Math 366 Practice Problems for Exam 2 **Spring 2009**

1.Constructive Proof of an Existential Statement.

(a) Prove that there is an even integer n such that $n \, mod \, 3 = 1$.

(b) Prove that there exists a rational number q such that $9\,q^2 = 4$.

(c) Prove that there exist two real numbers whose product is less than their sum.

(d) Prove that there exist two real numbers which are not equal to each other and whose product is equal to their sum.

(e) Prove that there is an odd integer n such that $n > 1$ and n has the form $3k+1$ for some integer k.

2. Direct Proof of a Universal Statement.

(a) Prove that if n is an integer which is divisible by 6 then n is divisible by 3.

(b) Prove that for any integers a, b, c, and d, if a divides b and c divides d then $a \cdot c$ divides $b \cdot d$.

(c) Prove that the product of two odd integers is odd.

(d) Prove that for any sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

(e) Prove that if n is an integer which is divisible by 5 then $3n$ is divisible by 15.

(f) Prove that for any rational numbers a and b, if $a \neq 0$ then there is a rational number x such that $ax + b = 0$.

(g) Show that the reciprocal of any nonzero rational number is rational.

3. Proof by Cases.

(a) Prove that for every integer n, n and $n+2$ have the same parity (i.e. either n and $n+2$ are both even or n and $n+2$ are both odd).

(b) Prove that for any integer n, $n^2 + n$ is even.

(c) Prove that for any integers n and m, $n^2 + 3m \neq 2$. Hint: Use an argument by cases depending on what the remainder is when n is divided by 3.

(d) Prove that for any sets A, B, and C, if $A \subseteq C$ then $A \cup (B \cap C) \subseteq C$.

4. Mathematical Induction.

(a) Prove that for any integer n, if $n \geq 0$ then 4 divides $5^n - 1$.

(b) Prove that for any integer n, if $n \geq 1$ then 4 divides $6^n - 2^n$.

(c) Show that $2n + 1 < 2^n$ for every integer n with $n \geq 3$.

(d) Using the fact that $2n + 1 < 2^n$ for every integer n with $n \geq 3$, show that for every integer n, if $n \geq 5$ then $n^2 < 2^n$.

5. Strong Mathematical Induction and the Well-Ordering Principle.

(a) Suppose $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$c_0 = 0, c_1 = 1,$$

$$c_k = 2c_{k-1} - c_{k-2} + 2 \text{ for all integers } k \geq 2.$$

Prove that $c_n = n^2$ for all integers $n \geq 0$.

(b) Suppose $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$c_0 = 2, c_1 = 5,$$

$$c_k = 5c_{k-1} - 6c_{k-2} \text{ for all integers } k \geq 2.$$

Prove that $c_n = 2^n + 3^n$ for all integers $n \geq 0$.

6. Proofs by Contradiction and Contraposition.

(a) Prove that there is no smallest real number x such that $1 < x < 2$..

(b) For any integer n, if n^2 is not divisible by 3 then n is not divisible by 3.

7. Computations with Sets.

(a) Let $A = \{a, c, d\}$ and $B = \{b, c, f\}$ be subsets of the universal set $U = \{a, b, c, d, e, f, g\}$. Compute $A \cup B$, $A \cap B$, $A - B$, A^c, and $A \times B$ using "bracket" notation.

(b) Let $A = \{1, 3\}$ and $B = \{2, 3\}$ be subsets of the universal set $U = \{0, 1, 2, 3, 4\}$. Compute $A \cup B$, $A \cap B$, $A - B$, A^c, and $A \times B$ using "bracket" notation.