ERGODICITY AND MIXING OF NONCOMMUTING EPIMORPHISMS

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Abstract. We study mixing properties of epimorphisms of a compact connected finite-dimensional abelian group X. In particular, we show that a set F, $|F| > \dim X$, of epimorphisms of X is mixing iff every subset of F of cardinality $(\dim X)+1$ is mixing. We also construct examples of free nonabelian groups of automorphisms of tori which are mixing, but not mixing of order 3, and show that, under some irreducibility assumptions, ergodic groups of automorphisms contain mixing subgroups and free nonabelian mixing subsemigroups.

1. Introduction

1.1. **Mixing sets.** Let X be a compact abelian group, $\mathcal{B}$ the completion of the Borel σ-algebra of X, and m the normalized Haar measure on X. A finite set F, $|F| > 1$, of epimorphisms (i.e., continuous surjective self-homomorphisms) of X is called *mixing* if for any collection of measurable sets $B_\gamma \in \mathcal{B}$, $\gamma \in F$,

$$m\left(\bigcap_{\gamma\in F} \gamma^{-n}(B_\gamma)\right) \to \prod_{\gamma\in F} m(B_\gamma) \quad \text{as } n \to \infty.$$

Such set is sometimes also called mixing shape. It is clear that if F is mixing, then every subset of F is mixing as well. However, in general, the assumption that all proper subsets of F are mixing does not imply that F is mixing. For example, it was shown by F. Ledrappier that there exist commuting automorphisms γ_1 and γ_2 of a compact totally disconnected abelian group such that the sets $\{id, \gamma_1\}$ $\{id, \gamma_2\}$, $\{\gamma_1, \gamma_2\}$ are mixing, but the set $\{id, \gamma_1, \gamma_2\}$ is not mixing (see [13] and [21, Chapter VIII]). Also, if one does not assume commutativity, similar examples exist for connected groups as well (see Corollary 1.11 below).

K. Schimdt has shown that when the group X is connected and the epimorphisms which form the set F commute, the situation is quite different (see [20]):

Theorem 1.1 (Schmidt)**.** *Let X be a compact connected abelian group and F a finite set of commuting epimorphisms of X. Then the set F is mixing iff every subset of F of cardinality 2 is mixing.*

In this paper, we prove a noncommutative analog of Theorem 1.1:

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Theorem 1.2. *Let X be a compact connected abelian group such that $\dim X = d < \infty$ and F a finite set of epimorphisms of X with $|F| > \dim X$. Then the set F is mixing iff every subset E of F with $|E| = d+1$ is mixing.*

Theorem 1.2 and Theorem 1.1 (in the finite-dimensional case) follow from Theorem 1.3 below. We also show that the bound $d+1$ in Theorem 1.2 is sharp (see Corollary 1.11 below).

1.2. **Mixing sets and spectrum.** Let X be a compact connected abelian group with $\dim X = d < \infty$. We denote by $\hat{X}$ the character group of X. Under the above assumptions, $\hat{X}$ is a discrete abelian torsion free group of rank d. Hence, we may assume that

$$\mathbb{Z}^d \subset \hat{X} \subset \mathbb{Q}^d.$$

(Conversely, any abelian group A such that $\mathbb{Z}^d \subset A \subset \mathbb{Q}^d$ corresponds to a compact connected abelian group of dimension d.)

Any continuous endomorphism T of X defines an endomorphism $\hat{T}$ of $\hat{X}$ that extends to a linear map of $\mathbb{Q}^d$. Note that T is surjective iff $\hat{T}$ is nondegenerate (i.e., $\det \hat{T} \neq 0$).

We establish the following criterion for mixing in terms of eigenvectors of the corresponding linear maps of $\mathbb{Q}^d$.

Theorem 1.3. *A set $\{T_1, \dots, T_s\}$ of epimorphisms of X is mixing iff for every $l \geq 1$, every subset $\{k_1, \dots, k_r\} \subset \{1, \dots, s\}$, and every $\lambda \in \mathbb{C}$, there are no λ-eigenvectors of $\hat{T}^l_{k_1}, \dots, \hat{T}^l_{k_r}$ that are linearly dependent over $\mathbb{Q}$.*

Remark 1.4. It follows from the proof that in Theorem 1.3 one can replace "for every $l \geq 1$" by "for every $l \geq 1$ such that $\phi(l) \leq (\dim X)^2$", where ϕ denotes the Euler's totient function. Moreover, this estimate is sharp (see Example 6.1 below).

We state some corollaries of Theorem 1.3. Note that Corollary 1.5 is just another formulation of Theorem 1.1 in the finite-dimensional case, and Corollary 1.7 implies Theorem 1.2.

Corollary 1.5. *For commuting epimorphisms $T_1, \dots, T_s$ of X, the following statements are equivalent:*

(a) *The set $\{T_1, \dots, T_s\}$ is mixing.*
(b) *For every $i \neq j$, the set $\{T_i, T_j\}$ is mixing.*
(c) *For every $i \neq j$, the linear map $\hat{T}_i^{-1}\hat{T}_j$ does not have roots of unity as eigenvalues.*

For two (not necessarily commuting) epimorphisms, we have the following criterion for mixing:

Corollary 1.6. *The set of epimorphisms $\{T_1, T_2\}$ of X is mixing iff there is no closed subgroup $Y \neq X$ such that for some $l \geq 1$, Y is $\{T_1^l, T_2^l\}$-invariant and $T_1^l = T_2^l$ on X/Y.*

Corollary 1.6 may fail if the group X is disconnected or infinite-dimensional (see Example 6.2 below).

Denote by $\mathrm{Spec}(T)$ the set of eigenvalues of $\hat{T}$. The following corollary of Theorem 1.3 characterizes mixing in terms of spectrum:

Corollary 1.7. *Let $T_1, \ldots, T_s$ be epimorphisms of X.*

(a) *If for every $l \geq 1$ and $i, j = 1, \ldots, s$, $i \neq j$,*
$$\mathrm{Spec}(T_i^l) \cap \mathrm{Spec}(T_j^l) = \emptyset,$$
then $\{T_1, \ldots, T_s\}$ is mixing.

(b) *If for some $l \geq 1$ and $S \subset \{1, \ldots, s\}$ such that $|S| > d$,*
$$\bigcap_{i \in S} \mathrm{Spec}(T_i^l) \neq \emptyset,$$
then $\{T_1, \ldots, T_s\}$ is not mixing.

(c) *If for every $l \geq 1$ and $S \subset \{1, \ldots, s\}$ such that $|S| > d$,*
$$\bigcap_{i \in S} \mathrm{Spec}(T_i^l) = \emptyset,$$
then $\{T_1, \ldots, T_s\}$ is mixing iff every subset of cardinality d is mixing.

Remark 1.8. In Corollary 1.7, one can replace "for every $l \geq 1$" by "for every $l \geq 1$ such that $\phi(l) \leq (\dim X)^2$". Moreover, this estimate is sharp (see Example 6.1 below).

Corollary 1.7(a) shows that, if epimorphisms $T_1, \ldots, T_s$ are "spectrally independent", then for every $B_1, \ldots, B_s \in \mathcal{B}$,
$$\lim_{n \to \infty} m(T_1^{-n} B_1 \cap \cdots \cap T_s^{-n} B_s) = m(B_1) \cdots m(B_s).$$
Although this limit does not exist in general (consider, for example, $T_1 = id$ and $T_2 = -id$), the proof of Theorem 1.3 implies the following corollary.

Corollary 1.9. *For any finite set $\{T_1, \ldots, T_s\}$ of epimorphisms of X, there exists $l \geq 1$ such that for every $k \in \mathbb{Z}/l\mathbb{Z}$ and $f_1, \ldots, f_s \in L^\infty(X)$, the limit*
$$\lim_{\substack{n \to \infty \\ n = k (\mathrm{mod}\, l)}} \int_X f_1(T_1^n x) \cdots f_s(T_s^n x)\, dm(x) \tag{1.1}$$
exists.

Remark 1.10. (i) It follows from the proof that the integer l appearing in Corollary 1.9 can be chosen so that $\phi(l) \leq (\dim X)^2$. Moreover, this estimate is sharp (see Example 6.1 below).

(ii) Corollary 1.9 is, in general, false if the group X is either infinite-dimensional or disconnected (see Example 6.3 below).

(iii) Existence of the Cesàro limit

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M+1}^{N} f_1(T_1^n x) \cdots f_s(T_s^n x)$$

in $L^2(X)$ for a certain class of epimorphisms of a compact abelian group X was proved by D. Berend in [3]. Corollary 1.9 strengthens Berend's result in the case when the group X is connected and finite-dimensional.

We call an automorphism T of X *unipotent* if the matrix $\hat{T}$ is unipotent.

Corollary 1.11. (a) *For every $s = 2, \dots, d+1$ there exists a set F with $|F| = s$ consisting of unipotent automorphisms of $\mathbb{T}^d$ such that F is not mixing, but every proper subset of F is mixing.*

(b) *For every $s = 2, \dots, d+1$ there exists a set of mixing epimorphisms F of $\mathbb{T}^d$ with $|F| = s$ such that F is not mixing, but every proper subset of F is mixing.*

The following corollary relates the notion of "mixing sets" (terminology from [21]) with the notion of "jointly mixing automorphisms" which was introduced in [2] and used in [5]. Epimorphisms $T_1, \dots, T_{s-1}$ are called *jointly mixing* if the set $\{T_1, \dots, T_{s-1}, id\}$ is mixing in our terminology.

Corollary 1.12. *The set $\{T_1, \dots, T_{s-1}, id\}$ of epimorphisms of X is mixing iff every T_i is mixing and $\{T_1, \dots, T_{s-1}\}$ is mixing.*

1.3. **Mixing groups and semigroups.** A semigroup Γ of epimorphisms of X is called *mixing* if for every $A, B \in \mathcal{B}$,

$$m(A \cap \gamma^{-1} B) \to m(A)m(B)$$

as $\gamma \to \infty$.

A semigroup Γ of of epimorphisms of X is *mixing of order s* if for every $B_1, \dots, B_s \in \mathcal{B}$,

$$m(\gamma_1^{-1} B_1 \cap (\gamma_2\gamma_1)^{-1} B_2 \cap \cdots \cap (\gamma_s \cdots \gamma_1)^{-1} B_s) \to m(B_1) \cdots m(B_s)$$

as the product $\gamma_j \cdots \gamma_i \to \infty$ for $1 < i \le j \le s$. Note that mixing corresponds to mixing of order 2.

We recall a classical result of Rokhlin (see [17]):

Theorem 1.13 (Rokhlin). *If a continuous epimorphism T of a compact abelian group is ergodic, then it is* mixing of all orders, *that is, for every $s \ge 1$, $B_1, \dots, B_s \in \mathcal{B}$, and $n_1, \dots, n_s \in \mathbb{N}$ such that $|n_i - n_j| \to \infty$ for $i \ne j$,*

$$m(T^{-n_1} B_1 \cap \cdots \cap T^{-n_s} B_s) \to m(B_1) \cdots m(B_s).$$

This result was extended to finitely generated abelian groups of automorphisms by K. Schmidt and T. Ward in [22]:

Theorem 1.14 (Schmidt, Ward)**.** *Let X be a compact connected abelian group and $\Gamma \subset \mathrm{Aut}(X)$, $\Gamma \simeq \mathbb{Z}^n$. Then Γ consists of ergodic automorphisms iff it is mixing of all orders.*

Note that the ergodic properties of the actions in Theorems 1.13 and 1.14 are quite different. The epimorphism T in Theorem 1.13 has completely positive entropy (see [18]), but the entropy of Γ-action in Theorem 1.14 is zero if $n > 1$ (see [21, Ch. V]).

While it is true that an arbitrary group Γ of automorphisms is mixing provided that every element of infinite order is ergodic (see Corollary 4.3 below), the statement about higher order of mixing fails if Γ is not virtually abelian. As an easy corollary of Corollary 1.7(b), we deduce the following result:

Corollary 1.15 (Bhattacharya)**.** *Let X be a compact connected abelian group with $\dim X = d < \infty$. Then every subgroup of $\mathrm{Aut}(X)$ which is not virtually abelian is not mixing of order $d+1$.*

Note that there exist free nonabelian semigroups of epimorphisms which are mixing of all orders (see Examples 6.6 and 6.7 below).

Corollary 1.15 was first proved by Bhattacharya in [6]. He also discovered some interesting rigidity properties of mixing subgroups which are not virtually abelian. However, it is not obvious whether such subgroups exist. In this direction, we show:

Theorem 1.16. *For every $d \geq 2$, $d \neq 3, 5, 7$, there exists a not virtually abelian mixing subgroup of $\mathrm{Aut}(\mathbb{T}^d)$ which is not mixing of order 3.*

At present, we don't know whether there are such examples for $d = 3, 5, 7$.

Mixing property is much better understood for $\mathbb{Z}^n$-actions by automorphisms of a compact abelian group X. When X is connected, 2-mixing implies mixing of all orders (see Theorem 1.14). If X is totally disconnected, then for every $s \geq 2$, there are examples that are s-mixing but not $(s+1)$-mixing (see [9]). It is also known that a $\mathbb{Z}^n$-action is s-mixing iff every subset of $\mathbb{Z}^n$ of cardinality s is mixing (see [14]).

1.4. **Ergodicity and mixing.** In this subsection we discuss some analogs of Rokhlin's theorem (Theorem 1.13) for general groups of automorphisms. Namely, given a compact abelian group X and a subgroup Γ of $\mathrm{Aut}(X)$, we investigate whether ergodicity implies mixing and mixing of higher orders. Recall that Γ is called *ergodic* if every measurable Γ-invariant subset of X has measure 0 or 1. Ergodicity is a weaker notion than mixing. In fact, if Γ contains a mixing automorphism, then it is ergodic. D. Berend showed in [1] that the converse is also true in the case when Γ is abelian:

Theorem 1.17 (Berend)**.** *Let X be a compact connected finite-dimensional abelian group and Γ an ergodic abelian semigroup of epimorphisms of X. Then Γ contains an ergodic epimorphism.*

Note that by Rokhlin's theorem, an ergodic epimorphism is mixing of all orders.

On the other hand, if Γ is not abelian, it may contain no mixing elements (see [1] or Examples 6.8 and 6.9 below). A somewhat stronger version of ergodicity — "hereditary ergodicity", which we will presently introduce, is more closely related to mixing and will allow us to naturally generalize Berend's theorem.

