Higher dimensional graph theory

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Chromatic polynomial $\chi_G(x)$ of graphs.

A *coloring* of G with x colors is a map $c : V(G) \to \{1, \ldots, x\}$. A coloring c is *proper* if for any edge $e = (v_1, v_2)$: $c(v_1) \neq c(v_2)$.

$\chi_G(x) :=$ *# of proper colorings of G in x colors.*

Properties .

- $\chi_G = \chi_{G-e} - \chi_{G/e}$;
- $\chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2}$, for a disjoint union $G_1 \sqcup G_2$;
- $\chi_\bullet = x$;
- $\chi_G(x) = \sum_{F \subseteq E(G)} (-1)^{|F|} x^{k(F)}$,

  where the sum runs over all spanning subgraphs F and $k(F)$ is the number of connected components of F.

Dichromatic polynomial $Z_G(x,t)$ of graphs.

$$Z_G(x,t) := \sum_{c \in Col_G(x)} t^{\#\text{ edges colored not properly by } c}$$

Properties .

- $\chi_G(x) = Z_G(x,0)$;
- $Z_G = Z_{G-e} + (t-1)Z_{G/e}$;
- $Z_{G_1 \sqcup G_2} = Z_{G_1} \cdot Z_{G_2}$, for a disjoint union $G_1 \sqcup G_2$;
- $Z_\bullet = x$;
- $Z_G(x,t) = \sum_{F \subseteq E(G)} x^{k(F)}(t-1)^{|F|}$;
- $Z_G(x,t)$ is the partition function of the Potts model in statistical mechanics.

Tutte polynomial $T_G(x, v)$ of graphs.

$$T_G(x,y) \ :== (x-1)^{-k(G)}(y-1)^{-v(G)} Z_G((x-1)(y-1), y) \ .$$

Properties .

- $T_G = T_{G-e} + T_{G/e}$ if e is neither a bridge nor a loop ;
- $T_G = xT_{G/e}$ if e is a bridge ;
- $T_G = yT_{G-e}$ if e is a loop ;
- $T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2}$ for a disjoint union $G_1 \sqcup G_2$and a one-point join $G_1 \cdot G_2$;
- $T_\bullet = 1$;
- $$T_G(x,y) \ := \sum_{F \subseteq E(G)} (x-1)^{k(F)-k(G)}(y-1)^{e(F)-v(F)+k(F)}$$

Specializations of $T_G(x, v)$.

- $\chi_G(x) = (-1)^{|V(G)|}(-x)^{k(G)} T_G(1-x, 0)$;
- $T_G(1,1) = \#$ of spanning trees of G ;
- $T_G(2,1) = \#$ of spanning forests of G ;
- $T_G(1,2) = \#$ of spanning connected subgraphs of G ;
- $T_G(2,2) = 2^{|E(G)|} = \#$ of spanning subgraphs of G ;
- *Flow polynomial*:
  $F_G(y) = (-1)^{|E(G)E+|V(G)|+k(G)} T_G(0, 1-y)$;
- For planar G: $\boxed{T_G(x,y) = T_{G^*}(y,x)}$

Cayley's and Kalai's formulas for # of spanning trees.

A.Cayley, 1889 (C.Borchardt, 1860): # of spanning trees of K_n $= n^{n-2}$.

G.Kalai, 1983: # of j dimensional spanning trees of an $(n-1)$ dimensional simplex $= n^{\binom{n-2}{j}}$

Cellular spanning trees.

K finite cell (CW) complex of dimension k.
$K_{(j)}$ j-skeleton of K.
Spanning subcomplex S of dimension j: $K_{(j-1)} \subseteq S \subseteq K_{(j)}$.
$\mathcal{S}_j$ set of all spanning subcomplexes of dimension j.
$f_j(S)$ # of j-cells of S.
$\widetilde{\beta}_j(S)$ reduced j-th Betti number $= rank(\widetilde{H}_j(S;\mathbb{Z}))$.

Definition. A j-dimensional *Cellular Spanning Tree* (j-CST) S of K is a j-dimensional spanning subcomplex such that:

$$\widetilde{H}_j(S) = 0, \qquad \widetilde{\beta}_{j-1}(S) = 0, \ (\ |\widetilde{H}_{j-1}(S)| < \infty \ ).$$

$\mathcal{T}_j(K)$ set of all j-CST's of K.

$$\widetilde{\tau}_j(K) := \sum_{S \in \mathcal{T}_j(K)} |\widetilde{H}_{j-1}(S)|^2$$

Kalai's formula.

Kalai's theorem (1983). *If K is a simplex with n vertices, $k = n - 1$, then*

$$\sum_{S \in \mathcal{T}_j(K)} |\widetilde{H}_{j-1}(S)|^2 = \widetilde{\tau}_j(K) = n^{\binom{n-2}{j}} .$$

Example. $n = 6, j = 2$.
46608 contractible 2-CST's;   12 homeomorphic to $\mathbb{R}P^2$.
$H_1(\mathbb{R}P^2) = \mathbb{Z}_2 \Longrightarrow 46608 + 12 * 4 = 46656 = 6^6$.

Cellular matrix-tree theorems, A. Duval, C. Klivans, J. Martin, (2009)

Bott polynomial.

R.Bott, 1952: $$R_K(\lambda) := \sum_{S\in\mathcal{S}_k} (-1)^{f_k(K)-f_k(S)} \lambda^{\beta_k(S)} \ .$$

Z.Wang, 1994: For $k = 1$, the Bott polynomial is essentially the flow polynomial of the graph K.

