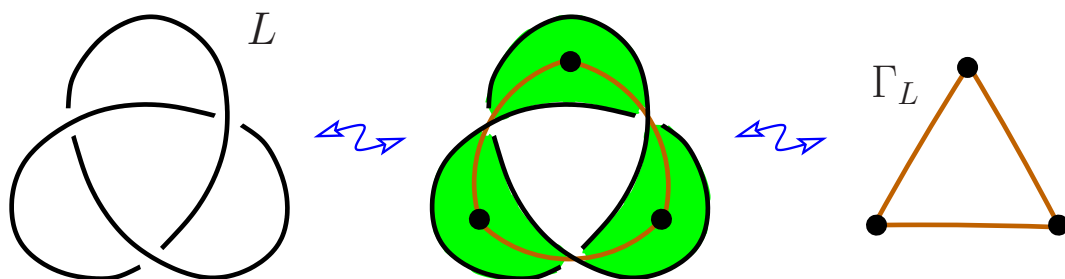# Thistlethwaite's theorem for virtual links

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Up to a sign and a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_{\Gamma_L}(-t, -t^{-1})$.

L

Γ_L

$$V_L(t) = t + t^3 - t^4$$

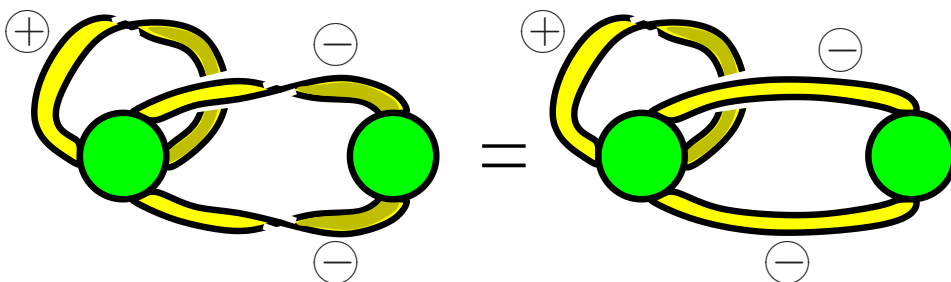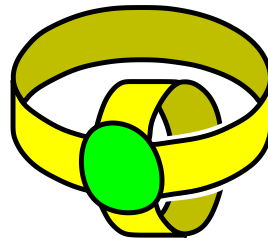$$= -t^2(-t^{-1} - t + t^2)$$
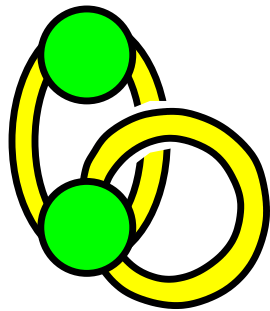
$$T_{\Gamma_L}(x, y) = y + x + x^2$$

