University of South Alabama

The Tutte polynomial, its applications and generalizations

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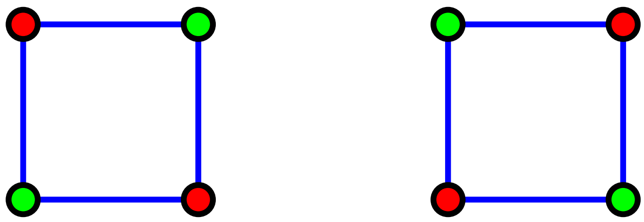
3:30 — 4:30 p.m.

Chromatic polynomial

$$C_\Gamma(q) := \#\text{ of proper colorings of } V(\Gamma) \text{ in } q \text{ colors}$$

Example. $C_{\square}(q) = q(q-1)(q^2-3q+3)$

$q = 2:$



Properties: $C_\Gamma = C_{\Gamma - e} - C_{\Gamma/e}, \quad C_{\Gamma_1 \sqcup \Gamma_2} = C_{\Gamma_1} \cdot C_{\Gamma_2}, \quad C_\bullet = q.$

$$\mathcal{S} := \Big\{ V(\Gamma) \to \{1, \ldots, q\} \Big\}$$

$$C_\Gamma(q) = \sum_{\sigma \in \mathcal{S}} \prod_{(a,b) \in E(\Gamma)} \Big(1 - \delta(\sigma(a), \sigma(b)) \Big)$$

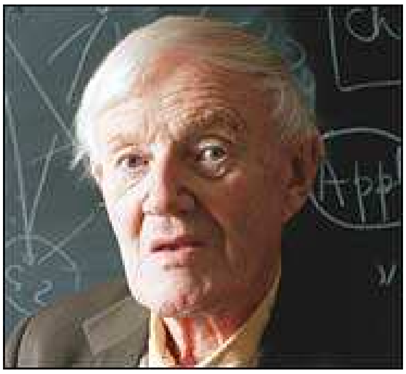
Dichromatic polynomial

$$Z_\Gamma(q, v) := \sum_{\sigma \in \mathcal{S}} \prod_{(a,b) \in E(\Gamma)} (1 + v\delta(\sigma(a), \sigma(b)))$$

$$C_\Gamma(q) = Z_\Gamma(q, -1). \qquad Z_\Gamma(q, v) = \sum_{F \subseteq E(\Gamma)} q^{k(F)} v^{e(F)}.$$

Properties: $Z_\Gamma = Z_{\Gamma - e} + v Z_{\Gamma/e}, \quad Z_{\Gamma_1 \sqcup \Gamma_2} = Z_{\Gamma_1} \cdot Z_{\Gamma_2}, \quad Z_\bullet = q.$

The Tutte polynomial



Let • Γ be a graph;

- $v(\Gamma)$ be the number of its vertices;
- $e(\Gamma)$ be the number of its edges;
- $k(\Gamma)$ be the number of components of Γ;
- $r(\Gamma) := v(\Gamma) - k(\Gamma)$ be the *rank* of Γ;
- $n(\Gamma) := e(\Gamma) - r(\Gamma)$ be the *nullity* of Γ;

$$T_\Gamma(x,y) \ := \sum_{F\subseteq E(\Gamma)} (x-1)^{r(\Gamma)-r(F)}(y-1)^{n(F)}$$

$$Z_\Gamma(q,v) = q^{k(\Gamma)}v^{r(\Gamma)}T_\Gamma(1+qv^{-1},1+v)$$

Properties.

