

G-MORSE THEORY REVISITED

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ELEMENTARY MORSE THEORY .

M closed manifold,

$h : M \rightarrow \mathbb{R}$ smooth function

$X : M \rightarrow TM$ smooth vector field.

h **Morse function**

$dh_x = 0 \Rightarrow$

$$(h, M, x) \sim \left\{ \begin{array}{l} h := h(x) - 1/2 \sum_{i=1}^q x_i^2 + 1/2 \sum_{i=q+1}^n x_i^2 \\ \mathbb{R}^q \times \mathbb{R}^{n-q} \\ 0 \end{array} \right\}$$

$Ind(x) = q$

$$\boxed{Cr(h) = \{x \in M, dh_x = 0\}}, Cr(h) = \sqcup_q Cr(h)_q$$

$X -h$ **gradient like vector field**

a) $X(h)(x) < 0, x \in M \setminus Cr(h)$

b) $(X, M, x) \sim (-grad_g h, M, x)$ for some Riemannian metric $g \Rightarrow Cr(h) = \mathcal{X}$

Stable/Unstable set

x critical point i.e $x \in Cr(h)$

$$W_x^\pm = \{y \in M \mid \lim_{t \rightarrow \pm\infty} \gamma_y(t) = x\} \quad i_x^\pm : W_x^\pm \rightarrow M$$

$$\Rightarrow W_x^\pm \sim \mathbb{R}^{Ind(x)/n - Ind(x)}$$

(X, h) Morse pair

a) h Morse function,

b) X $(-h)$ -gradient like vector field

(X, h) Morse Smale pair

a) (X, h) Morse pair

b) $x, y \in Cr(h) \Rightarrow W_x^- \pitchfork W_y^+$

Proposition 1. 1. *Given f smooth function there exists h a Morse function arbitrary close from f in C^r topology .*

2. Given h a Morse function there exists X $(-h)$ -gradient like vector field.

3. Given (X, h) a Morse pair there exists X' arbitray close from X in C^0 topology so that (X', h) is a Morse Smale pair. One can actually choose X' to be equal to X away from an arbitrary neighborhood of $Cr(h)$

Let (X, h) be a Morse-Smale pair

$$x, y \in Cr(h) \Rightarrow \mathcal{T}(x, y) := W_x^- \cap W_y^+ / \mathbb{R}$$

smooth manifold of dimension $ind(x) - ind(y) - 1$

Consider $\hat{\mathcal{T}}(x, z)$, \hat{W}_x^- and $\hat{i}_x^- : \hat{W}_x^- \rightarrow M$ defined by

$$\hat{\mathcal{T}}(x, z) := \bigsqcup_r \bigsqcup_{y_1, \dots, y_r} \mathcal{T}(x, y_1) \times \cdots \times \mathcal{T}(y_{r-1}, y_r) \times \mathcal{T}(y_r, z)$$

$$\hat{W}_x^- := \bigsqcup_r \bigsqcup_{y_1, \dots, y_r} \mathcal{T}(x, y_1) \cdots \mathcal{T}(y_{r-1}, y_r) \times W_{y_r}^-$$

$$\hat{i}_x^- \big|_{\bigsqcup_{y_1, \dots, y_r} \mathcal{T}(x, y_1) \cdots \mathcal{T}(y_{r-1}, y_r) \times W_{y_r}^-} := i_{y_r}^- \cdot Pr_{W_{y_r}^-}.$$

Theorem 1.1. *$\hat{\mathcal{T}}(x, z)$ is a smooth compact manifold with corners whose k -corner is*

$$\hat{\mathcal{T}}(x, z)_r = \bigsqcup_{y_1, \dots, y_r} \mathcal{T}(x, y_1) \times \cdots \times \mathcal{T}(y_{r-1}, y_r) \times \mathcal{T}(y_r, z).$$

2. *\hat{W}_x^- is a smooth compact manifold with corners whose k -corner is*

$$(\hat{W}_x^-)_r = \bigsqcup_{y_1, \dots, y_r} \mathcal{T}(x, y_1) \cdots \mathcal{T}(y_{r-1}, y_r) \times W_{y_r}^-$$

3. *\hat{i}_x^- is a smooth map.*

APPLICATIONS

1) Smooth cell structure of M

Corollary 1. 1. *The partition of M into W_x^- , $x \in Cr(h)$, is a smooth CW-complex whose closed cells are given by*

$$\hat{i}_x^- : \hat{W}_x^- \rightarrow M.$$

2. *There exists compatible smooth triangulations of M , $\hat{T}(x, y)$, \hat{W}_x^- . In particular the open simplexes of W_x^- define the collection of open simplexes of M*

