# Math 787.03 Mock Exam 2, Summer, 2003
# Solutions

August 12, 2003

Four complete solutions are sufficient to pass. Please use only one side of every sheet, and use distinct sheets for distinct problems.

1. Find the smallest constant $C > 0$ such that

$$(a+b+c+d+e)^2 \leq C(a^2+b^2+c^2+d^2+e^2)$$

for every $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, and $e \geq 0$.

**Solution.** This is from the Spring 93 qualifying exam. For every such $a$, $b$, $c$, $d$, $e$, we have by the Schwarz inequality,

$$(a \cdot 1 + \cdots + e \cdot 1)^2 \leq 5(a^2 + \cdots + e^2).$$

So the smallest such constant $C$ is less than or equal to 5, i.e, $C \leq 5$. On the other hand, taking $a = \cdots = e = 1$ gives

$$(a + \cdots + e)^2 = 5(a^2 + \cdots + e^2).$$

So no constant smaller than 5 can work, i.e., $C \geq 5$. Thus $C = 5$.

2. Assume that $f \in C^\infty([0,1])$ satisfies the following conditions:

   (a) $f$ is not identically 0

   (b) $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$

   (c) for a sequence of real numbers $a_n$, the series $\sum_{n=1}^\infty a_n f^{(n)}(x)$ converges uniformly on $[0,1]$.

Prove that $\lim_{n\to\infty} n! a_n = 0$.

**Solution.** See Kaczor and Nowak, V.2, 3.2.38 and its solution.

3. Let $a > 0$ and $f$ be continuously differentiable in $[0,a]$. Show that

$$|f(0)| \leq \frac{1}{a}\int_0^a |f(x)|\,dx + \int_0^a |f'(x)|\,dx.$$

**Solution.** This is from the Autumn 92 exam. By the Mean Value Theorem for Integrals (which is applicable here because $f'$ is continuous hence integrable on $[0, a]$) there is a $\xi \in (0, a)$ such that

$$af(\xi) = \int_0^a f(x)\,dx.$$

On the other hand the Fundamental Theorem of Calculus gives

$$f(\xi) = f(0) + \int_0^\xi f'(x)\,dx.$$

Putting these together gives

$$af(0) = \int_0^a f(x)\,dx - a\int_0^\xi f'(x)\,dx.$$

Now

$$a|f(0)| \le \int_0^a |f(x)|\,dx + a\int_0^\xi |f'(x)|\,dx \le \int_0^a |f(x)|\,dx + a\int_0^a |f'(x)|\,dx.$$

Dividing by $a$ (which is allowable and will preserve the inequality since $a > 0$) gives the result. One could also solve this using integration by parts.

4. Prove that if $\sum_{n=1}^\infty a_n$ is a convergent series of real numbers and the sequence of real numbers $\{b_n\}$ is monotonic and bounded, then the series $\sum_{n=1}^\infty a_n b_n$converges.

   **Solutions.** (This is Abel's test for convergence; see Rudin, Principles of Mathematical Analysis, Chapter 3, exercise 8, or Kaczor and Nowak, V.2, problem 3.2.15 for Abel's test for uniform convergence.) Here are two solutions; the first uses Dirichlet's test and the second uses only the summation by parts formula.

   **Solution 1.** First assume $\{b_n\}$ is non-increasing. Since $\{b_n\}$ is a bounded sequence, there is a (finite) number $b$ such that $b_n - b \ge 0$ and $b_n \to b$. Let $c_n = b_n - b$. Then $c_n \ge 0$ and $c_n \to 0$. Now we apply Dirichlet's test for convergence:

   **Theorem 1 (Dirichlet's test)** *Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers whose partial sums form a bounded sequence, and let $\{c_n\}_{n=1}^\infty$ be a non-increasing sequence of non-negative (real) numbers which converges to 0. Then $\sum_{k=1}^\infty a_k c_k$ converges.*

   Note $a_n b_n = a_n c_n + b a_n$. Since $\sum_1^\infty a_n c_n$ converges by the Dirichlet test and $\sum_1^\infty b a_n$ converges, $\sum_1^\infty a_n b_n$ converges also.

   If $\{b_n\}$ is non-decreasing then let $c_n = b - b_n$ Then $c_n \ge 0$, $\lim_{n\to\infty} c_n = 0$, and $a_n b_n = -a_n c_n + b a_n$. Now proceed as in the non-increasing case.