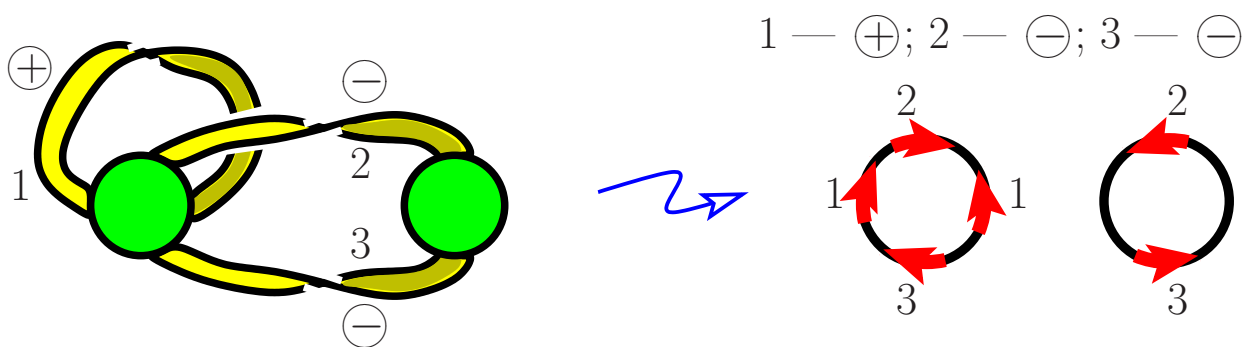$$T_{\Gamma_L}(-t, -t^{-1}) = -t^{-1} - t + t^2$$
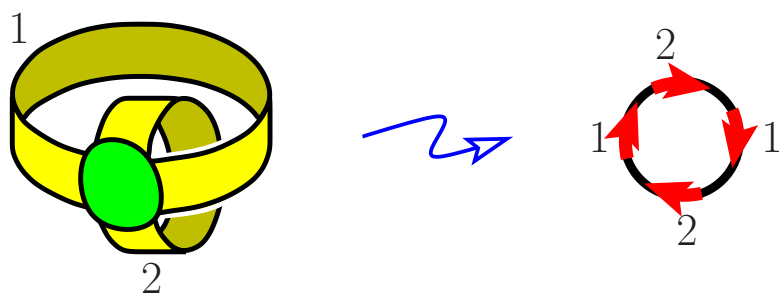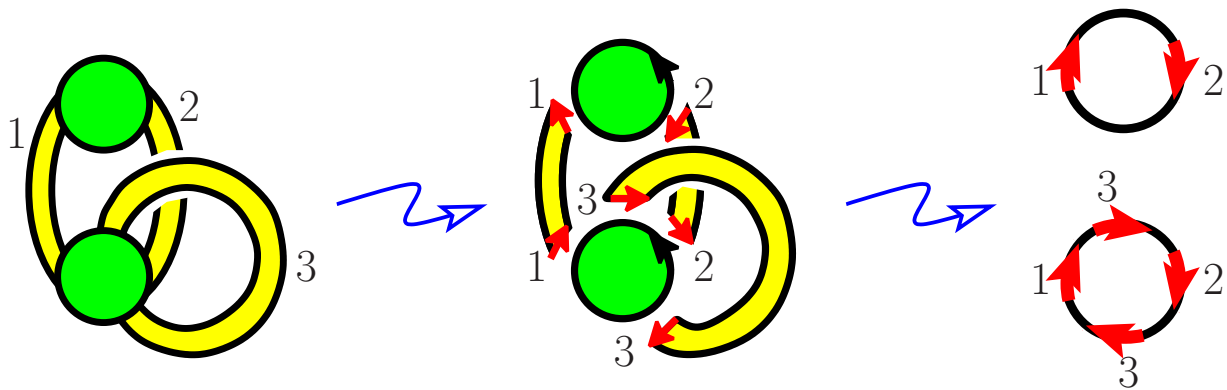
Ribbon graphs

A ribbon graph G is a surface represented as a union of vertices-discs and edges-ribbons

- discs and ribbons intersect by disjoint line segments,

- each such line segment lies on the boundary of precisely one vertex and precisely one edge;

- every edge contains exactly two such line segments.
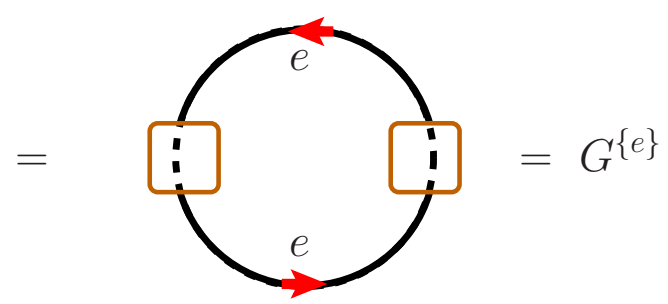
Examples

Arrow presentation

Generalized duality with respect to the set of edges

$G =$

double
the edge

forming the
new vertex

$=$ $= G^{\{e\}}$

$G =$

drawing the
boundary components

$=$ $= G^{\{e\}}$

e

e

Examples.

