OSU, Combinatorics seminar 1/21/2016

Ribbon graphs and Δ-matroids

Abstract Graphs $G=(E,V)$ vs Matroids

(connected)

$E = E(G)$

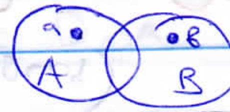
{edges of a spanning tree} $\in \mathcal{B}$

Matroids

Exchange axioms for bases:

$(E, \mathcal{B})$, $\mathcal{B} \subseteq 2^E$

1) $\forall B \in \mathcal{B}$, any proper subset of $B \notin \mathcal{B}$
2) $\forall A, B \in \mathcal{B}$, $\forall a \in A \setminus B$, $\exists b \in B \setminus A \mid A \setminus \{a\} \cup \{b\} \in \mathcal{B}$



Ribbon graphs

$G = (E, V)$

top. discs

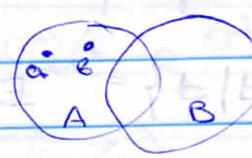
- vertices $\cap$ edges = disjoint line segments
- each segment $\in \partial(v)$ $\exists! v$
  $\in \partial(e)$ $\exists! e$
- $e \ni$ two segments

Δ-Matroids $(E, \mathcal{F})$

André Bouchet '87 feasible sets

Symmetric exchange axiom

$\forall A, B \in \mathcal{F}$, $\forall a \in A \Delta B$, $\exists b \in A \Delta B \mid A \Delta \{a, b\} \in \mathcal{F}$



A quasi-tree = a ribbon graph with a single boundary component

Theorem Let $G = (E, V)$ be a ribbon graph. Then $(E$, spanning quasi-trees$)$ is a Δ-matroid.

Δ-matroid $(E, \mathcal{F}) =: D$

$e \in E$ is a coloop iff $e \in F$ for $\forall F \in \mathcal{F}$

$e \in E$ is a loop iff $\forall F \in \mathcal{F}$, $e \notin F$.

Deletion

contraction

if e is not a coloop; $D \setminus e := (E \setminus e, \{F \mid F \in \mathcal{F} \text{ and } F \subset E \setminus e\})$

is e is not a loop: $D/e := (E \setminus e, \{F \setminus e \mid F \in \mathcal{F} \text{ and } e \in F\})$

Minors

min and max matroids

$D = (E, \mathcal{F})$ is a Δ-matroid

$D_{min} := (E, \mathcal{F}_{min})$, $\mathcal{F}_{min} := \cancel{sa}\ \{F \mid F \in \mathcal{F}, F \text{ is of minimal cardinality}\}$

$D_{max} := (E, \mathcal{F}_{max})$, $\mathcal{F}_{max} := \{F \mid F \in \mathcal{F}, F \text{ —''— maximal —''—}\}$

$D(G)_{min} = \mathcal{C}(G)$

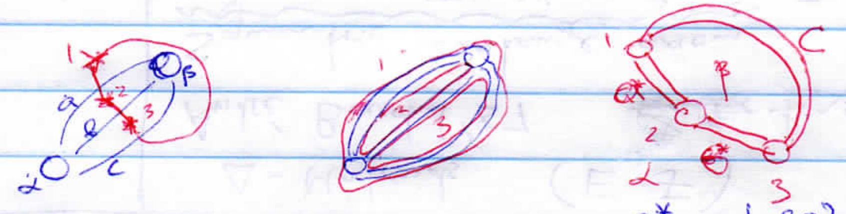
$D(G)_{max} = (\mathcal{C}(G^*))^*$

D_{min} and D_{max} are matroids, $w(D) := r(D_{max}) - r(D_{min})$

Duality

~~Partial du~~

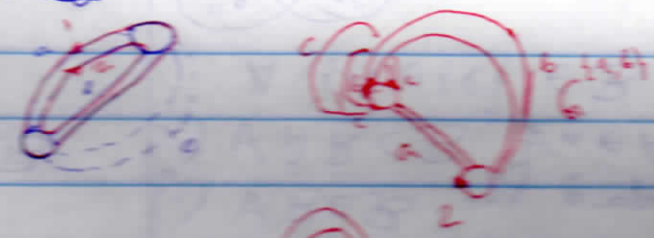
G



$G^* = G^{\{a,b,c\}}$

Partial duality (2009)

$G^{\{a,b\}}$



G $\quad$ G^e

Properties 1) $G^E = G^*$, $G^\emptyset = G$

2) $(G^A)^B = G^{A \Delta B}$

3) $G/e = G^e \setminus e$

4) preserves orientability.

Twists of Δ-matroids (Bouchet '87)

$D = (E, \mathcal{F})$, $A \subset E$

$D * A = (E, \{A \Delta F \mid F \in \mathcal{F}\})$

Theorem

$$D_G * A = D_{G^A}$$

Polynomials

Krushkal polynomial

$$K_{G,\Sigma}(X,Y,A,B) := \sum_{F \subseteq E(G)} X^{k(F)-k(G)} Y^{k(\Sigma \setminus F)-k(\Sigma)} A^{s(F)/2} B^{s^\perp(F)/2}$$

where $s(F) := \beta_1(\tilde{F}; \mathbb{Z}_2)$ $s^\perp(F) := \beta(\Sigma \setminus F; \mathbb{Z}_2)$

$$T_G(x,y) = (y-1)^{s(\Sigma)/2} K_{G,\Sigma}(x-1, y-1, y-1, (y-1)^{-1})$$

Th. $K_G(X,Y,A,B) = K_{G^*}(Y,X,B,A)$

Theorem G ribbon graph

$$\cancel{K_G(X,Y,A,B) = \sum_{F \subseteq E(G)} X^{r_G(G)-r_G(F)} Y^{r_{G^*}(E)-r_{G^*}(F)} A}$$

$w(F) := w(D|F)$

Theorem $D = (E(G), \mathcal{F}(G))$ for a ribbon graph G

$$K_G(X,Y,A,B) = \sum_{F \subset E} X^{r_{min}(E)-r_{min}(F)} Y^{r^*_{min}(E)-r^*_{min}(\bar{F})} A^{w_D(F)/2} B^{w_{D^*}(\bar{F})/2}$$