Legendrian and transverse knots and their invariants

UIC Colloquium at Knots in Chicago

Sept. 10, 2010

3pm - 4pm.

1) Contact structure
2) Low dimensional contact topology
3) Legendrian knots
4) Transverse knots

① **Contact structure** [M^m is a smooth manifold]

is a maximal non-integrable field of tangent hyperplanes $H_x \subset T_x M$

$$H_x = \{ \vec{v} \in T_x M : \alpha(\vec{v}) = 0 \} \quad \text{where } \alpha \in \Omega^1(M)$$

non-integrability : $d\alpha|_{H_x}$ is a non-degenerate bilinear form

Maximal integrable manifold has dimension n (Legendrian manifolds)

$$\Downarrow$$

$$m = 2n+1 \qquad \underbrace{\alpha \wedge d\alpha \wedge d\alpha \ldots \wedge d\alpha}_{n} \neq 0$$

Darboux Theorem

$\exists$ local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ such that $\alpha = \sum y_i \, dx_i - dz$

$$H_x = \{ dz = \sum y_i \, dx_i \}$$

Standard contact $\mathbb{R}^{2n+1}$: $\alpha = \sum y_i \, dx_i - dz$

Contact topology
top: $\cong$ homeomorphism
dif top: $\cong$ diffeomorphism
: $\cong$ contactomorphism

② **Low dimensional contact topology** $n = 1$

Standard contact $\mathbb{R}^3$: $\alpha = y \, dx - dz$

$\cancel{dy}$ $y = \frac{dz}{dx}$, $z = f(x)$

$$\big\} \text{ 1-graph}$$

$$(x, f(x), f'(x)) \quad \text{Legendrian curve.}$$

Rotationally standard contact structure

$$\alpha_{rot} = dz - y\,dx + x\,dy$$

Exercise Find a contactomorphism that sends α_{rot} to α_{std}

Answer $(x, y, z) \to (-x, 2y, -xy - z)$

$$\alpha_{std} = 2y\,d(-x) - d(-xy - z) =$$

$$= -2y\,dx + x\,dy + y\,dx + dz = \alpha_{rot}.$$

(1″

Contact Bundle

$$\begin{array}{ccc} \{H_x\} & \subset & TM \\ & \searrow & \downarrow \\ & & M \end{array}$$

Standard contact $\mathbb{R}^{2n+1}$ is parallelizable

$$\begin{array}{ccc} \mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \supset S^1 \\ \downarrow & & \\ \mathbb{R}^3 & & \end{array}$$

③ **Legendrian knots** $L \subset \mathbb{R}^3$

Unit tangent vectors

$$L = \{S^1 \to \mathbb{R}^3\} \xrightarrow{\mu} S^1 \subset \mathbb{R}^2$$

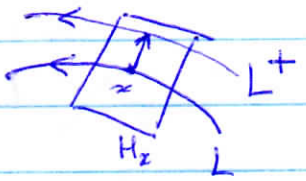
Maslov number $m := \deg \mu$

Thurston - Bennequin number: $tb(L) = lk(L, L^+)$

Legendrian framing

$\alpha \wedge d\alpha \to$ orientation of $\mathbb{R}^3$

$d\alpha \to$ orientation of H_x



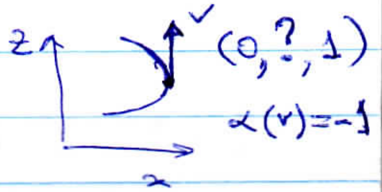
(Front)

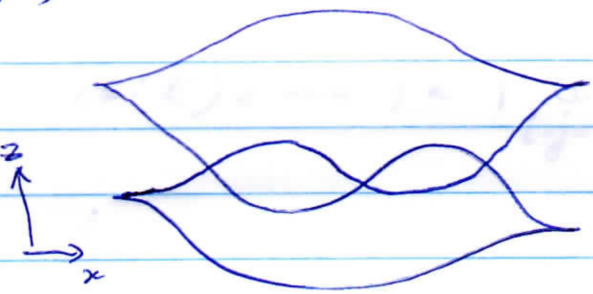
③ Diagrams of Legendrian knots

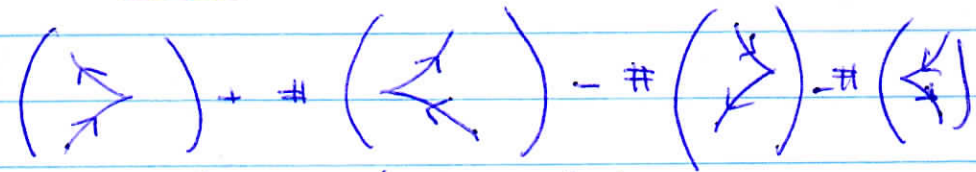
$\mathbb{R}^3_{(x,y,z)}$ $\quad \alpha = y\,dx - dz$

