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A Tutte Polynomial for Coloured Graphs

BÉLA BOLLOBÁS[1,2] and OLIVER RIORDAN[1]

[1] Department of Mathematical Sciences,
University of Memphis, Memphis TN 38152, USA
(e-mail: bollobas@msci.memphis.edu)

[1] Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England
(e-mail: B.Bollobas@dpmms.cam.ac.uk)

[2] Institute for Advanced Study,
Olden Lane, Princeton NJ 08540, USA

We define a polynomial W on graphs with colours on the edges, by generalizing the spanning tree expansion of the Tutte polynomial as far as possible: we give necessary and sufficient conditions on the edge weights for this expansion not to depend on the order used. We give a contraction-deletion formula for W analogous to that for the Tutte polynomial, and show that any coloured graph invariant satisfying such a formula can be obtained from W. In particular, we show that generalizations of the Tutte polynomial obtained from its rank generating function formulation, or from a random cluster model, can be obtained from W. Finally, we find the most general conditions under which W gives rise to a link invariant, and give as examples the one-variable Jones polynomial, and an invariant taking values in $\mathbb{Z}/22\mathbb{Z}$.

1. Introduction

1.1. Basic definitions

Throughout this paper, we shall consider the set $\mathcal{G}$ of finite multigraphs, with loops allowed. Usually, we shall call an element of $\mathcal{G}$ a *graph*, but sometimes we shall write *multigraph* for emphasis.

The *Tutte polynomial*, or *dichromate* of [34], is an isomorphism-invariant function $T : \mathcal{G} \longrightarrow \mathbb{Z}[x, y]$, which arises in many different ways. We shall consider four different definitions of the Tutte polynomial.

The first definition, due to Tutte [34], is based on a *spanning tree expansion*. We take an order ϕ on $E(G)$, and, for each spanning tree S of G, use ϕ and S to classify the edges of G into four types. We then assign a weight to each edge, depending on its type, and multiply these weights to find the weight of S. Finally, the Tutte polynomial of G is obtained by summing over all spanning trees S of G. This definition is described precisely in Section 2. Tutte proved the rather surprising result that the polynomial obtained is independent of the order ϕ, and hence defines a graph invariant.

The second definition we shall consider is in terms of contraction-deletion formulae. We shall say that an edge e of G is *ordinary* if e is neither a bridge nor a loop. The Tutte polynomial $T(G; x, y)$ can then be defined by the recurrence relations

$$T(G;x,y) = \begin{cases} xT(G/e;x,y) & \text{if } e \text{ is a bridge,} \\ yT(G-e;x,y) & \text{if } e \text{ is a loop,} \\ T(G-e;x,y) + T(G/e;x,y) & \text{if } e \text{ is ordinary,} \end{cases} \tag{1.1}$$

together with the condition

$$T(E_n) = 1, \tag{1.2}$$

where E_n is the graph with n vertices and no edges. Note that some work is required to show that these conditions do have a solution. One approach is to take an alternative definition of T, for instance via spanning trees, and show that it satisfies (1.1). We give a more direct approach in Section 2.

The third definition we consider is essentially equivalent to the second. We shall call a graph G *basic* if G has no ordinary edges. We can define T by the single relation

$$T(G;x,y) = T(G-e;x,y) + T(G/e;x,y), \tag{1.3}$$

for every ordinary edge e of a graph G, together with the boundary condition

$$T(G;x,y) = x^b y^l, \tag{1.4}$$

for any basic graph G consisting of b bridges and l loops.

The fourth and final definition we shall consider is via the *Whitney–Tutte dichromatic polynomial* [33],

$$Q(G;t,z) = \sum_{H \subseteq G} t^{k(H)} z^{n(H)}, \tag{1.5}$$

where the sum is over all spanning subgraphs H of G, $k(H)$ is the number of components of H, and $n(H)$ is the nullity of H. The Tutte polynomial is then given by

$$T(G;x,y) = (x-1)^{-k(G)} Q(G; x-1, y-1).$$

We shall discuss the relationships between the above definitions in Section 2. In this paper we consider generalizations of the Tutte polynomial to coloured graphs. To be specific, by a *coloured graph* we mean a pair (G, c), where G is a multigraph, and c is a function from $E(G)$ to an arbitrary set Λ of colours.

1.2. Earlier work

Before turning to coloured graphs, it is natural to ask whether the relations (1.1) can be generalized on uncoloured graphs.

In fact, a slight extension of a result of Oxley and Welsh [22] shows that there is a unique map $U : \mathcal{G} \to \mathbb{Z}[x, y, \alpha, \sigma, \tau]$ such that

$$U(E_n) = U(E_n; x, y, \alpha, \sigma, \tau) = \alpha^n$$

for every $n \geqslant 1$, and for every $e \in E(G)$ we have

$$U(G) = \begin{cases} xU(G/e) & \text{if } e \text{ is a bridge,} \\ yU(G-e) & \text{if } e \text{ is a loop,} \\ \sigma U(G-e) + \tau U(G/e) & \text{if } e \text{ is ordinary.} \end{cases} \tag{1.6}$$

Furthermore,

$$U(G) = \alpha^{k(G)} \sigma^{n(G)} \tau^{r(G)} T(G; x/\tau, y/\sigma). \tag{1.7}$$

This answers the question of how far the recurrence relations (1.1) can be extended on uncoloured graphs by introducing new coefficients. This result may be perceived as somewhat negative, in that the function obtained is no more general than the Tutte polynomial. However, the result is still important for two reasons. The first is that it shows that this approach does not give rise to a more general invariant. The second is that it illustrates the generality and importance of the Tutte polynomial, showing that any graph invariant satisfying contraction-deletion relations of this form may be read out of the Tutte polynomial. Particular examples are the chromatic polynomial, the flow polynomial, the number of spanning trees or forests, and the number of connected subgraphs. Many more are given by Welsh [36]. This fact illustrates why the Tutte polynomial is such an important graph invariant.

Returning to coloured graphs, the invariants we are interested in are those which are like the Tutte polynomial in one of two senses. Either they should obey recurrence relations analogous to (1.6), but with each of the variables x, y, σ, τ replaced by one variable for each colour, or they should have a spanning tree expansion analogous to that of the Tutte polynomial. In fact, we shall see in Section 3.1 that these conditions are equivalent.

The first such invariants were defined for *weighted graphs*, where the set Λ of colours is just the set of real numbers. The most direct way to obtain such invariants is to take the rank generating formulation (1.5) of the Tutte polynomial, and add coefficients depending on the weights. For example, for a graph G with weights $w(e) \in \mathbb{R}$ on the edges, Traldi [32] defined the dichromatic polynomial $Q(G; t, z)$ to be

$$Q(G; t, z) = \sum_{H \subseteq G} \left(\prod_{e \in E(G)} w(e) \right) t^{k(H)} z^{n(H)}. \tag{1.8}$$

Traldi showed that this polynomial obeys recurrence relations as above, with the coefficients certain functions of the weights, t and z, and that the polynomial has a spanning tree expansion. In fact, as remarked at the end of [32], this polynomial can be considered as a special case of the random cluster model of Fortuin and Kasteleyn [7], which we describe precisely in Section 4. Again, this comes from an explicit definition, similar to (1.8), which can be thought of in terms of the number of components of a random subgraph of G

The next group of coloured Tutte polynomials were defined for signed graphs, where $\Lambda = \{+, -\}$, motivated by the close connection between signed plane graphs and link diagrams. These started from the very simple description of the Jones polynomial given by Kauffman [13], in terms of state models. This was translated by Thistlethwaite [29] to a single variable signed graph polynomial defined by a spanning tree expansion, and extended to a three variable polynomial by Kauffman [14]. The latter polynomial specializes to the Tutte polynomial on graphs with constant sign. On signed plane graphs, considered as link diagrams, it is exactly the original Kauffman state model, before the conditions guaranteeing Reidemeister invariance are imposed. Kauffman showed that this polynomial satisfies recurrence relations as described above, with the same coefficients as in the spanning tree expansion. Motivated by the connection with link diagrams, the coefficients for contracting a positive edge and deleting a negative edge, for example, were taken to be the same, since these operations on graphs correspond to the same operation on link diagrams.