$T_\Gamma = T_{\Gamma-e} + T_{\Gamma/e}$ if e is neither a bridge nor a loop ;
$T_\Gamma = xT_{\Gamma/e}$ if e is a bridge ;
$T_\Gamma = yT_{\Gamma-e}$ if e is a loop ;
$T_{\Gamma_1\sqcup\Gamma_2} = T_{\Gamma_1\cdot\Gamma_2} = T_{\Gamma_1}\cdot T_{\Gamma_2}$ for a disjoint union, $\Gamma_1\sqcup\Gamma_2$
$T_\bullet = 1$. and a one-point join, $\Gamma_1\cdot\Gamma_2$;

$T_\Gamma(1,1)$ is the number of spanning trees of Γ ;
$T_\Gamma(2,1)$ is the number of spanning forests of Γ ;
$T_\Gamma(1,2)$ is the number of spanning connected subgraphs of Γ ;
$T_\Gamma(2,2) = 2^{|E(\Gamma)|}$ is the number of spanning subgraphs of Γ .

The Potts model

C.Domb (1952). $q = 2$ the Ising model; W.Lenz (1920).

Atoms are located at the sites of vertices $V(\Gamma)$.
Nearest neighbors are indicated by edges $E(\Gamma)$.
An atom exists in one of q different states (*spins*).
A *state*, $\sigma \in \mathcal{S}$, is an assignments of spins to all vertices $V(\Gamma)$.
Neighboring atoms interact with each other only is their spins are the same.
The energy of the interaction is $-J$ (*coupling constant*).
The model is called *ferromagnetic* if $J > 0$ and *antiferromagnetic* if $J < 0$.

Energy of a state σ (*Hamiltonian*),

$$H(\sigma) = -J \sum_{(a,b)\in E(\Gamma)} \delta(\sigma(a), \sigma(b)).$$

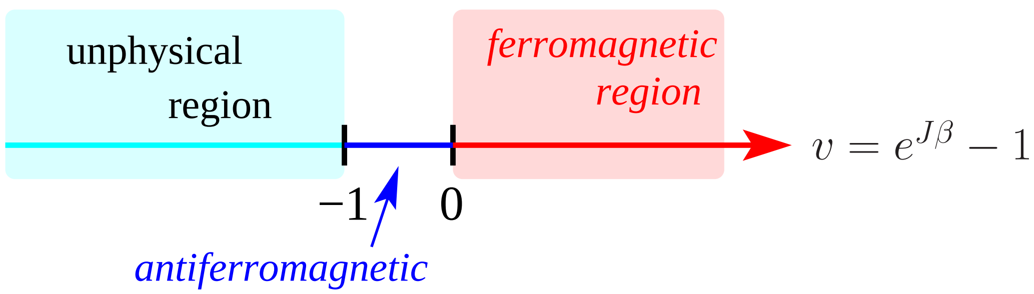
Boltzmann weight of σ:

$$e^{-\beta H(\sigma)} = \prod_{(a,b)\in E(\Gamma)} e^{J\beta\delta(\sigma(a),\sigma(b))} = \prod_{(a,b)\in E(\Gamma)} \Big(1+(e^{J\beta}-1)\delta(\sigma(a),\sigma(b))\Big),$$

where the *inverse temperature* $\beta = \frac{1}{\kappa\, T}$, T is the temperature, $\kappa = 1.38 \times 10^{-23}$ joules/Kelvin is the *Boltzmann constant*.

The Potts partition function

$$Z_\Gamma^{\text{Potts}} := \sum_{\sigma\in\mathcal{S}} e^{-\beta H(\sigma)} = Z_\Gamma(q, e^{J\beta} - 1)$$

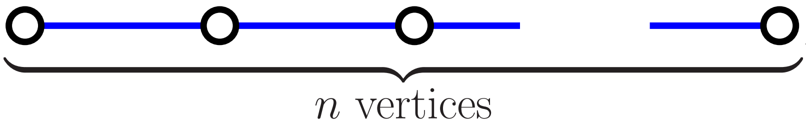


Probability of a state σ: $\quad P(\sigma) := e^{-\beta H(\sigma)}/Z_\Gamma$.
Expected value of a function $f(\sigma)$:

$$\langle f \rangle := \sum_\sigma f(\sigma) P(\sigma) = \sum_\sigma f(\sigma) e^{-\beta H(\sigma)}/Z_\Gamma \ .$$

