Random gluing polygons

Sergei Chmutov
joint work with Boris Pittel

Ohio State University, Mansfield

Stochastic Topology and Thermodynamic Limits
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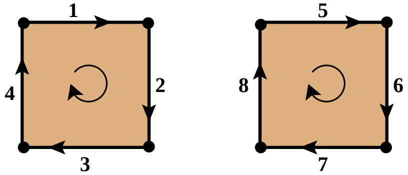
9:00 — 9:45 a.m.

Polygons. Notations.

$n :=$ # (oriented) polygons
$N :=$ total (even) number of sides
$n_j :=$ # j-gons, $\quad \sum n_j = n, \quad \sum j n_j = N$
$[N] := \{1, 2, \ldots, N\}$
$\alpha \in S_N$ is a permutation of $[N]$ cyclically permutes edges of polygons according to their orientations.

Example.

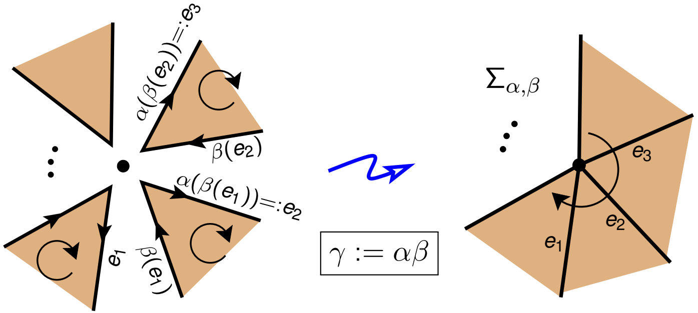


$\alpha = (1234)(5678)$

n_j equals the number of cycles of α of length j.

Gluing polygons. Permutations.

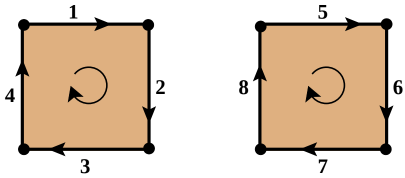
$\beta \in S_N$ is an involution without fixed points;
β has $N/2$ cycles of length 2.

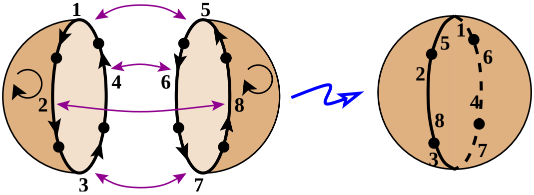


vertices of $\Sigma_{\alpha,\beta}$ = # cycles of γ.
connected components of $\Sigma_{\alpha,\beta}$ = # orbits of the subgroup generated by α and β.

Gluing polygons. Example.

$n = 2$, $N = 8$, $\qquad \alpha = (1234)(5678)$



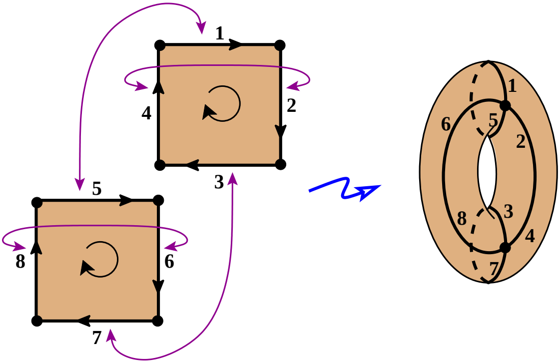


$\beta = (15)(28)(37)(46)$

$\gamma = (16)(25)(38)(47)$

Gluing polygons. Example.

$n = 2$, $N = 8$, $\alpha = (1234)(5678)$

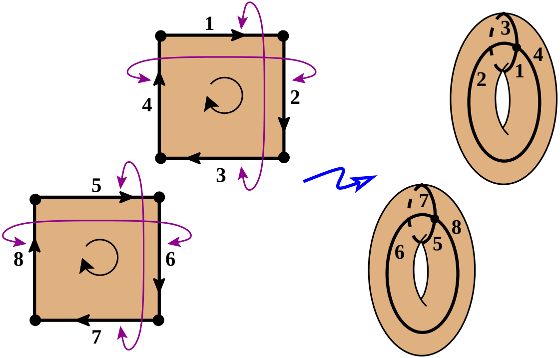


$\beta = (15)(24)(37)(68)$

$\gamma = (1652)(3874)$

Gluing polygons. Example.

$n = 2$, $N = 8$, $\qquad \alpha = (1234)(5678)$

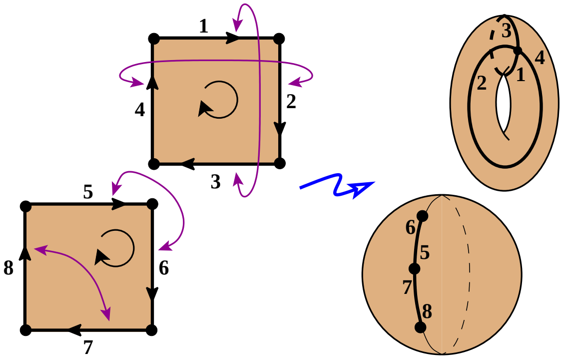


$\beta = (13)(24)(57)(68)$

$\gamma = (1432)(5876)$

Gluing polygons. Example.