Also for signed graphs, Murasugi [20] used a rank generating function formulation to define an invariant that is essentially the special case of Traldi's dichromatic polynomial given by taking the weights of positive and negative edges to be x and x^{-1} respectively.

In the papers mentioned above, general polynomials satisfying certain conditions were produced. However, no attempt was made to find the *most* general invariant satisfying these conditions, or to prove that any such invariant must be of a certain form. First Zaslavsky [40] and then Schwärzler and Welsh [26] set themselves exactly such tasks, for very different conditions. Zaslavsky considered invariants satisfying recurrence relations similar to (1.6), with arbitrary coefficients in an arbitrary field K. In fact, the form of the relations he considered is closer to the third definition of the Tutte polynomial, from (1.3) and (1.4), but this makes no difference, as explained in the next section. Zaslavsky also considered matroids rather than graphs, but, as we shall see, this too makes very little difference. He showed that all such invariants are of one of seven types, and described exactly the possible coefficients. He also showed that such invariants have spanning tree expansions.

For signed graphs, Schwärzler and Welsh [26] started from relations with arbitrary coefficients, this time in a ring. However, in their quest for the most general invariant satisfying such relations they make some unstated assumptions. As a result, their description of the most general invariant (Proposition 2.1 in [26]) is incorrect. In particular, it contradicts Theorem 6; for more detail see Remark 5 following this result. Nevertheless, Schwärzler and Welsh obtain an invariant that includes both Kauffman's Tutte polynomial for signed graphs, and Murasugi's polynomial. They also obtain the Jones polynomial by making appropriate substitutions and then checking Reidemeister invariance.

1.3. Summary of results

In this paper we consider two very precise questions, asking for the most general coloured graph invariant satisfying certain conditions. Such questions are important even if the answers turn out to give nothing new, since we then know that a certain approach goes so far, and no further. In fact, we do obtain a new invariant W, which, as an automatic consequence of our approach, includes the previous generalizations as special cases.

The first condition we consider is that of having a spanning tree expansion. We start by taking independent variables for each edge type (defined with respect to an order ϕ and a spanning tree T) and colour, obtaining a function $W(G, c, \phi)$ depending both on the coloured graph (G, c) and on the order ϕ on $E(G)$. We then determine the minimum conditions that must be imposed on these variables to make this invariant independent of the order ϕ. More precisely, we define an order-independent invariant $W(G, c)$ taking values in a quotient ring, such that any other invariant with a spanning tree expansion can be obtained by composing $W(G, c)$ with a suitable ring homomorphism. This invariant retains many properties of the Tutte polynomial. In particular, on graphs whose edges are all the same colour, it essentially coincides with the Tutte polynomial.

The second question we consider is that of satisfying recurrence relations analogous to (1.6), but with different coefficients for edges of different colours. Answering this question will give a coloured graph invariant with an analogous property to that of the function U described above – it will include *any* coloured graph invariant satisfying recurrence relations of this form.

Somewhat surprisingly, it turns out that we already have the invariant we are looking for – it is the same function W as above. Of course, we must do some work to show this. We show on the one hand that W obeys relations of this form, with certain constraints on the variables. On the other hand, we show that these constraints are necessary, in the sense that, unless they are satisfied, there is no invariant satisfying the recurrence relations. This shows that W is the universal invariant we are seeking, and may be used to calculate any coloured graph invariant satisfying recurrence relations of this form, no matter what the coefficients. This includes all the previous generalizations of the Tutte polynomial to coloured, weighted or signed graphs mentioned above.

At this point we note that, like the Tutte polynomial, the invariant W depends only on the cycle matroid of a graph G, and that W can be extended to all matroids. This is because W is determined by the sets of edges that include a cycle, and we only use properties of such sets which hold for general matroids. Having extended W to matroids, we can compare our results with those of Zaslavsky [40]; by not imposing the restriction that the coefficients lie in a field, we obtain one universal invariant, from which the seven classes of invariant described by Zaslavsky can be obtained. This demonstrates the advantages of our unrestricted approach.

As an application, we use W to find the most general link invariant that can be defined via a spanning tree expansion on signed plane graphs. Again, our aim is to make no unnecessary assumptions. Thus, even if we do not obtain a new invariant, we shall have proved that what we do get is the most general link invariant satisfying these conditions. To do this we must first describe exactly the relationship between link diagrams and signed plane graphs, and the exact graph equivalents of Reidemeister moves. This is simple in principle, but there are some complications concerning diagrams with more than one component, which we have not seen described in print. These complications cannot be avoided, as they arise naturally in the expansion considered. Using this correspondence, we obtain an invariant taking values in a certain quotient of $\mathbb{Z}[A, B, d]$, which, since the Jones polynomial has a spanning tree expansion [29], can be used to calculate the Jones polynomial. There seems no reason why this invariant should not distinguish some pairs of links with the same Jones polynomial, but, for the few pairs we have tried so far, it does not.

We remark that all the problems considered here have a property that seems to arise frequently when finding necessary and sufficient conditions for something. This is that, in all cases, the stated conditions are trivially necessary, and what is surprising, and requires some work to prove, is that they are also sufficient. Of course, it is easy to find necessary conditions. The key is to find the *right* conditions, so that they are not too numerous, but they are also sufficient.

The rest of the paper is organized as follows. In Section 2, we give some background concerning the Tutte polynomial, relating the various different definitions considered above. In Section 3, we define a polynomial W on coloured graphs using a spanning tree expansion, keeping the maximum generality allowed by independence of the order used. We then consider recurrence relations similar to (1.6), and show that the most general solution is given by the same function W. In Section 4, we consider the polynomial Z defined by Fortuin and Kasteleyn [7] by considering the number of components of a random subgraph of a graph G. Schwärzler and Welsh [26] showed that their invariant, defined for graphs with only two colours on the edges, can be used to recover a special case of the polynomial Z. Here we show that from W we can recover Z in its full generality. In Section 5, we consider a generalization of the rank generating function, given by Traldi [32], and show that it can also be obtained from W.

In Section 6, we turn to links, establishing the most general conditions under which W is well defined on link diagrams. We give two specific examples of link invariants which can be obtained from W: the Jones polynomial, and a function taking values in $\mathbb{Z}/22\mathbb{Z}$. We also show that the signed graph polynomial defined by Kauffman [14] from his bracket is less general than W restricted to signed graphs.

At this point we shall start again from the beginning, describing the original development of the Tutte polynomial, and the relationships between its various different definitions.

2. The Tutte polynomial

In 1912 Birkhoff [3] proved a determinant formula for the number $p_G(\lambda)$ of (proper) vertex colourings of a graph G with λ distinguishable colours, $1, 2, \ldots, \lambda$, say. He proved also that if G has m_{ij} spanning subgraphs of rank i and nullity j then

$$p_G(\lambda) = \sum_{i,j} (-1)^{i+j} m_{ij} \lambda^{|G|-i}. \tag{2.1}$$

Although our terminology and notation are standard, let us point out that we write $|G|$ for the order of G, $e(G)$ for its size and $k(G)$ for the number of its components; the *rank* of G is $r(G) = |G| - k(G)$, and its *nullity* is $n(G) = e(G) - |G| + k(G) = e(G) - r(G)$. Thus, in particular, $\sum_{i+j=\ell} m_{ij} = \binom{e(G)}{\ell}$.

