Math 152.02

Practice Midterm 1 - Solutions

1. (a) Suppose that $f(x)$ is an even function and that $\int_0^4 f(x)dx = 6$, and that $\int_{-2}^2 f(x)dx = 2$. Find $\int_2^4 f(x)dx$.

$$\begin{aligned}\int_2^4 f(x)dx &= \int_0^4 f(x)dx - \int_0^2 f(x)dx \\ &= \int_0^4 f(x)dx - \frac{1}{2}\int_{-2}^2 f(x)dx \quad \text{(since } f \text{ is even)} \\ &= 6 - \frac{2}{2} \\ &= \boxed{5}\end{aligned}$$

(b) Suppose that $g(x)$ is an odd function and that $\int_0^4 g(x)dx = 3$, and $\int_0^2 g(x)dx = 1$. Find $\int_{-2}^4 g(x)dx$.

$$\begin{aligned}\int_{-2}^4 g(x)dx &= \int_{-2}^0 g(x)dx + \int_0^4 g(x)dx \\ &= -\int_0^2 g(x)dx + \int_0^4 g(x)dx \quad \text{(since } g \text{ is odd)} \\ &= -1 + 3 \\ &= \boxed{2}\end{aligned}$$

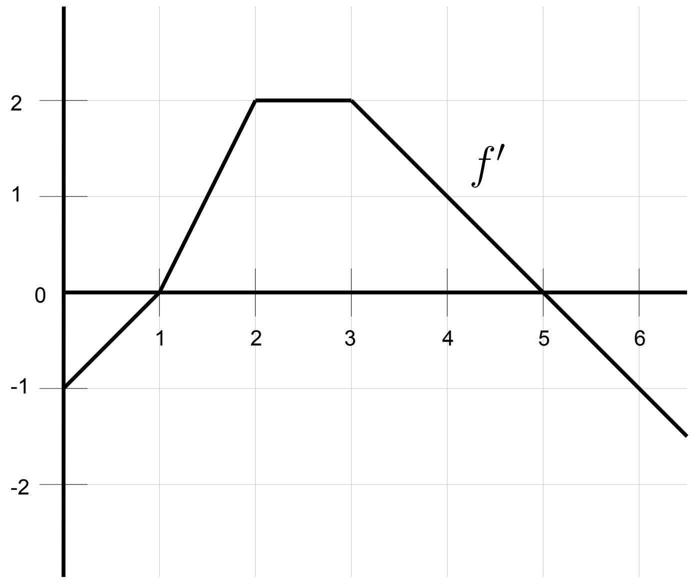
2. Draw an area that represents the definite integral $\int_0^3 \sqrt{9-x^2}\,dx$. Evaluate this integral.

This is a quarter of circle with radius 3 so

$$\int_0^3 \sqrt{9-x^2}\,dx = \pi 3^2 \cdot \frac{1}{4} = \boxed{\tfrac{9\pi}{4}}$$

3. Suppose $f'(x)$ is given by the graph below, and that $f(0) = \frac{3}{2}$. Complete the table of values for $f(x)$.

x	0	1	3	5
$f(x)$	$\frac{3}{2}$	1	4	6



4. Find the following indefinite integrals.

(a) $\int \pi x^2 + \frac{1}{\pi x^2}\, dx$

$$\begin{aligned}\int \pi x^2 + \frac{1}{\pi x^2}\, dx &= \int \pi x^2 + \frac{1}{\pi} \cdot x^{-2}\, dx \\ &= \frac{\pi x^3}{3} + \frac{x^{-1}}{\pi(-1)} + C \\ &= \boxed{\frac{\pi x^3}{3} - \frac{1}{\pi x} + C}\end{aligned}$$

(b) $\int \sqrt{x} - \frac{1}{x\sqrt{x}}\, dx$

$$\begin{aligned}\int \sqrt{x} - \frac{1}{x\sqrt{x}}\, dx &= \int x^{\frac{1}{2}} - x^{-\frac{3}{2}}\, dx \\ &= \frac{2}{3} \cdot x^{\frac{3}{2}} - \left(-\frac{2}{1}\right) \cdot x^{-\frac{1}{2}} + C \\ &= \boxed{\frac{2}{3} \cdot x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + C}\end{aligned}$$

(c) $\int e^t + e^3\, dt$

$$\int e^t + e^3\, dt = \boxed{e^t + e^3 t + C}$$

(d) $\int \frac{2}{x} + \frac{x}{2}\, dx$

$$\begin{aligned}\int \frac{2}{x} + \frac{x}{2}\, dx &= \int 2x^{-1} + \frac{1}{2} \cdot x\, dx \\ &= 2\ln x + \frac{1}{2} \cdot \frac{x^2}{2} + C \\ &= \boxed{2\ln x + \frac{x^2}{4} + C}\end{aligned}$$