2) The geometric complex

Fix orientations $\mathcal{O} := \{o_x, x \in Cr(h)\}$ of $T_x(W_x^-)$

Define the complex $(C^*(X), \delta_{h, \mathcal{O}}^*(t))$ by:

$$C^q(X) := \text{Maps}(Cr(h)_q, \mathbb{R})$$

$$((\delta_{h, \mathcal{O}}^q(t))(f))(y) := \sum_{x \in Cr(h)_q} e^{-t(h(x)-h(y))} \delta(x, y) f(x)$$

$$y \in Cr(h)_{q+1}, \delta(x, y) = \#(\mathcal{T}(y, x)), f \in C^q(X)$$

3) Integration Theory

$$\text{Int}_h^g(t)(\omega) := \int \hat{W}_x^-(i_x^-)^*(e^{(-t(h-h(x)))}\omega)$$

$$\text{Int}_h^*(t) : (\Omega^*(M), d_h^*(t) = d^* + tdh\wedge) \rightarrow (C^*(X), \delta_{h,\mathcal{O}}^*(t))$$

4) WHS-theorem

Theorem 2. *Given a Riemannian metric g (which is admissible for (X, h)) there exists $T > 0$ and a canonical decomposition*

$$(\Omega^*(M), d_h^*(t)) = (\Omega_{sm}^*(M), d_h^*(t)) \oplus (\Omega_{la}^*(M), d_h^*(t))$$

so that

$$\text{Int}_h(t) : (\Omega_{sm}^*(M), d_h^*(t)) \rightarrow (C^*(X, \mathcal{O}), \delta_{h,\mathcal{O}}^*(t))$$

is an $O(1/t)$ isometry provided $t > T$.

ELEMENTARY G-MORSE THEORY .

$\tilde{M} = (M, \mu : G \times M \rightarrow M)$ closed G - manifold,

Let $x \in M$.

$$H := G_x = \{g \in G, \mu(g, x) = x\}$$

$$\Sigma(x) := \mu(G, x) = G/H \text{ orbit of } x \in M$$

$$\rho^x : H \rightarrow Gl(V = T_x(M)/T_x(\Sigma)) \text{ isotropy repr.}$$

$$(U, \Sigma) \sim (G \times_H V, G/H)$$

$$G \times_H V \rightleftharpoons G/H$$

$h : M \rightarrow \mathbb{R}$ a smooth G -invariant function

$X : M \rightarrow TM$ smooth G -equivariant vector field.

$$x \in Cr(h) \Rightarrow \Sigma(x) \subset Cr(h)$$

$$x', x'' \in \Sigma \Rightarrow \rho_{x'} \equiv \rho_{x''}$$

G - **Morse function** h satisfies:

$$\mathbf{a)} \rho^x = \rho_-^x \oplus \rho_+^x, \rho_{\pm}^x : H \rightarrow O(V_{\pm}), V = V_- \oplus V_+$$

$$x \in Cr(h) \Rightarrow \Sigma(x) \subset Cr(h)$$

$$\mathbf{b)} \Sigma \subset Cr(h) \Rightarrow$$

$$(h, M, \Sigma) \sim \left\{ \begin{array}{l} h := h(x) - 1/2||v_-||^2 + 1/2||v_+||^2 \\ E(\rho_- \oplus \rho_+) \\ \Sigma \end{array} \right\}$$

$$Ind(\Sigma) := dim(V_-)$$

G - normal Morse function

h satisfies

- a) h is a G - Morse function,
- b) ρ_x^- trivial for any $x \in Cr(h)$

$X -h$ gradient like vector field and g - equivariant

$$\mathcal{X} := \{x \in M \mid X(x) = 0\} \Rightarrow \mathcal{X} = Cr(h)$$

$$\Sigma \subset Cr(h)$$

$$W_\Sigma^\pm = \{y \in M \mid \lim_{t \rightarrow \pm\infty} \gamma_y(t) \in \Sigma\} = \bigcup_{x \in \Sigma} W_x^\pm$$

$$i_\Sigma^\pm : W_\Sigma^\pm \rightarrow M \quad p_\Sigma^\pm : W_\Sigma^\pm \rightarrow \Sigma$$

p_Σ^\pm is a G - bundle eqv. to

$G \times_{G_x} W_x^\pm \rightarrow \Sigma(x)$ equivalent to $G \times_{G_x} V_\pm \rightarrow G/G_x$, for some
(and then any) $x \in \Sigma$.