The investigation of the coefficients m_{ij} was continued by Whitney in his thesis at Harvard, the results of which were published in several papers, including [38] and [39]. (In [38] Whitney rediscovered (2.1) as well.) Let

$$m_i = \sum_j (-1)^j m_{ij},$$

so that

$$p_G(\lambda) = \sum_i (-1)^i m_i \lambda^{|G|-i}.$$

Whitney [39] proved that m_i is the number of i-sets of edges containing no broken cycles. A *broken cycle* is defined in terms of a total (linear) order on $E = E(G)$: it is the edge set of a cycle from which the last edge has been deleted. Note that, *a priori*, m_i depends on which order we choose on E, but the result implies that it is, in fact, independent of the order.

Let $G_1, G_2, \ldots$ be an enumeration of all (isomorphism classes of finite) nonseparable graphs. Another result of Whitney is that $m_{ij} = m_{ij}(G)$ is a polynomial in the numbers $N_1, N_2, \ldots,$ where N_h is the number of subgraphs of G isomorphic to G_h.

Also,

$$m_{ij}(G_1 \cup G_2) = \sum_{p,q} m_{pq}(G_1) m_{i-p,j-q}(G_2).$$

Even more importantly, in [39] Whitney remarked that R. M. Foster used the following formula to compute $m_{ij}(G)$: if $e \in E(G)$, then

$$m_{ij}(G) = \begin{cases} m_{ij}(G-e) + m_{i-1,j}(G/e) & \text{if } e \text{ is not a loop,} \\ m_{ij}(G-e) + m_{i,j-1}(G-e) & \text{if } e \text{ is a loop.} \end{cases} \tag{2.2}$$

Here $G - e$ is the graph obtained from G by *deleting* e, and G/e is obtained from G by *contracting* e. Formula (2.2) is, perhaps, the first occurrence of a *contraction-deletion formula*; formulae of this type are of the utmost importance in the study of polynomials related to the Tutte polynomial.

The use of contraction-deletion formulae practically compels us to consider graphs with *loops and multiple edges*, and this is precisely what we shall do. As stated in the Introduction, we shall write $\mathcal{G}$ for the set of (isomorphism classes of finite) graphs with loops and multiple edges. We shall be interested in functions of graphs that are constant on isomorphism classes; that is, we wish to study functions defined on $\mathcal{G}$. Note that for $G \in \mathcal{G}$, $E(G/e)$ is naturally identified with $E(G-e) = E(G)\backslash\{e\}$.

With hindsight it is clear that the two-dimensional array $(m_{ij}(G))_{i,j}$ is best studied by taking the m_{ij} to be the coefficients of a polynomial in two variables, but this important step was taken only several years later, in 1947. Let $q_{k\ell} = q_{k\ell}(G)$ be the number of spanning subgraphs of G with k components and nullity ℓ. Then the *dichromatic polynomial* Q of the graph G can be written as

$$Q(G; t, z) = \sum_{k,\ell} q_{k\ell} t^k z^\ell,$$

so that Q maps $\mathcal{G}$ into $\mathbb{Z}[t, z]$. The polynomial Q is frequently called the *Whitney–Tutte dichromatic polynomial.* It makes little difference whether we use the coefficients $q_{k\ell}$ or m_{ij}, since we have $m_{ij}(G) = q_{|G|-i,j}(G)$. Thus, defining the *rank generating function* R of the graph G as

$$R(G; w, z) = \sum_{ij} m_{ij} w^i z^j, \tag{2.3}$$

we have $R(G; w, z) = w^{|G|} Q(G; w^{-1}, z)$, and $Q(G; t, z) = t^{|G|} R(G; t^{-1}, z)$. Using these relations, we see that (2.2) means exactly that

$$Q(G;t,z) = \begin{cases} Q(G-e;t,z) + Q(G/e;t,z) & \text{if } e \text{ is not a loop,} \\ (z+1)Q(G-e;t,z) & \text{if } e \text{ is a loop,} \end{cases} \tag{2.4}$$

and for $n = 1, 2, \ldots$ we also have

$$Q(E_n;t,z) = t^n, \tag{2.5}$$

where E_n is the empty graph of order n. It is immediate that the contraction-deletion formula (2.4) and the boundary conditions (2.5) determine Q. In 1947 Tutte [33] studied the dichromatic polynomial as an example of a W-*function*, a function W defined on $\mathcal{G}$, such that $W(G) = W(G-e) + W(G/e)$, whenever $e \in E(G)$ is not a loop.

As $m_{ij}(G) = q_{|G|-i,j}(G)$, Whitney's theorem given in (2.1) can be restated as follows:

$$p_G(\lambda) = \sum_{i,j}(-1)^{i+j} m_{ij}\lambda^{|G|-i} = \sum_{k,\ell}(-1)^{|G|+k+\ell} q_{k,\ell}\lambda^k = (-1)^{|G|}Q(G;-\lambda,-1). \tag{2.6}$$

This theorem of Whitney was extended by Tutte [33, 35] to a result concerning all vertex colourings of graphs, not only proper vertex colourings. Let $G = (V,E) \in \mathcal{G}$, $\lambda \in \mathbb{N}$, and set $F_\lambda = [\lambda]^V = \{1, 2, \ldots, \lambda\}^V$. Thus F_λ is the set of all vertex colourings of G with colours $1, 2, \ldots, \lambda$. For $f \in F_\lambda$ let $\phi(f)$ be the number of edges of G joining vertices of the same colour, so that f is a proper colouring if and only if $\phi(f) = 0$. Then

$$\sum_{f \in F_\lambda} x^{\phi(f)} = (x-1)^{|V|} Q(G; \frac{\lambda}{x-1}, x-1).$$

In a way reminiscent of Whitney's use of broken cycles, in 1954 Tutte [34] defined another polynomial whose definition depends *a priori* on an order we impose on the set of edges. Let $G = (V,E)$ be a connected graph (with loops and multiple edges). Let us endow E with a total order by choosing a $1-1$ map $\phi : E \to [|E|] = \{1, 2, \ldots, |E|\}$, and setting $x \prec_\phi y$, or simply $x \prec y$, if $\phi(x) < \phi(y)$. Given a spanning tree T of G (and our order given by ϕ), Tutte classified the edges of G as follows. An edge $e \in E(G)$ is *internally active* if it is the smallest edge between the two components of $T-e$. Also, an edge $f \in E(G) \backslash E(T)$ is *externally active* if it is the smallest edge in the unique cycle of $T \cup \{f\}$.

With these definitions, we are ready to introduce Tutte's *dichromate*, which is now always called the *Tutte polynomial* of G:

$$T(G;x,y) = \sum_{i,j} t_{ij} x^i y^j,$$

where t_{ij} is the number of spanning trees with precisely i internally active edges and precisely j externally active edges. Tutte proved the startling result that $T(G;x,y)$ is *independent of the order on E*.

It is natural to extend the domain of T to the entire set $\mathcal{G}$ by setting

$$T(G) = \prod_{i=1}^{k} T(G_i),$$

where $G_1, G_2, \ldots, G_k$ are the components of G.

