

University of Liverpool

**Graphs on surfaces and virtual link
theory**

Sergei Chmutov

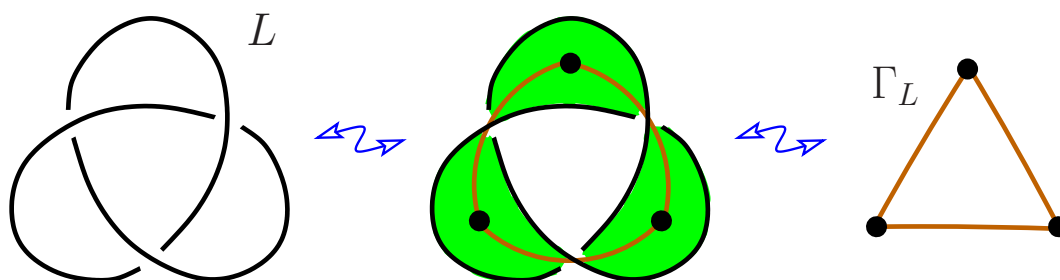
Max-Planck-Institut für Mathematik, Bonn

The Ohio State University, Mansfield

Friday, December 12, 2008

M. B. Thistlethwaite'87 [Th],
 L. Kauffman, K.Murasugi, F.Jaeger

Up to a sign and a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_{\Gamma_L}(-t, -t^{-1})$.



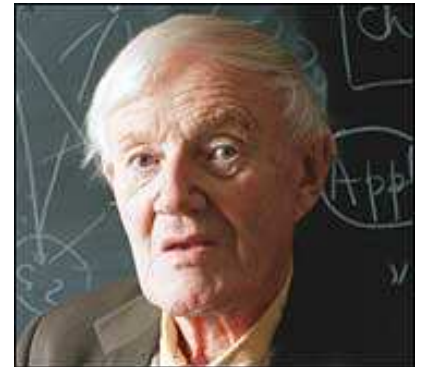
$$\begin{aligned} V_L(t) &= t + t^3 - t^4 \\ &= -t^2(-t^{-1} - t + t^2) \end{aligned}$$

$$\begin{aligned} T_{\Gamma_L}(x, y) &= y + x + x^2 \\ T_{\Gamma_L}(-t, -t^{-1}) &= -t^{-1} - t + t^2 \end{aligned}$$

The Tutte polynomial

Let \bullet F be a graph;

- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F ;
- $r(F) := v(F) - k(F)$ be the *rank* of F ;
- $n(F) := e(F) - r(F)$ be the *nullity* of F ;



$$T_{\Gamma}(x, y) := \sum_{F \subseteq E(\Gamma)} (x - 1)^{r(\Gamma) - r(F)} (y - 1)^{n(F)}$$

Properties.

$T_{\Gamma} = T_{\Gamma - e} + T_{\Gamma / e}$ if e is neither a bridge nor a loop ;

$T_{\Gamma} = x T_{\Gamma / e}$ if e is a bridge ;

$T_{\Gamma} = y T_{\Gamma - e}$ if e is a loop ;

$T_{\Gamma_1 \sqcup \Gamma_2} = T_{\Gamma_1 \cdot \Gamma_2} = T_{\Gamma_1} \cdot T_{\Gamma_2}$ for a disjoint union, $G_1 \sqcup G_2$
and a one-point join, $G_1 \cdot G_2$;

$T_{\bullet} = 1$.

$T_{\Gamma}(1, 1)$ is the number of spanning trees of Γ ;

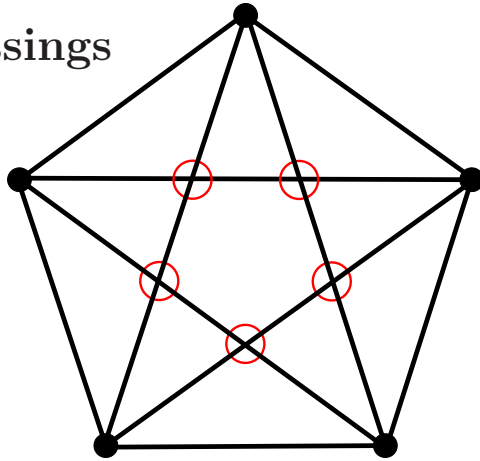
$T_{\Gamma}(2, 1)$ is the number of spanning forests of Γ ;