Let X be a compact abelian group, Y a closed subgroup of X, and $\Gamma \subset \mathrm{Aut}(X)$. We define

$$\Gamma_Y = \{\gamma \in \Gamma : \gamma \cdot Y \subset Y\}.$$

If Γ_Y has finite index in Γ, we call the subgroup Y *virtually* Γ*-invariant*. In the case when X contains no proper closed connected virtually Γ-invariant subgroups, we call the group Γ *strongly irreducible*. Note that for connected group X, strong irreducibility implies ergodicity (see Proposition 5.2 below), but the converse is not true (see Example 6.8 below). We call a subgroup $\Gamma \subset \mathrm{Aut}(X)$ *hereditarily ergodic* if for every closed connected virtually Γ-invariant subgroup Y of X, the action of Γ_Y on Y is ergodic.

It is not hard to check that for abelian groups of automorphisms of compact connected finite-dimensional group X, the notions of ergodicity and hereditary ergodicty coincide (this fails, in general, for infinite-dimensional groups X — see Example 6.10 below). Hence, Berend's theorem in this case states that hereditary ergodicity is equivalent to existence of an automorphism which is mixing of all orders. The following theorem generalizes this result to solvable groups of automorphisms.

Theorem 1.18. *Let X be a compact connected finite-dimensional abelian group and Γ a solvable subgroup of automorphisms of X. Then the following statements are equivalent:*

(a) Γ *is hereditarily ergodic.*
(b) Γ *contains an abelian subgroup which is mixing of all orders.*

Note that the assumption in Theorem 1.18 that the group Γ is solvable is essential (see Example 6.9 below). Also, Theorem 1.18 fails without the assumption that X is finite-dimensional (see Examples 6.10 and 6.11 below).

According to the *Rosenblatt's alternative* (see [19]), any finitely generated solvable group is either virtually nilpotent or contains a free nonabelian subsemigroup. In the latter case Theorem 1.18 can be strengthened as follows:

Theorem 1.19. *Let X be a compact connected finite-dimensional abelian group and Γ a solvable group of automorphisms of X, which is not virtually nilpotent. Then the following statements are equivalent:*

(a) Γ *is hereditarily ergodic.*
(b) Γ *contains a free nonabelian subsemigroup which is mixing of all orders.*

Combining Theorems 1.18 and 1.19, we deduce

Corollary 1.20. *Let X be a compact connected finite-dimensional abelian group and Γ a solvable strongly irreducible group of automorphisms of X. Then Γ contains an* *abelian subgroup which is mixing of all orders. Moreover, if* Γ *is not virtually nilpotent, then* Γ *contains a free nonabelian subsemigroup which is mixing of all orders.*

Without the assumption that the group Γ is solvable, Corollary 1.20 fails (see Example 6.9 below).

It follows from the *Tits alternative* (see [24] or [15, Section 5J]) that any finitely generated subgroup of $\mathrm{Aut}(X)$ is either virtually solvable or contains a nonabelian free group. Recently, E. Breuillard and T. Gelander proved a *topological Tits alternative* (see [7]): any finitely generated matrix group either contains a Zariski open solvable subgroup or a Zariski dense free subgroup. Utilizing this result, we obtain

Theorem 1.21. *Let* X *be a compact connected finite dimensional abelian group and* Γ *an ergodic (hereditarily ergodic, strongly irreducible) subgroup of* $\mathrm{Aut}(X)$ *which is not virtually solvable. Then* Γ *contains a free nonabelian ergodic (hereditarily ergodic, strongly irreducible) subgroup.*

Example 6.9 below illustrates that an ergodic group may contain no ergodic elements.

1.5. Some special cases of the above results appeared in [5]. Note that in [5] we used a slightly different definition for mixing (borrowed from [2]), but in this paper we adopt the definition from [21]. The relation between these two definitions is quite straightforward (see Corollary 1.12).

The paper is organized as follows. The main theorem (Theorem 1.3) is proved in Section 2. The rest of the results stated in Subsection 1.2 are proved in Section 3. The results about mixing groups of automorphisms (stated in Subsection 1.3) are proved in Section 4. The theorems from Subsection 1.4 are proved in Section 5. Section 6 contains some examples and counterexamples related to the results of this paper.

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2. Mixing and linear relations (proof of Theorem 1.3)

Let X be a compact connected abelian group of finite dimension d. We identify its character group $\hat{X}$ with a subgroup of $\mathbb{Q}^d$. Then every endomorphism T of X induces a linear map $\hat{T}$ of $\mathbb{Q}^d$.

We recall the well-known characterization of mixing:

Lemma 2.1. *The set* $\{T_1, \dots, T_s\}$ *of epimorphisms of* X *is mixing iff there are no* $x_1, \dots, x_s \in \mathbb{Q}^d$ *such that* $(x_1, \dots, x_s) \neq (0, \dots, 0)$ *and for infinitely many* $n \geq 1$,

$$\hat{T}_1^n x_1 + \cdots + \hat{T}_s^n x_s = 0.$$

As an application of Lemma 2.1, we show that the set of epimorphisms $\{T_1, \dots, T_s\}$, $s > d$, is not mixing provided that the linear maps $\hat{T}_1, \dots, \hat{T}_s$ have the same characteristic polynomial.

Proposition 2.2. *Let $T_1,\dots,T_s$ be epimorphisms of X and assume that there exists a polynomial $p(x)\in\mathbb{Q}[x]$ with $\deg p<s$ such that $p(\hat{T}_i)=0$ for $i=1,\dots,s$. Then $\{T_1,\dots,T_s\}$ is not mixing.*

Proof. To prove the proposition, it suffices to construct $(x_1,\dots,x_s)\in(\mathbb{Q}^d)^s-\{(0,\dots,0)\}$ such that

$$\hat{T}_1^n x_1+\cdots+\hat{T}_s^n x_s=0 \tag{2.1}$$

for infinitely many n.

Let $p(x)=p_1(x)^{e_1}\cdots p_l(x)^{e_l}$ where $p_i(x)\in\mathbb{Q}[x]$, $i=1,\dots,l$, are distinct and irreducible. Let $d_i=\deg p_i$ and $\lambda_{i,j}$, $j=1,\dots,d_i$, be the roots of p_i. Let $P_{i,j,k}\in\mathrm{M}(d,\mathbb{Q}(\lambda_{i,j}))$ be the projection on the root space of T_k corresponding to $\lambda_{i,j}$. Then

$$\hat{T}_k^n P_{i,j,k}=\lambda_{i,j}^n\sum_{u=0}^{e_i-1} n^u A_{i,j,k,u}$$

for some $A_{i,j,k,u}\in\mathrm{M}(d,\mathbb{Q}(\lambda_{i,j}))$. Since the coefficients of the $(e_i d)\times sd$ matrix

$$B_{i,j}=(A_{i,j,k,u}: u=0,\dots,e_i-1,\ k=1,\dots,s)$$

lie in $\mathbb{Q}(\lambda_{i,j})$, we have

$$\mathrm{rank}_{\mathbb{Q}}(B_{i,j})\le e_i d\cdot[\mathbb{Q}(\lambda_{i,j}):\mathbb{Q}]=e_i d d_i.$$

It follows that for the $(le_i d)\times sd$ matrix

$$C=(B_{i,1}: i=1,\dots l),$$

$$\mathrm{rank}_{\mathbb{Q}}(C)\le\sum_{i=1}^{l} e_i d d_i=(\deg p)\cdot d.$$

Hence, there exists a vector $x=(x_1,\dots,x_s)\in(\mathbb{Q}^d)^s-\{(0,\dots,0)\}$ such that $Cx=0$. Then

$$\sum_{k=1}^{s}\hat{T}_k^n P_{i,1,k}x_k=0 \tag{2.2}$$

for every $i=1,\dots,l$. For fixed i,k, the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ permutes transitively the roots $\lambda_{i,j}$ and the matrices $P_{i,j,k}$, $j=1,\dots,e_i$. Hence, if follows from (2.2) that for every $j=1,\dots,e_i$,

$$\sum_{k=1}^{s}\hat{T}_k^n P_{i,j,k}x_k=0. \tag{2.3}$$

Summing (2.3) over i and j, we deduce (2.1). ☐

To analyze the equation in Lemma 2.1, we use the following statement sometimes referred to as Kronecker's lemma (see [10, p. 27]):

Lemma 2.3 (Kronecker). *If λ is an algebraic integer such that all of its conjugates have absolute value one, then λ is a root of unity.*

We also mention an equivalent formulation of Kronecker's lemma, which we use latter: if λ is an element of a number field K and λ is not a root of unity, then there exists an absolute value $|\cdot|_v$ on K such that $|\lambda|_v \neq 1$.

Note that $\{T_1, \dots, T_s\}$ is mixing iff $\{T_1^l, \dots, T_s^l\}$ is mixing for some (all) $l \geq 1$. This observation implies that in the proof of Theorem 1.3 we may assume without loss of generality that

$$\lambda, \mu \in \bigcup_{k=1}^{s} \mathrm{Spec}(T_k) \text{ and } \lambda^{-1}\mu \text{ a root of unity } \Rightarrow \lambda = \mu. \tag{2.4}$$

Under this assumption, Theorem 1.3 can be restated as follows

Theorem 2.4. *Let $T_1, \dots, T_s$ be epimorphisms of X that satisfy (2.4). Then the set $\{T_1, \dots, T_s\}$ is mixing iff for every subset $\{k_1, \dots, k_r\} \subset \{1, \dots, s\}$ and every $\lambda \in \mathbb{C}$, there are no λ-eigenvectors of $\hat{T}_{k_1}, \dots, \hat{T}_{k_r}$ that are linearly dependent over $\mathbb{Q}$.*

Proof. Suppose that there exist a nonempty subset $S \subset \{1, \dots, s\}$, $\alpha_k \in \mathbb{Q} - \{0\}$, $k \in S$, and eigenvalues w_k for $\hat{T}_k$, $k \in S$, with the same eigenvalue λ such that

$$\sum_{k \in S} \alpha_k w_k = 0.$$

This implies that the subspace

$$V = \left\{ (v_k) \in (\mathbb{C}^d)^{|S|} : \sum_{k \in S} \alpha_k \hat{T}_k^n v_k = 0 \text{ for all } n \geq 1 \right\}$$

is not trivial. Since this subspace is defined over $\mathbb{Q}$, it contains a nonzero rational vector $(x_k : k \in S)$ that gives a nonzero solution of the equation

$$\sum_{k=1}^{s} \hat{T}_k^n x_k = 0. \tag{2.5}$$

Hence, by Lemma 2.1, the set $\{T_1, \dots, T_s\}$ is not mixing.

Conversely, suppose that the set $\{T_1, \dots, T_s\}$ is not mixing. Then by Lemma 2.1, there exists $(x_1, \dots, x_s) \in (\mathbb{Q}^d)^s - \{(0, \dots, 0)\}$ such that (2.5) holds for infinitely many $n \geq 1$.

Let

$$p_k(x) = p_{k,1}(x)^{m_{k,1}} \cdots p_{k,l_k}(x)^{m_{k,l_k}}$$

be the characteristic polynomial of $\hat{T}_k$, $k = 1, \dots, s$, where $p_{k,i}(x) \in \mathbb{Q}[x]$ are distinct and irreducible over $\mathbb{Q}$. Let $d_{k,i} = \deg(p_{k,i})$. For a root λ of p_k, denote by V_k^λ the root subspace of T_k with respect to λ. Then

$$\mathbb{C}^d = \bigoplus_{\lambda:\, p_k(\lambda)=0} V_k^\lambda.$$

Note that for fixed k and i, the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ permutes transitively the spaces V_k^λ where λ satisfies $p_{k,i}(\lambda) = 0$. This implies that the subspaces

$$V_{k,i} = \bigoplus_{\lambda:\, p_{k,i}(\lambda)=0} V_k^\lambda$$

are rational. Then

$$\mathbb{Q}^d = \bigoplus_{i=1}^{l_k} V_{k,i}(\mathbb{Q}), \tag{2.6}$$

and there exist vectors $x_{k,i} \in V_{k,i}(\mathbb{Q})$, not all zero, such that

$$\sum_{k=1}^{s} \sum_{i=1}^{l_k} \hat{T}_k^n x_{k,i} = 0 \tag{2.7}$$

for infinitely many $n \geq 1$. For a root λ of $p_{k,i}$, let P_k^λ denote the projection from $V_{k,i}$ on the root space V_k^λ. Since

$$\hat{T}_k|_{V_k^\lambda} = \lambda(id + N_k^\lambda)$$

where $N_k^\lambda : V_k^\lambda \to V_k^\lambda$ is nilpotent linear map such that $(N_k^\lambda)^{m_{k,i}} = 0$, we have

$$\hat{T}_k^n P_k^\lambda = \lambda^n \sum_{u=0}^{m_{k,i}-1} \binom{n}{u} A_{k,u}^\lambda$$

where $A_{k,u}^\lambda : V_{k,i} \to V_k^\lambda$ are linear maps and $A_{k,0}^\lambda = P_k^\lambda$. With respect to a rational basis on $V_{k,i}$, $A_{k,u}^\lambda$ is represented by $d \times (\dim V_{k,i})$ matrix with coefficients in $\mathbb{Q}(\lambda)$. Then (2.7) is equivalent to

$$\sum_{k=1}^{s} \sum_{i=1}^{l_k} \sum_{\lambda:\, p_{k,i}(\lambda)=0} \sum_{u=0}^{m_{k,i}-1} \lambda^n \binom{n}{u} A_{k,u}^\lambda x_{k,i} = 0. \tag{2.8}$$

Denote by K the number field generated by the eigenvalues of T_i, $i = 1, \ldots, s$, and let $\mathcal{V}_K$ be the set of absolute values of K.

