Solution Hints for Spring 96 Exam

Revised 6/03

1. A decreasing sequence which is bounded below has a limit. It's obvious all the x_n's are positive, so the x_n's are bounded below by zero, but to show it's decreasing you'll find we need a better lower bound. Rearranging the expression for x_{n+1} in terms of x_n gives

$$-2 = x_n^2 - 2x_{n+1}x_n.$$

Completing the square gives

$$x_{n+1}^2 - 2 = (x_n - x_{n+1})^2 \geq 0.$$

So if $n > 1$, then $x_n^2 \geq 2$ and since $x_n > 0$ we actually have $x_n \geq \sqrt{2}$. (Another way to see this is to apply the inequality relating the arithmetic and geometric means: for $n > 1$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \sqrt{x_n \cdot \frac{2}{x_n}} = \sqrt{2}$.)

Then

$$x_{n+1} - x_n = \frac{1}{x_n} - \frac{1}{2}x_n = \frac{1}{2x_n}(2 - x_n^2) \leq 0.$$

The sequence is decreasing and bounded below. This proves it has a limit. To find the limit, use the standard trick: take the limit on both sides of the equation (now that we know it exits). Then

$$L = \frac{1}{2}(L + \frac{2}{L}).$$

Solving for L gives $L = \pm\sqrt{2}$, and since the limit is positive we get $L = \sqrt{2}$.

2. The idea for this one is that the terms $e^{\frac{1}{n^\alpha}}$ are getting close to 1 as $n \to \infty$, so one might suspect that the derivative of the exponential function might somehow measure how quickly $(e^{\frac{1}{n^\alpha}} - 1)^3$ goes to zero.

Try using the limit comparison theorem with $(\frac{1}{n^\alpha})^3$ (think of this as the "h" (cubed) in the difference quotient that defines the derivative): since $\alpha > 0$ we have $\frac{1}{n^\alpha} \to 0$ as $n \to \infty$ so

$$\lim_{n\to\infty} \frac{(e^{\frac{1}{n^\alpha}} - 1)^3}{(\frac{1}{n^\alpha})^3} = \lim_{h\to 0} \left(\frac{e^h - 1}{h}\right)^3 = 1.$$

By the limit comparison theorem, the series $\sum(e^{\frac{1}{n^\alpha}} - 1)$ converges if and only if $\sum \frac{1}{n^{3\alpha}}$ converges. Since $\sum \frac{1}{n^{3\alpha}}$ converges if and only if $3\alpha > 1$, the original series converges if and only if $\alpha > \frac{1}{3}$.

3. a. Inspecting the graph of $\frac{1}{t}$ and recalling that $\log(n) = \int_1^n \frac{1}{t}\,dt$ shows that

$$\log(n) \leq \sum_{k=1}^{n} \frac{1}{k} \leq \log(n) + 1.$$

Dividing by $\log(n)$ and using the "squeeze theorem" shows that that $\lim_{n\to\infty} \frac{1}{\log(n)} \sum_{k=1}^n \frac{1}{k} = 1$.

b. From part a we have $0 \leq \sum_{k=1}^n \frac{1}{k} - \log(n) \leq 1$ so the sequence D_n is bounded below. We have

$$D_{n+1} - D_n = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{t}\,dt \leq \frac{1}{n+1} - \frac{1}{n+1} = 0.$$

So the sequence D_n is decreasing. Since it is bounded below, it converges.

4. Note

$$\begin{aligned}|f_n(x) - f(x)| &= \left|\int_{-\infty}^{\infty} f(x-y) n e^{-\pi n^2 y^2}\,dy - \int_{-\infty}^{\infty} f(x) e^{-\pi y^2}\,dy\right| \\ &= \left|\int_{-\infty}^{\infty} \left(f(x - \frac{u}{n}) - f(x)\right) e^{-\pi u^2}\,du\right| \\ &\le \int_{-\infty}^{\infty} \left|\left(f(x - \frac{u}{n}) - f(x)\right)\right| e^{-\pi u^2}\,du.\end{aligned}$$

The idea is to split the integral into two parts: in the first part $e^{-\pi u^2}$ is small and in the second part $|f(x + \frac{u}{n}) - f(x)|$ is small (here you need the uniform continuity of f). Let M be an upper bound for $|f|$. Given $\epsilon > 0$,

i. Choose $N > 0$ so that $\displaystyle\int_{(-\infty,-N]\cup[N,\infty)} e^{-\pi u^2}\,du \le \frac{\epsilon}{4M}$.

ii. Having fixed N, choose δ such that for all $x, y \in \mathbf{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{4N}$ (using the uniform continuity of f).

iii. Having fixed N and δ, choose L such that for $u \in [-N, N]$, $n > L$ implies that $|\frac{u}{n}| < \delta$.