(X, h) G -Morse pair

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(X, h) G - normal Morse pair

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G -Morse Smale pair (normal Morse Smale pair)

(X, h) satisfies

- a) (X, h) a G -Morse pair (G - normal Morse pair)
- b) For any $\Sigma, \Sigma', x \in \Sigma \Rightarrow W_x^- \pitchfork W_{\Sigma'}^+$ It suffices to check for one x .

Let (X, h) be a G -Morse-Smale pair

$\Sigma, \Sigma' \subset Cr(h) \Rightarrow$

$$\mathcal{T}(\Sigma, \Sigma') := W_{\Sigma}^- \cap W_{\Sigma'}^+ / \mathbb{R}$$

smooth manifold of dimension $dim(\Sigma) + ind(\Sigma) - ind(\Sigma') - 1$

$\Sigma, \Sigma' \subset Cr(h), x \in \Sigma \Rightarrow$

$$\mathcal{T}(x, \Sigma') := W_x^- \cap W_{\Sigma'}^+ / \mathbb{R}$$

smooth manifold of dimension $ind(\Sigma) - ind(\Sigma') - 1$ on which G_x acts.

$$u : \mathcal{T}(\Sigma, \Sigma') \rightarrow \Sigma, \quad l : \mathcal{T}(\Sigma, \Sigma') \rightarrow \Sigma'$$

smooth maps, u a smooth G - bundle isomorphic to
 $G \times_{G_x} \mathcal{T}(x, \Sigma') \rightarrow G/G_x$

ρ_x^- trivial $\Rightarrow p_{\Sigma}^-$ and u are trival bundles.

Define:

$\hat{\mathcal{T}}(\Sigma, \Sigma'), \hat{W}_{\Sigma}^-$ G -spaces,

$\hat{u} : \hat{\mathcal{T}}(\Sigma, \Sigma') \rightarrow \Sigma, l : \hat{\mathcal{T}}(\Sigma, \Sigma') \rightarrow \Sigma'$ G -equiv. maps,

$\hat{p}_{\Sigma} : \hat{W}_{\Sigma}^- \rightarrow \Sigma, \hat{i}_{\Sigma}^- : \hat{W}_{\Sigma}^- \rightarrow M$ G - equiv.maps.

$$\hat{\mathcal{T}}(\Sigma, \Sigma') := \sqcup_r \bigsqcup_{\Sigma_1, \dots, \Sigma_r} \mathcal{T}(\Sigma, \Sigma_1) \times_{\Sigma_1} \cdots \times_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} \mathcal{T}(\Sigma_r, \Sigma')$$

$$\hat{W}_{\Sigma}^- := \sqcup_r \bigsqcup_{\Sigma_1, \dots, \Sigma_r} \mathcal{T}(\Sigma, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} W_{\Sigma_r}^-$$

$$\hat{i}_{\Sigma}^- \Big|_{\bigsqcup_{\Sigma_1, \dots, \Sigma_r} \mathcal{T}(\Sigma, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} W_{\Sigma_r}^-} := i_{\Sigma_r}^- \cdot Pr_{W_{\Sigma_r}^-}.$$

Theorem 3. 1. $\hat{\mathcal{T}}(\Sigma, \Sigma')$ is a smooth compact manifold with corners whose k -corner is

$$\hat{\mathcal{T}}(\Sigma, \Sigma')_r = \bigsqcup_{y_1, \dots, y_r} \mathcal{T}(\Sigma, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} \mathcal{T}(\Sigma_r, \Sigma').$$

The maps \hat{u} and \hat{l} are smooth with \hat{u} a G -bundle with fiber

$$T_x := \bigsqcup_{\Sigma_1, \dots, \Sigma_r} \mathcal{T}(x, \Sigma_1) \times_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} \mathcal{T}(\Sigma_r, \Sigma').$$

This bundle is isomorphic to $G \times_{G_x} T_x \rightarrow G/G_x$ and is trivial if ρ_- is trivial.

2. \hat{W}_{Σ}^- is a smooth compact manifold with corners whose k -corner is

$$(\hat{W}_{\Sigma}^-)_r = \bigsqcup_{y_1, \dots, y_r} \mathcal{T}(\Sigma, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} W_{\Sigma_r}^-.$$

The map \hat{p}_{Σ}^- a smooth G -bundle with fiber

$$F_x = \bigsqcup_{y_1, \dots, y_r} \mathcal{T}(x, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} W_{\Sigma_r}^-.$$

This bundle is isomorphic to $G \times_{G_x} F_x \rightarrow G/G_x$ and is trivial if ρ_- is trivial.