$n = 2$, $N = 8$, $\qquad \alpha = (1234)(5678)$

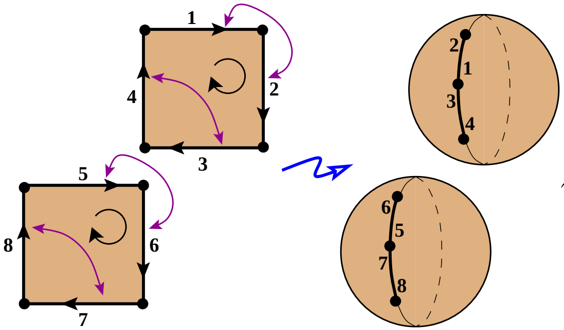


$\beta = (13)(24)(56)(78)$

$\gamma = (1432)(57)(6)(8)$

Gluing polygons. Example.

$n = 2$, $N = 8$, $\quad \alpha = (1234)(5678)$



$\beta = (12)(34)(56)(78)$

$\gamma = (13)(2)(4)(57)(6)(8)$

Gluing polygons. Example.

$n = 2$, $N = 8$, $\alpha = (1234)(5678)$

There are $7!! = 105$ possibilities for choosing β.

surface $\Sigma_{\alpha,\beta}$	S^2	T^2	$2T^2$	$T^2 + S^2$	$2S^2$
# gluings	36	60	1	4	4

Random gluing.

$\mathbf{n} := \{n_j\}$ is a partition of $n = \sum n_j$.

Let $\mathcal{C}_{\mathbf{n}}$ be the conjugacy class of α, all permutations in S_N with the cycle structure $\mathbf{n}$.

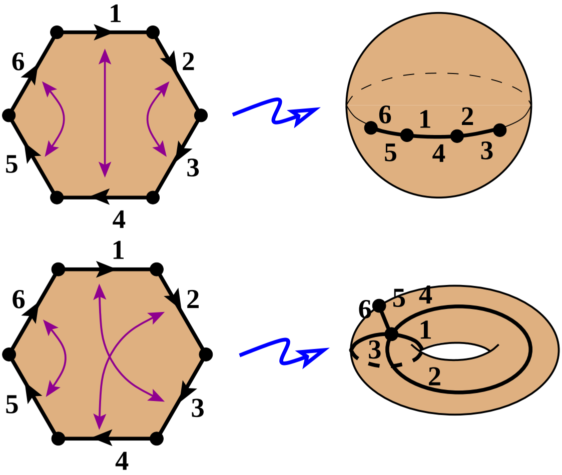
Let $\mathcal{C}_{N/2}$ be the conjugacy class of β, all permutations in S_N with all cycles length 2.

A ***random surface*** is the surface $\Sigma_{\alpha,\beta}$ obtained by gluing according to the permutations α and β that are independently chosen uniformly at random from the conjugacy classes $\mathcal{C}_{\mathbf{n}}$ and $\mathcal{C}_{N/2}$ respectively.

Harer-Zagier formula. $n = 1$.

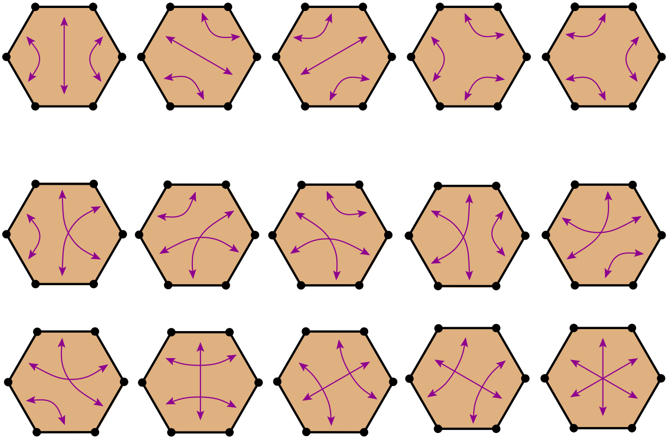
$\underline{n = 1}$, $\qquad \alpha = (123\ldots N)$.

Example: $N = 6$



$\underline{n = 1, N = 6}$ $\quad V_{\mathbf{n}}$ =# vertices of $\Sigma_{\alpha,\beta}$. $\quad |\mathcal{C}_{N/2}| = 5!! = 15$.

$V_{\mathbf{n}} = 4$



$V_{\mathbf{n}} = 2$

Generating function: $T_N(y) := \sum\limits_{\beta} y^{V_{\mathbf{n}}}$. $\quad T_6(y) = 5y^4 + 10y^2$.

Harer-Zagier formula.