$T_{\Gamma}(1, 2)$ is the number of spanning connected subgraphs of Γ ;

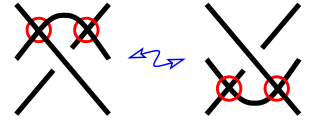
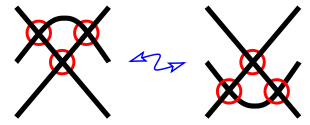
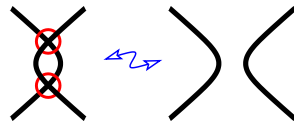
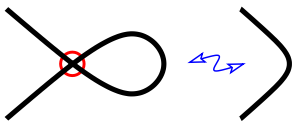
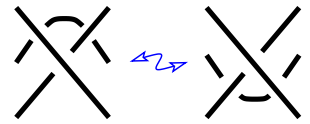
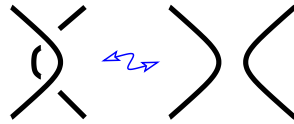
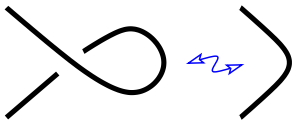
$T_{\Gamma}(2, 2) = 2^{|E(\Gamma)|}$ is the number of spanning subgraphs of Γ .

Virtual links

Virtual crossings

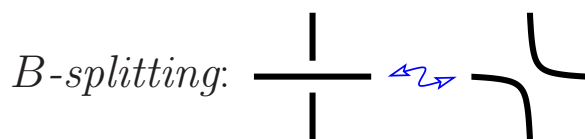
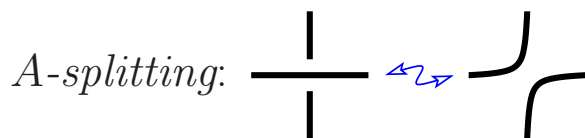


Reidemeister moves



The Kauffman bracket

Let L be a virtual link diagram.



A *state* S is a choice of either A - or B -splitting at every classical crossing.

$\alpha(S) = \#(\text{of } A\text{-splittings in } S)$

$\beta(S) = \#(\text{of } B\text{-splittings in } S)$

$\delta(S) = \#(\text{of circles in } S)$

$$[L](A, B, d) := \sum_S A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$$

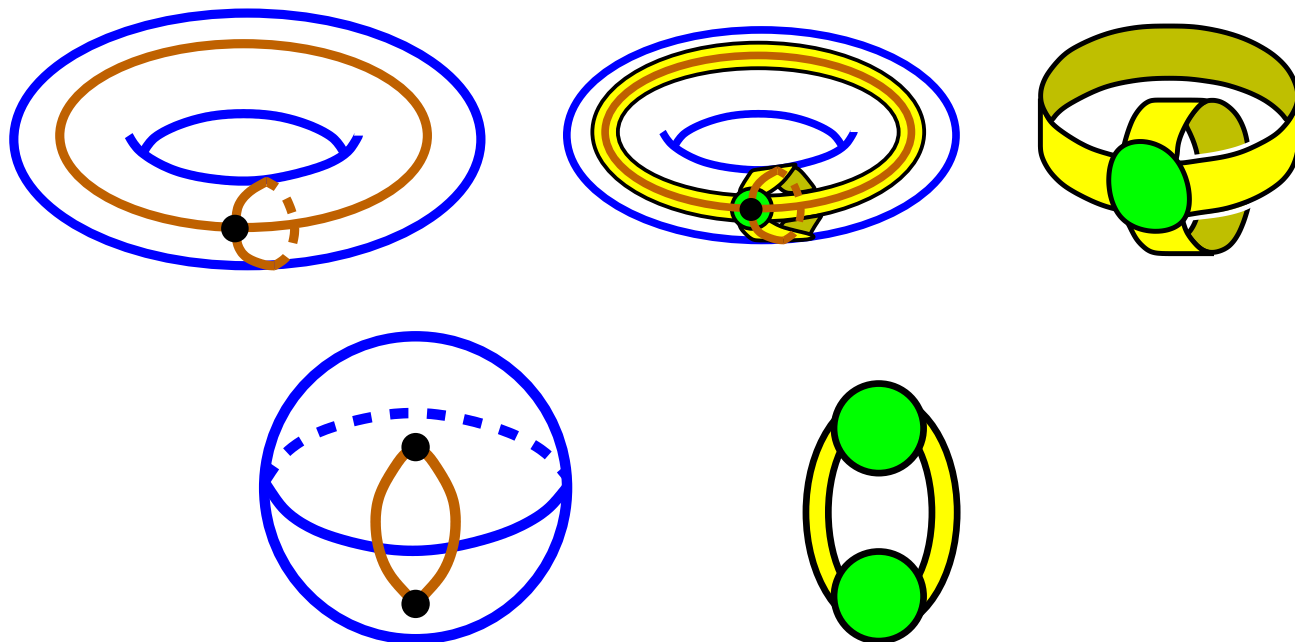
$$J_L(t) := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Example

