787.03 Summer 2003 Problem Set 1

Due Monday, June 30, 2003.

These problems are from either the Berkely book (B), section 1.1 (Elementary Calculus) or Kaczor and Nowak, Vol. I (KNI), chapter 1.2 (Elementary Inequalities). I typed them up in case you don't have the books yet. I tried to look for typos, but there may be mistakes still. If a problem seems impossible, you can look at the solution hints in the back of the books (they're at Long's Bookstore and on reserve in the Science and Engineering library).

1. (B, 1.1.2) Let $f : [0,1] \to \mathbf{R}$ be continuously differentiable, with $f(0) = 0$. Prove that
$$\sup_{0\le x\le 1} |f(x)| \le \sqrt{\int_0^1 (f'(x))^2\, dx}.$$

2. (B, 1.1.6) Suppose $f : \mathbf{R} \to \mathbf{R}$ is a nonconstant function such that $f(x) \le f(y)$ whenever $x \le y$. Prove that there exist $a \in \mathbf{R}$ and $c > 0$ such that $f(a+x) - f(a-x) \ge cx$ for all $x \in [0,1]$.

3. (B, 1.1.17) Let f be a C^2 function on the real line. Assume f is bounded with bounded second derivative. Let
$$A = \sup_{x\in\mathbf{R}} |f(x)|, \qquad B = \sup_{x\in\mathbf{R}} |f''(x)|.$$
Prove that
$$\sup_{x\in\mathbf{R}} |f'(x)| \le 2\sqrt{AB}.$$

4. (B, 1.1.25) 1. For $0 \le \theta \le \frac{\pi}{2}$, show that
$$\sin\theta \ge \frac{2}{\pi}\theta.$$
2. By using Part 1, or any other method, show that if $\lambda < 1$, then
$$\lim_{R\to\infty} R^{\lambda} \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta = 0.$$

5. (B, 1.1.30) Let S be the set of all real C^1 functions f on $[0,1]$ such that $f(0) = 0$ and
$$\int_0^1 f'(x)^2\, dx \le 1.$$
Define
$$J(f) = \int_0^1 f(x)\, dx.$$
Show that J is bounded on S, and compute its supremum. Is there a function $f_0 \in S$ at which J attains its maximum value? If so, what is f_0?

6. (KNI, 1.2.1) Show that if $a_k > -1$, $k = 1, \dots, n$ are all positive or negative, then

$$(1 + a_1) \cdot (1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + \cdots + a_n$$

(Note that if $a_1 = \cdots = a_n$ then we get the well known Bernoulli inequality: $(1 + a)^n \geq 1 + na$, $a > -1$.)

7. (KNI, 1.2.21) Let $p_1, p_2, \dots, p_n$ be positive numbers and let a_k be arbitrary real numbers. Find the minimum of $\sum_{k=1}^{n} p_k a_k^2$ subject to the constraint $\sum_{k=1}^{n} a_k = 1$.

8. (KNI, 1.2.29) For positive a, b, c, verify the following claims:

   (a) $\dfrac{bc}{a} + \dfrac{ac}{b} + \dfrac{ab}{c} \geq (a + b + c)$

   (b) $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \geq \dfrac{1}{\sqrt{bc}} + \dfrac{1}{\sqrt{ca}} + \dfrac{1}{\sqrt{ab}}$

   (c) $\dfrac{2}{b+c} + \dfrac{2}{c+a} + \dfrac{2}{a+b} \geq \dfrac{9}{a+b+c}$

   (d) $\dfrac{b^2 - a^2}{c+a} + \dfrac{c^2 - b^2}{a+b} + \dfrac{a^2 - c^2}{b+c} \geq 0$

   (e) $\dfrac{1}{8}\dfrac{(a-b)^2}{a} \leq \dfrac{a+b}{2} - \sqrt{ab} \leq \dfrac{1}{8}\dfrac{(a-b)^2}{b}$ provided $b \leq a$.

9. (KNI, 1.2.48) Show that for positive $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$, we have

$$\sqrt[n]{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)} \geq \sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n}.$$