ function on R, $[k_{\lambda\mu}]$ a non-singular symmetric matrix, $[k^{\lambda\mu}]$ its reciprocal and $[a_{\lambda\mu}]$ a non-zero symmetric matrix such that $k^{\lambda\mu} a_{\lambda\mu} = 0$. Consider the linear differential equation in R^2*

$$(26) \qquad \ddot{C}^\lambda(t) = A(t)C^\lambda(t) + k^{\lambda\nu} a_{\nu\mu} C^\mu(t)\,, \quad \lambda = 2, 3\,.$$

If C^λ_1, C^λ_2 ($\lambda = 2, 3$) are two solutions of (26), periodic with a common period $T > 0$, then the determinant

$$\begin{vmatrix} C_1^2(t) & C_2^2(t) \\ C_1^3(t) & C_2^3(t) \end{vmatrix}$$

vanishes for some real t.

Proof. The matrix $[k^{\lambda\nu} a_{\nu\mu}]$ represents a non-zero endomorphism Q of R^2 satisfying trace $Q = 0$ and self-adjoint with respect to the (possibly indefinite) inner product determined by $k_{\lambda\mu}$. Equation (26) is equivalent to

$$\ddot{C}(t) = A(t)C(t) + Q(C(t)) . \tag{27}$$

Given two T-periodic solutions C_1, C_2 of (27), we are now to show that the vectors $C_1(t)$, $C_2(t)$ are dependent for some t. In a suitably chosen basis e_2, e_3 of R^2 the matrix for Q is

$$\begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$$

for some a, b with $a^2 + b^2 > 0$ (according as $k_{\lambda\mu}$ is or is not definite, we use here a basis of eigenvectors of Q or just any orthonormal basis). Equation (27) takes now the form

$$\ddot{y}^2 = (A + a)y^2 + by^3 , \qquad \ddot{y}^3 = (A - a)y^3 - by^2 . \tag{28}$$

Let $C_i = y_i^\lambda e_\lambda$, $i = 1, 2$. The function

$$\sigma = a(\dot{y}_1^2 y_2^3 - y_1^2 \dot{y}_2^3 - y_1^3 \dot{y}_2^2 + \dot{y}_1^3 y_2^2) + b(y_1^2 \dot{y}_2^2 - \dot{y}_1^2 y_2^2 + y_1^3 \dot{y}_2^3 - \dot{y}_1^3 y_2^3)$$

is T-periodic, hence $\dot{\sigma}(t) = 0$ for some t. On the other hand, (28) yields

$$\dot{\sigma} = 2(a^2 + b^2)\begin{vmatrix} y_1^2 & y_2^2 \\ y_1^3 & y_2^3 \end{vmatrix} ,$$

which completes the proof.

Theorem 3. *Let M be a complete, analytic, four-dimensional, essentially conformally symmetric Ricci-recurrent manifold. If the fundamental group of M is Abelian, then M is non-compact.*

Proof. Suppose on the contrary that M is a compact, complete, analytic, four-dimensional, essentially conformally symmetric Ricci-recurrent manifold with Abelian fundamental group. In virtue of Theorem 1 we may assume that the universal covering $\tilde{M}$ of M is the Euclidean R^4 together with the metric $\tilde{g}$ defined in (i) of Theorem 1. Moreover, M may be identified with the orbit space $\tilde{M}/\Gamma$, Γ being a properly discontinuous group of fixed point free isometries of $\tilde{M}$, isomorphic to $\pi_1 M$. The subgroup $G = \Gamma \cap \ker h_{13}$ is of finite index in Γ, since $\operatorname{im} h_{13}$ is finite (cf. Remark 1), and therefore the obvious map $\tilde{M}/G \to \tilde{M}/\Gamma = M$ is a finite covering. Thus G is a properly discontinuous Abelian group of isometries of $\tilde{M}$ and $\tilde{M}/G$ is compact. By Lemma 1, G is isomorphic to Z^4. If h_2 were trivial on G, then $(x, u, w) \mapsto x$ would define in view of Theorem 2 an unbounded function on $\tilde{M}/G$. Hence, by Remark 1, $h_2(G)$ is infinite cyclic, so that we have the following splitting exact sequence:

$$0 \to \ker h_2 \cap G \to G \underset{h_2}{\to} h_2(G) \to 0$$

and

(29) $$\ker h_2 \cap G \cong Z^3 .$$

Given $a = (1, 0, I, C(t), r) \in \ker h_2 \cap G$, we have

(30) either $C(t) \neq 0$ for all t or $C = 0$ identically,

and

(31) $$C(t + T) = C(t) \quad \text{for each} \quad t \in R \quad \text{and each} \quad T \in h_2(G) .$$

In fact, if $C(x) = 0$ but C did not vanish identically, then $\dot{C}(x) \neq 0$, since C is a solution of the second order linear differential equation (10). Choosing $u \in R^2$ such that $\langle \dot{C}(x), u \rangle = r$ (in the notations of Remark 1) we have, by (23), $a(x, u, 0) = (x, u, 0)$, so that $a = 1$ and $C = 0$ identically, a contradiction, which proves (30). Now choose $a_1 = (1, T, I, C_1(t), r_1) \in G$. By commutativity, $aa_1 = a_1 a$, which in view of (22) implies (31).