(α, β, δ)	$(3, 0, 1)$	$(2, 1, 2)$	$(2, 1, 2)$	$(1, 2, 1)$
	$(2, 1, 2)$	$(1, 2, 1)$	$(1, 2, 3)$	$(0, 3, 2)$

$$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d; \quad J_L(t) = 1$$

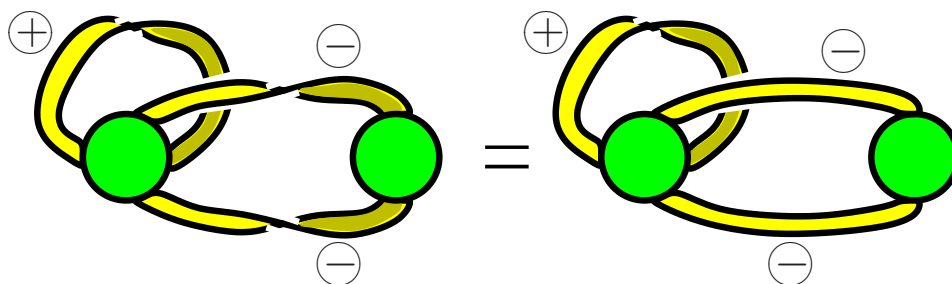
Graphs on surfaces



Ribbon graphs

A ribbon graph G is a surface represented as a union of vertices-discs  and edges-ribbons 

- discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.



The Bollobás-Riordan polynomial

Let \bullet F be a ribbon graph;

- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F ;
- $r(F) := v(F) - k(F)$ be the *rank* of F ;
- $n(F) := e(F) - r(F)$ be the *nullity* of F ;
- $\text{bc}(F)$ be the number of boundary components of F ;
- $s(F) := \frac{e_-(F) - e_-(\bar{F})}{2}$.

$$R_G(x, y, z) :=$$

$$\sum_F x^{r(G) - r(F) + s(F)} y^{n(F) - s(F)} z^{k(F) - \text{bc}(F) + n(F)}$$

Relations to the Tutte polynomial.

$$R_G(x - 1, y - 1, 1) = T_G(x, y)$$

If G is planar (genus zero):

