On the Gross-Mansour-Tucker conjecture

Sergei Chmutov

Ohio State University, Mansfield

Joint work with Fabien Vignes-Tourneret

arXiv:2101.09319v1 [math.CO]

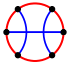
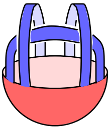
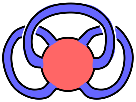
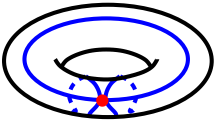
8th European Congress of Mathematics
Minisymposium: Graphs, Polynomials, Surfaces, and Knots (MS - ID 49)

Monday, June 21, 2021

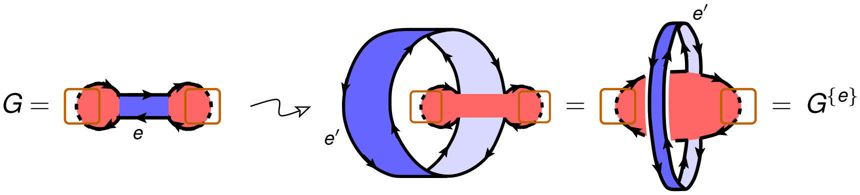
Definition. A ***ribbon graph*** G is a surface with boundary represented as the union of two sets of closed topological discs called ***vertex-discs*** $V(G)$ and ***edge-ribbons*** $E(G)$, satisfying the following conditions:

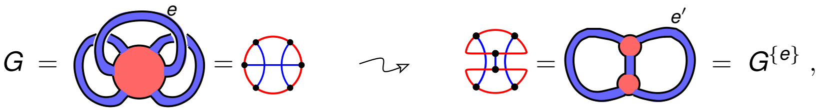
- the vertex-discs and edge-ribbons intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex-disc and precisely one edge-ribbon;
- every edge-ribbon contains exactly two such line segments.

Example.

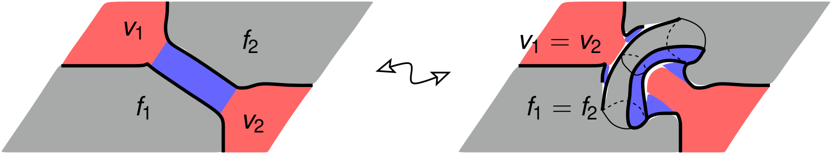
 $\rightsquigarrow$  $=$  $= G, \quad \Sigma_G =$ 

Partial duality of ribbon graphs.





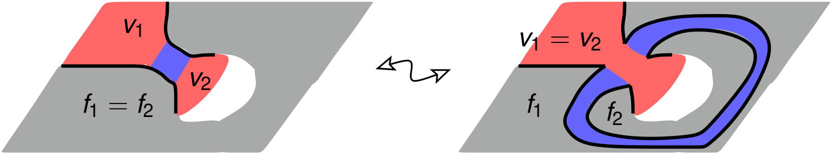
Type *pp* | Type *uu*



$v_1 \neq v_2, f_1 \neq f_2$ | $v_1 = v_2, f_1 = f_2$

$$g \longleftrightarrow g+1$$

Type *pu* | Type *up*



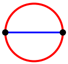
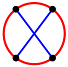
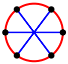
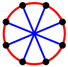
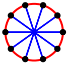
$v_1 \neq v_2, f_1 = f_2$ | $v_1 = v_2, f_1 \neq f_2$

$$g \longleftrightarrow g$$

J. L. Gross, T. Mansour, T. W. Tucker, *Partial duality for ribbon graphs, I: Distributions*, European Journal of Combinatorics **86** (2020) 103084, 1–20.

GMT-conjecture. *For any ribbon graph there is a subset of edges partial duality relative to which changes the genus.*

The ***ribbon-join*** (I. Moffatt) $G_1 \vee G_2$ is obtained by gluing together a vertex-disc of G_1 and a vertex-disc of G_2 along some arcs on their boundaries.

$B_1 =$ , $B_2 =$ , $B_3 =$ , $B_4 =$ , $B_5 =$ , $\ldots$

Q. Yan, X. Jin, *Counterexamples to a conjecture by Gross, Mansour, and Tucker on partial-dual genus polynomials of ribbon graphs*, European Journal of Combinatorics **93** (2021) 103285. Paper ID 103285. Preprint `arXiv:2004.12564v1 [math.CO]` 27 Apr 2020.

Theorem. *The genus of any partial dual to B_{2n+1} is equal to n.*

Definition. A connected ribbon graph G ***join-prime*** if it cannot be represented as the ribbon-join of two graphs G_1, G_2 with at least one edge-ribbon each: $G \neq G_1 \vee G_2$.

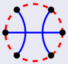
—, F. Vignes-Tourneret, *On a conjecture of Gross, Mansour and Tucker*, Preprint `arXiv:2101.09319v1 [math.CO]` 22 Jan 2021. To apper in European Journal of Combinatorics.

Theorem. *For any join-prime ribbon graph different from partial duals of B_{2n+1}, there are partial duals of different genus.*

Lemma. *Let G be a one-vertex join-prime ribbon graph and $e \in E(G)$. Suppose that the genus of partial duals of G stay the same, $g(G) = g(G^A)$ for all subsets $A \subseteq E(G)$. Then*

1. *e is attached to different face-discs $f_1 \neq f_2$. That is e has to be of Type up.*
2. *Any edge-ribbon interlaced with e is attached to the same face-discs f_1 and f_2.*
3. *Any edge-ribbon not interlaced with e is attached to a pair of face-discs different from $\{f_1, f_2\}$.*

Proof.



□

The non-orientable counterpart of the GMT conjecture.

- Maya Thompson (Royal Holloway University of London).
- Q. Yan, X. Jin, *Partial-dual genus polynomials and signed intersection graphs*, Preprint `arXiv:2102.01823v1 [math.CO]` 3 Feb 2021.

The only non-orientable join-prime ribbon graph whose partial duals have the same Euler genus is the one-vertex ribbon graph with one twisted edge.

THANK YOU!!!