Since (2.8) holds for infinitely many n, it is equivalent to the system of equations

$$\sum_{k,i,\lambda,j}{}' \sum_{u=0}^{m_{k,i}-1} \lambda^n \binom{n}{u} A_{k,u}^\lambda x_{k,i} = 0, \quad \delta > 0, \tag{2.9}$$

where the sum $\sum'$ is taken over those λ's such that $p_{k,i}(\lambda)=0$ and $|\lambda|_v=\delta$, $v\in\mathcal{V}_K$. Conjugating (2.9) by $\sigma\in\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, we deduce that (2.9) is equivalent to the system of equations

$$\sum_{k,i,\lambda,j}{}'' \sum_{u=0}^{m_{k,i}-1} \lambda^n \binom{n}{u} A^\lambda_{k,u} x_{k,i}=0, \quad \delta_{\sigma,v}>0,\, \sigma\in\mathrm{Gal}(\mathbb{C}/\mathbb{Q}),\, v\in\mathcal{V}_K, \tag{2.10}$$

where the sum $\sum''$ is taken over λ's such that $p_{k,i}(\lambda)=0$ and $|\lambda^\sigma|_v=\delta_{\sigma,v}$ for every $\sigma\in\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and $v\in\mathcal{V}_K$.

If

$$\lambda,\mu\in\bigcup_{k=1}^s \mathrm{Spec}(T_k)$$

and $|\lambda^\sigma|_v=|\mu^\sigma|_v$ for every $\sigma\in\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and $v\in\mathcal{V}_K$, then $\lambda^{-1}\mu$ is a root of unity by Lemma 2.3, and by (2.4), $\lambda=\mu$. Hence, (2.10) is equivalent to the system of equations

$$\sum_{k,i:\,p_{k,i}(\lambda)=0} \sum_{u=0}^{m_{k,i}-1} \binom{n}{u} A^\lambda_{k,u} x_{k,i}=0, \quad \lambda\in\bigcup_{k=1}^s \mathrm{Spec}(T_k). \tag{2.11}$$

Let $m_\lambda=\max\{m_{k,i} : p_{k,i}(\lambda)=0\}$. Since (2.11) holds for infinitely many n, it is equivalent to

$$\sum_{k,i:\,p_{k,i}(\lambda)=0} A^\lambda_{k,u} x_{k,i}=0, \quad \lambda\in\bigcup_{k=1}^s \mathrm{Spec}(T_k),\, u=0,\dots,m_\lambda-1. \tag{2.12}$$

For every $k=1,\dots,s$ and $i=1,\dots,l_k$, choose $\lambda_{k,i}$ such that $p_{k,i}(\lambda_{k,i})=0$. If $p_{k,i}$'s have a common root for different k's, we choose the same $\lambda_{k,i}$. Let

$$\Lambda=\{\lambda_{k,i} : k=1,\dots,s,\, i=1,\dots,l_k\}.$$

Note that for $\sigma\in\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, we have

$$\sigma(V^\lambda_k)=V^{\sigma(\lambda)}_k,\ \sigma(P^\lambda_k)=P^{\sigma(\lambda)}_k,\ \sigma(N^\lambda_k)=N^{\sigma(\lambda)}_k,\ \sigma(A^\lambda_{k,u})=A^{\sigma(\lambda)}_{k,u}.$$

Since the polynomial $p_{k,i}$ is irreducible, the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts transitively on the set of roots of $p_{k,i}$. Hence, if (2.12) holds for $\lambda=\lambda_{k,i}$, then it holds for all λ's such that $p_{k,i}(\lambda)=0$. Therefore, (2.12) is equivalent to

$$\sum_{k,i:\,p_{k,i}(\lambda)=0} A^\lambda_{k,u} x_{k,i}=0, \quad \lambda\in\Lambda,\, u=0,\dots,m_\lambda-1. \tag{2.13}$$

Since polynomials $p_{k,i}$, $i=1,\dots,l_k$, have no common roots, it follows that for every $k=1,\dots,s$ and $\lambda\in\Lambda$, there is at most one i such that $p_{k,i}(\lambda)=0$. Hence, the system of equations (2.13) splits into independent systems of equations

$$\sum_{k,i:\,p_{k,i}(\lambda)=0} A^{\lambda}_{k,u} x_{k,i} = 0, \quad u = 0,\dots,m_\lambda - 1 \tag{2.14}$$

indexed by $\lambda \in \Lambda$. Therefore, (2.13) has a nontrivial solution iff for some $\lambda \in \Lambda$, (2.14) has a nontrivial solution.

Let $\lambda \in \Lambda$ be such that (2.14) has a nontrivial solution and $u_0 \in \{0,\dots,m_\lambda - 1\}$ be maximal index such that (2.14) contains nonzero terms. Since

$$A^{\lambda}_{k,u} = \lambda^{-u}(\hat{T}_k - \lambda)^u|_{V_k^\lambda}, \quad u \ge 1,$$

it follows that (nonzero) vectors $A^{\lambda}_{k,u_0} x_{k,i}$ are eigenvectors of $\hat{T}_k$ with eigenvalue λ which are linearly dependent over $\mathbb{Q}$. This proves the theorem. $\square$

3. Proofs of Corollaries formulated in Subsection 1.2

Proof of Corollary 1.5. It is clear that (a)$\Rightarrow$(b).

Suppose that $\hat{T}_i^{-1}\hat{T}_j$ has a root of unity as an eigenvalue. Since $\hat{T}_i$ and $\hat{T}_j$ commute, this implies that for some $l \ge 1$, the subspace

$$V = \{v \in \mathbb{C}^d : \hat{T}_i^l v = \hat{T}_j^l v\}$$

is not $\{0\}$. Since V is $\{\hat{T}_i^l, \hat{T}_j^l\}$-invariant and $\hat{T}_i^l|_V = \hat{T}_j^l|_V$, it follows that $\hat{T}_i^l$ and T_j^l have common eigenvector with the same eigenvalue. Hence, by Theorem 1.3, $\{T_i, T_j\}$ is not mixing. This shows that (b)$\Rightarrow$(c).

To prove that (c)$\Rightarrow$(a), suppose that (c) holds, but $\{T_1,\dots,T_s\}$ is not mixing. Then by Theorem 1.3, there exist a nonempty $S \subset \{1,\dots,s\}$, $\alpha_k \in \mathbb{Q} - \{0\}$, $k \in S$, and eigenvectors $w_k \in \mathbb{C}^d$ of $\hat{T}_k^l$, $k \in S$, with the same eigenvalue λ such that

$$\sum_{k\in S} \alpha_k w_k = 0.$$

Hence, we have a nonzero vector space

$$V = \left\{ (v_k) \in (\mathbb{C}^d)^{|S|} : \sum_{k\in S} \alpha_k v_k = 0, \ \ \hat{T}_k^l v_k = \lambda v_k \ \text{ for } k \in S \right\}.$$

Since $\hat{T}_i$'s commute, this vector space is $\{\hat{T}_1^l,\dots,\hat{T}_s^l\}$-invariant, and it contains a common eigenvector $v = (v_k : k \in S)$:

$$\hat{T}_k^l v = \lambda_k v, \quad k \in S.$$

Let $S_0 = \{k \in S : v_k \ne 0\}$. Note that $|S_0| > 1$. For $k \in S_0$, $\lambda_k = \lambda$. Hence, $\hat{T}_i^{-l}\hat{T}_j^l = (\hat{T}_i^{-1}\hat{T}_j)^l$ has eigenvalue 1 for $i, j \in S_0$. This contradicts (c). Hence, (c)$\Rightarrow$(a). $\square$

Proof of Corollary 1.6. If $\{T_1, T_2\}$ is mixing on X, then clearly, $\{T_1^l, T_2^l\}$ is mixing on X/Y. Hence, one direction of the corollary is obvious.

Suppose that $\{T_1, T_2\}$ is not mixing. Then, by Theorem 1.3, $\hat{T}_1^l$ and $\hat{T}_2^l$ have common eigenvector with the same eigenvalue for some $l \geq 1$. This eigenvector is contained in the rational subspace

$$V = \{v \in \mathbb{C}^d : \hat{T}_1^l v = \hat{T}_2^l v,\ \hat{T}_1^l \hat{T}_2^l v = \hat{T}_2^l \hat{T}_1^l v\}.$$

Consider the subgroup

$$Y = \{x \in X : \chi(x) = 1 \quad \text{for} \ \ \chi \in V \cap \hat{X}\}.$$

Since V is rational, $V \cap \hat{X} \neq 0$ and $Y \neq X$, and since the subspace V is $\{\hat{T}_1^l, \hat{T}_2^l\}$-invariant, the subgroup Y is $\{T_1^l, T_2^l\}$-invariant. The character group of X/Y is $V \cap \hat{X}$, and $\hat{T}_1^l = \hat{T}_2^l$ on $V \cap \hat{X}$. Hence, it follows that $T_1^l = T_2^l$ on X/Y. This proves the corollary. □

Proof of Corollary 1.7. Under the assumption in (a), for every $\lambda \in \mathbb{C}$, there is at most one $\hat{T}_i^l$ with the eigenvalue λ. Hence, the maps $\hat{T}_i^l$ cannot have linearly dependent eigenvectors with the same eigenvalue, and by Theorem 1.3, the set $\{T_1, \dots, T_s\}$ is mixing.

Suppose that there exist $S \subset \{1, \dots, s\}$ with $|S| = r > d$, $l \geq 1$, and $\lambda \in \mathbb{C}$ such that

$$\lambda \in \mathrm{Spec}(\hat{T}_k^l) \quad \text{for } k \in S.$$

We are going to show now that the set $\{T_k^l : k \in S\}$ is not mixing. This will imply that the set $\{T_1, \dots, T_s\}$ is not mixing as well.

Denote by $q(x) \in \mathbb{Q}[x]$ the minimal polynomial of λ and consider a rational subspace

$$W_k = \{v \in \mathbb{C}^d : q(\hat{T}_k^l) v = 0\}.$$

Note that W_k contains all μ-eigenspaces of $\hat{T}_k^l$ such that $q(\mu) = 0$. In particular,

$$\dim W_k \geq \deg(q).$$

Denote by P_k the projection from W_k to the λ-eigenspace of $\hat{T}_k^l$. According to Theorem 1.3, it suffices to show that there exist $x_k \in W_k(\mathbb{Q})$, not all zero, such that

$$\sum_{k \in S} P_k x_k = 0. \tag{3.1}$$

Choose a rational basis in W_k. With respect to this basis, the linear map P_k is represented by $d \times (\dim W_k)$ matrix with coefficients in $\mathbb{Q}(\lambda)$. Consider the $d \times (\dim W_k^\lambda \cdot |S|)$-matrix

$$P = (P_k : k \in S).$$

Since the coefficients of P are in $\mathbb{Q}(\lambda)$,

$$\mathrm{rank}_{\mathbb{Q}}(P) \leq d \cdot [\mathbb{Q}(\lambda) : \mathbb{Q}] = d \cdot \deg(q) < |S| \cdot \dim W_k.$$

Hence, there exists nonzero vector

$$x = (x_k : k \in S) \in \prod_{k \in S} W_k(\mathbb{Q}) \simeq \mathbb{Q}^{|S|\cdot \dim W_k}$$

such that $P \cdot x = 0$. Hence, (3.1) has a nonzero solution. This proves (b).

Now we prove (c). Suppose that for every $l \geq 1$ and $S \subset \{1, \ldots, s\}$ such that $|S| > d$,

$$\bigcap_{i \in S} \mathrm{Spec}(T_i^l) = \emptyset,$$

Then if $\hat{T}_i^l$'s have linearly dependent eigenvectors with the same eigenvalue, there is a subset of $\hat{T}_i^l$'s of cardinality at most d with the same property. Hence, it follows from Theorem 1.3 that $\{T_1, \ldots, T_s\}$ is mixing iff every subset of cardinality d is mixing. $\square$

Proof of Corollary 1.9. We choose $l \geq 1$ so that $T_1^l, \ldots, T_s^l$ satisfy condition (2.4). It suffices to prove the corollary for $k = 0$, and to simplify calculations, we also assume that $l = 1$. The proof of the general case easily reduces to this situation.

We use the notation introduced in the proof of Theorem 2.4.

For $\chi_1, \ldots, \chi_s \in \hat{X}$,

$$\int_X \chi_1(T_1^n x) \cdots \chi_s(T_s^n x)\, dm(x) = \begin{cases} 1 \text{ if } \hat{T}_1^n \chi_1 + \cdots + \hat{T}_s^n \chi_s = 0, \\ 0 \text{ if } \hat{T}_1^n \chi_1 + \cdots + \hat{T}_s^n \chi_s \neq 0. \end{cases} \tag{3.2}$$

Denote by $Q_{k,i}$ the projection on the space $V_{k,i}$ with respect to the decomposition (2.6). If

$$\hat{T}_1^n \chi_1 + \cdots + \hat{T}_s^n \chi_s = 0 \tag{3.3}$$

for infinitely many n, then by the proof of Theorem 1.3,

$$\sum_{k,i:p_{k,i}(\lambda)=0} A_{k,u}^{\lambda} Q_{k,i} \chi_k = 0 \quad \text{for } \lambda \in \Lambda,\ u = 0, \ldots, m_\lambda - 1. \tag{3.4}$$

Conversely, (3.4) implies that (3.3) holds for every $n \geq 1$. Denote by Δ the set of $(\chi_1, \ldots, \chi_s) \in \hat{X}^s$ such that (3.4) holds. We claim that for every $f_1, \ldots, f_s \in L^\infty(X)$,

$$\lim_{n\to\infty} \int_X f_1(T_1^n x) \cdots f_s(T_s^n x)\, dm(x) = \sum_{(\chi_1, \ldots, \chi_s) \in \Delta} \hat{f}_1(\chi_1) \cdots \hat{f}_s(\chi_s).$$

When $f_1, \ldots, f_s$ are characters, this follows from (3.2). For general L^∞-functions, the claim is proved by the standard approximation argument. $\square$

Proof of Corollary 1.11(a). Let us choose linearly dependent over $\mathbb{Q}$ vectors $v_1, \ldots, v_s \in \mathbb{Z}^d$ such that every proper subset of $\{v_1, \ldots, v_s\}$ is linearly independent over $\mathbb{Q}$. There exist nilpotent matrices $N_1, \ldots, N_s \in \mathrm{M}(d, \mathbb{Z})$ such that

$$\mathrm{Ker}(N_i) = \langle v_i \rangle .$$

Set $T_i = id + N_i$. Then it follows from Theorem 1.3 that the set $\{T_1, \dots, T_s\}$ is not mixing, but its every proper subset is mixing. □

To prove Corollary 1.11(b), we need a lemma:

Lemma 3.1. *For every $d \geq 1$, there exists an irreducible monic polynomial $p(x) \in \mathbb{Z}[x]$ which has real roots with different absolute values.*