Once we know that $T(G;x,y)$ is independent of the order, it is easy to show that T satisfies the contraction-deletion relations (1.1). The simplest case is when e is a loop. Then the spanning trees of G are exactly the spanning trees of $G-e$. As e itself forms a cycle, it is always externally active, contributing a factor of y. Also, deleting e does not affect the activity of any other edge f, as e can never lie in the cut or cycle determined by f. When e is a bridge the argument is similar, using the fact that the spanning trees of G are in bijection with those of G/e, where the bijection is given by contracting the edge e in each tree. Finally, when e is neither a bridge nor a loop, we make use of the order invariance, by taking e to be the last edge in the order ϕ. In this case e is never active. Also, the spanning trees of G containing e are in bijection with those of G/e, and the spanning trees of G not containing e are in bijection with those of $G-e$. Together with the fact that the presence of e cannot alter the activity of another edge f, as $f \prec e$, this shows that (1.1) is satisfied in this case as well.

Note that, when e is a bridge, we can replace the condition $T(G;x,y) = xT(G/e;x,y)$ by the condition $T(G;x,y) = xT(G-e;x,y)$. This is because the graphs G/e and $G-e$ have the same blocks, and the Tutte polynomial of a graph depends only on its blocks. We shall use the formulation with G/e, as it has the advantage of giving the Tutte polynomial of a connected graph in terms of the Tutte polynomials of smaller connected graphs.

It is trivial that $T(G;x,y)$ is determined by the recursion formula (1.1), together with the boundary condition (1.2) that $T(E_n;x,y) = 1$ for every n. As mentioned in the previous section, an alternative approach is to use only the relation (1.3) for ordinary edges, and the boundary condition (1.4). It is easy to see that these two approaches are equivalent. On the one hand, any solution to (1.1) and (1.2) clearly satisfies (1.3) and (1.4). On the other hand, suppose T' is a solution to (1.3) satisfying (1.4), so T' trivially satisfies (1.2), and let G be a graph with a bridge e. Then G and G/e have the same ordinary edges, as do the graphs obtained from them by applying the same sequence of contractions and deletions to each. We may thus apply the same sequence of reductions to G and G/e, using (1.3) to express $T'(G)$ as $\sum T'(B_i)$, for some sequence $(B_i)_1^k$ of basic graphs, and $T'(G/e)$ as $\sum T'(B_i/e)$, for the same sequence $(B_i)_1^k$. Since the boundary condition (1.4) ensures that $T(B_i) = xT(B_i/e)$, we thus have that $T'(G) = xT'(G/e)$. A similar argument shows that $T'(G) = yT'(G-e)$ when e is a loop, and thus that T' satisfies (1.1).

Tutte's dichromatic polynomial Q and dichromate (Tutte polynomial) T are related in a very simple way:

$$Q(G;t,z) = T(G;t+1,z+1)t^{k(G)}, \tag{2.7}$$

as can be seen from the recurrence relations obeyed by each, or from (1.7). In particular, (2.6) and (2.7) imply that

$$p_G(\lambda) = (-1)^{r(G)}\lambda^{k(G)}T(G;1-\lambda,0). \tag{2.8}$$

The dichromatic polynomial and the dichromate are only two prominent members of a family of polynomials defined in various natural ways which can be obtained from each other by simple substitutions. In fact, taking the standard definitions of rank, nullity, deletion and contraction for *matroids*, the Tutte polynomial is easily extended to suitable classes of matroids. Going a little further, Brylawski and Oxley [4] define a

Tutte–Grothendieck invariant for a class $\mathscr{K}$ of matroids closed under isomorphism and taking minors, as a function f from $\mathscr{K}$ into a ring R such that, if $M \in \mathscr{K}$ and $e \in M$, then

$$f(M) = f(M - e) + f(M/e)$$

if e is neither a loop nor an isthmus (bridge), and

$$f(M) = f(M(e))f(M - e)$$

otherwise, where $M(e)$ is the submatroid on $\{e\}$. Brylawski and Oxley show that every Tutte–Grothendieck invariant is essentially an evaluation of the Tutte polynomial.

One frequently encounters polynomials that satisfy contraction-deletion formulae with coefficients other than 1: for example, the chromatic polynomial, or the polynomials f and f^* defined by Negami [21]. Since these polynomials are special cases of the polynomial U mentioned in the previous section, they are also simple transforms of the Tutte polynomial. Note that, in these cases, as for the Tutte polynomial, we could use $G - e$ instead of G/e when e is a bridge. For U we would get the same polynomial but with a change of variables, as the map U satisfies $U(G) = x\alpha^{-1}U(G - e)$ when e is a bridge.

Before proceeding further, let us show from first principles, without any reference to other polynomials or counting functions, that the contraction-deletion formula (1.1) and the boundary condition (1.2) do define a graph polynomial $T(G)$. The proof below is very simple, but as we spell it out in some detail, it is not that short. A similar argument could be used to prove the existence of $U(G)$.

Theorem 1. *There is a unique graph polynomial $T(G) = T(G; x, y)$ such that the contraction-deletion formula (1.1) holds and $T(E_n) = 1$ for every n.*

Proof. As the uniqueness of $T(G)$ is immediate, we have to show only its existence. We can view (1.1), together with the condition $T(E_n) = 1$, as a procedure for calculating $T(G)$: apply (1.1) to any edge $e \in E(G)$, then to any edge of each non-empty graph in the resulting expression, and so on, until $T(G)$ has been expressed in terms of x and y only. At first sight, it appears that for a fixed graph G this procedure may give many different results, depending on the edges chosen at each stage. In fact, we shall prove by induction on $e(G)$ that all these possible results are the same. Thus this procedure defines a graph polynomial $T(G)$, which satisfies the conditions of the theorem.

For $e(G) \leqslant 1$ the assertion is trivial, so assume that $e(G) \geqslant 2$, and that $T(G')$ is well defined for every graph G' with $e(G') < e(G)$. For $e \in E(G)$, let $T_e(G) = T_e(G; x, y)$ be given by (1.1); all we need to show is that $T_e(G)$ is independent of e, *i.e.*, that

$$T_e(G) = T_f(G)$$

for any distinct edges $e, f \in E(G)$.

Let us write $T_{e,f}(G)$ for the expression obtained for $T(G)$ by first applying (1.1) to $e \in E(G)$ and then to $f \in E(G-e)$ or $f \in E(G/e)$. For example, if e is a bridge of G and f is a loop then $T_{e,f}(G) = xyT(G/e - f)$, if e and f are bridges then $T_{e,f}(G) = x^2T(G/e/f)$, if e and f are loops then $T_{e,f}(G) = y^2T(G - e - f)$, if e is a bridge and f is an ordinary edge of G/e (neither a bridge, nor a loop) then $T_{e,f}(G) = xT(G/e-f) + xT(G/e/f)$, and so on. Let $T_{f,e}(G)$ be defined similarly.

As, by our induction hypothesis, $T_e(G) = T_{e,f}(G)$ and $T_f(G) = T_{f,e}(G)$, it suffices to check that $T_{e,f}(G) = T_{f,e}(G)$. Now, this is clearly true if e and f are parallel (*i.e.*, have precisely the same end-vertices), or if the 'nature' of e is the same in G, G/f and $G-f$, and the 'nature' of f is the same in G, G/e and $G-e$. Here the 'nature' of an edge is whether it is a bridge, a loop or an ordinary edge. If e is a loop in G then it is also a loop in G/f and $G-f$, and if it is a bridge in G then it is also a bridge in G/f and $G-f$. Also, if e is an ordinary edge in G and f is not parallel to e, then e is an ordinary edge in G/f as well, and it is an ordinary edge in $G-f$ unless e is a bridge in $G-f$ (and so f is a bridge in $G-e$).

