

Mathematics 152.02 Solutions to Review for Final Exam

1. Suppose that the function $f(t)$ represents the price of a barrel of oil in dollars at time t , measured in days, where $f(0)$ is the price at midnight on December 31, 2005. In words, what is represented by $\frac{1}{90} \int_0^{90} f(t) dt$?

$$\frac{1}{90} \int_0^{90} f(t) dt \text{ is the average price of a barrel of oil in dollars during the first 90 days of 2006.}$$

2. Suppose that $f(x)$ is an odd function, $g(x)$ is an even function, and that the following values are known:

$$\int_0^3 f(x) dx = 2 \quad \int_3^5 f(x) dx = 1 \quad \int_0^3 g(x) dx = 4 \quad \int_3^5 g(x) dx = 1.$$

Find $\int_{-3}^5 f(x) + g(x) dx$.

$$\begin{aligned} \int_{-3}^5 f(x) + g(x) dx &= \int_{-3}^5 f(x) dx + \int_{-3}^5 g(x) dx \\ &= \int_{-3}^3 f(x) dx + \int_3^5 f(x) dx + \int_{-3}^3 g(x) dx + \int_3^5 g(x) dx \\ &= 0 + \int_3^5 f(x) dx + 2 \int_0^3 g(x) dx + \int_3^5 f(x) dx \\ &= 0 + 1 + 2 \cdot 4 + 1 \\ &= \boxed{10} \end{aligned}$$

3. Find the solution to the initial value problem

$$f'(t) = \frac{t^3 + 1}{t^3 - 4t} \quad f(1) = 2$$

$$\begin{aligned} f'(t) &= \frac{t^3 + 1}{t^3 - 4t} \\ &= 1 + \frac{4t + 1}{t^3 - 4t} \\ &= 1 + \frac{4t + 1}{t(t+2)(t-2)} \\ &= 1 + \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-2} \\ &= 1 - \frac{1}{4t} - \frac{7}{8(t+2)} + \frac{9}{8(t-2)} \end{aligned}$$

Thus

$$\begin{aligned} f(t) &= \int \left(1 - \frac{1}{4t} - \frac{7}{8(t+2)} + \frac{9}{8(t-2)} \right) dx \\ &= t - \frac{\ln|t|}{4} - \frac{7 \ln|t+2|}{8} + \frac{9 \ln|t-2|}{8} + C \end{aligned}$$

$f(1) = 2$ implies that

$$\begin{aligned} 2 &= 1 - \frac{\ln|1|}{4} - \frac{7 \ln|1+2|}{8} + \frac{9 \ln|1-2|}{8} + C \\ C &= 2 - 1 + \frac{\ln|1|}{4} + \frac{7 \ln|1+2|}{8} - \frac{9 \ln|1-2|}{8} \\ C &= 2 - 1 + \frac{0}{4} + \frac{7 \ln 3}{8} - \frac{9 \cdot 0}{8} \\ C &= 1 + \frac{7 \ln 3}{8} \end{aligned}$$

$$f(t) = t - \frac{\ln|t|}{4} - \frac{7 \ln|t+2|}{8} + \frac{9 \ln|t-2|}{8} + 1 + \frac{7 \ln 3}{8}$$

4. For each of the following improper integrals, say whether it converges or diverges. If it converges, find its value.

$$\int_0^{\infty} \frac{1}{1+x^2} dx \quad \int_2^{\infty} \frac{1}{x} dx \quad \int_{0.5}^2 \frac{1}{1-x} dx \quad \int_1^{\infty} \frac{1}{x^3} dx$$

$$\begin{aligned} \int_0^{\infty} \frac{1}{1+x^2} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx \\ &= \lim_{b \rightarrow \infty} \arctan(x) \Big|_0^b \\ &= \lim_{b \rightarrow \infty} \arctan(b) - \arctan(0) \\ &= \boxed{\frac{\pi}{2} \text{ convergent}} \end{aligned}$$

$$\begin{aligned} \int_2^{\infty} \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \ln|x| \Big|_2^b \\ &= \lim_{b \rightarrow \infty} \ln|b| - \ln(2) \\ &= \boxed{\text{divergent}} \end{aligned}$$

$$\begin{aligned} \int_{0.5}^2 \frac{1}{1-x} dx &= \int_{0.5}^1 \frac{1}{1-x} dx + \int_1^2 \frac{1}{1-x} dx \\ &= \lim_{b \rightarrow 1^-} \int_{0.5}^b \frac{1}{1-x} dx + \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{1-x} dx \\ &= \lim_{b \rightarrow 1^-} (-\ln|1-x| \Big|_{0.5}^b) + \lim_{c \rightarrow 1^+} (-\ln|1-x| \Big|_c^2) \\ &= \lim_{b \rightarrow 1^-} (-\ln|1-b| + \ln|1-0.5|) + \lim_{c \rightarrow 1^+} (-\ln|1-2| + \ln|1-c|) \\ &= \boxed{\text{divergent}} \end{aligned}$$