Proof. Consider the polynomial

$$p(x) = (x - q) \cdots (x - dq) + q$$

where q is a prime number. Note that this polynomial is irreducible by the Eisenstein criterion (see [12, IV §3]). Let us assume that d is even (the argument for odd d is analogous). Then for sufficiently large q, we have

$$p((4i + 1)q/2) \geq (q/2)^d + q > 0, \quad i = 0, \dots, d/2,$$
$$p((4i + 3)q/2) \leq -(q/2)^d + q < 0, \quad i = 0, \dots, d/2 - 1.$$

This implies that $p(x)$ has d distinct positive real roots. □

Proof of Corollary 1.11(b). Let $p(x)$ be as in Lemma 3.1 and let $T_1 \in \mathrm{M}(d, \mathbb{Z})$ has $p(x)$ as its characteristic polynomial. Denote by λ_i, $i = 1, \dots, d$, the roots of $p(x)$ and by σ_i the embedding $\mathbb{Q}(\lambda_1) \to \mathbb{R}$ such that $\lambda_1 \mapsto \lambda_i$. Let $\{v_1, \dots, v_d\}$ be an integral basis of $\mathbb{Q}(\lambda_1)$. It is well-known that

$$\det(v_k^{\sigma_i} : i, k = 1, \dots, d) \neq 0. \tag{3.5}$$

Let

$$A = \mathrm{diag}(1, \dots, 1, 2, \dots, s - 1).$$

Note that A has minimal polynomial

$$q(x) = \prod_{l=1}^{s-1} (x - l) = \sum_{j=1}^{s} q_j x^{j-1}$$

and $q_j \neq 0$ for all j. Put

$$w_j = A^{j-1} \cdot {}^t(v_1, \dots, v_d), \quad j = 1, \dots, d.$$

It follows from (3.5) that $w_j^{\sigma_i}$, $i = 1, \dots, d$, are linearly independent over $\mathbb{Q}$. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ permutes the vectors $w_j^{\sigma_i}$, $i = 1, \dots, d$. Therefore,

$$\mathbb{Q}^d = \left\{ \sum_{i=1}^{d} a^{\sigma_i} w_j^{\sigma_i} : a \in \mathbb{Q}(\lambda_1) \right\}. \tag{3.6}$$

for every $j = 1, \dots, d$. Define $T_j \in \mathrm{M}(d, \mathbb{R})$ such that

$$T_j w_j^{\sigma_i} = \lambda_i w_j^{\sigma_i}, \quad i = 1, \dots, d.$$

Then $\det(T_j) \neq 0$ and $T_j^{\sigma} = T_j$ for every $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Thus, $T_j \in \mathrm{GL}(d, \mathbb{Q})$. Multiplying T_j's and λ_i's by an integer we may assume that T_j's have integer entries.

We claim that there is $(x_1,\dots,x_s)\in(\mathbb{Q}^d)^s-\{(0,\dots,0)\}$ such that

$$\sum_{j=1}^{s} T_j^n x_j = 0 \tag{3.7}$$

for every $n\ge 1$, and for any $J\subsetneq\{1,\dots,s\}$, there is no $(x_j : j\in J)\in(\mathbb{Q}^d)^{|J|}-\{(0,\dots,0)\}$ such that

$$\sum_{j\in J} T_j^n x_j = 0 \tag{3.8}$$

for infinitely many n. By Lemma 2.1, this implies that $\{T_1,\dots,T_s\}$ is not mixing, but its every proper subset is mixing.

Put $x_j = q_j\sum_{i=1}^d w_j^{\sigma_i}\in\mathbb{Q}^d$. Note that

$$\sum_{i=1}^{d} w_j^{\sigma_i} = A^j\left(\sum_{i=1}^{d} w_1^{\sigma_i}\right)\neq 0$$

by (3.5). We have

$$\sum_{j=1}^{s} T_j^n x_j = \sum_{i=1}^{d}\sum_{j=1}^{s} q_j\lambda_i^n w_j^{\sigma_i} = \sum_{i=1}^{d}\lambda_i^n\sum_{j=1}^{s} q_j A^{j-1} w_1^{\sigma_i} = 0.$$

This proves (3.7).

It follows from (3.6) that (3.8) is equivalent to existence of

$$(a_j : j\in J)\in\mathbb{Q}(\lambda_1)^{|J|}-\{(0,\dots,0)\}$$

such that

$$\sum_{j\in J}\sum_{i=1}^{d} T_j^n a_j^{\sigma_i} w_j^{\sigma_i} = 0$$

for infinitely many n. Then

$$\sum_{i=1}^{d}\lambda_i^n\sum_{j\in J} a_j^{\sigma_i} w_j^{\sigma_i} = 0.$$

Since λ_i's have different absolute values, this implies that

$$\sum_{j\in J} a_j w_j = \left(\sum_{j\in J} a_j A^{j-1}\right) w_1 = 0. \tag{3.9}$$

Let

$$r(x) = \sum_{j\in J} a_j x^{j-1} = b\prod_{j=1}^{l}(x-\mu_j).$$

for $l \le s-1$ and $b, \mu_j \in \mathbb{C}$. We have $r(A)w_1 = 0$ and $\tilde{r}(A)w_1 = 0$ with

$$\tilde{r}(x) = \prod (x - \mu_j)$$

where the product is taken over μ_j which are eigenvalues of A. In particular, $\tilde{r}(x) \in \mathbb{Q}[x]$. Then $\tilde{r}(A)w_1^{\sigma_i} = 0$ for every $i = 1, \ldots, d$. It follows from (3.5) that $\tilde{r}(A) = 0$. Since the minimal polynomial $q(x)$ of A has degree $s-1$, this implies that $r(x)$ is a scalar multiple of $q(x)$. In particular, $a_j \neq 0$ for every $j = 1, \ldots, s$, which is a contradiction. □

Proof of Corollary 1.12. It is clear that if the set $\{T_1, \ldots, T_{s-1}, id\}$ is mixing, then every T_i is mixing and $\{T_1, \ldots, T_{s-1}\}$ is mixing as well.

Conversely, suppose that the set $\{T_1, \ldots, T_{s-1}, id\}$ is not mixing. Then for some $l \ge 1$, the linear maps $\hat{T}_1^l, \ldots, \hat{T}_{s-1}^l, id$ have linearly dependent eigenvectors with the same eigenvalue λ. Note that if $\lambda = 1$, then some $\hat{T}_k^l$ has eigenvalue one, and T_k is not mixing. Otherwise, it follows that the linear maps $\hat{T}_1^l, \ldots, \hat{T}_{s-1}^l$ have linearly dependent eigenvectors with the same eigenvalue. Hence, by Theorem 1.3, the set $\{T_1, \ldots, T_{s-1}\}$ is not mixing. □

4. Mixing groups and semigroups

Proposition 4.1. *Let X be any compact abelian group and Γ a torsion free subgroup of* $\mathrm{Aut}(X)$*. Then Γ is mixing iff every element $\gamma \in \Gamma - \{e\}$ is ergodic.*

Proof. If the action of Γ on X is mixing, then the action of every infinite subgroup of Γ is mixing as well, and in particular, every $\gamma \in \Gamma - \{e\}$ is ergodic.

Conversely, suppose that the action of Γ on X is not mixing. Then for some $(\chi, \psi) \in \hat{X}^2 - \{(0,0)\}$, the set

$$S = \{\gamma \in \Gamma : \hat{\gamma}\chi = \psi\}$$

is infinite. For every $\gamma \in S^{-1}S$, we have $\hat{\gamma}\chi = \chi$, and the action of such γ on X is not ergodic. This proves the proposition. □

Now we assume that X *is connected and* $\dim X = d < \infty$. We are going to show that under these assumptions, the torsion free condition in Proposition 4.1 can be omitted. But first, we need the following lemma (see [1, Lemma 4.3] for a different proof).

Lemma 4.2. *Every torsion subgroup (i.e., every element is of finite order) of* $\mathrm{GL}(d, \mathbb{Q})$ *is finite.*

In the proofs below, we use some basic facts about algebraic groups and Zariski topology, which can be found in [15] and [23].

Proof. Let Γ be a torsion subgroup of $\mathrm{GL}(d, \mathbb{Q})$. The eigenvalues of a matrix in Γ are roots of unity each having degree at most d over $\mathbb{Q}$. Hence, their order is bounded, and there exists $n \geq 1$ such that $\Gamma^n = \{e\}$. Let $G \subset \mathrm{SL}(d, \mathbb{C})$ be the Zariski closure of Γ. Then its connected component G^o has finite index in G, and $G^n = \{e\}$. For $g \in G$, let $g = g_s g_u$ be the Jordan decomposition of g. Since g_u is unipotent and $g_u^n = e$, it follows that $g_u = e$ and every element of G is semisimple. Hence, G^o is a torus and since $(G^o)^n = \{e\}$, we deduce that $G^o = \{e\}$ and Γ is finite. □

Proposition 4.3. *Let X be a compact connected finite-dimensional abelian group and Γ an infinite subgroup of* $\mathrm{Aut}(X)$. *Then the following statements are equivalent:*

(a) *The action of Γ on X is mixing.*
(b) *Every infinite cyclic subgroup of Γ is ergodic on X.*
(c) *For every $\gamma \in \Gamma$ of infinite order, the linear map $\hat{\gamma}$ does not have roots of unity as eigenvalues.*

Proof. It is well-known that (a)⇒(b) and (b)⇔(c). To show that (b)⇒(a), we observe that if the action of Γ on X is not mixing, then for some $\chi \in \hat{X} - \{0\}$, the subgroup $\{\gamma \in \Gamma : \hat{\gamma}\chi = \chi\}$ is infinite (see the proof of Proposition 4.1), and it suffices to show that this subgroup contains an element of infinite order. This follows from Lemma 4.2. □

Note that Proposition 4.3((a)⇔(b)) fails in general if X is disconnected or infinite-dimensional (see Example 6.4 below). Also, it fails for semigroups (see Example 6.5 below).

The following lemma is used in the proof of Corollary 1.15.

Lemma 4.4. *Every solvable mixing subgroup of* $\mathrm{Aut}(X)$ *is a finite extension of abelian group.*

Proof. Let Γ be a solvable mixing subgroup of $\mathrm{Aut}(X)$. We show that $\hat{\Gamma} \subset \mathrm{GL}(d, \mathbb{Q})$ is a finite extension of abelian group. Let $G \subset \mathrm{GL}(d, \mathbb{C})$ be the Zariski closure of Γ. Then G is solvable too. The connected component G^o has finite index in G, and is is conjugate to a subgroup of the upper triangular subgroup (see [23, Section 6.3]). In particular, the commutant $[G^o, G^o]$ is a unipotent subgroup. The subgroup $\hat{\Gamma}_0 = G^o \cap \hat{\Gamma}$ has finite index in $\hat{\Gamma}$. Since Γ is mixing, it follows from Proposition 4.3 that $[\hat{\Gamma}_0, \hat{\Gamma}_0] = 1$. This proves the corollary. □

Proof of Corollary 1.15. Note that the subgroup Γ is isomorphic to a subgroup of $\mathrm{GL}(d, \mathbb{Q})$. By the Tits alternative (see [24] or [15, Section 5J]), Γ is either finite extension of solvable group or contains a nonabelean free subgroup. Thus, we may assume that Γ contains a nonabelean free subroup. Let γ and δ be free generators and let $T_i = \delta^{-i}\gamma\delta^i$. Then

$$T_i^n T_j^{-n} = \delta^{-i}\gamma^n\delta^{i-j}\gamma^{-n}\delta^j \to \infty \quad \text{for } i \neq j.$$

On the other hand, linear maps $\hat{T}_i$ have the same characteristic polynomial. Hence, it follows from Corollary 1.7(b) (or Proposition 2.2) that the set $\{T_1, \ldots, T_{d+1}\}$ is not mixing. This implies that Γ is not mixing of order $d + 1$. □

Using Proposition 4.3, we develop two approaches to construction of mixing subgroups. The first approach is based on the result of Y. Benoist [4] on asymptotic cones of discrete groups (see Proposition 4.5) and the second approach is based on the theory of division algebras (see Corollary 4.8).

Proposition 4.5. *For every even $d \geq 2$, there exists a mixing subgroup of* $\mathrm{Aut}(\mathbb{T}^d)$ *which is Zariski dense in* $\mathrm{SL}(d,\mathbb{C})$.

Proof. We start by reviewing a result of Y. Benoist from [4], which will be used in the proof.

For $g \in \mathrm{SL}(d,\mathbb{R})$, let us denote by $\lambda_1(g),\dots,\lambda_d(g)$ the eigenvalues of g such that $|\lambda_1(g)| \geq \cdots \geq |\lambda_d(g)|$ and

$$\ell_g = (\log|\lambda_1(g)|,\dots,\log|\lambda_d(g)|).$$

The vector ℓ_g belongs to the set

$$\mathfrak{a}^+ \stackrel{def}{=} \left\{(x_1,\dots,x_d) \in \mathbb{R}^d : \sum_{i=1}^d x_i = 0,\ x_1 \geq \cdots \geq x_d\right\}.$$

Let Γ be a subgroup of $\mathrm{SL}(d,\mathbb{R})$. The *limit cone* ℓ_Γ of Γ is the smallest closed cone in $\mathfrak{a}^+$ that contains all ℓ_γ, $\gamma \in \Gamma$. Since Γ is a group, the limit cone ℓ_Γ is stable under the involution

$$i(x_1,\dots,x_d) = (-x_d,\dots,-x_1).$$

It was shown by Y. Benoist in [4] that if Γ is Zariski dense, then the asymptotic cone ℓ_Γ is convex, has nonempty interior, and is equal to the *asymptotic cone* of Γ. The asymptotic cone is the cone consisting of limit directions of the set

$$\{\log(\mu(\gamma)) : \gamma \in \Gamma\} \subset \mathfrak{a}^+$$

where $\mu(g)$ denotes the A^+ component of g with respect to KA^+K-decomposition ($K = \mathrm{SO}(n)$, $A^+ =$ positive Weyl chamber). In the case when Γ is a lattice, the asymptotic cone is always equal to $\mathfrak{a}^+$. In particular,

$$\ell_{\mathrm{SL}(d,\mathbb{Z})} = \mathfrak{a}^+.$$

If Γ is a Zariski dense subgroup, Y. Benoist also showed in [4] that for every closed convex i-invariant cone $\mathcal{C} \subset \ell_\Gamma$ with nonempty interior, there exists a Zariski dense subgroup $\Gamma_0 \subset \Gamma$ such that $\ell_{\Gamma_0} = \mathcal{C}$.