Expected energy:

$$\langle H \rangle = \sum_\sigma H(\sigma) e^{-\beta H(\sigma)}/Z_\Gamma = -\frac{d}{d\beta} \ln Z_\Gamma \ .$$

Example. $\Gamma =$ 

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$$T_\Gamma = x^{n-1}, \qquad \begin{aligned} Z_\Gamma &= q v^{n-1}(1 + q v^{-1})^{n-1} = q(q+v)^{n-1} \\ &= q(q - 1 + e^{\beta J})^{n-1} \ . \end{aligned}$$

Expected energy: $\quad \langle H \rangle = (n-1) \dfrac{-J e^{\beta J}}{q - 1 + e^{\beta J}}$.

Expected energy per atom as $n \to \infty$:

$$\lim_{n \to \infty} \frac{\langle H \rangle}{n} = \frac{-J e^{\beta J}}{q - 1 + e^{\beta J}} \ .$$

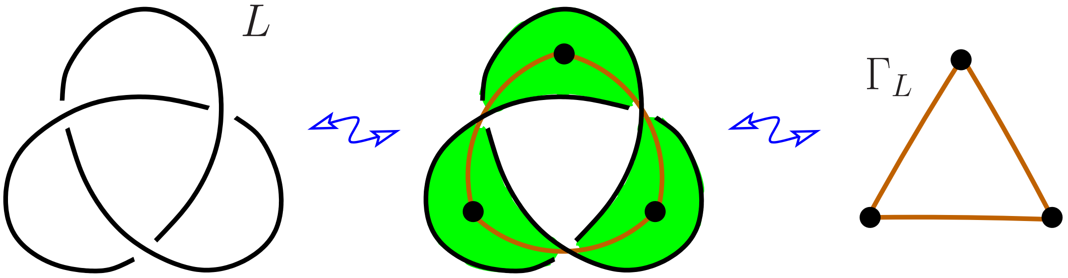
$\underline{T \to \infty}$ $(\beta \to 0)$: The energy per atom $\to -J/q$.
$\underline{T \to 0}$ $(\beta \to \infty)$.

$J < 0$ (antiferromagnetic): The energy per atom $\to 0$. In general, $e^{\beta J} \to 0$ and the partition function $\to Z_\Gamma(q, -1) = C_\Gamma(q)$.

$J > 0$ (ferromagnetic): The energy per atom $\to -J$.

Morwen Thistlethwaite (1987)

Up to a sign and a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_{\Gamma_L}(-t,-t^{-1})$.



$$V_L(t) = t + t^3 - t^4 = -t^2(-t^{-1} - t + t^2)$$

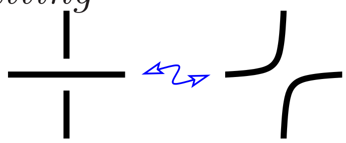
$$T_{\Gamma_L}(x,y) = y + x + x^2$$

$$T_{\Gamma_L}(-t,-t^{-1}) = -t^{-1} - t + t^2$$

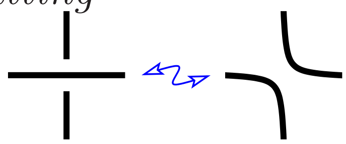
The Kauffman bracket

Let L be a virtual link diagram.

A-splitting



B-splitting



A *state* S is a choice of either A- or B-splitting at every classical crossing.

$\alpha(S) = \#(\text{of } A\text{-splittings in } S)$

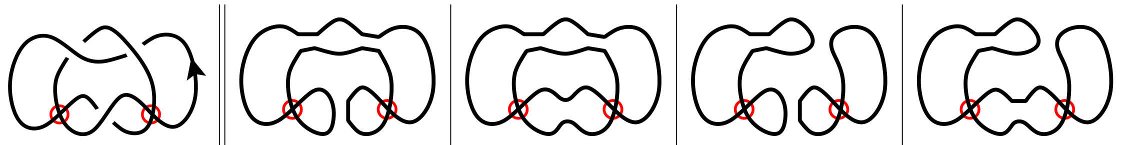
$\beta(S) = \#(\text{of } B\text{-splittings in } S)$