Hence, it suffices to check that $T_{e,f}(G) = T_{f,e}(G)$ in the case when G is obtained from a graph $G_1 \cup G_2$, with $V(G_1) \cap V(G_2) = \emptyset$, by adding to it edges e and f, both joining a component C_1 of G_1 to a component C_2 of G_2. Then

$$T_{e,f}(G) = T(G-e) + T(G/e) = xT(G-e/f) + T(G/e/f) + T(G/e-f),$$

and

$$T_{f,e}(G) = T(G-f) + T(G/f) = xT(G-f/e) + T(G/f/e) + T(G/f-e).$$

Since $G/e/f$ is the same graph as $G/f/e$, it suffices to show that

$$T(G/e-f) = T(G/f-e),$$

and

$$T(G-e/f) = T(G-f/e).$$

In fact, these two statements are equivalent, as the graphs involved are exactly the same. In particular, both $G/e-f = G-f/e$ and $G/f-e = G-e/f$ are of the form $G_1 \cup G_2$, with G_1 and G_2 sharing precisely one vertex (albeit different pairs of vertices in the two graphs). Now, the conditions on T imply easily that, if $H = H_1 \cup H_2$ with $|V(H_1) \cap V(H_2)| \leqslant 1$ and T is well defined on H, then $T(H) = T(H_1)T(H_2)$. From the induction hypothesis, T is well defined on the relevant graphs, so we have $T(G/e-f) = T(G_1)T(G_2) = T(G/f-e)$, completing the proof. □

The Tutte polynomial extends not only the chromatic polynomial, as shown by (2.8), but also the flow polynomial of a graph. Given a graph G and an additively written finite abelian group A, an *A-flow* on G is a flow on G with values in A that satisfies Kirchhoff's current law at each vertex. Writing $q_G(A)$ for the number of nowhere-zero A-flows on G, it turns out that $q_G(A)$ depends only on the cardinality of A, so we may define $q_G(|A|) = q_G(A)$. The function $q_G(\lambda)$ is a polynomial in λ, called the *flow polynomial* of G. The flow polynomial is essentially the Tutte polynomial evaluated at $x = 0$:

$$q_G(\lambda) = (-1)^{n(G)} T(G; 0, 1-\lambda).$$

As mentioned in the previous section, many specific evaluations of the Tutte polynomial have attractive interpretations in terms of various graph invariants. For example, if G is a connected graph, then $T(G;1,1)$ is the number of spanning trees, $T(G;2,1)$ is the number of forests, $T(G;1,2)$ is the number of connected spanning subgraphs, and $T(G;2,2)$ is the number of spanning subgraphs.

Although we shall not consider such questions here, we note that much work has been done concerning the complexity of evaluating the Tutte polynomial of a graph G. For example, Annan [2] considers the complexity of calculating the coefficients of $T(G;x,y)$, while Jaeger, Vertigan and Welsh [10] consider the complexity of evaluating $T(G;x,y)$ for given x, $y \in \mathbb{Q}$. In most cases, these calculations turn out to be #P-hard, but random algorithms for approximating the values have been described, by Alon, Frieze and Welsh [1], for example. For a survey of such results, see [37]. On a slightly different note, Sekine, Imai and Tani [27] give practical algorithms for calculating the Tutte polynomials of moderately sized graphs, including all graphs with at most 14 vertices and at most 91 edges.

It is difficult to overestimate the importance of the Tutte polynomial and its extensions, as they are fundamental in such diverse fields as graph theory, knot theory, percolation theory, coding theory and statistical mechanics. More often than not, one needs Tutte polynomials of graphs with additional structures and a variety of boundary conditions. As described in the Introduction, our main aim is to determine the most general forms of the Tutte polynomial under certain conditions.

Perhaps the two most important properties of the Tutte polynomial are the existence of a contraction-deletion formula and the existence of a spanning tree expansion for connected graphs. In addition, it also has an expansion in terms of the rank and nullity of its subgraphs. In generalizing the Tutte polynomial, one can make use of any of these properties. As the Tutte polynomial itself has a history of appearing in a variety of different guises, we shall investigate the relationships between these apparently different generalizations.

Given a set $\mathscr{G}_c$ of graphs with colours on the edges, what is the most general form of a map $f : \mathscr{G}_c \to R$, where R is a ring, such that f satisfies an appropriate contraction-deletion formula, and what is the most general form that has a spanning tree expansion for connected graphs? In the next section, we shall answer these questions for coloured graphs. In the subsequent sections, we shall study the connection with other generalizations of the Tutte polynomial, and with polynomials arising in knot theory.

3. A Tutte polynomial for coloured graphs

In this section we shall consider functions having a spanning tree expansion, or satisfying recurrence relations, with coefficients that are different for different edges e of a graph G. There are two possible approaches. The first is to take the coefficients as functions of the actual edge e, obtaining a graph function that is not an isomorphism invariant. The second approach, which seems more natural, is to consider graphs with colours on the edges, and to take the coefficients as functions of the colour $c(e)$ of the edge e. In this way we shall obtain functions of coloured graphs invariant under those graph isomorphisms which map each edge to an edge of the same colour.

Most of the time it will make little difference which approach we consider, and we shall phrase all our arguments in terms of the second approach. When there is a significant difference, we shall point this out.

3.1. Spanning tree expansions

For the rest of this paper, by a *coloured graph* we mean a graph $G = (V, E)$ together with a function $c : E \to \Lambda$, where Λ is a set, the *set of colours*. If $c(e) = \lambda \in \Lambda$ then the edge e has *colour* λ. We shall write $\mathcal{G}_c$ for the set of all such pairs (G, c).

As in Section 2, we endow E with a *total order* by taking a bijection ϕ from E to $\{1, \dots |E|\}$. For $e, f \in E$, we write $e \prec_\phi f$, or simply $e \prec f$, if $\phi(e) < \phi(f)$. Let $\mathcal{G}_{c,o}$ be the set of all triples (G, c, ϕ), where (G, c) is a coloured graph, and ϕ is an order on $E(G)$.

In order to ensure that our graphs have spanning trees rather than only spanning forests, throughout this section we shall restrict our attention to the sets $\mathcal{G}_c^*$ and $\mathcal{G}_{c,o}^*$ of coloured graphs (with an order) whose underlying graph G is connected. Furthermore, for notational simplicity, we shall identify spanning subgraphs of G with their edge sets.

Let $T \subseteq E$ be a spanning tree of G. For each edge $e \in T$, by the *cut* of $T - e$ we mean the set of edges of G going between the two components of $T - e$, and we denote it by $\mathrm{cut}(T - e)$. Also, if $e \in E - T$ then we write $\mathrm{cyc}(T \cup e)$ for the unique cycle in $T \cup e$, and we call it the *cycle* of $T \cup e$.

We say that an edge $e \in T$ is *internally active* (with respect to T) if it is the first edge in $\mathrm{cut}(T - e)$, in the order ϕ. Otherwise, $e \in T$ is *internally inactive*. Also, $e \in E - T$ is *externally active* if e is the first edge in $\mathrm{cyc}(T \cup e)$, again in the order ϕ, and *externally inactive* otherwise.

In this section we shall use the notions of internal and external activity to define a coloured Tutte polynomial $W(G, c, \phi)$ in as general a way as possible. We shall then establish the least restrictive conditions under which this polynomial is independent of the order ϕ.

We start by defining the *weight* $w(G, c, \phi, T, e)$ of an edge e with respect to a spanning tree T as follows. If e has colour λ, then

$$w(G, c, \phi, T, e) = \begin{cases} X_\lambda & \text{if } e \text{ is internally active,} \\ Y_\lambda & \text{if } e \text{ is externally active,} \\ x_\lambda & \text{if } e \text{ is internally inactive,} \\ y_\lambda & \text{if } e \text{ is externally inactive.} \end{cases}$$

Initially we shall define a polynomial in the independent variables $\{X_\lambda, Y_\lambda, x_\lambda, y_\lambda : \lambda \in \Lambda\}$. Later we shall impose relations on these variables to ensure that the polynomial is independent of the order ϕ.