$$R_G(x - 1, y - 1, z) = T_G(x, y)$$

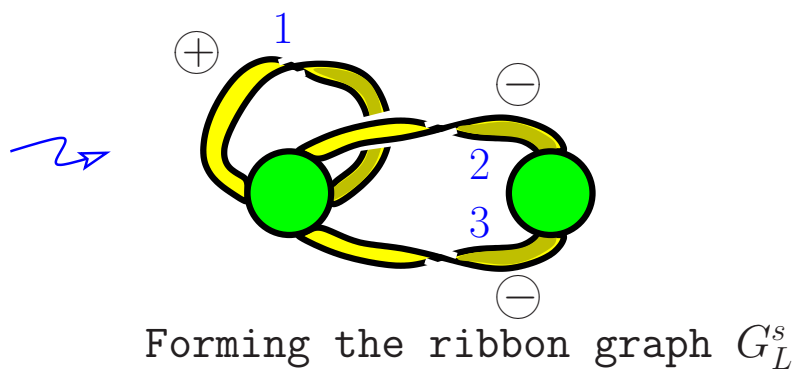
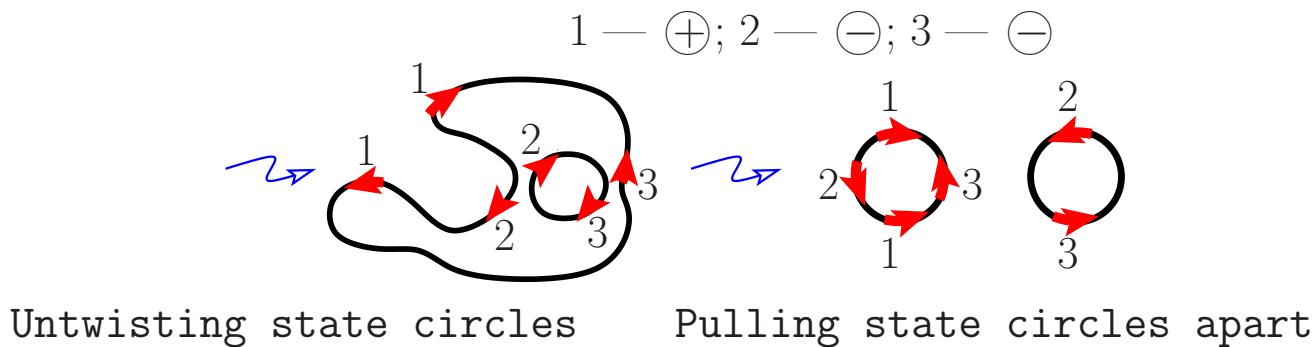
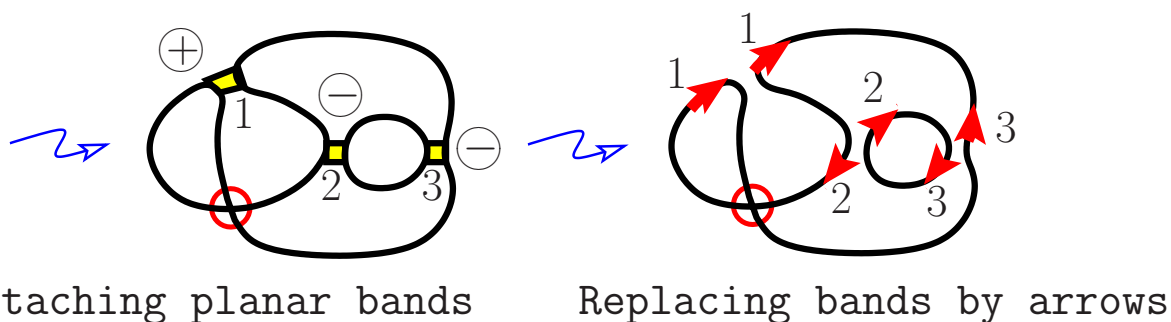
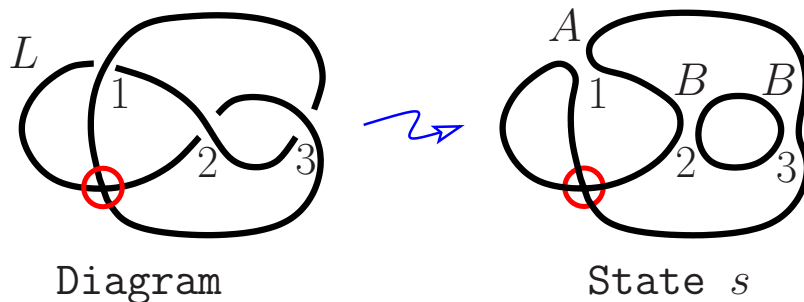
Example.