We assert now that for any $a_1, a_2 \in \ker h_2 \cap G$, say $a_i = (1, 0, I, C_i(t), r_i)$, $i = 1, 2$, we have

(32) C_1 and C_2 are linearly dependent over reals.

In fact, $a_1 a_2 = a_2 a_1$ implies by (22) the relation $\omega(C_1, C_2) = 0$. We have thus

(33) $$a_1^k a_2^l = (1, 0, I, kC_1 + lC_2, kr_1 + lr_2) .$$

Suppose (32) fails. From (31), (30) and Lemma 4 it follows that for some $x \in R$, $C_1(x)$ and $C_2(x)$ are linearly dependent non-zero vectors. If both derivatives $\dot{C}_1(x)$, $\dot{C}_2(x)$ vanished, (32) would follow, so that we may assume $\dot{C}_1(x) \neq 0$ and

(34) $$C_2(x) = \xi C_1(x) \neq 0 ,$$

ξ being irrational in view of (30) and (33), since C_1 and C_2 are assumed independent. Thus

(35) $$\dot{C}_2(x) \neq \xi \dot{C}_1(x) .$$

As in Lemma 2, we may choose sequences k_m, l_m of integers such that

(36) $$k_m + l_m \xi \to 0 \quad \text{as} \quad m \to \infty .$$

Since ξ is irrational, we may claim $k_m \neq 0 \neq l_m$. We assert

(37) $$|k_m r_1 + l_m r_2| \leqq S \| k_m \dot{C}_1(x) + l_m \dot{C}_2(x) \|$$

for some constant S and certain norm $\| \ \|$ in R^2. To verify (37), consider two cases. If $\dot{C}_1(x)$, $\dot{C}_2(x)$ are dependent, say $\dot{C}_2(x) = \eta \dot{C}_1(x)$, $\eta \neq \xi$ by (35), then the sequence $z_m = (k_m r_1 + l_m r_2)/(k_m + l_m \eta)$ converges to $(r_2 - \xi r_1)/(\eta - \xi)$, so that $|z_m| \leqq B$ for some $B > 0$. Relation (37) holds now for any norm, with $S = B \|\dot{C}_1(x)\|^{-1}$. On the other hand, if $\dot{C}_1(x)$, $\dot{C}_2(x)$ are independent, choose a positive definite inner product in R^2 so as to make them orthonormal and take into account the induced norm. Clearly (37) is satisfied by $S = |r_1| + |r_2|$. Thus, assuming C_1, C_2 independent over reals, we obtain (37), so that by Lemma 3 there exists a bounded sequence $\bar{u}_m$ in R^2 such that $\langle k_m \dot{C}_1(x) + l_m \dot{C}_2(x), \bar{u}_m \rangle = k_m r_1 + l_m r_2$. Without loss of generality we may assume that $\bar{u}_m$ converges, say to u. The sequence $u_m = \bar{u}_m - \frac{1}{2}(k_m C_1(x) + l_m C_2(x))$ also tends to u in view of (36) and (34). Therefore $(x, u_m, 0) \to (x, u, 0)$ and

$a_1^{k_m}a_2^{l_m}(x, u_m, 0) \to (x, u, 0)$ by (23) and (33). This implies that a_1 and a_2 are dependent over integers, hence so are C_1 and C_2, a contradiction. Thus we have proved (32).

Using (29) we may choose $a_1, a_2, a_3 \in \ker h_2 \cap G$ independent over integers, say $a_i = (1, 0, I, C_i, r_i)$. Changing their order if necessary and using (32) we obtain $C_2 = \xi C_1$ and $C_3 = \eta C_1$ for some real ξ and η. Applying Lemma 2 to the vectors $(1, r_1)$, (ξ, r_2), (η, r_3) of R^2, we find sequences of integers k_m, l_m, p_m, not all tending to zero and such that $k_m + l_m\xi + p_m\eta \to 0$ and $k_m r_1 + l_m r_2 + p_m r_3 \to 0$ as $m \to \infty$. From (33) it follows that $a_1^{k_m}a_2^{l_m}a_3^{p_m}(0, 0, 0) \to (0, 0, 0)$, which contradicts proper discontinuity of G. This completes the proof.

Corollary 2. *Let M be a compact four-dimensional analytic manifold whose fundamental group is a finite extension of an Abelian group (e.g., any compact four-manifold admitting a flat positive definite Riemannian metric). Then M admits no complete, analytic, essentially conformally symmetric Ricci-recurrent Riemannian metric. In particular (cf. [3], Corollary 1) M admits no complete, analytic, essentially conformally symmetric metric of index one.*

The assertion of Corollary 2 can be obtained from Theorem 3 by considering a suitable finite covering of M.

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