For a spanning tree T, we now define the *weight* of T as

$$w(G, c, \phi, T) = \prod_{e \in E} w(G, c, \phi, T, e).$$

Finally, the *coloured Tutte polynomial* $W_0(G, c, \phi) \in \mathbb{Z}[X_\lambda, Y_\lambda, x_\lambda, y_\lambda : \lambda \in \Lambda]$ is defined as the sum over all spanning trees T of $w(G, c, \phi, T)$. From now on we shall write $\mathbb{Z}_\Lambda$ for $\mathbb{Z}[X_\lambda, Y_\lambda, x_\lambda, y_\lambda : \lambda \in \Lambda]$.

Note that if Λ consists of one element or, equivalently, if we substitute

$$\begin{aligned} X_\lambda &= x, \\ Y_\lambda &= y, \\ x_\lambda &= 1, \\ y_\lambda &= 1, \end{aligned} \tag{3.1}$$

for all λ, then the weight of a tree T is just $x^i y^j$, where i and j are the numbers of internally and externally active edges, respectively. Thus, in this case, W_0 is just the usual Tutte polynomial $T(G; x, y)$ for G as an uncoloured graph.

Before we turn to the conditions for independence of the order, let us consider what happens if G is a *plane graph*, and we replace it by its dual G'. For the Tutte polynomial $T(G)$ we have $T(G; x, y) = T(G'; y, x)$, and this extends to the polynomial $W_0(G, c, \phi)$ we have just defined for coloured plane graphs. To ensure that $W_0(G', c, \phi)$ makes sense, we shall identify the edge set E' of G' with $E = E(G)$ in the obvious way. Now a spanning subgraph of G, that is, a set $F \subseteq E$, is connected if and only if its complement has no cycle when considered as a subgraph of G'. Dually, a subgraph of G is acyclic if and only if its complement is a connected subgraph of G'. Thus $T \subseteq E$ is a spanning tree in G if and only if $E - T$ is a spanning tree in G'. Also, since the cut $T - e$ in G and the cycle of $E - T \cup e$ in G' consist of the same edges, an edge e is active in G with respect to T if and only if it is active in G' with respect to $E - T$. Combining these observations with the definition of $W_0(G, c, \phi)$, we obtain

$$W_0(G, c, \phi)(X_\lambda, Y_\lambda, x_\lambda, y_\lambda)_{\lambda \in \Lambda} = W_0(G', c, \phi)(Y_\lambda, X_\lambda, y_\lambda, x_\lambda)_{\lambda \in \Lambda}. \tag{3.2}$$

We now turn to the dependence of the map $W_0 : \mathscr{G}^*_{c,o} \to \mathbb{Z}_\Lambda$ on the order. There is no reason why $W_0(G, c, \phi)$ should not depend on ϕ, and as the calculations in Section 3.3 will show, this is indeed the case. However, we know that some evaluations of W_0 do not depend on the order, for example (3.1). For a fixed (G, c), which evaluations of $W_0(G, c, \phi)$ are independent of ϕ will depend very much on the structure of (G, c). For example, if (G, c) is monochromatic, then any evaluation will do. (This can be deduced from Tutte's result, and also follows from Theorem 2 below.) Also, if G has a bridge e of colour λ, then since any spanning tree T uses e, and e is the only edge in $\mathrm{cut}(T - e)$, and hence is active, any evaluation with $X_\lambda = 0$ will make $W_0(G, c, \phi)$ zero for any order. We would like to know for which evaluations $W_0(G, c, \phi)$ is independent of ϕ for *all* coloured graphs. The most general form of this question is: for which ring homomorphisms $f : \mathbb{Z}_\Lambda \to R$, where R is any ring, is the map $f \circ W_0$ independent of ϕ? This is the same question as the following: for which ideals $I \subseteq \mathbb{Z}_\Lambda$ is the induced map $\overline{W}_0 : \mathscr{G}^*_{c,o} \to \mathbb{Z}_\Lambda / I$ independent of the order? We now state our first main result, which answers this question precisely, and allows us to describe the most general connected graph invariant with a spanning tree expansion.

Theorem 2. *Let $I \subseteq \mathbb{Z}_\Lambda$ be an ideal, and let $\overline{W}_0 : \mathscr{G}^*_{c,o} \to \mathbb{Z}_\Lambda / I$ be the composition*

$$\mathscr{G}^*_{c,o} \xrightarrow{W_0} \mathbb{Z}_\Lambda \xrightarrow{q} \mathbb{Z}_\Lambda / I,$$

where q is the quotient map. Then $\overline{W}_0(G,c,\phi) = \overline{W}_0(G,c,\phi')$ holds for all connected coloured graphs (G,c), and all pairs of orders ϕ, ϕ' on $E(G)$, if and only if

$$X_\lambda y_\mu - y_\lambda X_\mu - x_\lambda Y_\mu + Y_\lambda x_\mu \in I, \tag{3.3}$$

$$Y_\nu(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu) \in I, \tag{3.4}$$

and

$$X_\nu(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu) \in I, \tag{3.5}$$

for all λ, μ, $\nu \in \Lambda$.

Note that if we consider uncoloured graphs, and coefficients $X_e, \ldots$ depending on the edge e, rather than on its colour, we obtain a result exactly like Theorem 2, but with λ, μ, ν replaced by all triples e, f, g of *distinct* edges.

Returning to coloured graphs, from now on we shall write I_0 for the minimal ideal $I \subseteq \mathbb{Z}_\Lambda$ satisfying (3.3)–(3.5). Also, given an ideal $I \supseteq I_0$, so that $\overline{W}_0(G,c,\phi) \in \mathbb{Z}_\Lambda/I$ does not depend on ϕ, we shall write W for the map $W : \mathcal{G}_c^* \to \mathbb{Z}_\Lambda/I$ defined by $W(G,c) = \overline{W}_0(G,c,\phi)$ for any order ϕ. If the ideal I is not specified, then we take $I = I_0$. The content of Theorem 2 is that the map $W : \mathcal{G}_c^* \to \mathbb{Z}_\Lambda/I_0$ obtained in this case is the most general connected graph invariant with a spanning tree expansion of the form described in Section 3.1, with arbitrary coefficients. Indeed, any such invariant W' is, by definition, of the form $W' = f \circ W_0$, for some ring homomorphism f. Theorem 2 then tells us that the kernel of f contains I_0, so W' can be written in the form $f' \circ W$. We have thus answered the first question raised in the Introduction, finding an invariant with a spanning tree expansion from which all others may be obtained.

In the case where $\mathbb{Z}_\Lambda/I$ is an integral domain, the conditions above simplify, since from (3.4) and (3.5) we can conclude that either $x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu \in I$ for all λ and μ, or X_λ and Y_λ are in I for all λ. Rephrasing this slightly, we have the following immediate consequence of Theorem 2.

Corollary 3. *Let $I \subseteq \mathbb{Z}_\Lambda$ be any ideal, and let $\overline{W}_0$ be as before. Then $\overline{W}_0(G,c,\phi) = \overline{W}_0(G,c,\phi')$ holds for all connected coloured graphs (G,c), and all pairs of orders ϕ, ϕ' on $E(G)$, provided that either*

$$X_\lambda y_\mu - y_\lambda X_\mu = x_\lambda Y_\mu - Y_\lambda x_\mu = x_\lambda y_\mu - y_\lambda x_\mu \tag{3.6}$$

holds in $\mathbb{Z}_\Lambda/I$ for all colours λ and μ, or

$$X_\lambda = Y_\lambda = 0 \tag{3.7}$$

holds in $\mathbb{Z}_\Lambda/I$ for all colours λ.