$1 - \oplus;\ 2 - \oplus$
$E' = \{1\}$

$1 - \ominus;\ 2 - \oplus$

$G = $ ... $\leadsto$ ... $=$

$= $ ... $= \ominus \quad \oplus = G^{\{1\}}$

$1 - \oplus;\ 2 - \oplus;\ 3 - \oplus$
$E' = \{3\}$

$1 - \oplus;\ 2 - \oplus;\ 3 - \ominus$

$G = $ ... $\leadsto$ ... $=$

$= $ ... $= \quad \oplus \quad \oplus \quad \ominus = G^{\{3\}}$

$$E' = \{1, 2\} \qquad 1 - \ominus;\ 2 - \oplus;\ 3 - \ominus$$

$$= G^{\{1,2\}}$$
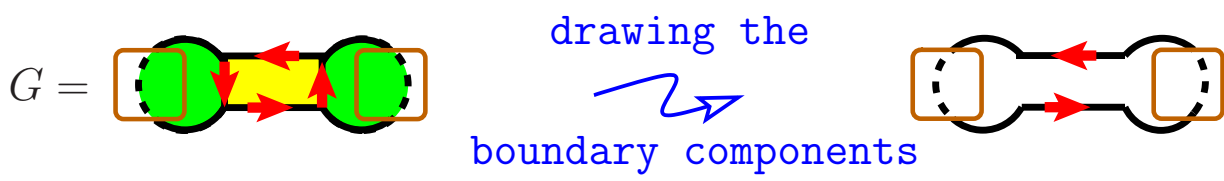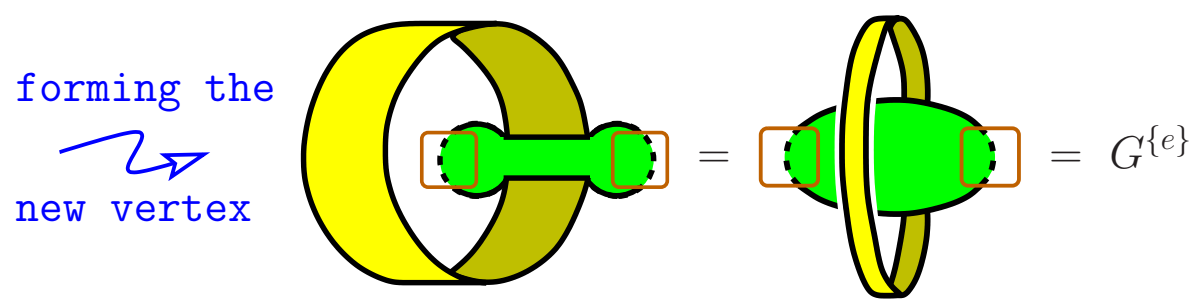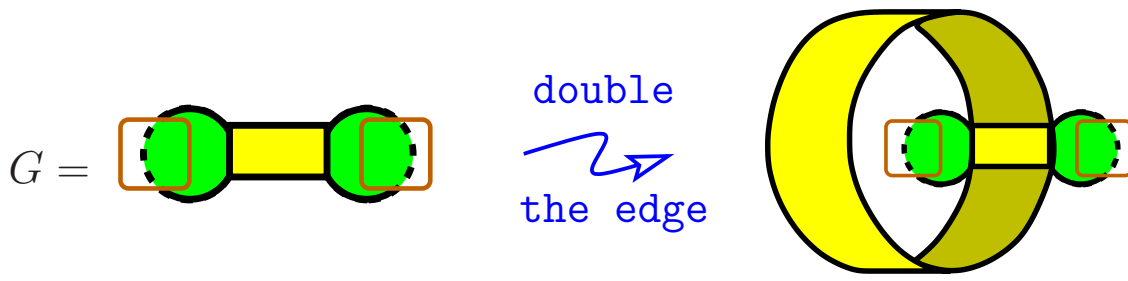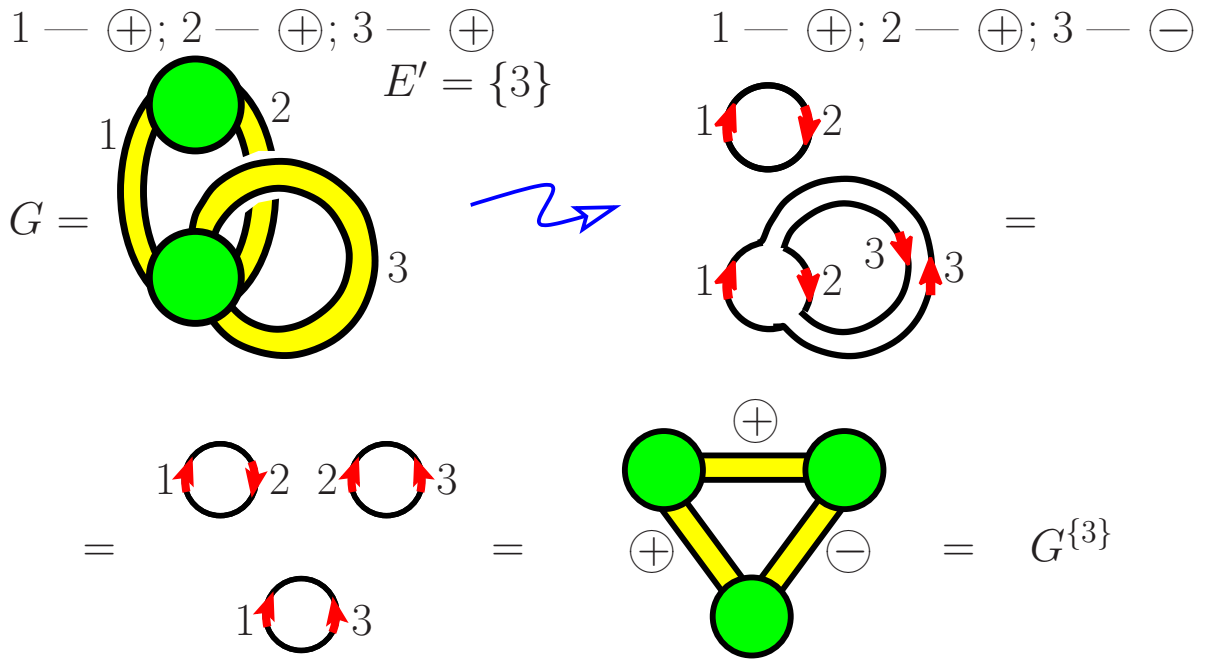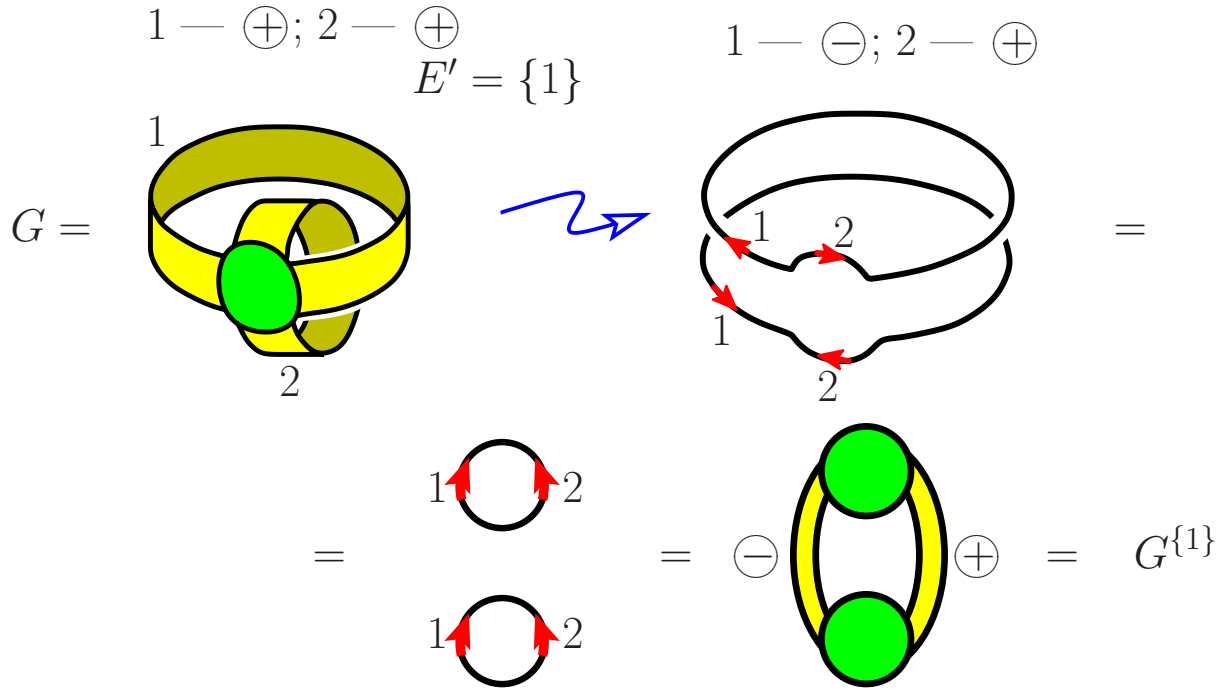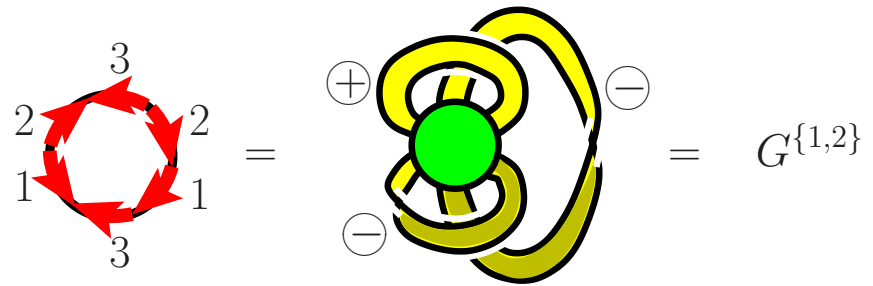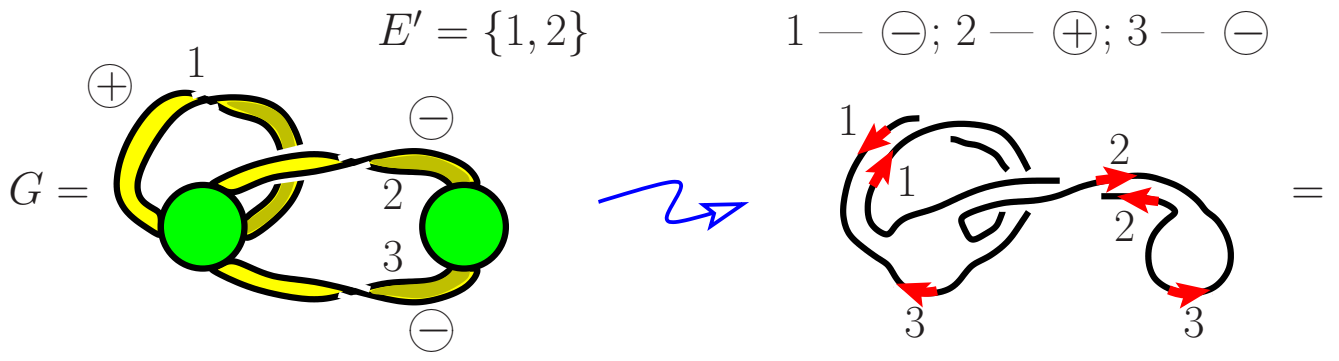
The Bollobás-Riordan polynomial