(k, r, n, bc, s)	$(1, 1, 1, 2, 1)$	$(1, 1, 0, 1, 0)$	$(1, 1, 0, 1, 0)$	$(2, 0, 0, 2, -1)$
	$(1, 1, 2, 1, 1)$	$(1, 1, 1, 1, 0)$	$(1, 1, 1, 1, 0)$	$(2, 0, 1, 2, -1)$

- $r(F) := v(F) - k(F)$;
- $n(F) := e(F) - r(F)$;
- $bc(F)$ is the number of boundary components;
- $s(F) := \frac{e_-(F) - e_-(\bar{F})}{2}$.

$$R_G(x, y, z) = x + 2 + y + xyz^2 + 2yz + y^2z .$$

Construction of a ribbon graph from a virtual link diagram



Theorem [Ch]

Let L be a virtual link diagram with e classical crossings, G_L^s be the signed ribbon graph corresponding to a state s , and $v := v(G_L^s)$, $k := k(G_L^s)$. Then $e = e(G_L^s)$ and

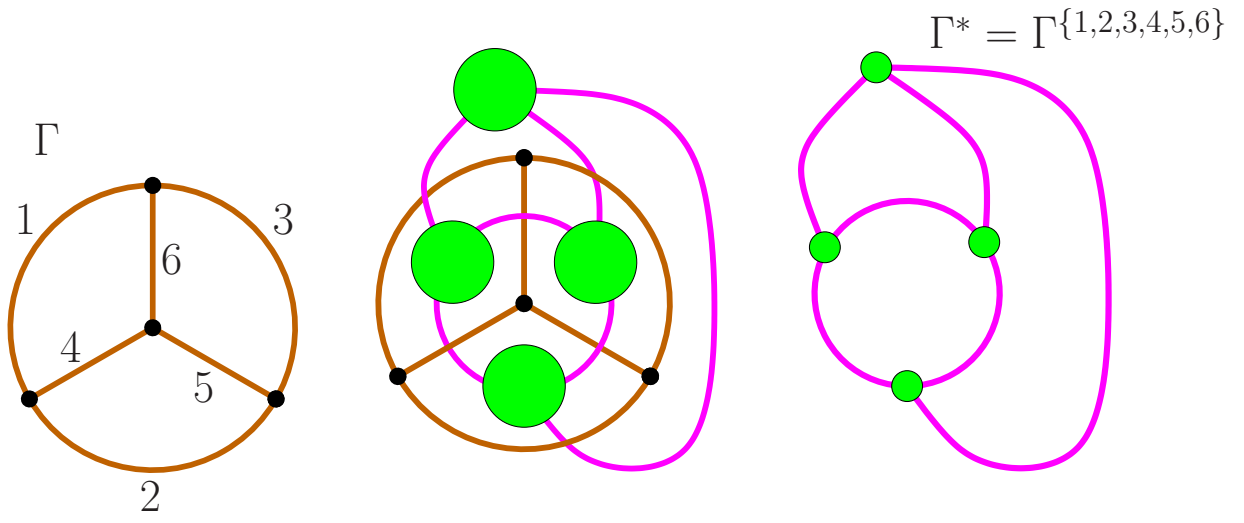
$$[L](A, B, d) = A^e \left(x^k y^v z^{v+1} R_{G_L^s}(x, y, z) \Big|_{x=\frac{Ad}{B}, y=\frac{Bd}{A}, z=\frac{1}{d}} \right) .$$

Idea of the proof.