Suppose that $d = 2k$. For $(x_1,\dots,x_{2k}) \in \mathfrak{a}^+$,

$$x_k \leq -k^{-1} \sum_{i=k+1}^{2k} x_i, \quad x_{k+1} \geq k^{-1} \sum_{i=1}^{k} x_i,$$

and for any $\delta \in (0, k^{-1})$

$$\mathcal{C}_\delta = \left\{ (x_1, \dots, x_{2k}) \in \mathfrak{a}^+ : \begin{array}{l} x_k \geq 0 \geq x_{k+1} \\ x_k \geq -\delta \sum_{i=k+1}^{2k} x_i \\ x_{k+1} \leq -\delta \sum_{i=1}^{k} x_i \end{array} \right\}$$

is closed convex i-invariant cone with nonempty interior. Hence, there exists a Zariski dense subgroup Γ of $\mathrm{SL}(d, \mathbb{Z})$ such that $\ell_\Gamma = \mathcal{C}_\delta$. Since

$$\mathcal{C}_\delta \cap \{x_i = 0\} = 0$$

for every $i = 1, \dots, 2k$, the group Γ contains no element (except identity) with an eigenvalue of absolute value one. By Proposition 4.3, Γ is mixing. This proves the proposition. □

If Γ is a finitely generated subgroup of the automorphism group of a compact connected finite-dimensional abelian group X, then by the Selberg lemma (see [8, Theorem 4.1] or [15, Section 5I]), Γ contains a torsion free subgroup of finite index. Clearly, this subgroup is mixing iff Γ is mixing. For torsion free subgroup Proposition 4.3 can be restated as follows:

Proposition 4.6. *Let Γ be a torsion free subgroup of* $\mathrm{Aut}(X)$*. Then the action of* Γ *on X is mixing iff*

$$\hat{\Gamma} - \hat{\Gamma} \subset \{0\} \cup \mathrm{GL}(d, \mathbb{Q}).$$

Recall that the Jacobson radical of a ring (with a unit) R is the intersection of all maximal ideals of R. We denote by $R^\times$ the group of units of a ring R.

Let $A_\Gamma \subset \mathrm{M}(d, \mathbb{Q})$ be the $\mathbb{Q}$-span of $\hat{\Gamma}$, $J_\Gamma \subset A_\Gamma$ the Jacobson radical of A_Γ and

$$\pi : A_\Gamma \to A_\Gamma / J_\Gamma$$

the factor map.

Proposition 4.7. *Let Γ be a torsion free subgroup of* $\mathrm{Aut}(X)$*. Then the action of* Γ *on X is mixing iff*

$$\hat{\Gamma} \cap (1 + J_\Gamma) = 1 \quad \textit{and} \quad \pi(\hat{\Gamma}) - \pi(\hat{\Gamma}) \subset \{0\} \cup (A_\Gamma / J_\Gamma)^\times.$$

Proof. Recall that the Jacobson radial is nilpotent and $1 + J_\Gamma \subset A_\Gamma^\times$.

Suppose that action of Γ on X is mixing. Since $1 + J_\Gamma$ consists of unipotent matrices, it follows from Proposition 4.3 that $\hat{\Gamma} \cap (1 + J_\Gamma) = 1$. The second property follows Proposition 4.6.

Conversely, suppose that these properties are satisfied. If for some $a \in A_\Gamma$, $\pi(a)$ is invertible, then there exists $b \in A_\Gamma$ such that $ab \in 1 + J_\Gamma$ and it follows that a is invertible as well. Therefore,

$$\hat{\Gamma} - \hat{\Gamma} \subset J_\Gamma \cup A_\Gamma^\times$$

If $\gamma_1 - \gamma_2 \in J_\Gamma$ for some $\gamma_1, \gamma_2 \in \hat{\Gamma}$, then $\gamma_1^{-1}\gamma_2 \in 1 + J_\Gamma$ and $\gamma_1 = \gamma_2$. This shows that

$$\hat{\Gamma} - \hat{\Gamma} \subset \{0\} \cup \mathrm{GL}(d, \mathbb{Q})$$

and the action of Γ on X is mixing by Proposition 4.6. $\square$

Proposition 4.7 implies, in particular, that the action of Γ on X is mixing provided that the $\mathbb{Q}$-span of $\hat{\Gamma}$ is a division subalgebra D in $\mathrm{M}(d,\mathbb{Q})$. This is possible only when $d = (\dim D)^l$ for some $l \geq 1$. In particular, d must be a perfect square.

Corollary 4.8. *For every perfect square $d > 1$, there exists a mixing subgroup Γ of $\mathrm{Aut}(\mathbb{T}^d)$ such that the Zariski closure of Γ is conjugate to*

$$\{\mathrm{diag}(g,\dots,g) :\ g \in \mathrm{SL}(\sqrt{d},\mathbb{C})\}.$$

The subgroup Γ is not mixing of order $p+1$ where p is the smallest prime divisor of $\sqrt{d}$.

Proof. There exists a central division algebra D over $\mathbb{Q}$ such that $\dim_{\mathbb{Q}} D = d$ and D is split over $\mathbb{R}$. Denote by $\mathrm{SL}(1,D)$ the group consisting of elements of D whose reduced norm is equal to one. Consider the right and left regular representations

$$\begin{aligned} \rho :& D\otimes\mathbb{C} \to \mathrm{End}(D\otimes\mathbb{C}) :\ \rho(d)x = x\cdot d,\\ \lambda :& D\otimes\mathbb{C} \to \mathrm{End}(D\otimes\mathbb{C}) :\ \lambda(d)x = d\cdot x. \end{aligned}$$

Let $\mathcal{O}$ be an order in D. Note that $(D\otimes\mathbb{R})/\mathcal{O}$ can be identified with the torus $\mathbb{T}^d$ and with respect to a basis of $\mathcal{O}$, we have

$$\begin{aligned} G &\stackrel{def}{=} \rho(\mathrm{SL}(1,D\otimes\mathbb{R})) \subset \mathrm{SL}(d,\mathbb{R}),\\ \Gamma &\stackrel{def}{=} \rho(\mathrm{SL}(1,\mathcal{O})) \subset \mathrm{SL}(d,\mathbb{Z}). \end{aligned}$$

Since D splits over $\mathbb{R}$, $G \simeq \mathrm{SL}(k,\mathbb{R})$ where $d = k^2$. By the Borel–Harish–Chandra theorem (see [16, Ch. IV]), Γ is a lattice in G. This implies that the Zariski closure of Γ is $\rho(\mathrm{SL}(1,D\otimes\mathbb{C}))$. Note that

$$D\otimes\mathbb{C} \simeq \mathrm{M}(k,\mathbb{C}), \quad \mathrm{SL}(1,D\otimes\mathbb{C}) \simeq \mathrm{SL}(k,\mathbb{C}),$$

and as a $\rho(D\otimes\mathbb{C})$-module,

$$D\otimes\mathbb{C} \simeq \mathbb{C}^k \oplus\cdots\oplus \mathbb{C}^k.$$

Hence,

$$\rho(\mathrm{SL}(1,D\otimes\mathbb{C})) = \{\mathrm{diag}(g,\dots,g) :\ g\in\mathrm{SL}(k,\mathbb{C})\}$$

in a suitable basis.

Let $\mathrm{Tr} : D \to \mathbb{Q}$ denote the reduced trace of the division algebra D, and $\hat{\mathcal{O}} \subset D$ the dual order of $\mathcal{O}$, that is,

$$\hat{\mathcal{O}} = \{u \in D_{\mathbb{Q}} :\ \mathrm{Tr}(\mathcal{O}\cdot u) \in \mathbb{Z}\}.$$

Then the set of characters of $D/\mathcal{O}$ is indexed by $\hat{\mathcal{O}}$:

$$\chi_u(x) = \exp(2\pi i\,\mathrm{Tr}(x\cdot u)), \quad u \in \hat{\mathcal{O}},$$

and the dual action of Γ is

$$u \mapsto \gamma \cdot u, \quad u \in \hat{\mathcal{O}},\ \gamma \in \Gamma.$$

Hence, the $\mathbb{Q}$-span of $\hat{\Gamma}$ is equal to the $\mathbb{Q}$-span of $\lambda(\mathrm{SL}(1, \mathcal{O}))$, and since $\mathrm{SL}(1, \mathcal{O})$ is Zariski dense in $\mathrm{SL}(1, D \otimes \mathbb{C})$, it is equal to $\lambda(D)$. Now it follows from Proposition 4.7 that the (right) action of Γ on $D/\mathcal{O}$ is mixing.

The central division algebra D contains a splitting field F such that $F/\mathbb{Q}$ is a cyclic extension of degree k. Moreover, since D splits over $\mathbb{R}$, F can be taken to be real. Then F contains a Galois subfield E such that $|E : \mathbb{Q}| = p$. By Dirichlet theorem (see [11, Ch. 2]), E contains a unit γ of infinite order (unless $E/\mathbb{Q}$ is a complex quadratic extension, which is not the case). Since

$$\mathrm{N}(\gamma) = N_{E/\mathbb{Q}}(\gamma)^{\dim_E D} = \pm 1,$$

we may choose $\gamma \in \Gamma$. There exist $\alpha_i \in \mathbb{Q}$, $i = 0, \dots, p$, $\alpha_p = 1$, such that

$$\sum_{i=0}^{p} \alpha_i \gamma^i = 0.$$

We claim that there exists $\gamma_n \in \Gamma$ such that

$$\gamma_n^{-1} \gamma^i \gamma_n \to \infty \quad \text{ as } n \to \infty.$$

for $i = 1, \dots, p$. It suffices to check that the centralizer $C_\Gamma(\gamma^{p!})$ has infinite index in Γ. If this is not the case, then $C_\Gamma(\gamma^{p!})$ is a lattice in G, and it follows that $\gamma^{p!}$ lies in the center of D, which is a contradiction. We have

$$\sum_{i=0}^{p} (\gamma_n^{-1} \gamma^i \gamma_n) \alpha_i = 0.$$

Since $\ell \alpha_i \in \hat{\mathcal{O}}$ for some $\ell \in \mathbb{N}$, this proves that $\rho(\Gamma)$ is not mixing of order $p + 1$. ☐

Corollary 4.9. *For every $d \geq 2$, $d \neq 3, 5, 7, 9$, there exists a free nonabelian mixing subgroup of* $\mathrm{Aut}(\mathbb{T}^d)$ *which is not mixing of order 3.*

Proof. Let $d = d_1 + d_2$ where $d_1 > 1$ is a perfect square and $d_2 > 1$ is even. If $d \neq 9$, we may take $d_2 \geq 2$. By Proposition 4.5 and Corollary 4.8, $\mathrm{Aut}(\mathbb{T}^{d_i})$, $i = 1, 2$, contain not virtually abelian free mixing subgroups. Hence, by the Tits alternative, there exist injective homomorphisms

$$\phi_i : F_2 \to \mathrm{Aut}(\mathbb{T}^{d_i}), \quad i = 1, 2,$$

where F_2 denotes the free group with 2 generators such that the action of $\phi_i(F_2)$ on $\mathbb{T}^{d_i}$ is mixing. Let

$$\Gamma = \{(\phi_1(\delta), \phi_2(\delta)) : \gamma \in F_2\} \subset \mathrm{Aut}(\mathbb{T}^{d_1} \times \mathbb{T}^{d_2}) = \mathrm{Aut}(\mathbb{T}^d).$$

If Γ is not mixing, there exist $x, y \in \mathbb{Q}^d - \{0\}$ and $\gamma_n \in \Gamma$ such that $\gamma_n \to \infty$ and ${}^t\gamma_n x = y$. Write $x = x_1 + x_2$ and $y = y_1 + y_2$ with respect to the decomposition

$\mathbb{Q}^d = \mathbb{Q}^{d_1} \oplus \mathbb{Q}^{d_2}$ Then for some $i = 1, 2$, we have $x_i, y_i \in \mathbb{Q}^{d_i} - \{0\}$ and ${}^t\phi_i(\delta_n)x_i = y_i$ where $\delta_n \in F_2$ corresponds to γ_n. This is a contradiction since the subgroups $\phi_i(F_2)$ are mixing. It follows that Γ is mixing.

If $d \neq 9$, the group Γ preserves the direct product decomposition $\mathbb{T}^d = \mathbb{T}^{d-2} \times \mathbb{T}^2$. If the action Γ on $\mathbb{T}^d$ is mixing of order 3, then the restriction of this action to $\mathbb{T}^2$ is mixing as well. However, this contradicts Proposition 1.15. □

Corollary 4.10. *There exists a not virtually abelian mixing subgroup of* $\mathrm{Aut}(\mathbb{T}^9)$ *which is not mixing of order 3.*

Proof. Let $K = \mathbb{Q}(\alpha)$ with $\alpha = \zeta + \bar{\zeta}$ where ζ is a primitive root of unity of order 7, and $D \supset K$ be a central division algebra over $\mathbb{Q}$ with $\dim_{\mathbb{Q}} D = 9$ (such algebra can be constructed using the cross product construction). One can check that α is a root of $x^3 + x^2 - 2x - 1 = 0$. In particular, this implies that α and $\beta = -1 - \alpha$ are units in K and $\alpha, \beta \in \mathrm{SL}(1, D)$. Let $\mathcal{O}$ be an order in D that contains α and β. Using the argument from the proof of Corollary 4.8, one can find a sequence $\{\gamma_n\} \subset \mathrm{SL}(1, \mathcal{O})$ such that $\gamma_n^{-1}\alpha\gamma_n \to \infty$ as $n \to \infty$. Then

$$\gamma_n^{-1}\alpha\gamma_n + \gamma_n^{-1}\beta\gamma_n + 1 = 0$$

and $\gamma_n^{-1}\beta\gamma_n \to \infty$, $\gamma_n^{-1}(\alpha\beta^{-1})\gamma_n \to \infty$. As in the proof of Corollary 4.8, this implies that the action of $\mathrm{SL}(1, \mathcal{O})$ on $D/\mathcal{O}$ by right multiplication is not mixing of order 3. □

Question 4.11. *Does there exist a not virtually abelian mixing subgroup in* $\mathrm{Aut}(\mathbb{T}^d)$ *for* $d = 3, 5, 7$*?*

According to Corollary 1.15, $\mathrm{Aut}(\mathbb{T}^2)$ contains no free nonabelian subgroup which is mixing of order 3 (see also Proposition 2.31 in the electronic version of [5]).