**Solution 2.** (See Rudin, Principles of Mathematical Analysis, Theorm 3.42. This proof is essentially the proof of the Dirichlet test.) As above assume first that $\{b_n\}$ is non-increasing and converges to $b$. Put $c_n = b_n - b$ and note $c_n \geq 0$, $\{c_n\}$ is non-increasing and converges to 0. As above it suffices to show $\sum_{n=1}^{\infty} a_n c_n$ converges. Let $a_0 = 0$ and for $k = 0, 1, \ldots$ let $A(k) = \sum_{l=0}^{k} a_k$. Then for $1 \leq m < n$, the summation by parts formula gives

$$\sum_{k=m}^{n} a_k c_k = \sum_{k=m}^{n-1} A(k)(c_k - c_{k+1}) + A(n)c_n - A(m-1)c_m.$$

Fix $\epsilon > 0$. Since $\sum_1^{\infty} a_n$ converges, the sequence $\{|A(k)|\}$ is bounded by some finite positive number $M$. Choose $N$ such that $0 < c_N < \epsilon/2M$. Then

$$\left|\sum_{k=m}^{n} a_k c_k\right| \leq \left|\sum_{k=m}^{n-1} A(k)(c_k - c_{k+1})\right| + \left|A(n)c_n - A(m-1)c_m\right|.$$

Since $c_k - c_{k+1} \geq 0$ and $c_k \geq 0$, this is

$$\leq M\left(\sum_{k=m}^{n-1} (c_k - c_{k+1}) + c_n + c_m\right).$$

Since the sum "telescopes," this is

$$\begin{aligned} &= M\left(-c_n + c_m + c_n + c_m\right) \\ &= M2c_m \leq M2c_N \leq \epsilon. \end{aligned}$$

If $b_n$ is non-decreasing, then put $c_n = b - b_n$ and proceed as above.

5. Suppose that $f$ is a real valued function of one real variable such that $\lim_{x\to c} f(x)$ exists for all $c \in [a, b]$. Show that $f$ is Riemann integrable on $[a, b]$.

   **Solution.** This is Berkeley, problem 1.5.4. I think the solution is incorrect. They claim that the hypotheses imply that $f$ has only finitely many discontinuities. But the function $f\colon [0,1] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x = 1/n \text{ for some natural number } n \\ 0 & \text{otherwise.} \end{cases}$$

   has the property that $\lim_{x\to c} f(x) = 0$ for all $c \in [0, 1]$ but has infinitely many points of (removable) discontinuity.

   It's not hard to give a solution using the following:

**Theorem 2 (Rudin, PMA, Theorem 11.33 (b))** $f\colon [a,b] \to \mathbf{R}$ *is Riemann integrable if and only if it is bounded and the set of points where $f$ is discontinuous has Lebesgue measure zero.*

Or one could appeal to the weaker corollary that if a function is bounded and has at most countably many discontinuities on $[a,b]$, then $f$ is Riemann integrable on $[a,b]$ (since a countable set has Lebesgue measure zero).

We need to show that the hypotheses imply that i) $f$ has at most countably many discontinuities and ii) $f$ is bounded.

But in fact a function with only simple discontinuities has at most countably many discontinuities (I don't think we need to prove this to give a complete solution to this problem, although we should know how to do it as that would be a good exam question!). It remains to show that $f$ must be bounded. Let

$$E_n = \{x \in [0,1] : |f(x)| < n\}.$$

Since $\lim_{x\to c} f(x)$ exists for all $c \in [0,1]$,

$$\cup_{n=1}^{\infty} E_n = [0,1].$$

Since $|f|$ is continuous, each $E_n$ is open in $[0,1]$ (in the subspace topology). Since $[0,1]$ is compact, there is a finite subcover $E_{n_1} \cup \cdots \cup E_{n_k}$. Then $|f|$ is bounded by $\max(n_1, \cdots n_k) < \infty$.

6. Show that for any continuous function $f : [0,1] \to \mathbf{R}$ and any $\epsilon > 0$, there is a function of the form

$$g(x) = \sum_{k=0}^{n} C_k x^{4k}$$

for some natural number $n$, where $C_0$, $C_1$, $\ldots$, $C_n$ are rational numbers and $|g(x) - f(x)| < \epsilon$ for all $x \in [0,1]$.

**Solution.** See Berkeley problem 1.6.31 and its solution. The idea is to let $h(x) = f(\sqrt[4]{x})$ and note that $h \in C^0([0,1])$ so $h$ can be uniformly approximated by a polynomial $P$. Then for all $x \in [0,1]$, we have $x^4 \in [0,1]$ so $f(x) = h(x^4)$ can be uniformly approximated by $P(x^4)$. Then once we have fixed such a polynomial $P$ we can approximate the coeficients of $P$ with rational numbers. See the Berkeley book for the complete solution.