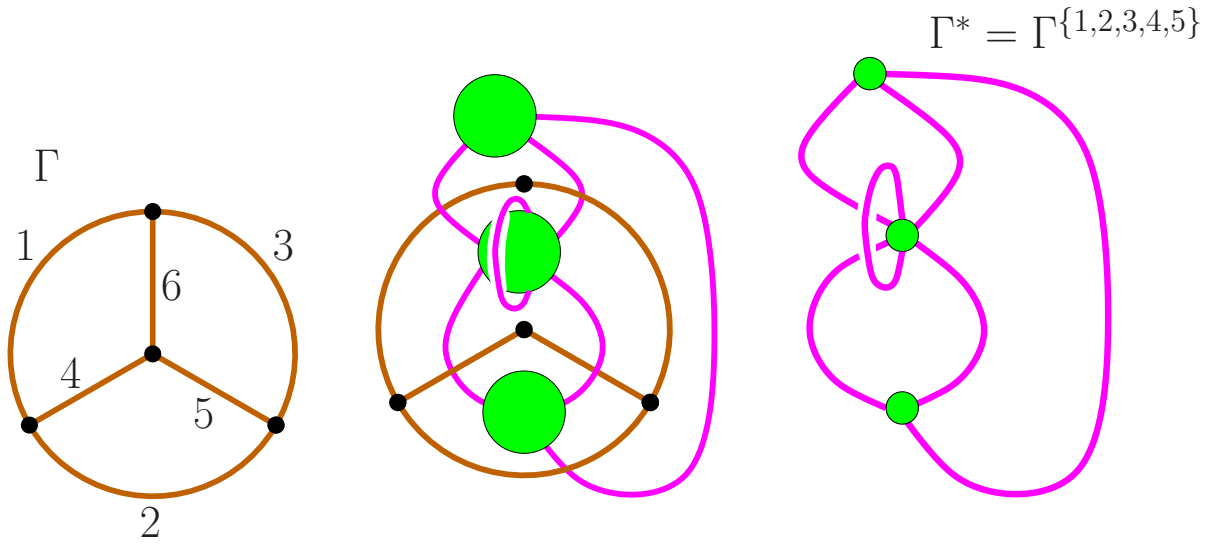
One-to-one correspondence between states s' of L and spanning subgraphs F' of G_L^s :

An edge e of G_L^s belongs to the spanning subgraph F' if and only if the corresponding crossing was split in s' differently comparably with s .

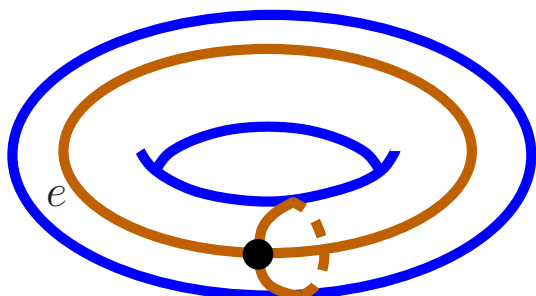
Duality



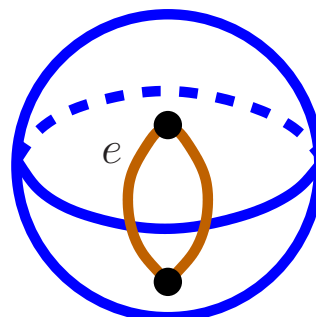
Generalized duality



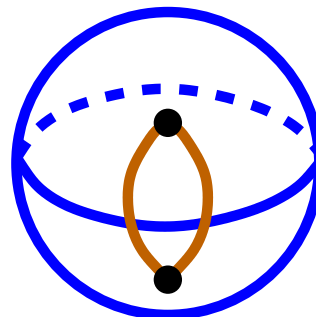
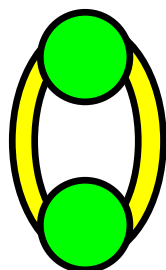
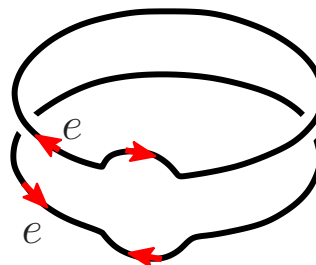
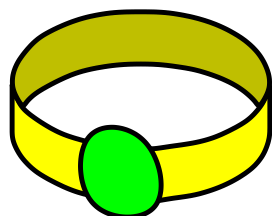
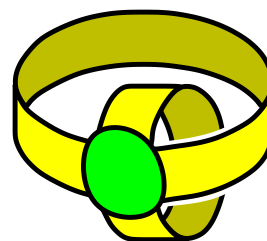
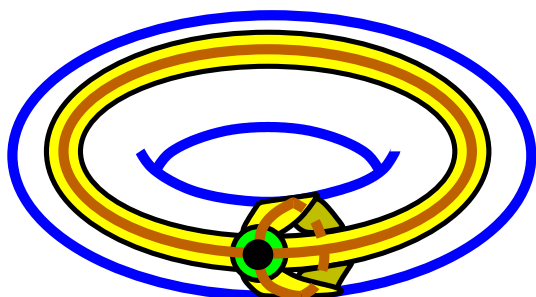
Examples