$\downarrow$

$\mathbb{R}^2_{x,z}$



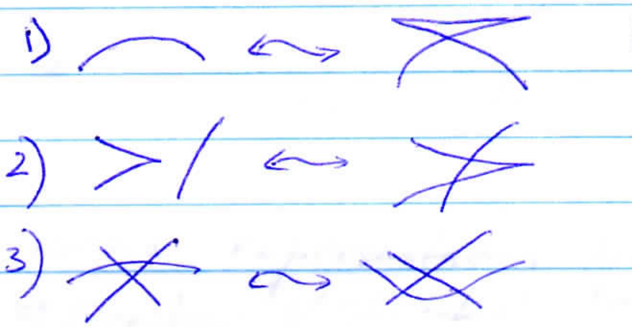


$m = \#$ 

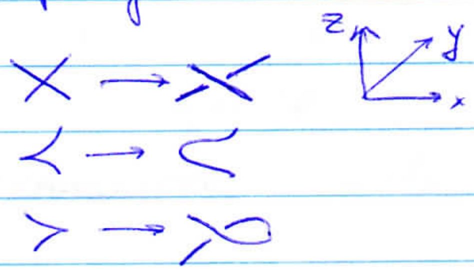
$tb = \#$ 

left cusps

Legendrian isotopy (Reidemeister moves)

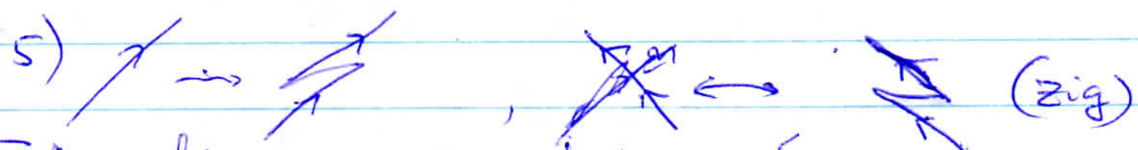


topological knots:



D. Fuchs, S. Tabachnikov '97:

Two top. knots given by Leg. fronts are equivalent iff the ~~two~~ fronts are related by 1) – 3) and by adding (~~del~~ removing) a zig-~~zag~~



F. Aicardi's move, Chekanov's pair

Estimates for $tb(K)$

—, V. Goryunov '97

$tb(K) < -\max \deg$ of x in $F(x,y)$

Khovanov homology:

L. Ng '05 $tb(K) < \min \{ k \mid \bigoplus_{i-j=k} HKh^{i,j}(K) \neq 0 \}$

Grid diagrams.

4. Transverse knots is a knot transversal to the contact structure.

Example 1) unit circle in (x,y)-plane to α_{rot}

2) L^+ is transverse (Leg. Reidemeister + zag 4)

Bennequin '83: $tb(L^+) = tb(L) + m(L)$

Legendrian L

Braid representation for transverse knots

S. Orevkov, V. Shevchishin '03

N. Wrinkle '02

Transverse Markov theorem

$B_1 \sim_{tran} B_2$ iff B_1 and B_2 are related by a sequence of braid conjugations and positive braid stabilizations / destabilizations

P. Ozsváth, Z. Szabó, D. Thurston '06

10_{132} $\sigma_3 \sigma_2^{-2} \sigma_3^{2} \sigma_2 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_1^{2}$ $\theta \in \widehat{HFK}$

$\sigma_3 \sigma_2^{-2} \sigma_3^{2} \sigma_2 \sigma_3^{-1} \sigma_1^{2} \sigma_2 \sigma_1^{-1}$

O. Plamenevskaya '04

$\psi(L) \in HKh(L)$ is a transverse invariant.