J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986) 457–485.

$$T_N(y) := \sum_{\beta} y^{V_{\mathbf{n}}}.$$

Generating function: $T(x,y) := 1 + 2xy + 2x \sum_{k=1}^{\infty} \frac{T_{2k}(y)}{(2k-1)!!} x^k.$

$$\boxed{T(x,y) = \left(\frac{1+x}{1-x}\right)^y}$$

—,B.Pittel, JCTA **120** (2013) 102–110: g_N = genus of $\Sigma_{\alpha,\beta}$.
Asymptotically as $N \to \infty$, g_N is normal
$\mathcal{N}((N - \log N)/2, (\log N)/4)$.

Main result.

—,B.Pittel, *On a surface formed by randomly gluing together polygonal discs*, Advances in Applied Mathematics, **73** (2016) 23–42.

$V_{\mathbf{n}}$ =# vertices of $\Sigma_{\alpha,\beta}$.

Theorem. *$V_{\mathbf{n}}$ is asymptotically normal with mean and variance* $\log N$ *both,* $V_{\mathbf{n}} \sim \mathcal{N}(\log N, \log N)$*,*
as $N \to \infty$, and uniformly on $\mathbf{n}$*.*

Previous results.

$\mathrm{E}[V_{\mathbf{n}}] \sim \log n \qquad \mathrm{Var}(\chi) \sim \log n$

- N. Pippenger, K. Schleich, *Topological characteristics of random triangulated surfaces*, Random Structures Algorithms, **28** (2006) 247–288.

  All polygons are triangles.

- A. Gamburd, *Poisson-Dirichlet distribution for random Belyi surfaces*, Ann. Probability, **34** (2006) 1827–1848.

  All polygons have the same number of sides, k.
  $2\,\mathrm{lcm}(2,k) \mid kn$

  γ *is asymptotically uniform on the alternating group* A_{kn}.

Key Theorem.

Depending on the parities of permutations $\alpha \in \mathcal{C}_{\mathbf{n}}$ and $\beta \in \mathcal{C}_{N/2}$ the permutation $\gamma = \alpha\beta$ is either even $\gamma \in A_N$ or odd $\gamma \in A_N^c := S_N - A_N$.
The probability distribution of γ is asymptotically uniform (for $N \to \infty$ uniformly in $\mathbf{n}$) on A_N or on A_N^c.
Let P_γ be the probability distribution of γ and let U be the uniform probability measure on A_N or on A_N^c.
Let $\|P_\gamma - U\| := (1/2)\sum_{s\in S_N} |P_\gamma(s) - U(s)|$ be the total variation distance between P_γ and U.

Theorem. $\|P_\gamma - U\| = O\big(N^{-1}\big)$.

Ideas of the proof.

P. Diaconis, M. Shahshahani, *Generating a random permutation with random tranpositions*, Z. Wahr. Verw. Gebiete, **57** (1981) 159–179.
Using the Fourier analysis on finite groups and the Plancherel Theorem:

$$\|P - U\|^2 \leq \frac{1}{4} \sum_{\rho \in \widehat{G},\ \rho \neq \mathrm{id}} \dim(\rho)\,\mathrm{tr}\big(\hat{P}(\rho)\hat{P}(\rho)^*\big);$$

here $\widehat{G}$ denotes the set of all irreducible representations ρ of G, "id" denotes the trivial representation, $\dim(\rho)$ is the dimension of ρ, and $\hat{P}(\rho)$ is the matrix value of the Fourier transform of P at ρ, $\hat{P}(\rho) := \sum_{g \in G} \rho(g)P(g)$.

Ideas of the proof.

For $G = S_N$, the irreducible representations ρ are indexed by partitions $\lambda \vdash N$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ of N.
Let $f^\lambda := \dim(\rho^\lambda)$ (given by the hook length formula) and χ^λ be the character of ρ^λ.

$$\|P_\gamma - U\|^2 \leq \frac{1}{4} \sum_{\lambda \neq (N),\,(1^N)} \left(\frac{\chi^\lambda(\mathcal{C}_{\mathbf{n}})\chi^\lambda(\mathcal{C}_{N/2})}{f^\lambda} \right)^2 .$$

Gamburd used estimate from S. V. Fomin, N. Lulov, *On the number of rim hook tableaux*, J. Math. Sciences, **87** (1997) 4118–4123, for $N = kn$,

$$|\chi^\lambda(\mathcal{C}_{N/k})| = O\big(N^{1/2-1/(2k)}\big)(f^\lambda)^{1/k}.$$

Ideas of the proof.

M. Larsen, A. Shalev, *Characters of symmetric groups: sharp bounds and applications*, Invent. Math., **174** (2008) 645–687.
Extension of the Fomin-Lulov bound for all permutations σ without cycles of length below m, and partitions λ:

$$|\chi^\lambda(\sigma)| \leq (f^\lambda)^{1/m+o(1)}, \quad N \to \infty.$$

$$\boxed{\|P_\gamma - U\|^2 = O(N^{-2}).}$$

Thanks.

THANK YOU!