Furthermore, if $\mathbb{Z}_\Lambda/I$ is an integral domain, then these conditions are necessary. □

Note that for the formulation with uncoloured graphs and coefficients $X_e, \ldots$ depending on the edge e, the equivalent of Corollary 3 does not hold. This is because, in this case, we cannot take ν equal to λ or μ in condition (3.4) or (3.5).

The force of Corollary 3 is shown by the following argument. Given any non-empty coloured graph (G, c), and any order ϕ on $E = E(G)$, there is some edge e with $\phi(e) = 1$. Suppose this edge has colour λ. Then e will be active (internally or externally) with respect to any spanning tree of G, and will contribute a factor of either X_λ or Y_λ to its weight. Thus if (3.7) holds, then $\overline{W}_0(G, c, \phi)$ will be zero in $\mathbb{Z}_\Lambda/I$ whenever G is non-empty, for any order ϕ. Corollary 3 thus says essentially that, in the integral domain case, condition (3.6) is necessary and sufficient for $\overline{W}_0$ to be order-independent.

We shall prove Theorem 2 in two sections, devoting the next section to the proof of the sufficiency of the conditions given for order independence, and the subsequent section to the proof of their necessity.

3.2. The sufficiency of the conditions for order independence

In order to prove that the conditions given in Theorem 2 are sufficient to ensure that $\overline{W}_0(G, c, \phi)$ is independent of the order ϕ, we shall follow the proof that the spanning tree expansion of the Tutte polynomial for uncoloured graphs is independent of the order.

Let ϕ and ϕ' be two orders on E obtained from each other by a *transposition*, so that they agree except on two edges, e and f, say, on which we have $\phi(e) = i$, $\phi(f) = i + 1$, and $\phi'(e) = i + 1$, $\phi'(f) = i$. Since any two orders can be obtained from each other by applying a sequence of transpositions, it suffices to find conditions ensuring that $W_0(G, c, \phi) - W_0(G, c, \phi') \in I$ for all connected coloured graphs (G, c) and pairs of orders $\{\phi, \phi'\}$ related by a transposition.

Now an edge $e_1 \in T$ is internally active if and only if there is no other edge $e_2 \in \mathrm{cut}(T - e_1)$ preceding e_1. Similarly, an edge $e_2 \notin T$ is externally active if and only if there is no other edge $e_1 \in \mathrm{cyc}(T \cup e_2)$ preceding e_2. We say that a pair of edges $\{e_1, e_2\}$ is *related* (with respect to T) if $e_1 \in T$, $e_2 \notin T$, and e_2 is in $\mathrm{cut}(T - e_1)$. Since this is the same condition as e_1 being in $\mathrm{cyc}(T \cup e_2)$, an edge is active if and only if it is not related to any edge preceding it. Hence the activities of all the edges of G are determined by comparing related pairs of edges.

The only edges whose orders change when we switch from ϕ to ϕ' are e and f; furthermore, the only *pair* of edges whose relative order changes is the pair $\{e, f\}$. Thus, for any tree T with respect to which e and f are not related, we have $w(G, c, \phi, T) = w(G, c, \phi', T)$. Hence it suffices to consider trees T for which e and f are related. Note that all such trees contain precisely one of e and f. Furthermore, there is a bijection from those containing e to those containing f, given by removing e and adding f; the fact that e and f are related ensures that the result is a spanning tree in which e and f are still related. Thus, to show that $W_0(G, c, \phi) - W_0(G, c, \phi') \in I$, it suffices to prove that, for pairs $\{T, T'\}$ where T is a spanning tree containing e and not f, with $e \in \mathrm{cyc}(T \cup f)$, and T' is the spanning tree $T - e \cup f$, we have

$$w(G, c, \phi, T) + w(G, c, \phi, T') - w(G, c, \phi', T) - w(G, c, \phi', T') \in I. \tag{3.8}$$

We would like to be able to ignore the weights of the other edges, and concentrate on e and f. It is true that the activity of an edge g different from e and f is not affected by switching from the order ϕ to the order ϕ', since comparing $\phi(g) \neq i, i + 1$ with i or $i + 1$ gives the same result. However, such an edge g may be active in one of T, T' and inactive in the other. Suppose then that $g \in T$ is an edge whose activity changes. Now g is preceded by an edge in precisely one of $\mathrm{cut}(T-g)$ and $\mathrm{cut}(T'-g)$. For these cuts to be different, g must belong to $\mathrm{cyc}(T \cup f)$. Since each cut then contains one of e and f, and g is the first edge in one of the cuts, we have $\phi(g) < i$. Also, the symmetric difference between these cuts is just $\mathrm{cut}(T-e)$, so some edge h in this cut satisfies $\phi(h) < \phi(g) < i$. Thus there are edges of order less than i both in $\mathrm{cyc}(T \cup f)$ (the edge g is such) and in $\mathrm{cut}(T-e)$ (the edge h is such). A similar argument holds if an edge $g \notin T$ changes activity between T and T'.

We now consider four cases, setting $\lambda = c(e)$, $\mu = c(f)$ and writing $\phi_0(F)$ for the minimal order of an element of $F \subseteq E$.

Case 1. Both $\phi_0(\mathrm{cyc}(T \cup f)) < i$ and $\phi_0(\mathrm{cut}(T-e)) < i$. (The remarks above show that if some edge other than e, f changes activity between T and T', then this case holds.) In this case e and f are inactive with respect to T, whether we consider the order ϕ or ϕ'. They are also inactive with respect to T', as $\mathrm{cyc}(T' \cup e) = \mathrm{cyc}(T \cup f)$, and $\mathrm{cut}(T'-f) = \mathrm{cut}(T-e)$. Now, since no other edge changes activity when we switch from ϕ to ϕ', we have

$$w(G,c,\phi,T) = w(G,c,\phi',T), \text{ and } w(G,c,\phi,T') = w(G,c,\phi',T'),$$

from which (3.8) follows.

For the remaining cases, we can simplify (3.8) by factoring out the product w_0 of the weights of the edges other than e and f, since, as shown above, these weights cannot change.

Case 2. We have $\phi_0(\mathrm{cut}(T-e)) < i$, but $\phi_0(\mathrm{cyc}(T \cup f)) \geqslant i$. Now f is the first edge in $\mathrm{cyc}(T \cup f)$ in the order ϕ', but not in the order ϕ. Similarly, e is the first edge in $\mathrm{cyc}(T' \cup e) = \mathrm{cyc}(T \cup f)$ only in ϕ. Thus (3.8) reduces to

$$w_0(x_\lambda y_\mu + Y_\lambda x_\mu - x_\lambda Y_\mu - y_\lambda x_\mu) \in I,$$

or

$$w_0(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu) \in I. \tag{3.9}$$

Now, as $\phi_0(\mathrm{cut}(T-e)) < i$, the edge h with $\phi(h) = 1$ is distinct from e and f. Let $c(h) = v$. As h is active, whether it lies in T or not, it contributes a factor of either X_v or Y_v to w_0. Thus (3.9) follows from (3.4) and (3.5).