Let
- F be a ribbon graph;
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;
- $\mathrm{bc}(F)$ be the number of boundary components of F;
- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

$$
R_G(x, y, z) :=
$$
$$
\sum_F x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-\mathrm{bc}(F)+n(F)}
$$

Relations to the Tutte polynomial.
$$
R_G(x-1, y-1, 1) = T_G(x, y)
$$

If G is planar (genus zero):
$$
R_G(x-1, y-1, z) = T_G(x, y)
$$

Example.

$(k, r, n, \mathrm{bc}, s)$	$(1, 1, 1, 2, 1)$	$(1, 1, 0, 1, 0)$	$(1, 1, 0, 1, 0)$	$(2, 0, 0, 2, -1)$
	$(1, 1, 2, 1, 1)$	$(1, 1, 1, 1, 0)$	$(1, 1, 1, 1, 0)$	$(2, 0, 1, 2, -1)$

- $r(F) := v(F) - k(F)$;

- $n(F) := e(G) - r(F)$;

- $\mathrm{bc}(F)$ is the number of boundary components;

- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

$$R_G(x, y, z) = x + 2 + y + xyz^2 + 2yz + y^2z \ .$$

Duality theorem

For any choice of the subset of edges E'. the restriction

of the polynomial $x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x,y,z)$ to the surface

$xyz^2 = 1$ is invariant under the generalized duality:

$$x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x,y,z)\Big|_{xyz^2=1} = x^{k(G')}y^{v(G')}z^{v(G')+1}R_{G'}(x,y,z)\Big|_{xyz^2=1}$$

where $G' := G^{E'}$.

Idea of the proof.

$$x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x,y,z) = \sum_F M_G(F)$$
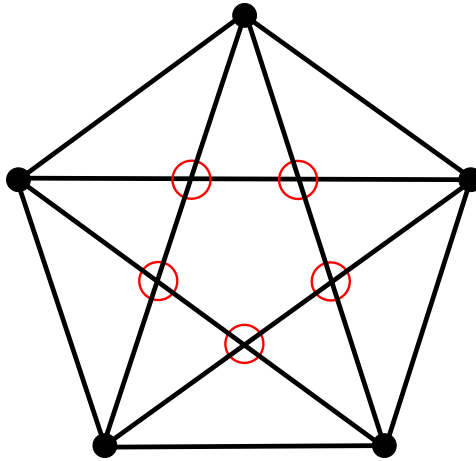
One-to-one correspondence $E(G) \supseteq F \leftrightarrow F' \subseteq E(G')$:

An edge e of G' belongs to the spanning subgraph
F' if and only if either $e \in E'$ and $e \notin F$, or
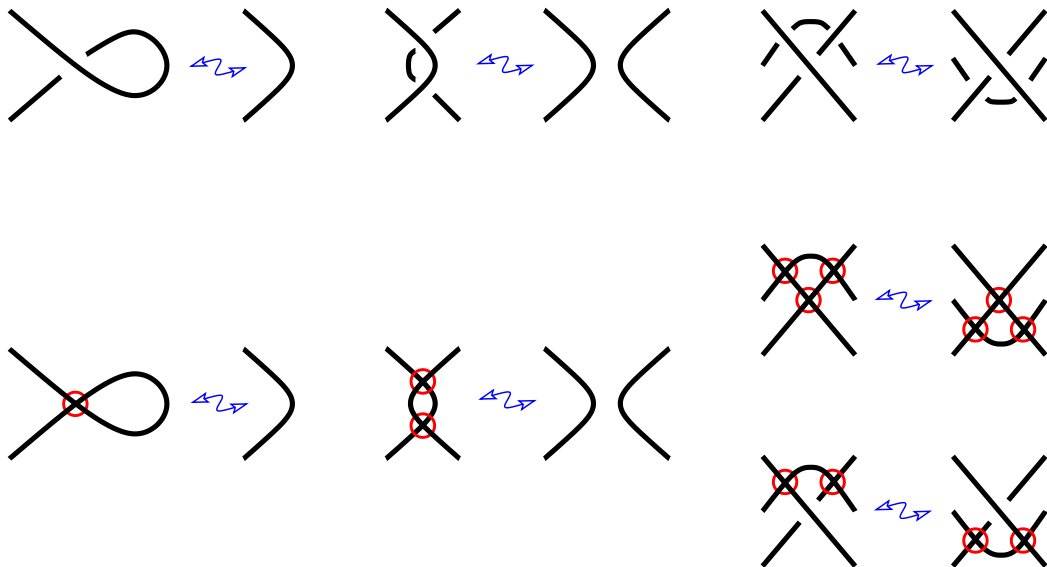*$e \notin E'$ and $e \in F$.*

$$M_G(F)\Big|_{xyz^2=1} = M_{G'}(F')\Big|_{xyz^2=1} ,$$
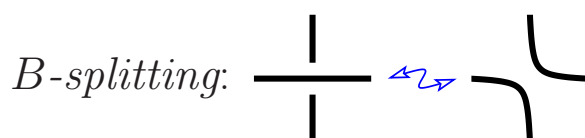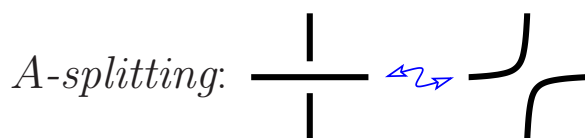
Virtual links

Virtual crossings

Reidemeister moves

The Kauffman bracket

Let L be a virtual link diagram.

A-splitting:

B-splitting:

A *state S* is a choice of either A- or B-splitting at every classical crossing.

$\alpha(S) = \#(\text{of } A\text{-splittings in } S)$

$\beta(S) = \#(\text{of } B\text{-splittings in } S)$

$\delta(S) = \#(\text{of circles in } S)$

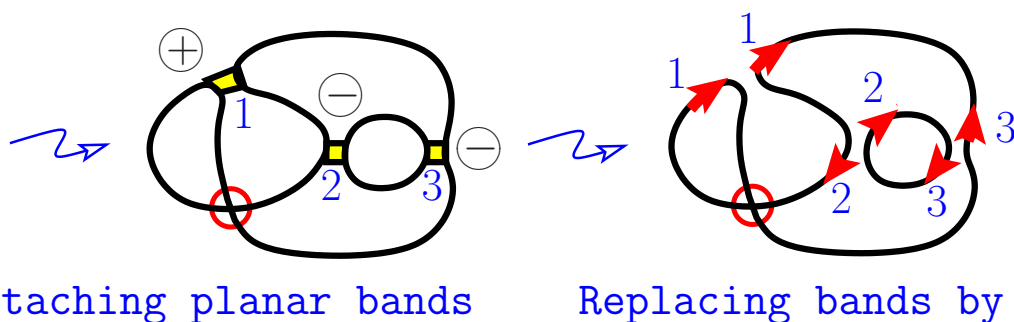$$[L](A, B, d) := \sum_S A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$$