Question 4.12. *Is there a free nonabelian mixing of order 3 subgroup in* $\mathrm{Aut}(\mathbb{T}^d)$ *for some* $d \geq 3$*?*

Note that there exist free nonabelian semigroups of epimorphisms of the torus $\mathbb{T}^d$ which are mixing of all orders (see Example 6.6 below).

5. Ergodicity and mixing

In this section, we prove Theorems 1.18, 1.19, and 1.21.

First, we recall the following well-known characterization of ergodicity (see, for example, [21, Chapter I]):

Proposition 5.1. *Let* Γ *be a groups of automorphisms of a compact abelian group* X. *Then the action of* Γ *on* X *is ergodic iff the action of* Γ *on* $\hat{X}$ *has no finite orbits except the trivial character.*

Using Proposition 5.1, we deduce

Proposition 5.2. *Let X be a compact connected abelian group and $\Gamma \subset \mathrm{Aut}(X)$. Then if the action of Γ on X is strongly irreducible, then it is ergodic.*

Note that the converse of Proposition 5.2 is not true (see Example 6.8 below).

Proof. Suppose that the action of Γ on X is not ergodic. Then there exist $\chi \in \hat{X} - \{0\}$ and a subgroup Λ of finite index in Γ such that $\Lambda \cdot \chi = \chi$. Consider the subgroup

$$A = \{\psi \in \hat{X} : \, k\psi \in \mathbb{Z} \cdot \chi \quad \text{for some } k \geq 1\}.$$

Using that $\hat{X}$ is torsion free, one can check that Λ acts trivially on A. In particular, $A \neq \hat{X}$. Also, it is clear that $\hat{X}/A$ is torsion free. Hence, there exists a proper closed connected Λ-invariant subgroup

$$\{x \in X : \, \chi(x) = 1 \quad \text{for all } \chi \in A\},$$

and the action of Γ on X is not strongly irreducible. □

In the proofs of the Theorems 1.18, 1.19, and 1.21, we will need the following three lemmas.

Lemma 5.3. *Let Γ be a group and $\rho_i : \Gamma \to (\mathbb{C}, +)$, $i = 1, \dots, t$, nontrivial homomorphisms. Then there exists $\gamma \in \Gamma$ such that $\rho_i(\gamma) \neq 0$ for every $i = 1, \dots, t$. Moreover, the set*

$$R = \{\gamma \in \Gamma : \, \rho_i(\gamma) \neq 0 \textit{ for every } i = 1, \dots, t\}.$$

generates Γ.

Proof. Consider a homomorphisms $\rho : \Gamma \to \mathbb{C}^t$ defined by

$$\rho(\gamma) = (\rho_1(\gamma), \dots, \rho_t(\gamma)).$$

Then $\Delta = \rho(\Gamma)$ is a subgroup of $\mathbb{C}^t$ such that $\pi_i(\Delta) \neq 0$, $i = 1, \dots, t$, where $\pi_i : \mathbb{C}^t \to \mathbb{C}$ the coordinate projection. It suffices to show that

$$\Delta \not\subseteq \bigcup_{i=1}^{t} \pi_i^{-1}(0).$$

Suppose that this is not the case. Then the Zariski closure $\bar{\Delta}$ of Δ is a linear subspace of $\mathbb{C}^t$ and

$$\bar{\Delta} = \bigcup_{i=1}^{t} (\bar{\Delta} \cap \pi_i^{-1}(0)).$$

However, this equality is impossible because $\bar{\Delta} \cap \pi_i^{-1}(0)$ are proper linear subspaces of $\bar{\Delta}$. This contradiction proves the first part of the lemma.

To prove the second part, take any $\gamma \in \Gamma$ and $\delta \in R$. Then for $k \geq 1$,

$$\rho(\gamma\delta^k) = \rho(\gamma) + k\rho(\delta),$$

and taking k such that

$$k \neq -\pi_i(\rho(\gamma))/\pi_i(\rho(\delta)) \quad \text{for every } i = 1, \dots, t,$$

we have $\delta^k \in \Gamma_0$ and $\gamma\delta^k \in R$. Hence, $\gamma \in \langle R \rangle$. This proves the lemma. □

Lemma 5.4. *Let Γ be a solvable subgroup of $\mathrm{GL}(d, \mathbb{Q})$. Then there exist a subgroup Λ such that $|\Gamma : \Lambda| < \infty$ and the commutant Λ' is unipotent, and a flag*

$$\mathbb{Q}^d = V_1 \supset V_2 \supset \cdots \supset V_{s+1} = \{0\}$$

consisting of rational Λ-invariant subspaces such that $\Lambda|_{V_i/V_{i+1}}$ is abelian for all $i = 1, \dots, s$.

Proof. There exists a subgroup Λ of finite index in Γ which can be conjugated (over $\mathbb{C}$) to a subgroup of the group of the upper triangular matrices (see, for example, the proof of Lemma 4.4). Then the commutant Λ' is a unipotent subgroup. Hence, the subspace $V^{\Lambda'}$ of Λ'-invariant vectors is not trivial. Since Λ' is normal in Λ, this subspace is Λ-invariant. Also, it is clear that $V^{\Lambda'}$ is rational, and $\Gamma|_{V^{\Lambda'}}$ is abelian. Now the lemma follows by induction on dimension. □

For a subgroup $\Gamma \subset \mathrm{GL}(d, \mathbb{Q})$, we denote by $\bar{\Gamma}$ its Zariski closure and by $\bar{\Gamma}^\circ$ the connected component of the closure.

Lemma 5.5. *Every subgroup Γ of $\mathrm{GL}(d, \mathbb{Q})$ contains a finitely generated subgroup Λ such that $\bar{\Lambda} = \bar{\Gamma}$.*

Proof. Take a finitely generated subgroup Δ such that $\dim \bar{\Delta}^\circ$ is maximal among all finitely generated subgroups. Then for every $\gamma \in \Gamma$,

$$\overline{\langle \Delta, \gamma \rangle}^o = \bar{\Delta}^o.$$

In particular, $\gamma^{-1}\bar{\Delta}^o\gamma \subset \bar{\Delta}^o$, and the group $\Gamma \cap \bar{\Delta}^o$ is normal in Γ. Also, since $\overline{\langle \Delta, \gamma \rangle}$ has finitely many connected components, $\gamma^k \in \bar{\Delta}^o$ for some $k \geq 1$ and the group $\Gamma/(\Gamma \cap \bar{\Delta}^o)$ consists of elements of finite order. The algebraic group $\bar{\Gamma}/\bar{\Delta}^o$ is defined over $\mathbb{Q}$ and it embeds via a $\mathbb{Q}$-map into $\mathrm{GL}(n)$ for some $n \geq 2$. Under this map, the subgroup $\Gamma/(\Gamma \cap \bar{\Delta}^o)$ is embedded into $\mathrm{GL}(n, \mathbb{Q})$. Hence, it follows from Lemma 4.2 that $\Gamma/(\Gamma \cap \bar{\Delta}^o)$ is finite. This implies that $\bar{\Delta}$ has finite index in $\bar{\Gamma}$. Since Γ is dense in $\bar{\Gamma}$, every coset of $\bar{\Delta}$ in $\bar{\Gamma}$ contains a representative from Γ. Now the required group Λ can be taken to be generated by Δ and these coset representatives. □

Proof of Theorem 1.18. (b)⇒(a): Suppose that there exists a closed connected virtually Γ-invariant subgroup Y of X such that the action of Γ_Y on Y is not ergodic. Then by Proposition 5.1, there exists a subgroup Λ with $|\Gamma : \Lambda| < \infty$ and $\chi \in \hat{Y} - \{0\}$ such that $\Lambda\chi = \chi$. The character group $\hat{Y}$ is equal to $\hat{X}/A(Y)$ where

$$A(Y) = \{\chi \in \hat{X} :\ \chi(Y) = 1\}.$$

Since Y is connected, $\hat{Y}$ is torsion free, and it follows that $\hat{X}/A(Y)$ embeds in $(\hat{X} \otimes \mathbb{Q})/(A(Y) \otimes \mathbb{Q})$. Therefore, the character χ gives a nonzero vector in $(\hat{X} \otimes \mathbb{Q})/(A(Y) \otimes \mathbb{Q})$ which is fixed by Λ. This implies that every element of Λ has eigenvalue one, and by Proposition 4.3, Λ contains no mixing subgroup. Since Λ has finite index in Γ, Γ does not contain any mixing subgroups as well.

(b)$\Rightarrow$(a): First, we can pass to a finite index subgroup Λ of Γ as in Lemma 5.4. Then for every $i = 1, \dots, s$,

$$V_i/V_{i+1} = \bigoplus_{j=1}^{n_i} (V_i/V_{i+1})_{\alpha_{i,j}},$$

where $(V_i/V_{i+1})_{\alpha_{i,j}}$ denotes the weight space corresponding to a homomorphism $\alpha_{i,j} : \Lambda \to \mathbb{C}^\times$.

Suppose that for some $\alpha_{i,j}$, the set $\alpha_{i,j}(\Lambda)$ consists of roots of unity. Since $\alpha_{i,j}(\Lambda)$ consists of eigenvalues of matrices in $\mathrm{GL}(d, \mathbb{Q})$, it follows that for every $\alpha \in \alpha_{i,j}(\Lambda)$, $[\mathbb{Q}(\alpha) : \mathbb{Q}] \le d$ and $\alpha^N = 1$ where $N \ge 1$ depends only on d. Hence, passing again, if necessary, to a finite index subgroup if needed, we can assume that $\alpha_{i,j}(\Lambda) = \{1\}$. Then there exists $v \in V_i(\mathbb{Q}) - V_{i+1}(\mathbb{Q})$ such that

$$\Lambda \cdot v = v + V_{i+1}.$$

Let

$$Y = \{x \in X : \chi(x) = 0 \quad \text{for all } \chi \in \hat{X} \cap V_{i+1}\}.$$

Y is a closed subgroup of X with the character group equal to $\hat{X}/(\hat{X} \cap V_{i+1})$. Since the character group of Y is torsion free, the group Y is connected. Take $l \ge 1$ such that $lv \in \hat{X}$. This gives a nontrivial character of Y which is fixed by Λ. Hence, the action of Λ on Y is not ergodic, which contradicts hereditary ergodicity.

It follows that there exists a finitely generated subgroup Λ_0 of Λ such that for every $\alpha_{i,j}$, the set $\alpha_{i,j}(\Lambda_0)$ contains an element which is not a root of unity. Denote by K the field generated by the sets $\alpha_{i,j}(\Lambda_0)$, $i = 1, \dots, s$, $j = 1, \dots, n_i$. Since Λ_0 is finitely generated, $[K : \mathbb{Q}] < \infty$. By Kronecker's lemma (Lemma 2.3), for every $\alpha_{i,j}$ there exists a an absolute value $|\cdot|_{i,j}$ of the field K such that $|\alpha_{i,j}(\Lambda_0)|_{i,j} \ne 1$. Consider the set of nontrivial homomorphisms

$$\rho_{i,j}(\lambda) = |\alpha_{i,j}(\lambda)|_{i,j} : \Lambda_0 \to \mathbb{R}^+, \quad i = 1, \dots, s,\ j = 1, \dots, n_i. \tag{5.1}$$

By Lemma 5.3, there exists $\gamma \in \Lambda_0$ such that $\rho_{i,j}(\gamma) \ne 1$ for all $\rho_{i,j}$'s. In particular, γ has no roots of unity as eigenvalues. Hence, it is ergodic, and moreover, it is mixing of all orders by Rokhlin's theorem (Theorem 1.13). □

Proof of Theorem 1.19. Note that (b)$\Rightarrow$(a) follows from Theorem 1.18, and it suffices to prove that (a)$\Rightarrow$(b).

By Lemma 5.5, there exists a finitely generated subgroup Δ of Γ such that $\bar{\Delta} = \bar{\Gamma}$. Since Γ is not virtually nilpotent, Δ is not virtually nilpotent as well. By Theorem 1.18, Γ contains a mixing transformation γ_0. Then the group generated by Δ and γ_0 is finitely generated, not virtually nilpotent, and it satisfies (a). Hence, we can assume that Γ is finitely generated.

Let Λ be a finite index subgroup of Γ as in Lemma 5.4 and

$$\alpha_{i,j} : \Gamma \to \mathbb{C}^\times, \quad i = 1, \dots, s, \ j = 1, \dots, n_i,$$

the weights of the action of Λ on V_i/V_{i+1}. Denote by K the field generated by the sets $\alpha_{i,j}(\Lambda)$, $i = 1, \dots, s$, $j = 1, \dots, n_i$. Since Λ is finitely generated, K has finite degree over $\mathbb{Q}$. As in the proof of Theorem 1.18, we deduce from (a) that for every $\alpha_{i,j}$ there exists a absolute value $|\cdot|_{i,j}$ of the field K such that the homomorphism

$$\rho_{i,j}(\lambda) = |\alpha_{i,j}(\lambda)|_{i,j} : \Lambda \to \mathbb{R}^+$$

is not trivial. Set

$$R = \{\lambda \in \Lambda : \ \rho_{i,j}(\lambda) \neq 0 \text{ for all } i = 1, \dots, s \text{ and } j = 1, \dots, n_i\}.$$

By Lemma 5.3, R generates Λ. Note that every $\lambda \in R$, $\hat{\lambda}$ does not have roots of unity as eigenvalues, and by Rokhlin's Theorem (Theorem 1.13), λ is mixing of all orders.

Claim. There there exist $\delta \in R$ and $\mu \in \Lambda'$ such that the semigroup $S = \langle \delta, \delta\mu \rangle$ is free.