3. \hat{i}_{Σ}^- is a smooth map.

Suppose (X, h) is G -normal Morse Smale. Choose for any $\Sigma \subset Cr(h)$ a point $x \in \Sigma$

By "passing to the orbit spaces" $i_{\Sigma}^{-} : W_{\Sigma}^{-} \rightarrow M \Rightarrow$

$$i_x^{-} : W_x^{-} \rightarrow M/G$$

Corollary 2. *The manifolds W_x^{-} have canonical compactifications to smooth manifolds with corners F_x and the maps i_x^{-} have (smooth) completions $\hat{i}_x^{-} : F_x \rightarrow M/G$ providing a (smooth) cell complex structure for M/G .*

Theorem 4. 1. *(K.H.Meyer) Given a G -invariant smooth function f there exists h a normal G -Morse function arbitrary close to f in C^0 topology.*

2. *Given h a G -Morse function there exists $(-h)$ -gradient like vector fields.*

3. *([B]) Given (X, h) a normal G -Morse pair, there exists X' arbitrary close to X in the C^0 topology so that (X', h) is a G -normal Morse Smale pair. One can choose X' to be equal to X away from an arbitrary G -invariant neighborhood of $Cr(h)$.*

APPLICATIONS:

1) Smooth cell structure

Corollary 3. *Suppose (X, h) is a G - Morse Smale pair.*

- 1. The partition of M into W_Σ^- is a smooth G - handle body whose G - handles are $\hat{i}_\Sigma^- : \hat{W}_\Sigma^- \rightarrow M$ where \hat{W}_Σ^- are the total spaces of the bundles $\hat{p}_\Sigma : \hat{W}_\Sigma^- \rightarrow \Sigma$.*
- 2. If the pair is normal this partition induces a smooth cell structure for M/G and compatible smooth triangulations of F_x, T_x lead to compatible smooth triangulations of M/G ; equivalently to a smooth G -triangulation of \tilde{M} . The open simplexes of M/G are open simplexes of W_x^- .*

2)The geometric complex

The smooth maps i_Σ^- combined with integration along the fibers (\int^F) provide for any critical orbit Σ the linear maps

$$\boxed{Int_\Sigma^*(t) : \Omega^*(M) \rightarrow \Omega^{*-Ind(\Sigma)}(\Sigma, o_\Sigma)}$$

by $\boxed{Int_\Sigma^*(t)(\omega) := \int^F (\hat{i}_x^-)^*(e^{-t(h-h(\Sigma))}\omega)}$

Choose orientations $\mathcal{O} : \{O_x \text{ orientation in } T_x(W_x^-)\}$.

$\mathcal{O} \Rightarrow$ "orientations" o_Σ of the bundles $\hat{p}_\Sigma : \mathcal{W}_\Sigma^- \rightarrow \Sigma$.

The integration along the fibers induces the linear maps

$$\partial_{\Sigma, \Sigma'} : \Omega^*(\Sigma', o'_{\Sigma'}) \rightarrow \Omega^{*-Ind(\Sigma)+Ind(\Sigma')+1}(\Sigma', o'_{\Sigma'}).$$

$$\partial_{\Sigma, \Sigma} : \Omega^*(\Sigma, o_\Sigma) \rightarrow \Omega^{*+1}(\Sigma, o_\Sigma) = d^*|_\Sigma$$

Define:

$$C^q(X) = \sum_{\Sigma} \Omega^{q-Ind(\Sigma)}(\Sigma, o_\Sigma)$$

and

$$\delta_{h, \mathcal{O}}^q(t)(\omega) := \oplus_{\Sigma'} e^{-t(h(x')-h(x))} \delta_{\Sigma, \Sigma'}(\omega)$$

for $\omega \in \Omega^q(\Sigma, o_\Sigma)$.

4 Integration Theory

$$Int_h^*(t) : (\Omega^*(M), d_h^*(t) = d^* + tdh \wedge) \rightarrow (C^*(X, \mathcal{O}), \delta_{h, \mathcal{O}}^*(t))$$

$$Int_h(t)(\omega) = \oplus_{\Sigma} Int_{\Sigma}^*(t)(i_x^-)^* \omega$$

$(\Omega^*(M), d_h^*(t))$ and $(C^*(X), \delta_{h, \mathcal{O}}^*(t))$ are complexes of G -representations and $Int_h^*(t)$ is G -equivariant.