$$J_L(t) := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Example

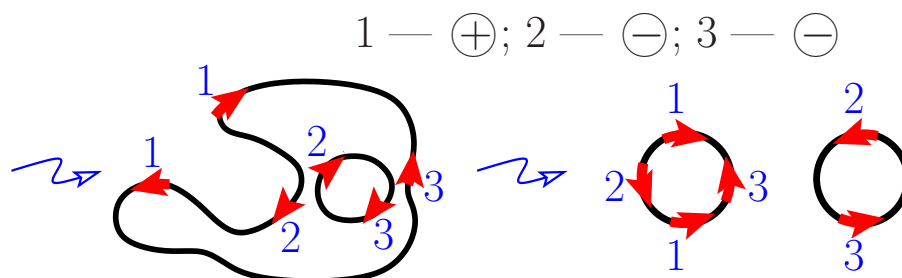|				
(α, β, δ)	$(3, 0, 1)$	$(2, 1, 2)$	$(2, 1, 2)$	$(1, 2, 1)$
	$(2, 1, 2)$	$(1, 2, 1)$	$(1, 2, 3)$	$(0, 3, 2)$

$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d;$ $\qquad J_L(t) = 1$

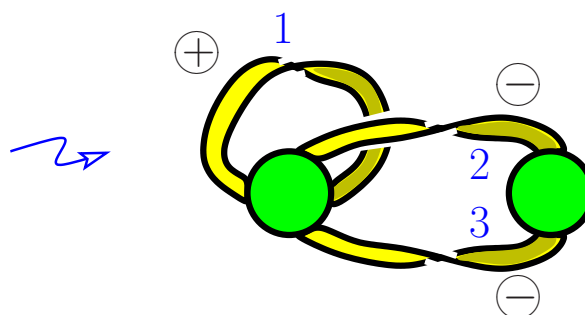# Construction of a ribbon graph from a virtual link diagram
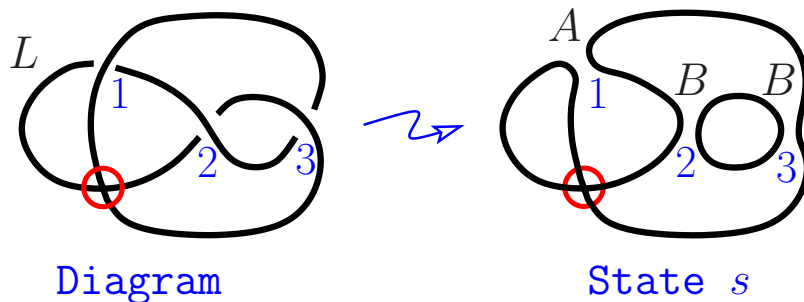
Diagram

State s

Attaching planar bands

Replacing bands by arrows

$1 - \oplus; 2 - \ominus; 3 - \ominus$

Untwisting state circles

Pulling state circles apart

Forming the ribbon graph G_L^s

Theorem

Let L be a virtual link diagram with e classical crossings,

G_L^s be the signed ribbon graph corresponding to a state s, and

$v := v(G_L^s)$, $k := k(G_L^s)$. Then $e = e(G_L^s)$ and

$$[L](A, B, d) = A^e \left(x^k y^v z^{v+1} R_{G_L^s}(x, y, z) \Big|_{x=\frac{Ad}{B}, \ y=\frac{Bd}{A}, \ z=\frac{1}{d}} \right) .$$

Idea of the proof.

One-to-one correspondence between states s' of L and spanning subgraphs F' of G_L^s:

An edge e of G_L^s belongs to the spanning subgraph
F' if and only if the corresponding crossing was
split in s' differently comparably with s.

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[CP]   S. Chmutov, I. Pak, *The Kauffman bracket of virtual links and the Bollobás-Riordan polynomial*, preprint `arXiv:math.GT/0609012`, Moscow Mathematical Journal **7**(3) (2007) 409–418.

[CV]   S. Chmutov, J. Voltz, *Thistlethwaite's theorem for virtual links*, preprint `arXiv:math.GT/0704.1310`. To appear in Journal of Knot Theory and its Ramifications.

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