$\delta(S) = \#(\text{of circles in } S)$

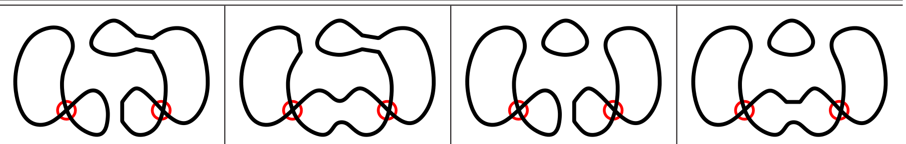
$$[L](A, B, d) \ := \sum_S A^{\alpha(S)}\, B^{\beta(S)}\, d^{\delta(S)-1}$$

$$J_L(t) \ := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Example



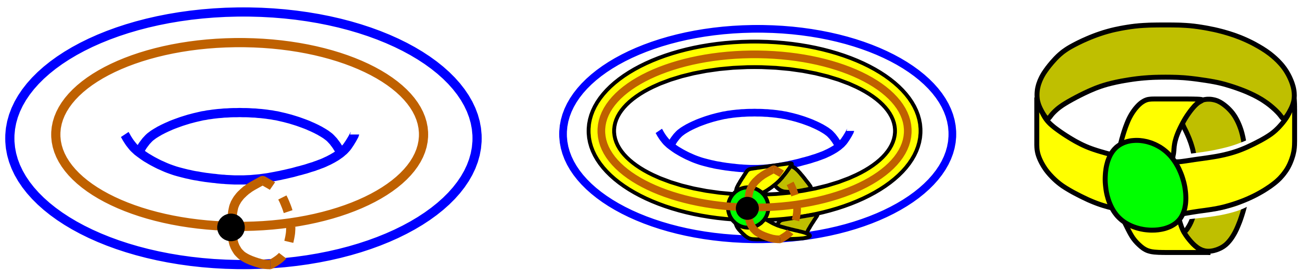
(α, β, δ)	$(3, 0, 1)$	$(2, 1, 2)$	$(2, 1, 2)$	$(1, 2, 1)$

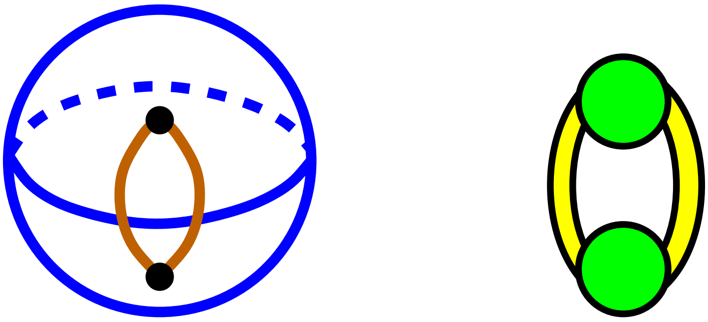


	$(2, 1, 2)$	$(1, 2, 1)$	$(1, 2, 3)$	$(0, 3, 2)$

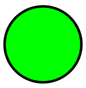
$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d$; $\qquad J_L(t) = 1$

Graphs on surfaces

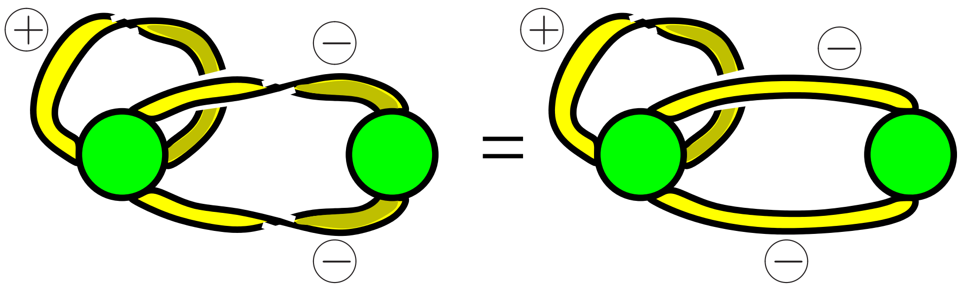




Ribbon graphs

A ribbon graph G is a surface represented as a union of vertices-discs  and edges-ribbons 

- discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.