Consider the derived series of Λ:

$$\Lambda \supset \Lambda' \supset \Lambda^{(2)} \supset \cdots \supset \Lambda^{(k+1)} = \{e\}.$$

Suppose that $\Lambda^{(i)}/\Lambda^{(i+1)}$ is finitely generated for $i = 0, \dots, l-1$, but $\Lambda^{(l)}/\Lambda^{(l+1)}$ is not finitely generated. Then $\Lambda/\Lambda^{(l)}$ is polycyclic, and in particular, finitely presented. Applying [19, Lemma 4.9], we deduce that there exists a finite subset T of $\Lambda^{(l)}/\Lambda^{(l+1)}$ such that $\Lambda^{(l)}/\Lambda^{(l+1)}$ is generated by $\lambda T \lambda^{-1}$, $\lambda \in \Lambda/\Lambda^{(l)}$. Since $\Lambda/\Lambda^{(l)}$ is polycyclic, $\Lambda'/\Lambda^{(l)}$ is finitely generated. Also, Λ' is nilpotent (see Lemma 5.4). This implies that the set $\lambda T \lambda^{-1}$, $\lambda \in \Lambda'/\Lambda^{(l)}$ generates a finitely generated subgroup of $\Lambda^{(l)}/\Lambda^{(l+1)}$. Since R generates Λ, there exist $\lambda_1, \dots, \lambda_r \in R$ such that

$$\Lambda = \lambda_1^{\mathbb{Z}} \cdots \lambda_r^{\mathbb{Z}} \Lambda'.$$

It follows that there exists a finite set $Q \subset \Lambda^{(l)}/\Lambda^{(l+1)}$ such that the group $\Lambda^{(l)}/\Lambda^{(l+1)}$ is generated by

$$\lambda_1^{n_1} \cdots \lambda_r^{n_r} q \lambda_r^{-n_r} \cdots \lambda_1^{-n_1}, \quad q \in Q, \ n_1, \dots, n_r \in \mathbb{Z}.$$

Hence, since $\Lambda^{(l)}/\Lambda^{(l+1)}$ is not finitely generated, we deduce that there exists $\delta \in R$ and $\mu \in \Lambda^{(l)}/\Lambda^{(l+1)}$ such that

$$\delta^n \mu \delta^{-n}, \quad n \in \mathbb{Z},$$

generates an infinitely generated subgroup. Now the claim follows from [19, Lemma 4.8].

Next, we consider the case when the all groups $\Lambda^{(i)}/\Lambda^{(i+1)}$ are finitely-generated. Then

$$\Lambda^{(i)}/\Lambda^{(i+1)} \simeq \mathbb{Z}^{d_i} \oplus A_i$$

where A_i is a finite abelian group. Denote by Δ_i the preimage of A_i under the factor map $\Lambda \to \Lambda/\Lambda^{(i+1)}$. Note that Δ_i is a normal subgroup of Λ. There exists a finite index subgroup Λ_0 of Λ such that the action of Λ_0 on $\Lambda^{(i)}/\Delta_i \simeq \mathbb{Z}^{d_i}$ is conjugate (over $\mathbb{C}$) to an action by upper triangular matrices. For $\gamma \in \Lambda_0$, we denote by $\beta_{i,j}(\gamma)$, $j = 1, \dots, n_i$, the eigenvalues of the corresponding upper triangular matrix. Note that the maps $\beta_{i,j} : \Lambda_0 \to \mathbb{C}^\times$ are homomorphism.

Suppose that for every $i = 1, \dots, k$ and $j = 1, \dots, n_i$, the set $\beta_{i,j}(\Lambda_0)$ consists of roots of unity. Since the sets $\beta_{i,j}(\Lambda_0)$ consist of algebraic numbers of degree at most n_i. It follows that there exists $N \geq 1$ such that for every $\beta \in \beta_{i,j}(\Lambda_0)$, we have $\beta^N = 1$. Hence, by passing to a finite index subgroup, we may assume that $\beta_{i,j}(\Lambda_0) = 1$ for all $\beta_{i,j}$'s. Also, passing to a finite index subgroup, we may assume that Λ_0 acts trivially on $\Delta_i/\Lambda^{(i+1)}$. Each of the linear maps

$$\Lambda^{(i)}/\Delta_i \to \Lambda^{(i)}/\Delta_i : x \mapsto \gamma x \gamma^{-1}, \quad \gamma \in \Lambda_0,$$

is unipotent. This implies that the corresponding action Λ_0 is unipotent. Then this action is conjugate to the action by a group of unipotent upper triangular matrices. Then the linear maps

$$\Lambda^{(i)}/\Delta_i \to \Lambda^{(i)}/\Delta_i : x \mapsto \gamma x \gamma^{-1} x^{-1} = [\gamma, x], \quad \gamma \in \Lambda_0,$$

generate a nilpotent subalgebra, and it follows that

$$[\Lambda_0, \dots, \Lambda_0, \Lambda^{(i)}] \subset \Delta_i \quad (n_i \text{ terms}).$$

Since Λ_0 acts trivially on $\Delta_i/\Lambda^{(i+1)}$, we also have

$$[\Lambda_0, \dots, \Lambda_0, \Lambda^{(i)}] \subset \Lambda^{(i+1)} \quad (n_i + 1 \text{ terms}).$$

This implies that

$$[\Lambda_0, \dots, \Lambda_0, \Lambda] = 1,$$

and in particular, Λ_0 is nilpotent, which is a contradiction.

We have shown that for some $i = 1, \dots, k$, $j = 1, \dots, n_i$, and $\gamma \in \Lambda_0$, the number $\beta_{i,j}(\lambda)$ is not a root of unity. Note that the numbers $\beta_{i,j}(\lambda)$, $j = 1, \dots, n_i$, are algebraic integers, and they are permuted by the action of the Galois group. Hence, by Kronecker's lemma (Lemma 2.3), $|\beta_{i_0,j_0}(\lambda)| \neq 1$ for some $i_0 = 1, \dots, k$ and $j_0 = 1, \dots, n_{i_0}$. Note that since the action of λ on Λ/Λ' is trivial, $i_0 > 1$. By Lemma 5.3, there exists $\lambda \in \Lambda_0$ such that $|\beta_{i_0,j_0}(\lambda)| \neq 1$ and $\rho_{i,j}(\lambda) \neq 1$ for all $\rho_{i,j}$'s as in (5.1).

By [19, Theorem 4.17], there exists $\mu \in \Lambda^{(i_0)} \subset \Lambda'$ such that the semigroup $S = \langle \lambda^n, \lambda^n \mu \rangle$ is free for sufficiently large $n \geq 1$. This proves the claim.

It remains to show that the action of the semigroup S on X is mixing of all orders. Suppose that, in contrary, there exist $x_j \in \hat{X} \otimes \mathbb{Q}$ and $\gamma_j^{(n)} \in \Gamma$, $j = 1, \dots, t$, such that $\gamma_k^{(n)} \dots \gamma_i^{(n)} \to \infty$ for $1 < i \leq k \leq t$ and

$$\hat{\gamma}_1^{(n)} x_1 + (\hat{\gamma}_1^{(n)} \hat{\gamma}_2^{(n)}) x_2 + \dots + (\hat{\gamma}_1^{(n)} \dots \hat{\gamma}_t^{(n)}) x_t = 0. \tag{5.2}$$

Denote by $p_i : V_j \to V_i/V_{i+1}$, $i = 1, \dots, s$, the projection maps. Since the action of Γ on V_i/V_{i+1} is abelian and $\mu \in \Lambda'$, it follows that μ acts trivially on V_i/V_{i+1}. Hence, for every $v \in V_i/V_{i+1}$,

$$\hat{\gamma}_1^{(n)} \dots \hat{\gamma}_k^{(n)} v = \delta^{l_1(n)+\dots+l_k(n)} v.$$

with $l_j(n) \to \infty$. Now we deduce from (5.2) that

$$\delta^{l_1(n)} p_1(x_1) + \delta^{l_1(n)+l_2(n)} p_1(x_2) + \dots + \delta^{l_1(n)+\dots+l_t(n)} p_1(x_t) = 0.$$

According to our choice of δ, the map δ has no roots of unity as eigenvalues for the action on V_1/V_2. Therefore, it follows from Rokhlin's Theorem (Theorem 1.13) that $p_1(x_j) = 0$ and $x_j \in V_2$ for $j = 1, \dots, t$. Applying the same argument to the spaces V_i/V_{i+1} for $i = 2, \dots, s$, we deduce that $x_j = 0$ for every $j = 1, \dots, t$. This proves that the action of S on X is mixing of all orders. □

Proof of Theorem 1.21. Passing to a finite index subgroup, we may assume that the Zariski closure $\bar{\Gamma}$ is connected. By Lemma 5.5, there exists a finitely generated subgroup Λ in Γ such that $\bar{\Lambda} = \bar{\Gamma}$. In particular, Λ is not virtually solvable.

Suppose that Λ contains a Zariski open solvable group $\Delta = \Lambda \cap U$, where U is an open subset of $\bar{\Lambda}$. Then for $\gamma_1, \gamma_2 \in \Lambda$ such that $\gamma_1^{-1}\gamma_2 \notin \Delta$,

$$\Lambda \cap \gamma_1 U \cap \gamma_2 U = \emptyset,$$

and since Λ is dense,

$$\gamma_1 U \cap \gamma_2 U = \emptyset.$$

Hence, we have a disjoint union

$$\bar{\Lambda} = \bigcup_{\gamma \in \Gamma/\Delta} \gamma \Delta U.$$

This implies that $\Delta = \Lambda$ and gives a contradiction. Therefore, by [7, Theorem 1.1], the group Λ contains nonabelian free subgroup Δ such that $\bar{\Delta} = \bar{\Lambda} = \bar{\Gamma}$.

Suppose that the action of Γ on X is ergodic, but the action of Δ on X is not ergodic. By Proposition 5.1, there exists $\chi \in \hat{X} - \{0\}$ such that $\Delta\chi$ is finite. Then $\bar{\Delta}\chi = \bar{\Gamma}\chi$ is finite, and this gives a contradiction.

For every closed connected subgroup Y of X and

$$A(Y) = \{\chi \in \hat{X} : \chi(Y) = 1\},$$

we have

$$A(Y) = \hat{X} \cap (A(Y) \otimes \mathbb{Q}).$$

Since $\bar{\Delta} = \bar{\Gamma}$, this implies that if Y is Δ-invariant, then it is Γ-invariant. In particular, this shows that if Γ is strongly irreducible, then Δ is strongly irreducible as well.

Suppose that the action of Δ on X is not hereditarily ergodic, i.e., there exist a closed connected virtually Δ_0-invariant subgroup Y, where Δ_0 is a subgroup of finite index in Δ, and $\chi \in \hat{Y} - \{0\}$ such that $\Delta_0\chi$ is finite. Then we deduce as above that Y is invariant under $\Gamma_0 = \Gamma \cap \bar{\Delta}_0$ which has finite index in Γ. The character group of Y can be identified with $\hat{X}/A(Y)$. Moreover, since Y is connected, $\hat{X}/A(Y)$ is torsion-free, and the map

$$i : \hat{X}/A(Y) \to (\hat{X} \otimes \mathbb{C})/(A(Y) \otimes \mathbb{C})$$

is injective. Using that $\Delta_0\chi$ is finite, we deduce that $\Gamma_0 \cdot i(\chi)$ and $\bar{\Gamma}_0 \cdot i(\chi)$ are finite. It follows that $\Gamma_0\chi$ is finite, and the action of Γ on X is not hereditarily ergodic. This proves the theorem. □

6. Examples

Example 6.1 (cf. Theorem 1.3 and Corollary 1.9)**.** *For*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

we have

(a) *$S^4 = id$ and $T^3 = id$. In particular, the set $\{T, S\}$ is not mixing on $\mathbb{T}^2$.*

(b) *For every $l \ge 1$ such that $\phi(l) < (\dim \mathbb{T}^2)^2$, the linear maps $\hat{S}^l$ and $\hat{T}^l$ don't have a common eigenvalue.*

(c) *There exists $f \in L^\infty(\mathbb{T}^2)$ such that the limit*

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} f(S^{ln}x) f(T^{ln}x)\, dm(x)$$

does not exist for every $l \ge 1$ with $\phi(l) < (\dim \mathbb{T}^2)^2$.

Claim (a) is straightforward.

The eigenvalues of $\hat{S}$ and $\hat{T}$ are the primitive roots of unity of order 4 and 3 respectively. Therefore,

$$\mathrm{Spec}(S^l) \cap \mathrm{Spec}(T^l) = \emptyset$$

unless l is divisible by 12. Since $\phi(l) \ge 4$ for all $l \ge 12$, this implies (b).

To prove (c), we take $x_0 \in \mathbb{T}^2$ such that the points

$$x_0, Sx_0, S^2x_0, S^3x_0, Tx_0, T^2x_0$$

are distinct and a neighborhood $U \subset \mathbb{T}^2$ of x_0 such that

$$S^nU \cap T^nU = \emptyset \Leftrightarrow S^nx_0 \ne T^nx_0.$$

Then for f equal to the characteristic function of U, we have

$$\int_{\mathbb{T}^2} f(S^nx) f(T^nx)\, dm(x) = \begin{cases} m(U), & \text{if 12 divides } n, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\phi(l) < 4$ implies that $l < 12$, this proves (c).

Example 6.2 (cf. Corollary 1.6)**.** *There exist (i) infinite-dimensional, (ii) disconnected, compact abelian group X and epimorphisms S and T of X such that $\{S, T\}$ is not mixing, but there is no proper closed subgroup Y of X such that for some $l \ge 1$, Y is $\{S^l, T^l\}$-invariant and $S^l|_{X/Y} = T^l|_{X/Y}$.*

We utilize an example constructed by D. Berend in [2] for a different purpose. Let

$$X = \prod_{n\in\mathbb{Z}} Y$$

for a compact abelian group Y (with appropriate choice of Y, X can be made infinite-dimensional or disconnected). Note that for every $\chi \in \hat{X}$, there exists a finite $D_\chi \subset \mathbb{Z}$ and $\chi_n \in \hat{Y} - \{0\}$, $n \in D_\chi$, such that

$$\chi\left((y_n)_{n\in\mathbb{Z}}\right) = \prod_{n\in D_\chi} \chi_n(y_n).$$

Consider the following permutations of $\mathbb{Z}$:

$$\begin{aligned} \sigma(n) &= n+1, \\ \pi(n) &= \begin{cases} n & \text{for } l_{2k} \le |n| \le l_{2k+1}, \\ -n & \text{for } l_{2k+1} \le |n| \le l_{2k+2}, \end{cases} \\ \tau &= \pi^{-1}\sigma\pi, \end{aligned}$$

where $\{l_i\}_{i\ge 1}$ is an increasing sequence of integers such that $l_0 = 0$ and

$$l_{i+1}/l_i \to \infty \quad \text{as } i \to \infty. \tag{6.1}$$

Permutations σ and τ define automorphisms S and T of X which act on X by permuting coordinates.