5. Write down the left and right Riemann sums for three equal intervals for the integral

$$\int_1^2 \frac{1}{x}\,dx.$$

Without using your calculator explain why

$$\frac{47}{60} \geq \ln(2) \geq \frac{37}{60}.$$

Let $f(x) = \frac{1}{x}$

$$\begin{aligned}
\text{LEFT}(3) &= \sum_{i=0}^{3-1} f\left(1 + i \cdot \frac{2-1}{3}\right) \cdot \frac{2-1}{3} \\
&= f(1) \cdot \frac{1}{3} + f\left(\frac{4}{3}\right) \cdot \frac{1}{3} + f\left(\frac{5}{3}\right) \cdot \frac{1}{3} \\
&= \frac{1}{1} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3} + \frac{3}{5} \cdot \frac{1}{3} \\
&= \frac{20}{60} + \frac{15}{60} + \frac{12}{60} \\
&= \boxed{\frac{47}{60}}
\end{aligned}$$

$$\begin{aligned}
\text{RIGHT}(3) &= \sum_{i=1}^{3} f\left(1 + i \cdot \frac{2-1}{3}\right) \cdot \frac{2-1}{3} \\
&= f\left(\frac{4}{3}\right) \cdot \frac{1}{3} + f\left(\frac{5}{3}\right) \cdot \frac{1}{3} + f(2) \cdot \frac{1}{3} \\
&= \frac{3}{4} \cdot \frac{1}{3} + \frac{3}{5} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} \\
&= \frac{15}{60} + \frac{12}{60} + \frac{10}{60} \\
&= \boxed{\frac{37}{60}}
\end{aligned}$$

$\frac{d}{dx} \ln x = \frac{1}{x}$ so by the Fundamental Theorem of Calculus $\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 = \ln 2$. On the interval $[1, 2]$ the function $\frac{1}{x}$ is a decreasing so

$$\text{LEFT}(3) \geq \int_1^2 \frac{1}{x}\,dx \geq \text{RIGHT}(3)$$

hence

$$\frac{47}{60} \geq \ln(2) \geq \frac{37}{60}.$$

6. Use the Fundamental Theorem of Calculus to compute the following definite integrals.

(a) $\displaystyle\int_1^3 \frac{1}{t^2}\,dt$

$$\begin{aligned}\int_1^3 \frac{1}{t^2}\,dt &= \int_1^3 t^{-2}\,dt\\ &= \left.\frac{t^{-1}}{-1}\right|_1^3\\ &= -\frac{1}{3} - \left(-\frac{1}{1}\right)\\ &= \boxed{\frac{2}{3}}\end{aligned}$$

(b) $\int_0^{\pi} (3\sin x + x + 5)\,dx$

$$\begin{aligned}\int_0^{\pi} (3\sin x + x + 5)\,dx &= \left.-3\cos x + \frac{x^2}{2} + 5x\right|_0^{\pi}\\ &= \left(-3\cos\pi + \frac{\pi^2}{2} + 5\pi\right) - \left(-3\cos 0 + \frac{0^2}{2} + 5\cdot 0\right)\\ &= 3 + \frac{\pi^2}{2} + 5\pi + 3\\ &= \boxed{6 + \frac{\pi^2}{2} + 5\pi}\end{aligned}$$

7. Find the derivative of $x\ln(x) - x$. Calculate $\int_2^3 \ln(x)dx$.

$$\frac{d}{dx}\left(x\ln(x) - x\right) = 1\cdot\ln x + x\cdot\frac{1}{x} - 1 = \boxed{\ln x}$$

By The Fundamental Theorem of Calculus and the differentiation above

$$\int_2^3 \ln(x)dx = \left.(x\ln(x) - x)\right|_2^3 = (3\ln(3) - 3) - (2\ln(2) - 2) = \boxed{3\ln(3) - 2\ln(2) + 1}$$

8. Find the derivative of $G(x) = \int_{\sin x}^{x^3} t\sqrt{t^4+1}dt$

$$\begin{aligned}\frac{d}{dx}G(x) &= \frac{d}{dx}\left(\int_{\sin x}^{x^3} t\sqrt{t^4+1}dt\right)\\ &= \frac{d}{dx}\left(\int_{\sin x}^{0} t\sqrt{t^4+1}dt + \int_0^{x^3} t\sqrt{t^4+1}dt\right)\\ &= \frac{d}{dx}\left(-\int_0^{\sin x} t\sqrt{t^4+1}dt + \int_0^{x^3} t\sqrt{t^4+1}dt\right)\\ &= \boxed{-\sin x\sqrt{(\sin x)^4+1}\cdot(\cos x) + x^3\sqrt{(x^3)^4+1}\cdot 3x^2}\end{aligned}$$

9. The average value of a function $f(x)$ for $1 \leq x \leq 5$ is equal to 7, and the average value of the same function $f(x)$ for $5 \leq x \leq 8$ is equal to 3. What is the average value of $f(x)$ for $1 \leq x \leq 8$?

$\frac{1}{5-1}\int_1^5 f(x)dx = 7$ so $\int_1^5 f(x)dx = 7 \cdot 4 = 28$.

$\frac{1}{8-5}\int_5^8 f(x)dx = 3$ so $\int_5^8 f(x)dx = 3 \cdot 3 = 9$.

Average value of f on $[1, 8]$ is

$$\begin{aligned} \frac{1}{8-1}\int_1^8 f(x)dx &= \frac{1}{8-1}\left(\int_1^5 f(x)dx + \int_5^8 f(x)dx\right) \\ &= \frac{1}{7}(28+9) \\ &= \boxed{\frac{37}{7}} \end{aligned}$$