The matrix-tree theorem in knot theory

University of South Alabama
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11:15–12:15
ILB 370

Let G be graph, $V(G) = \{v_1, v_2, \dots v_n\}$

$E(G)$ set of edges (oriented)

$e \rightsquigarrow w_e$ weight of e

(Combinatorial)
Laplacian of G $\qquad \mathbb{C}^{V(G)} = \{ f : V(G) \to \mathbb{C} \}$, $\dim \mathbb{C}^{V(G)} = n$

basis: $f_i(v_j) = \delta(i,j)$

$$\Lambda : \mathbb{C}^{V(G)} \longrightarrow \mathbb{C}^{V(G)}$$
$$f \longmapsto g, \quad g(v) := \sum_{(v^1, v^2) \in E(G)} w_e \cdot (f(v) - f(v'))$$

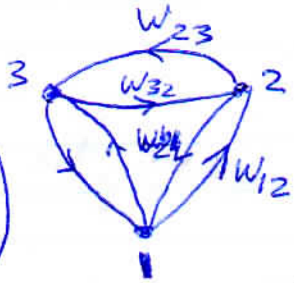
Matrix of Λ in the basis $f_1, \dots, f_n$

$$\Lambda = (\lambda_{ij}), \quad \lambda_{ij} = \begin{cases} -w_{(ij)} & \text{if } i \neq j \\ \sum_{k \neq i} w_{ik} & \text{if } i = j \end{cases}$$

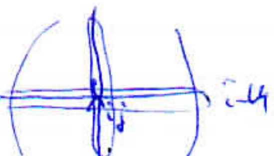
$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ - & - & - & - \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{pmatrix}$$

Example

$$\Lambda = \begin{pmatrix} w_{12} + w_{13} & -w_{12} & -w_{13} \\ -w_{21} & w_{21} + w_{23} & -w_{23} \\ -w_{31} & -w_{32} & w_{31} + w_{32} \end{pmatrix}$$



Cofactors of i-th row / j-th column $\Lambda^{(ij)} = (-1)^{i+j} \det$



Lemma $\quad \Lambda^{(ij)} = \Lambda^{(ij')}$

Matrix-tree theorem (oriented)

$$\boxed{\Lambda^{(ij)} = \sum_{T} \prod_{e \in T} w_e}$$

T: $\underbrace{\text{oriented spanning trees}}$ towards v_i

$\det \Lambda^{(11)} = w_{21} w_{31} + w_{21} w_{32} + w_{23} w_{31}$

<u>Example</u> K_n

$$\Lambda = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & & & & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & & & & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

<u>Cayley theorem</u>: n^{n-2} = # of spanning trees of K_n

<u>Exercise</u> $K_{p,q}$ # span. trees $= p^{q-1} \cdot q^{p-1}$

<u>Conway polynomial</u>

L link with m components

$\nabla(L) = c_0 + c_1 z + c_2 z^2 + \ldots$ the Conway polynomial

<u>Theorem</u> Hosokawa '58, Hartley '83, Hoste '85

$$c_0 = c_1 = \ldots = c_{m-2} = 0$$

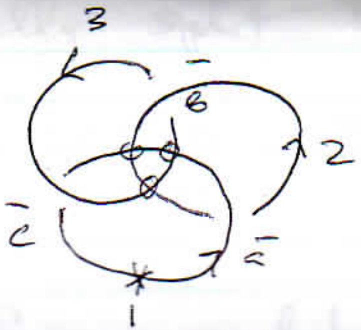
$$c_{m-1} = \det \Lambda^{(ij)}, \quad w_{ij} = lk_{ij}(L)$$

$$\nabla_{asc}(L) = \sum_{\substack{S \\ \text{ascending} \\ \text{one-component}}} \left(\prod_{i \in S} \varepsilon_i \right) \cdot z^{|S|}$$

L_S

- $\nearrow_i \in S \rightsquigarrow) \uparrow$
- one-component $\iff$ L_S is a knot.
- ascending $\iff$ at the first approach to each crossing of S we <u>jump</u> <u>down</u> to smooth it.

Example



$\nabla_{asc}(L) = z^3$

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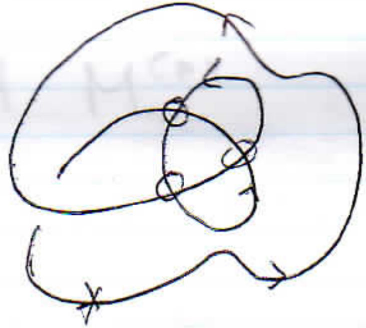
$S = \{a\}$

ascending but
not one-comp.