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^3} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^3} dx \\ &= \lim_{b \rightarrow \infty} \frac{1}{-2x^2} \Big|_1^b \\ &= \lim_{b \rightarrow \infty} \frac{1}{-2b^2} - \frac{1}{-2 \cdot 1^2} \\ &= \boxed{\frac{1}{2} \text{ convergent}} \end{aligned}$$

5. A solid has its base in the (x, y) -plane. The base of the solid is the region between the curve $y = 4 - x^2$ and the x -axis. Each cross-section of the solid perpendicular to the x -axis is a square. Find the volume of the solid.

$$A(x) = y^2 = (4 - x^2)^2$$

The roots of $4 - x^2$ are $x = 2$ and $x = -2$ so

$$\begin{aligned}
 V &= \int_{-2}^2 A(x) dx \\
 &= \int_{-2}^2 (4 - x^2)^2 dx \\
 &= \int_{-2}^2 16 - 8x^2 + x^4 dx \\
 &= 16x - \frac{8x^3}{3} + \frac{x^5}{5} \Big|_{-2}^2 \\
 &= \left(16 \cdot 2 - \frac{8 \cdot 2^3}{3} + \frac{2^5}{5} \right) - \left(16 \cdot (-2) - \frac{8 \cdot (-2)^3}{3} + \frac{(-2)^5}{5} \right) \\
 &= \boxed{\frac{512}{15}}
 \end{aligned}$$

6. Find the volume of the donut obtained by rotating a circle of radius 1 and center $(0, 2)$ around the x -axis.

$$\begin{aligned}
 A(x) &= \pi R(x)^2 - \pi r(x)^2 \\
 &= \pi(2 + \sqrt{1 - x^2})^2 - \pi(2 - \sqrt{1 - x^2})^2 \\
 &= \pi(4 + 4\sqrt{1 - x^2} + (1 - x^2)) - \pi(4 - 4\sqrt{1 - x^2} + (1 - x^2)) \\
 &= 8\pi\sqrt{1 - x^2}
 \end{aligned}$$

$$\begin{aligned}
 V &= \int_{-1}^1 A(x) dx \\
 &= \int_{-1}^1 8\pi\sqrt{1 - x^2} dx \\
 &\text{(trig substitution } x = \sin \theta, dx = \cos \theta d\theta \text{ so } \theta = \arcsin x) \\
 &= 8\pi \int_{\arcsin(-1)}^{\arcsin 1} (\sqrt{1 - \sin^2 \theta}) \cos \theta d\theta \\
 &= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sqrt{\cos^2 \theta}) \cos \theta d\theta \\
 &= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta \\
 &= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2} d\theta \\
 &= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2} \cos(2\theta) d\theta \\
 &= 8\pi \left(\frac{1}{2}\theta + \frac{1}{4} \sin(2\theta) \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= 8\pi \left(\frac{1}{2} \cdot \frac{\pi}{2} + \frac{1}{4} \sin \left(2 \cdot \frac{\pi}{2} \right) \right) - 8\pi \left(\frac{1}{2} \left(-\frac{\pi}{2} \right) + \frac{1}{4} \sin \left(2 \cdot \left(-\frac{\pi}{2} \right) \right) \right) \\
 &= \boxed{4\pi^2}
 \end{aligned}$$

7. Suppose that $g(x)$ is a function such that $g(x) \geq 0$ and $g'(x) \geq 1$ for all $x \geq 0$. Define a function $f(x)$ by the formula $f(x) = \int_0^x \sqrt{(g'(t))^2 - 1} dt$. Show that the arc length of the curve $y = f(x)$ between $x = a$ and $x = b$ is equal to $g(b) - g(a)$.

$$f(x) = \int_0^x \sqrt{(g'(t))^2 - 1} dt$$

$$f'(x) = \frac{d}{dx} \int_0^x \sqrt{(g'(t))^2 - 1} dt$$

(by the fundamental theorem of calculus)

$$= \sqrt{(g'(x))^2 - 1}$$

arc length of $f = \int_a^b \sqrt{1 + (f'(x))^2} dx$

(by the calculation above)

$$= \int_a^b \sqrt{1 + \left(\sqrt{(g'(x))^2 - 1}\right)^2} dx$$

$$= \int_a^b \sqrt{1 + (g'(x))^2 - 1} dx$$

$$= \int_a^b \sqrt{(g'(x))^2} dx$$

$$= \int_a^b g'(x) dx$$

(by the fundamental theorem of calculus)

$$= g(b) - g(a)$$

8. Find the arc length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ on the interval between the points $(0, 1)$ and $(1, (e + 1/e)/2)$.