Then $n > L$ implies that for $u \in [-N, N]$, $|f(x - \frac{u}{n}) - f(x)| < \frac{\epsilon}{4N}$, so

$$\begin{aligned}\left|\int_{-\infty}^{\infty} (f(x - \frac{u}{n}) - f(x)) e^{-\pi u^2}\,du\right| &\le 2M \int_{(-\infty,-N]\cup[N,\infty)} e^{-\pi u^2}\,du + \frac{\epsilon}{4N}\int_{-N}^{N} e^{-\pi u^2}\,du \\ &\le 2M \cdot \frac{\epsilon}{4M} + \frac{\epsilon}{4N}\int_{-N}^{N} 1\,du \le \frac{\epsilon}{2} + \frac{\epsilon}{2}.\end{aligned}$$

5. Integration by parts gives, for $t > 0$ and $M > 0$,

$$\int_0^M \frac{\cos x}{x+t}\,dx = \frac{\sin M}{M+t} + \int_0^M \frac{\sin x}{(x+t)^2}\,dx.$$

Since $\left|\frac{\sin x}{(x+t)^2}\right| \le \frac{1}{(x+t)^2}$, the integral converges by the comparison test. This shows that f is defined on $(0, \infty)$. Note

$$f(t) = \int_0^{\infty} \frac{\sin x}{(x+t)^2}\,dx.$$

To show that f is continuous, let $t_o \in (0, \infty)$. Then for all $t \in [t_o/2, \infty)$,

$$\left|\frac{\sin x}{(x+t)^2}\right| \le \frac{1}{(x + t_o/2)^2}.$$

Then the integrand is dominated by an integrable function (independent of t). By the Lebesgue dominated convergence theorem one can interchange the limit and integral:

$$\lim_{t\to t_o} f(t) = \lim_{t\to t_o} \int_0^{\infty} \frac{\sin x}{(x+t)^2}\,dx = \int_0^{\infty} \frac{\sin x}{(x+t_o)^2}\,dx.$$

This shows f is continuous on $(0, \infty)$.

6. The idea is given $\epsilon > 0$ to find how many of the c_i's are less than ϵ in absolute value. Say $|c_i| < \epsilon$ if $n > N$. Then in a partition P of $[0, 1]$ there are at most N subintervals containing the corresponding $x_1, \ldots, x_N$ where the $|c_i|$ may be be greater than ϵ. We can make the Riemann sum over these intervals small by choosing δ small enough. On the remaining subintervals (whose total length is at most 1) the function f is less than ϵ in absolute value.

Fix $\epsilon > 0$. Fix an N such that $i > N$ implies $|c_i| < \epsilon/2$. Let $C = \max(|c_1|, \ldots, |c_N|)$. Now let $\delta = \epsilon/(2NC)$. Let P be any partition of $[0, 1]$ with $\|P\| < \delta$. Let $I_1 \subset P$ be the subintervals which contain at least one of the $x_1, \ldots, x_N$ and let $I_2 \subset P$ be the subintervals which do not. There are at most N subintervals in I_1 and $|f| < \epsilon/2$ on the subintervals in I_2. Let t_k^* be any point in the k^{th} subinterval. The absolute value of the corresponding Riemann sum is

$$\left| \sum_{k=0}^{n} f(t_k^*)(t_{k+1} - t_k) \right| \leq \left(\sum_{I_1} + \sum_{I_2} \right) |f(t_k^*)|(t_{k+1} - t_k).$$

The sum over I_1 is less than or equal to $C \cdot N \cdot \epsilon/(2CN) = \epsilon/2$. The sum over in I_2 can be estimated by $(\epsilon/2) \sum_{I_2} (t_{i+1} - t_i) < \epsilon/2$ since the total length of the interval is 1.