$S = \{a, b\}$

One component
ascending

$l_{i/j} = \sum \varepsilon_{i/j}$

$l_{1/2} = -1, \quad l_{2/1} = 0$
$l_{1/3} = 0, \quad l_{3/1} = -1$
$l_{2/3} = -1 \quad l_{3/2} = 0$

$w_{ij} := lk_{j/i}(L)$

$$\Delta \begin{pmatrix} -1 & 0 & 1 \\ \hline 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

Algebraically split links $\ell_{ij} = 0$ ~~for~~ $\forall i,j$

__Theorem__ Traldi '84, Levine '97

$$C_{m-1} = C_m = \dots = C_{2m-3} = 0$$

$$C_{2m-2} = \det M^{(P)}$$

$M = (m_{ij})$, $m_{ij} = \sum_k \mu_{ijk}(L)$

triple Milnor #s

$$\mu_{ijk} = -\mu_{jik} = \mu_{jki}$$

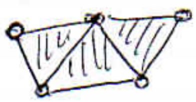
__Pfaffian__ __Matrix-tree__ __theorem__ (G. Masbaum 2002, A. Vaintrob)

if M is a skew-symmetric $2n \times 2n$ matrix then $\mathrm{Pf}(M) = \sqrt{\det(M)}$

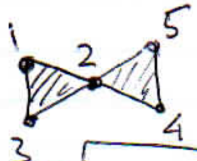
__Example:__

$$\det \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (af - be + cd)^2$$

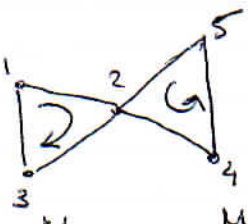
3-graphs = 2 dim $\overset{\text{pure}}{\text{simplicial}}$ complexes

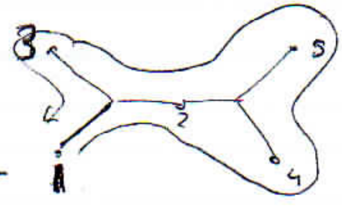


Spanning tree



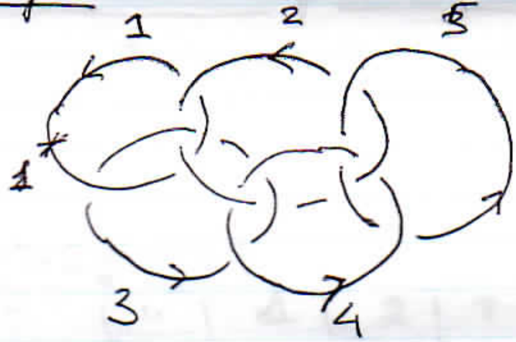
$$\boxed{\mathrm{Pf}(M^{(P)}) = \sum_{\substack{T \\ \text{spanning trees}}} \Big(\prod_{(ijk) \in E(T)} \mu_{ijk} \Big)}$$





$(1\,2\,4\,5\,3) = \sigma(\tilde{T})$

Example

$L =$ 

$$\nabla_{asc}(L) = z^8$$

$m = 5$

$2m-2 = 8$

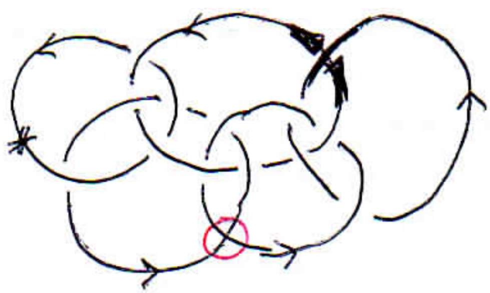
$m-1 = 4$

$$M = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

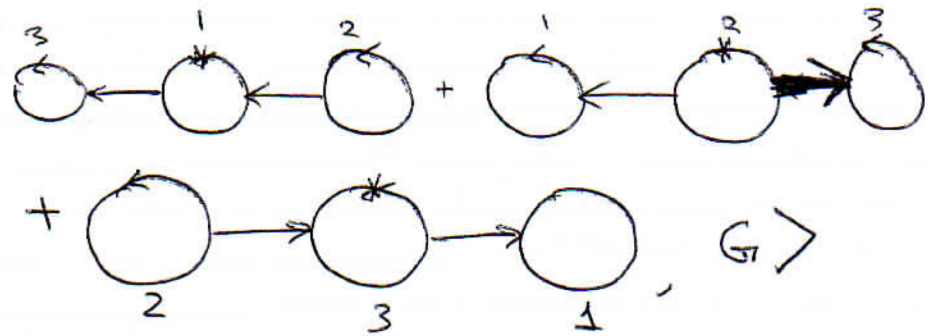
$\mu_{123} = -1$

$\mu_{234} = 1$

$\mu_{245} = 1$

$L =$ 

$$\nabla_{asc}(L) = 2z^8$$

Polyak: $\mu_{123} = \langle$ 

$, G \rangle$