First, we observe that $\{S, T\}$ is not mixing. In fact, for

$$B = \{(y_n)_{n\in\mathbb{Z}} : x_0 \in A\}$$

where A is a measurable subset of Y, we have

$$m(S^{-n}B \cap T^{-n}B) = \begin{cases} m(B) & \text{for } \pi(n) = n, \\ m(B)^2 & \text{for } \pi(n) \ne n. \end{cases} \tag{6.2}$$

Suppose that there exists a proper closed subgroup Y such that for some $l \ge 1$, Y is $\{S^l, T^l\}$-invariant and $S^l|_{X/Y} = T^l|_{X/Y}$. This is equivalent to existence of a proper subgroup Γ of $\hat{X}$ such that Γ is $\{\hat{S}^l, \hat{T}^l\}$-invariant and $\hat{S}^l|_\Gamma = \hat{T}^l|_\Gamma$. Consider the map

$$\hat{X} \to \{D \subset \mathbb{Z} : |D| < \infty\} : \chi \mapsto D_\chi$$

Since this map is $\langle \sigma, \tau\rangle$-equivariant, the image of Γ is a set Δ consisting of finite subsets of $\mathbb{Z}$ such that Δ is $\{\sigma^l, \tau^l\}$-invariant and $\sigma^l(D) = \tau^l(D)$ for every $D \in \Delta$. It follows that for every $k \ge 1$ and $D \in \Delta$, we have

$$\sigma^l \pi \sigma^{kl}(D) = \pi\sigma^{(k+1)l}(D). \tag{6.3}$$

Take $d \in D$. Because of (6.1), there exist infinitely many $k_i \ge 1$ such that

$$\pi(\sigma^{k_i l}(d)) = \sigma^{k_i l}(d) \quad \text{and} \quad \pi(\sigma^{(k_i+1)l}(d)) = -\sigma^{(k_i+1)l}(d).$$

Then by (6.3),

$$\sigma^{-(k_i+1)l}\pi^{-1}\sigma^l\pi\sigma^{k_i l}(d) = -d - 2(k_i+1)l \in D.$$

This contradicts finiteness of D. Hence, $\Delta = \{\emptyset\}$ and $\Gamma = \hat{X}$ which proves the claim.

Example 6.3 (cf. Corollary 1.9). *There exist (i) infinite-dimensional, (ii) disconnected, compact abelian group X, a Borel subset B of X, and epimorphisms T and S of X such that for every $l \geq 1$, the limit*

$$\lim_{n\to\infty} m(S^{-ln}B \cap T^{-ln}B)$$

does not exist.

Let S and T be as in Example 6.2. It follows from (6.1) that for every $l \geq 1$, there exist infinitely many $n_1, n_2 \geq 1$ such that $\pi(ln_1) = ln_1$ and $\pi(ln_2) \neq ln_2$. Hence, by formula (6.2), the limit does not exist.

Example 6.4 (cf. Proposition 4.3). *There exist (i) infinite-dimensional, (ii) disconnected, compact abelian group X and an infinite subgroup of* $\mathrm{Aut}(X)$ *such that the action of Γ on X is not mixing and every element of infinite order is ergodic.*

Take

$$X = \prod_{n\geq 1} Y \quad \text{for a compact abelian group } Y$$

(choosing Y appropriately, one can make X either disconnected or infinite dimensional). Take Γ to be the group of finitary permutations of the components of X. It is a torsion group which is not mixing.

To give a less trivial example, consider

$$\begin{aligned} \Gamma &= \mathbb{Z} \ltimes V \quad \text{with } V = \{\pm 1\}^{\mathbb{Z}}, \\ V_0 &= \{(v_i) \in V : v_i = 1 \quad \text{for } i \geq 1\}, \\ X &= \prod_{\Gamma/V_0} Y \quad \text{for a compact abelian group } Y. \end{aligned}$$

The group Γ acts on X permuting coordinates, and since V_0 does not contain non-trivial normal subgroup, Γ embeds in $\mathrm{Aut}(X)$. Every element of infinite order in Γ is mixing, but because V_0 is infinite, the action of Γ is not mixing.

Example 6.5 (cf. Proposition 4.3). *There exists a semigroup Γ of epimorphisms of the torus $\mathbb{T}^d$ which is not mixing, but its every finitely generated subsemigroup is mixing.*

Consider

$$\Gamma = \langle 2 \cdot \mathrm{SL}(d, \mathbb{Z}) \rangle .$$

Since

$$\begin{pmatrix} 2 & 2n \\ 0 & 2 \end{pmatrix} \in \Gamma$$

for every n, it is not mixing. On the other hand, if Γ_0 is a finitely generated subsemigroup and

$$\gamma_i = 2^{n_i}\delta_i \in \Gamma_0, \quad \delta_i \in \mathrm{SL}(d,\mathbb{Z}),$$

such that $\gamma_i \to \infty$, then $n_i \to \infty$ as well. This implies that for every $\chi \in \hat{\mathbb{T}}^d - \{0\} = \mathbb{Z}^d - \{0\}$,

$$\hat{\gamma}_i\chi \to \infty.$$

Hence, the action of Γ_0 on X is mixing.

Example 6.6 (cf. Corollary 1.15)**.** *There exists a free nonabelian semigroup Γ of epimorphisms of the torus $\mathbb{T}^d$ which is mixing of all orders.*

Take $\alpha, \beta \in \mathrm{SL}(d,\mathbb{Z})$ that generate a free group and let Γ be the semigroup generated by 2α and 2β. It was shown above that Γ is mixing. Suppose that Γ is mixing of order $s-1$, but not mixing of order s. Then there exist $x_i \in \mathbb{Z}^d - \{0\}$ and $\gamma_i^{(n)} \in \Gamma$, $i = 1, \dots, s$, such that $\gamma_j^{(n)} \dots \gamma_i^{(n)} \to \infty$ for $1 < i \le j \le s$ and

$$\hat{\gamma}_1^{(n)}x_1 + (\hat{\gamma}_1^{(n)}\hat{\gamma}_2^{(n)})x_2 + \dots + (\hat{\gamma}_1^{(n)} \dots \hat{\gamma}_s^{(n)})x_s = 0.$$

Since $\gamma_2^{(n)} \to \infty$,

$$\hat{\gamma}_2^{(n)} = 2^{k_n}\delta_n \quad \text{for } k_n \to \infty \text{ and } \delta_n \in \mathrm{SL}(d,\mathbb{Z}).$$

It follows that 2^{k_n} divides x_1 and $x_1 = 0$. This gives a contradiction. Hence, Γ is mixing of all orders.

Example 6.7 (cf. Corollary 1.15 and Lemma 4.4)**.** *For $d \ge 4$, there exists a free nonabelian semigroup of automorphisms of the torus $\mathbb{T}^d$ which generates a solvable group of degree 2 and is mixing of all orders.*

Write $d = d_1 + d_2$ with $d_1, d_2 \ge 2$, take hyperbolic matrices $A \in \mathrm{SL}(d_1,\mathbb{Z})$, $B \in \mathrm{SL}(d_2,\mathbb{Z})$, and consider the semigroup Γ generated by

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad C \in \mathrm{M}(d_1 \times d_2, \mathbb{Z}).$$

Suppose that there exist $x_i = (u_i, v_i) \in \mathbb{Z}^d$ and $\gamma_i^{(n)} \in \Gamma$, $i = 1, \dots, s$, such that $\gamma_j^{(n)} \dots \gamma_i^{(n)} \to \infty$ for $1 < i \le j \le s$ and

$$\hat{\gamma}_1^{(n)}x_1 + (\hat{\gamma}_1^{(n)}\hat{\gamma}_2^{(n)})x_2 + \dots + (\hat{\gamma}_1^{(n)} \dots \hat{\gamma}_s^{(n)})x_s = 0.$$

Then

$$\hat{\gamma}_1^{(n)} \dots \hat{\gamma}_i^{(n)} = \begin{pmatrix} {}^tA^{k_i(n)} & 0 \\ * & {}^tB^{l_i(n)} \end{pmatrix}$$

with

$$k_i(n) \to \infty, \;\; k_{i+1}(n) - k_i(n) \to \infty,$$
$$l_i(n) \to \infty, \;\; l_{i+1}(n) - l_i(n) \to \infty$$

as $n \to \infty$. We have

$$\sum_{i=1}^{s} {}^tA^{k_i(n)} u_i = 0,$$

and since A is hyperbolic, $u_i = 0$ for $i = 1, \dots, s$. Then

$$\sum_{i=1}^{s} {}^tB^{l_i(n)} v_i = 0,$$

and it follows that $x_i = 0$ for $i = 1, \dots, s$. This shows that Γ is mixing of all orders. Since matrices A and B are hyperbolic, the linear map

$$C \mapsto ACB^{-1}, \quad C \in \mathrm{M}(d_1 \times d_2, \mathbb{Z}),$$

has eigenvalues λ with $|\lambda| \neq 1$. Hence, by [19, Theorem 4.17], Γ contains a free nonabelian semigroup.

Example 6.8 (cf. Proposition 5.2)**.** *The action of*

$$\Gamma = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\} \subset \mathrm{SL}(d, \mathbb{Z})$$

on the torus $\mathbb{T}^d$ is ergodic, but not strongly irreducible and not hereditarily ergodic.

This is straightforward to check using Proposition 5.1.

Example 6.9 (cf. Theorem 1.18 and Corollary 1.20)**.** *There exists $\Gamma \subset \mathrm{Aut}(\mathbb{T}^3)$ such that the action of Γ on X is strongly irreducible and in particular hereditarily ergodic, but Γ contains no ergodic elements.*

Consider $\Gamma = \mathrm{SO}(2,1) \cap \mathrm{SL}(3, \mathbb{Z})$. If the action of Γ on $\mathbb{T}^3$ is not strongly irreducible, then there exists a subgroup Λ of finite index in Γ and a Λ-invariant subgroup A of $\mathbb{Z}^3$ such that $\mathbb{Z}^3/A$ is torsion-free. Then $A \otimes \mathbb{Q}$ is a proper Λ-invariant subspace. Since the action of $\mathrm{SO}(2,1)$ on $\mathbb{R}^3$ is irreducible, and Λ is Zariski dense in $\mathrm{SO}(2,1)$, this gives a contradiction. Hence, the action of Γ is strongly irreducible.

It is also easy to show that Γ contains no ergodic elements. Denote by B the standard bilinear form and suppose that $\gamma \in \Gamma$ has no roots of unity as eigenvalues. Let $v, w \in \mathbb{C}^3$ be eigenvectors of γ with eigenvalues λ, μ respectively. Then

$$B(v, v) = B(\gamma v, \gamma v) = \lambda^2 B(v, v)$$

and it follows that $B(v, v) = 0$. Similarly, $B(w, w) = 0$. Since B is nondegenerate, $B(v, w) \neq 0$. Then the computation as above shows that $\lambda\mu = 1$. This implies that γ acts trivially on the orthogonal complement of the subspace $\langle v, w \rangle$, which is a contradiction.

Example 6.10 (cf. Theorem 1.18)**.** *There exist a compact connected infinite-dimensional abelian group X and an automorphism T of X which is mixing, but not hereditarily ergodic.*

Let Y be any compact connected abelian group,

$$X = \prod_{n\in\mathbb{Z}} Y, \quad \text{and} \quad T : (y_n)_{n\in\mathbb{Z}} \mapsto (y_{n+1})_{n\in\mathbb{Z}}.$$

Then T is mixing, but T acts trivially on the connected subgroup

$$\{(y_n)_{n\in\mathbb{Z}} : y_n \text{ is constant}\}.$$

Hence, T is not hereditarily ergodic.

Example 6.11 (cf. Theorem 1.18)**.** *There exist an infinite-dimensional compact connected abelian group X and an abelian subgroup Γ of $\mathrm{Aut}(X)$ such that the action of Γ on X is hereditarily ergodic, but the action of every finitely generated subgroup of Γ is not ergodic. In particular, Γ contains no mixing elements.*

Take $T \in \mathrm{GL}(2,\mathbb{Z})$ with the characteristic polynomial $x^2 - x - 1$. Note that T acts ergodically on the torus $\mathbb{T}^2$. Consider

$$X = \prod_{n\geq 1} \mathbb{T}^2,$$
$$\Gamma = \prod_{n\geq 1} \langle T\rangle \quad \text{(direct product)}.$$

Define $T_i \in \Gamma$, $i \geq 1$, by

$$T_i \cdot (x_n)_{n\geq 1} = (x_1, \ldots, x_{i-1}, Tx_i, x_{i+1}, \ldots).$$

The character group of X is

$$\hat{X} = \oplus_{n\geq 1}\mathbb{Z}^2.$$

We claim that any Γ-invariant subgroup S of $\hat{X}$ is of the form $\oplus_{n\geq 1} S_n$ where S_n is a Γ-invariant subgroup of $\hat{Y}$. Indeed, this follows from the identity

$$(T_i^2 - T_i)\cdot (s_n)_{n\geq 1} = (0, \ldots, 0, s_i, 0, \ldots), \quad (s_n)_{n\geq 1} \in S.$$

This implies that any closed connected Γ invariant subgroup Y of X has the character group of the form

$$\hat{Y} = \oplus_{n\geq 1}\mathbb{Z}^2/S_n.$$

where S_n is a T-invariant subgroup of $\mathbb{Z}^2$ such that $\mathbb{Z}^2/S_n$ is torsion free, i.e., $S_n = 0$ or $S_n = \mathbb{Z}^2$. Since T acts ergodically on $\mathbb{T}^2$, the set $\mathbb{Z}^2 - \{0\}$ contains no finite $\hat{T}$-orbits. This implies that there are no finite Γ-orbits in $\hat{Y} - \{0\}$. Hence, the action of Γ is hereditarily ergodic.

It is easy to see that any finitely generated subgroup of Γ fixes some nonzero elements in $\hat{X}$. Hence, by Proposition 5.1, such subgroup is not ergodic.

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