The Bollobás-Riordan polynomial

Let
- F be a ribbon graph;
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;
- $\mathrm{bc}(F)$ be the number of boundary components of F;
- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

$$R_G(x, y, z) \ := \sum_F x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-\mathrm{bc}(F)+n(F)}$$

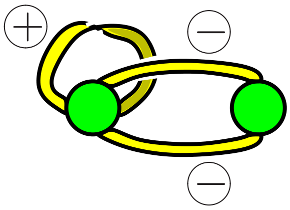
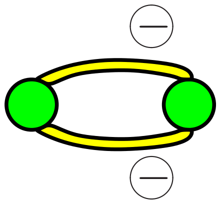
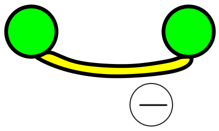
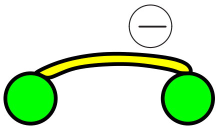
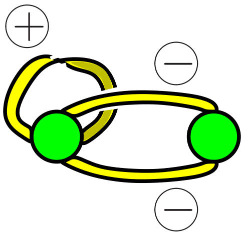
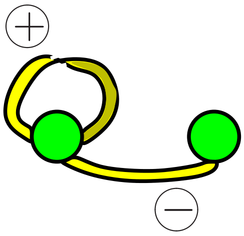
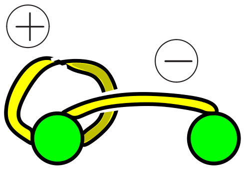
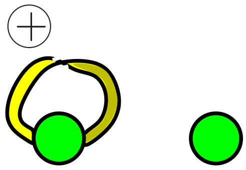
Relations to the Tutte polynomial.

$$R_G(x-1, y-1, 1) = T_G(x, y)$$

If G is planar (genus zero):

$$R_G(x-1, y-1, z) = T_G(x, y)$$

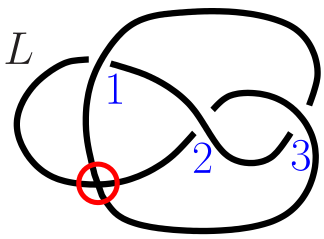
Example.

				
$(k, r, n, \mathrm{bc}, s)$	$(1, 1, 1, 2, 1)$	$(1, 1, 0, 1, 0)$	$(1, 1, 0, 1, 0)$	$(2, 0, 0, 2, -1)$
				
	$(1, 1, 2, 1, 1)$	$(1, 1, 1, 1, 0)$	$(1, 1, 1, 1, 0)$	$(2, 0, 1, 2, -1)$

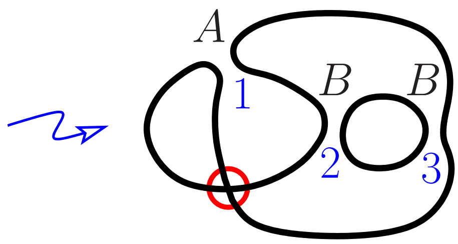
- $r(F) := v(F) - k(F)$;
- $n(F) := e(G) - r(F)$;
- $\mathrm{bc}(F)$ is the number of boundary components;
- $s(F) := \dfrac{e_-(F) - e_-(\overline{F})}{2}$.

$$R_G(x, y, z) = x + 2 + y + xyz^2 + 2yz + y^2z \ .$$

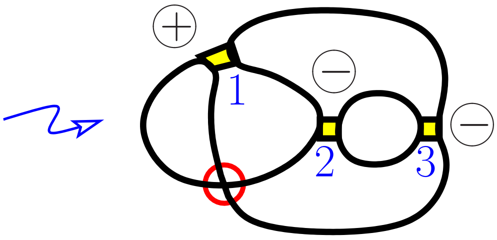
Construction of a ribbon graph from a virtual link diagram