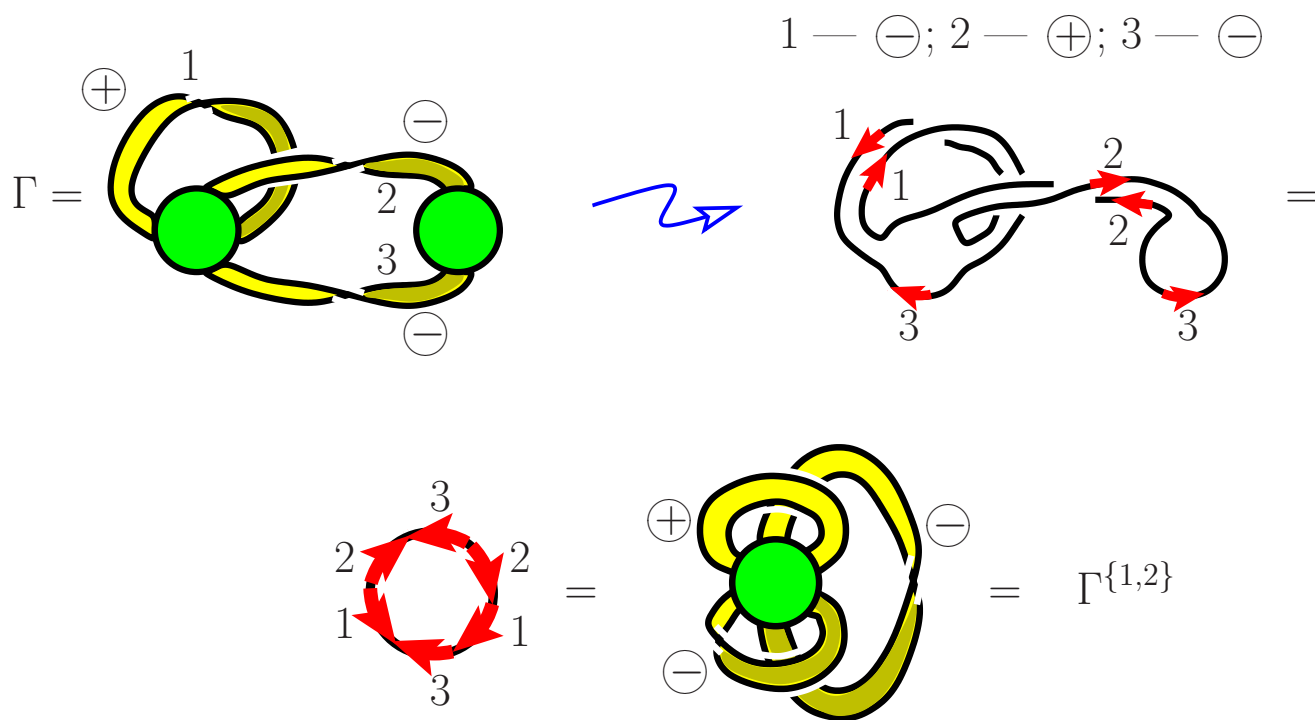
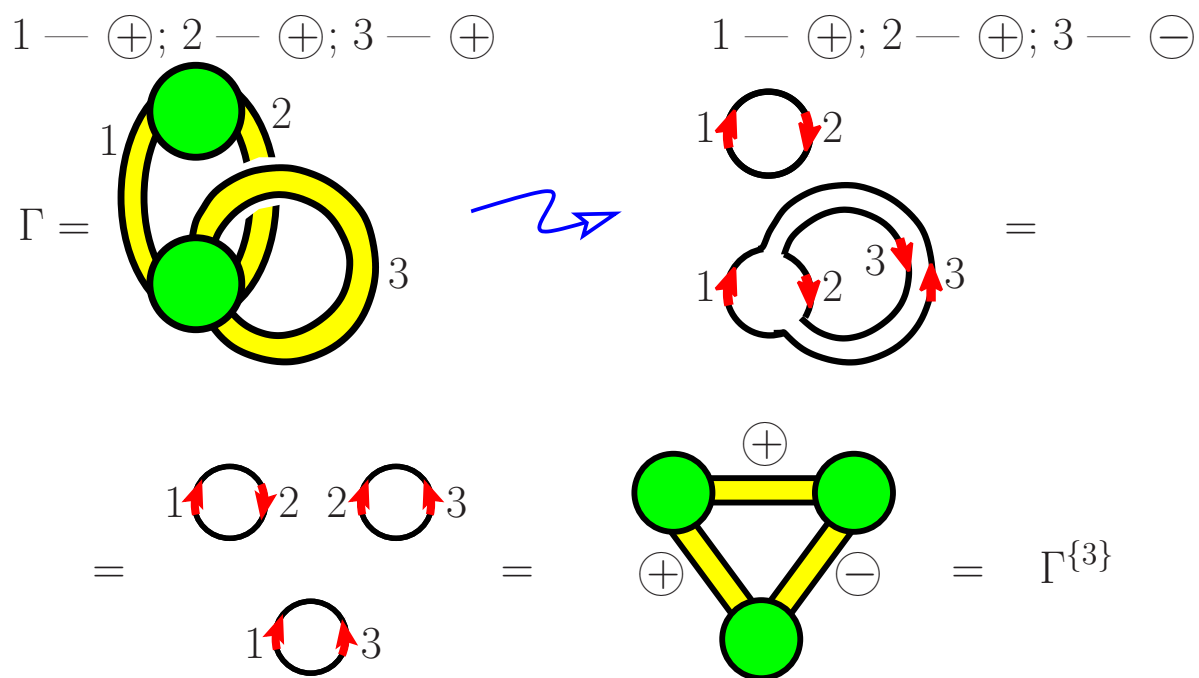