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}(e^x - e^{-x})$$

arc length = $\int_0^1 \sqrt{1 + (f'(x))^2} dx$

$$= \int_0^1 \sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2} dx$$

$$= \int_0^1 \sqrt{1 + \frac{1}{4}e^{2x} - \frac{1}{2}e^x e^{-x} + \frac{1}{4}e^{-2x}} dx$$

$$= \int_0^1 \sqrt{\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}} dx$$

$$= \int_0^1 \sqrt{\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)^2} dx$$

$$= \int_0^1 \frac{1}{2}e^x + \frac{1}{2}e^{-x} dx$$

$$= \frac{1}{2}e^x - \frac{1}{2}e^{-x} \Big|_0^1$$

$$= \left(\frac{1}{2}e^1 - \frac{1}{2}e^{-1}\right) - \left(\frac{1}{2}e^0 - \frac{1}{2}e^0\right)$$

$$= \boxed{\frac{e}{2} - \frac{1}{2e}}$$

9. Find the arc length of the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln(x)$ between $x = 1$ and $x = 2$.

$$f(x) = \frac{1}{2}x^2 - \frac{1}{4}\ln(x)$$

$$f'(x) = x - \frac{1}{4x}$$

$$\begin{aligned} \text{arc length} &= \int_1^2 \sqrt{1 + (f'(x))^2} dx \\ &= \int_1^2 \sqrt{1 + \left(x - \frac{1}{4x}\right)^2} dx \\ &= \int_1^2 \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}} dx \\ &= \int_1^2 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}} dx \\ &= \int_1^2 \sqrt{\left(x + \frac{1}{4x}\right)^2} dx \\ &= \int_1^2 x + \frac{1}{4x} dx \\ &= \left. \frac{x^2}{2} + \frac{\ln x}{4} \right|_1^2 \\ &= \left(\frac{2^2}{2} + \frac{\ln 2}{4} \right) - \left(\frac{1^2}{2} + \frac{\ln 1}{4} \right) \\ &= \boxed{\frac{3}{2} + \frac{\ln 2}{4}} \end{aligned}$$

10. Find the area of the region in the plane, which (in polar coordinates) is bounded by the lines $\theta = 0$, $\theta = \pi$ and by the portion of the logarithmic spiral $r = e^\theta$ for $0 \leq \theta \leq \pi$.

$$\begin{aligned} \text{Area} &= \int_0^\pi \frac{1}{2} (r(\theta))^2 d\theta \\ &= \int_0^\pi \frac{1}{2} (e^\theta)^2 d\theta \\ &= \frac{1}{2} \int_0^\pi e^{2\theta} d\theta \\ &= \left. \frac{1}{4} e^{2\theta} \right|_0^\pi \\ &= \frac{1}{4} e^{2\pi} - \frac{1}{4} e^{2 \cdot 0} \\ &= \boxed{\frac{e^{2\pi}}{4} - \frac{1}{4}} \end{aligned}$$

11. Find the center of mass of a hemisphere of radius R whose base lies in the (x, y) -plane. (Hint: take the center of the hemisphere to lie at the origin. By symmetry you may assume that the center of mass lies on the positive z -axis.)

$$\begin{aligned} A(z) &= \pi(r(z))^2 \\ &= \pi(\sqrt{R^2 - z^2})^2 \\ &= \pi(R^2 - z^2) \end{aligned}$$

$$\begin{aligned}
\bar{z} &= \frac{\int_0^R z \delta A(z) dz}{\int_0^R \delta A(z) dz} \\
&= \frac{\int_0^R z \delta \pi (R^2 - z^2) dz}{\int_0^R \delta \pi (R^2 - z^2) dz} \\
&= \frac{\delta \pi \int_0^R R^2 z - z^3 dz}{\delta \pi \int_0^R R^2 - z^2 dz} \\
&= \frac{\left. \frac{R^2 z^2}{2} - \frac{z^4}{4} \right|_0^R}{\left. R^2 z - \frac{z^3}{3} \right|_0^R} \\
&= \frac{\frac{R^4}{2} - \frac{R^4}{4}}{R^3 - \frac{R^3}{3}} \\
&= \frac{3}{8} R
\end{aligned}$$

By symmetry $\bar{x} = 0$ and $\bar{y} = 0$.

The center of mass is the point $(0, 0, \frac{3}{8} R)$.

12. Let C_1 and C_2 be two long cylinders of radius one in space, so that the axis of symmetry of C_1 is the x -axis and the axis of symmetry of C_2 is the y -axis. Find the volume of the intersection of C_1 and C_2 . (Hint: each cross-section perpendicular to the z -axis is a square.)

C_1 has equation $y^2 + z^2 = 1$ and C_2 has equation $x^2 + z^2 = 1$. For a fixed value of z the cylinders intersect in a square with sides of length $2y = 2x = 2\sqrt{1 - z^2}$.

$$\begin{aligned}
A(z) &= (2\sqrt{1 - z^2})^2 \\
&= 4 - 4z^2
\end{aligned}$$

$$\begin{aligned}
V &= \int_{-1}^1 A(z) dz \\
&= \int_{-1}^1 4 - 4z^2 dz \\
&= 4z - \frac{4z^3}{3} \Big|_{-1}^1 \\
&= \left(4 \cdot 1 - \frac{4 \cdot 1^3}{3} \right) - \left(4 \cdot (-1) - \frac{4 \cdot (-1)^3}{3} \right) \\
&= \boxed{\frac{16}{3}}
\end{aligned}$$