The Method of Undetermined Coefficients

The *Method of Undetermined Coefficients* is useful for certain important classes of integrals and as a tool for solving various kinds of differential equations. The table below is only a brief *introduction* to the method. It contains (indefinite) integrals, suggested forms for the antiderivatives and any restrictions that may apply. P, Q, R and S below are *polynomials*, $\deg P = n$, and a and b are *nonzero constants*. Notice that the suggested forms are not necessarily the simplest ones possible. However, any 'improvements' would have increased the number of possible cases and destroyed the elegance of the method.

Your friendly Mathematics instructor, *Miroslav*

	The integral ...	**is equal to ...**	**where ...**
1.	$\int P(x)\, e^{ax} dx$	$Q(x)\, e^{ax} + C$	$\deg Q = n$
2.	$\int P(x)\, \cos ax\, dx$	$Q(x)\, \cos ax + R(x)\, \sin ax + C$	$\deg Q \le n$, $\deg R \le n$
3.	$\int P(x)\, \sin ax\, dx$	$Q(x)\, \cos ax + R(x)\, \sin ax + C$	$\deg Q \le n$, $\deg R \le n$
4.	$\int P(x)\, e^{ax} \cos bx\, dx$	$Q(x)\, e^{ax} \cos bx + R(x)\, e^{ax} \sin bx + C$	$\deg Q \le n$, $\deg R \le n$
5.	$\int P(x)\, e^{ax} \sin bx\, dx$	$Q(x)\, e^{ax} \cos bx + R(x)\, e^{ax} \sin bx + C$	$\deg Q \le n$, $\deg R \le n$
6.	$\int \frac{Ax+B}{(x^2+px+q)^m}\, dx$	$\frac{Q(x)}{(x^2+px+q)^{m-1}} + \int \frac{\bar{C}x+D}{x^2+px+q}\, dx$	$p^2 - 4q < 0$, $m \ge 2$,

m is an integer, $m \ge 2$, $\deg Q \le 2m-3$, and A, B, $\bar{C}$, D are (real) constants

Example: Evaluate $\int (3x+2)\, \sin 3x\, dx$.

According to **3.** above, we have

$$\int (3x+2)\, \sin 3x\, dx = (Ax+B)\, \cos 3x + (\bar{C}x+D)\, \sin 3x + C\,.$$

Differentiating with respect to x, we obtain

$$(3x+2)\, \sin 3x = (3\bar{C}x + A + 3D)\, \cos 3x + (-3Ax - 3B + \bar{C})\, \sin 3x\ ,$$

so that $3\bar{C} = 0$, $A + 3D = 0$, $-3A = 3$, $-3B + \bar{C} = 2$, which implies $A = -1$, $B = -\frac{2}{3}$, $\bar{C} = 0$, $D = \frac{1}{3}$. Therefore,

$$\int (3x+2)\, \sin 3x\, dx = \left(-x - \frac{2}{3}\right) \cos 3x + \frac{1}{3} \sin 3x + C\ \therefore$$