Graph Γ on a torus



Dual graph $\Gamma^{\{e\}}$ with respect to the edge e is embedded into a sphere





Duality theorem [Ch]

For any choice of the subset of edges E' . the restriction of the polynomial $x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x, y, z)$ to the surface $xyz^2 = 1$ is invariant under the generalized duality:

$$x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x, y, z) \Big|_{xyz^2=1} = x^{k(G')}y^{v(G')}z^{v(G')+1}R_{G'}(x, y, z) \Big|_{xyz^2=1}$$

where $G' := G^{E'}$.

Idea of the proof.

$$x^{k(G)}y^{v(G)}z^{v(G)+1}R_G(x, y, z) = \sum_F M_G(F)$$

One-to-one correspondence $E(G) \supseteq F \leftrightarrow F' \subseteq E(G')$:

An edge e of G' belongs to the spanning subgraph F' if and only if either $e \in E'$ and $e \notin F$, or $e \notin E'$ and $e \in F$.

$$M_G(F) \Big|_{xyz^2=1} = M_{G'}(F') \Big|_{xyz^2=1},$$

Corollary

Let G be a connected plane ribbon graph, i.e. its underlying graph Γ is embedded into the plane. Then

$$T_\Gamma(x, y) = T_{\Gamma^*}(y, x)$$

Theorem of [CP]: The state s comes from a checkerboard coloring of the diagram L .

Theorem of [CV]: The state s is the Seifert state, i.e. all splittings preserve the orientation of L .

Theorem of [DFKLS]: The state $s = s_A$, i.e. all splittings are A -splittings.

Fabien Vignes-Tourneret (Vienna).

The multivariate signed Bollobás-Riordan polynomial, preprint arXiv:math.CO/0811.1584.

The combinatorial part \implies the multivariable Bollobás-Riordan polynomial.

Iain Moffatt (University of South Alabama, Mobile).

Partial duality and Bollobas and Riordan's ribbon graph polynomial, preprint arXiv:math.CO/0809.3014.

The duality theorem \iff the HOMFLYPT polynomial.

References

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- [Ch] S. Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial*, preprint [arXiv:math.CO/0711.3490](https://arxiv.org/abs/math.CO/0711.3490). To appear in Journal of Combinatorial Theory Ser.B.
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- [CV] S. Chmutov, J. Voltz, *Thistlethwaite’s theorem for virtual links*, Journal of Knot Theory and its Ramifications , **17**(10) (2008) 1189-1198. Preprint [arXiv:math.GT/0704.1310](https://arxiv.org/abs/math.GT/0704.1310).
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