\Rightarrow

$$\boxed{(Int_h^*(t)) = \oplus (Int_h^*(t))_\xi}$$

$$\boxed{(Int_h^*(t))_\xi : (\Omega^*(M)_\xi, d_h^*(t)) \rightarrow (C^*(X)_\xi, \delta_h(t))}$$

with $(C^*(X)_\xi, \delta_{h,\mathcal{O}}^*(t))$ finite dimensional cochain complex.

4)WHS-theorem

Theorem 5. *Given a Riemannian metric g admissible for (X, h) and ξ an irreducible representation there exists $T_\xi > 0$ and a canonical decomposition*

$$\boxed{(\Omega^*(M)_\xi, d_h^*(t)) = (\Omega_{\xi,sm}^*(M), d_h^*(t)) \oplus (\Omega_{\xi,la}^*(M), d_h^*(t))}$$

so that

$$(Int_h(t))_\xi : (\Omega_{\xi,sm}^*(M), d_h^*(t)) \rightarrow (C^*(X)_\xi, \delta_{h,\mathcal{O}}^*(t))$$

is an $O(1/t)$ isometry provided $t > T_\xi$.

PROOF OF THEOREMS 3,4)

$$K := W_{\Sigma}^{-} \cap M^{c-2\epsilon}, K \sim G \times_H K_x, K_x = W_x^{-} \cap M^{c-2\epsilon}$$

$$Z = (\bigcup_{h(\Sigma') \leq c} W_{\Sigma'}^{+} \cap M^{c-2\epsilon}), Z \subset M^{c-2\epsilon} \text{ Whitney stratif.}$$

$$Maps_G(K, M | K_x \pitchfork Z) \subset Maps_G(K, M)$$

$$Maps_H(K_x, M | K_x \pitchfork Z) \subset Maps_H(K_x, M)$$

$$Maps(K_x, M^H | K_x \pitchfork Z^H) \subset Maps(K_x, M^H)$$

MANIFOLDS WITH CORNERS \mathcal{P} :

\mathcal{P} is locally diffeomorphic $\mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n) \mid x_i \geq 0\}$.

\mathcal{P}_k points of \mathcal{P} which in some chart have exactly k vanishing coordinates. \mathcal{P}_k is a smooth manifold of dimension $(n - k)$.

$\partial\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \dots \cup \mathcal{P}_n$ is closed in \mathcal{P} and a topological manifold. $(\mathcal{P}, \partial\mathcal{P})$ is a topological manifold with boundary $\partial\mathcal{P}$.

A product of manifolds with boundary is a manifold with corners.

Remark 1. *If \mathcal{P} is a smooth manifold with corners, \mathcal{O}, \mathcal{S} smooth manifolds, $p : \mathcal{P} \rightarrow \mathcal{O}$ and $s : \mathcal{S} \rightarrow \mathcal{O}$ smooth maps so that p and s are transversal (p is transversal to s if its restriction to each k -boundary \mathcal{P}_k is transversal to s), then $p^{-1}(s(\mathcal{S}))$ is a smooth submanifold with corners of \mathcal{P} .*

PROOF OF THEOREM 3 1)

$c_N > \cdots > c_i > c_{i-1} > \cdots > c_1$, all critical values of h .

$\epsilon > 0$ small so that $c_i - \epsilon > c_{i-1} + \epsilon$ Denote:

$$M_i := h^{-1}(c_i),$$

$$M_i^\pm := h^{-1}(c_i \pm \epsilon_i)$$

$$M(i) := h^{-1}(c_{i-1}, c_{i+1}).$$

$P_i \equiv$ trajectories (possible broken) from M_i^- to M_i^-

P_i compact smooth manifold with boundary.

Let $h(\Sigma) = c_{r+1}$, $h(\Sigma') = c_{r-k-1}$ and take

$$\mathcal{P} := P_{r,r-k} := P_r \times P_{r-1} \times \cdots \times P_{r-k},$$

$$\mathcal{O} := \prod_{i=r}^{r-k} (M_i^+ \times M_i^-),$$

$$\mathcal{S} := S_\Sigma^- \times M_r^- \times \cdots \times M_{r-k+1}^- \times S_{\Sigma'}^+.$$

Define $p : \mathcal{P} \rightarrow \mathcal{O}$ $s : \mathcal{S} \rightarrow \mathcal{O}$.

Show p and s are smooth and transversal.

$$\Rightarrow \boxed{\hat{T}(\Sigma, \Sigma') = p^{-1}(s(\mathcal{S}))}$$