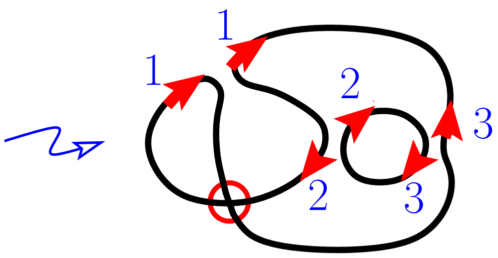
Diagram



State s

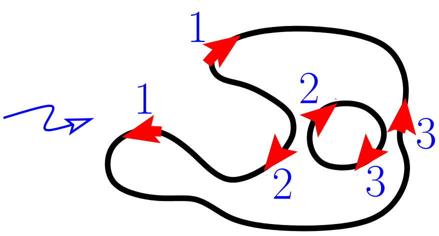


Attaching planar bands

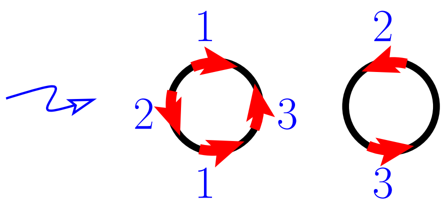


Replacing bands by arrows

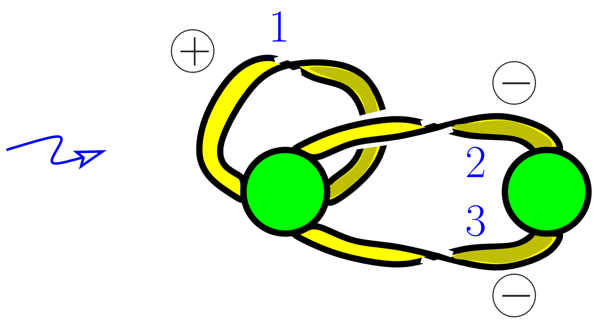
1 — ⊕; 2 — ⊖; 3 — ⊖



Untwisting state circles



Pulling state circles apart



Forming the ribbon graph G_L^s

Theorem

Let L be a virtual link diagram with e classical crossings, G_L^s be the signed ribbon graph corresponding to a state s, and $v := v(G_L^s)$, $k := k(G_L^s)$. Then $e = e(G_L^s)$ and

$$[L](A, B, d) = A^e \left(x^k y^v z^{v+1} R_{G_L^s}(x, y, z) \Big|_{x=\frac{Ad}{B},\ y=\frac{Bd}{A},\ z=\frac{1}{d}} \right) .$$

Idea of the proof.

One-to-one correspondence between states s' of L and spanning subgraphs F' of G_L^s:

> *An edge e of G_L^s belongs to the spanning subgraph F' if and only if the corresponding crossing was split in s' differently comparably with s.*

Further developments

Iain Moffatt:

- *Knot invariants and the Bollobás-Riordan polynomial of embedded graphs*, European Journal of Combinatorics, 29 (2008) 95-107. `arXiv:math/0605466`.
- *Partial duality and Bollobás and Riordan's ribbon graph polynomial*, Discrete Mathematics, 310 (2010) 174-183. `arXiv:0809.3014`.
- *A characterization of partially dual graphs*. `arXiv:0901.1868`.

Fabien Vignes-Tourneret:

- *The multivariate signed Bollobás-Riordan polynomial*, Discrete Mathematics, 309 (2009) 5968-5981. `arXiv:0811.1584`.
- (joint with T. Krajewski, V. Rivasseau) *Topological graph polynomials and quantum field theory, Part II: Mehler kernel theories*. `arXiv:0912.5438`. (non-commutative Grosse-Wulkenhaar quantum field theory)