Krushkal-Renardy polynomial.

V.Krushkal, D.Renardy, 2010: For $1 \leqslant j \leqslant k$,

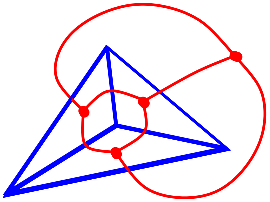
$$T^j_K(x,y) := \sum_{S\in\mathcal{S}_k} x^{\beta_{j-1}(S)-\beta_{j-1}(K)} y^{\beta_j(S)} .$$

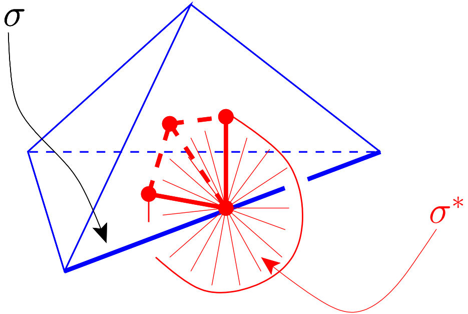
- $T^1_K(x,y) = T_{K_{(1)}}(x+1, y+1)$.
- For dual cellulations K and K^* of the sphere S^k,

$$\boxed{T^j_K(x,y) = T^{k-j}_{K^*}(y,x)}$$

Dual cellulations.

$$\{j-\text{cells of } K\} \quad \longleftrightarrow \quad \{(k-j)-\text{cells of } K^*\}$$
$$\sigma \quad \leftrightarrow \quad \sigma^*$$





Modified Krushkal-Renardy polynomial.

C.Bajo, B.Burdick, S.Ch., 2014:

$$\widetilde{T}^j_K(x,y) := \sum_{S\in\mathcal{S}_k} |\mathrm{tor}(H_{j-1}S)|^2\, x^{\beta_{j-1}(S)-\beta_{j-1}(K)} y^{\beta_j(S)}\ .$$

- If $\widetilde{\beta}_j(K)=0$, $1\leqslant j<k$, then $\widetilde{T}^j_K(0,0)=\widetilde{\tau}_j(K)$.
- For dual cellulations K and K^* of the sphere S^k,

$$\boxed{\widetilde{T}^j_K(x,y)=\widetilde{T}^{k-j}_{K^*}(y,x)}$$

- $R_K(\lambda)=(-1)^{\beta_k(K)}T^k_K(-1,-\lambda)\ .$

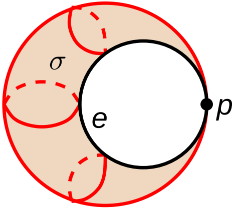
Special cells.

Let $\sigma \in K$ be a k-cell, $\overline{\sigma}$ be its closure in K, and $\partial\sigma := \overline{\sigma} - \sigma$ be its boundary.
$\overline{\sigma}$, $\partial\sigma$, $K - \sigma$, and $K/\overline{\sigma}$ inherit the cellular structure from K.

Definition.

- σ is a *loop* in K if $H_k(\overline{\sigma}) \cong \mathbb{Z}$;
- σ is a *bridge* in K if $\beta_{k-1}(K - \sigma) = \beta_{k-1}(K) + 1$;
- σ is *boundary regular* if $\widetilde{H}_{k-1}(\partial\sigma) \cong \mathbb{Z}$.

Example.

$K =$ 

dim	0	1	2
cell	p	e	σ

$$K \sim S^2 \vee S^1$$

$$H_0(K) = H_1(K) = H_2(K) = \mathbb{Z}$$

$\overline{\sigma} = K \Longrightarrow \sigma$ is a loop.
$\partial\sigma = e \cup p = S^1$. So $H_1(\partial\sigma) = \mathbb{Z}$, and σ is boundary regular.
$T_K^2(x, y) = 1 + y$.

Contraction — Deletion relations.

(i) *If σ is neither a bridge nor a loop and is boundary regular, then*

$$T^k_K(X,Y) = T^k_{K/\overline{\sigma}}(X,Y) + T^k_{K-\sigma}(X,Y)\ .$$

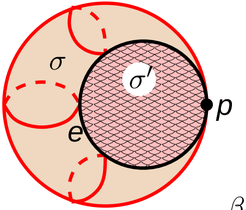
(ii) *If σ is a loop, then*

$$T^k_K(X,Y) = (Y+1)T^k_{K-\sigma}(X,Y)\ .$$

(iii) *If σ is a bridge and boundary regular, then*

$$T^k_K(X,Y) = (X+1)T^k_{K/\overline{\sigma}}(X,Y)\ .$$

Example.

$K =$ 

dim	0	1	2
cell	p	e	σ, σ'

$$K \sim S^2$$

$$H_0(K) = H_2(K) = \mathbb{Z}, \quad H_1(K) = 0$$

$$\beta_1(K - \sigma') = 1 \Longrightarrow \sigma' \text{ is a bridge.}$$

$\partial\sigma' = e \cup p = S^1$. So $H_1(\partial\sigma') = \mathbb{Z}$, and σ' is boundary regular.
$T^2_K(x, y) = (x + 1)T^2_{K/\overline{\sigma}'}(x, y)$. $\qquad K/\overline{\sigma}' = S^2$.
$T^2_{K/\overline{\sigma}'}(x, y) = y + 1$.

$$T^2_K(x, y) = (x + 1)(y + 1) = xy + x + y + 1$$

THANK YOU!