Case 3. We have $\phi_0(\mathrm{cyc}(T \cup f)) < i$, but $\phi_0(\mathrm{cut}(T-e)) \geqslant i$. This time e or f is internally active whenever it has order i, so (3.8) becomes

$$w_0(X_\lambda y_\mu + y_\lambda x_\mu - x_\lambda y_\mu - y_\lambda X_\mu) \in I. \tag{3.10}$$

As before, w_0 has a factor of X_v or Y_v for some v. Also note that, given (3.3), equation (3.10) is equivalent to (3.9). Thus (3.10) follows from (3.3), (3.4) and (3.5).

Case 4. Both $\phi_0(\mathrm{cyc}(T \cup f)) \geqslant i$ and $\phi_0(\mathrm{cut}(T - e)) \geqslant i$. This time (3.8) becomes

$$X_\lambda y_\mu + Y_\lambda x_\mu - x_\lambda Y_\mu - y_\lambda X_\mu \in I, \tag{3.11}$$

which follows from (3.3).

We have now shown that, provided (3.3), (3.4) and (3.5) hold for all colours λ, μ, v, we have $W_0(G, c, \phi) - W_0(G, c, \phi') \in I$ for all pairs of orders $\{\phi, \phi'\}$ related by a transposition, and hence for all pairs of orders. This proves the sufficiency part of Theorem 2. □

3.3. The necessity of the conditions for order independence

In this section we complete the proof of Theorem 2, by showing that the conditions given are necessary to ensure the order independence of $\overline{W}_0(G, c, \phi)$, by considering some specific small coloured graphs.

Example 1. We first take G to be a *double edge*, that is, a graph with two vertices v_1 and v_2, and two edges, e_1 and e_2, which both go from v_1 to v_2. Let e_1 have colour λ and e_2 colour μ.

Taking $\phi(e_1) = 1$, $\phi(e_2) = 2$ and $\phi'(e_1) = 2$, $\phi'(e_2) = 1$, we have

$$W_0(G, c, \phi) = X_\lambda y_\mu + Y_\lambda x_\mu$$

and

$$W_0(G, c, \phi') = x_\lambda Y_\mu + y_\lambda X_\mu.$$

We thus require

$$X_\lambda y_\mu + Y_\lambda x_\mu - x_\lambda Y_\mu - y_\lambda X_\mu \in I,$$

which is exactly (3.11). Hence we require

$$X_\lambda y_\mu - y_\lambda X_\mu - x_\lambda Y_\mu + Y_\lambda x_\mu \in I,$$

for all λ and μ, which is just (3.3).

Example 2. Next we consider a triple edge, where $c(e_1) = v$, $c(e_2) = \lambda$, and $c(e_3) = \mu$. For $\phi(e_1) = 1$, $\phi(e_2) = 2$, $\phi(e_3) = 3$ and $\phi'(e_1) = 1$, $\phi'(e_2) = 3$, $\phi'(e_3) = 2$, we obtain

$$W_0(G, c, \phi) = X_v y_\lambda y_\mu + Y_v x_\lambda y_\mu + Y_v Y_\lambda x_\mu,$$

and

$$W_0(G, c, \phi') = X_v y_\lambda y_\mu + Y_v x_\lambda Y_\mu + Y_v y_\lambda x_\mu.$$

Hence we require

$$Y_v x_\lambda y_\mu + Y_v Y_\lambda x_\mu - Y_v x_\lambda Y_\mu - Y_v y_\lambda x_\mu \in I,$$

i.e., that (3.4) holds:

$$Y_v(x_\lambda Y_\mu - Y_\lambda x_\mu - x_\lambda y_\mu + y_\lambda x_\mu) \in I.$$

Now, considering the dual graph to the triple edge, using the relation (3.2), a similar argument shows that we must also have

$$X_v(y_\lambda X_\mu - X_\lambda y_\mu - y_\lambda x_\mu + x_\lambda y_\mu) \in I.$$

Given (3.3), which we have already shown to be necessary, this is equivalent to (3.5). Thus we have shown that for the map $\overline{W}_0 : \mathscr{G}^*_{c,o} \to \mathbb{Z}_\Lambda/I$ to be independent of the order ϕ, we must have (3.3), (3.4) and (3.5). This completes the proof of Theorem 2. □

In the next section we shall extend W to the whole set $\mathscr{G}_c$, allowing disconnected graphs.

3.4. Disconnected graphs

So far we have been considering only connected graphs, so that the concept of spanning trees made sense. In order to extend W to disconnected graphs, we consider *spanning forests* instead, where a spanning forest $F \subseteq G$ is the union of a spanning tree for each component of G, so F is a forest with $k(F) = k(G)$. Given $(G, c, \phi) \in \mathscr{G}_{c,o}$, and a spanning forest $F \subseteq G$, we can define the activity of an edge $e \in E - F$ as before. This time, however, an edge $e \in F$ is active if it is the first edge in $\text{cut}(F - e)$, which we define as the set of all edges between the two components of $F - e$ joined by e. We define the weight $w(G, c, \phi, F, e)$ as before, and the weight of F as

$$w(G, c, \phi, F) = \alpha_{k(F)} \prod_{e \in E} w(G, c, \phi, F, e).$$

We now define a map $W_0' : \mathscr{G}_{c,o} \to \mathbb{Z}_{\Lambda,\alpha_i} = \mathbb{Z}_\Lambda[\alpha_i : i = 1, 2, \ldots]$, setting $W_0'(G, c, \phi)$ to be the sum of the weights of the spanning forests of F.

Suppose $(G, c, \phi) \in \mathscr{G}_{c,o}$, and that G has k components, $G_1, \ldots, G_k$. Then a spanning tree F of G is just the union of spanning trees $T_i \subseteq G_i$. Also, the activity of an edge $e \in G_i$ with respect to F is just its activity with respect to T_i, using the induced order on $E(G_i)$. Thus we have the following relationship between W_0' and W_0:

$$W_0'(G, c, \phi) = \alpha_{k(G)} \prod_{i=1}^{k} W_0(G_i, c, \phi_i), \tag{3.12}$$

where ϕ_i is the order induced on $E(G_i)$ by ϕ. Thus W_0' is essentially given by W_0, except that we can now choose a different normalization for graphs with different numbers of components. In some contexts restricting this normalization to be of the form $\alpha_n = C\alpha^{n-1}$ will give a polynomial with nicer properties. Since this does not destroy any information about the graph, we shall do this when convenient. For the moment, we leave W_0' in its most general form. From now on, we shall also write W_0 for the extended map W_0', even though there is a slight inconsistency with the normalization. Whenever it matters, it will always be clear which normalization we are considering.

As a consequence of the fact that it is only the orders induced on the components that matter when evaluating W_0, we can immediately extend Theorem 2 to graphs with more than one component to obtain the following result.

Corollary 4. *Let $I \subseteq \mathbb{Z}_{\Lambda,\alpha_i}$ be an ideal, and let $\overline{W}_0 : \mathscr{G}_{c,o} \to \mathbb{Z}_{\Lambda,\alpha_i}/I$ be the composition*

$$\mathscr{G}_{c,o} \xrightarrow{W_0} \mathbb{Z}_{\Lambda,\alpha_i} \xrightarrow{q} \mathbb{Z}_{\Lambda,\alpha_i}/I,$$

where q is the quotient map. Then $\overline{W}_0(G, c, \phi) = \overline{W}_0(G, c, \phi')$ holds for all coloured graphs

(G,c), and all pairs of orders ϕ, ϕ' on $E(G)$, if and only if $I_0' \subseteq I$, where I_0' is the ideal of $\mathbb{Z}_{\Lambda,\alpha_i}$ generated by $\bigcup_{i=1}^{\infty} \alpha_i I_0 \subset \mathbb{Z}_{\Lambda,\alpha_i}$.

Proof. For sufficiency, consider (G,c) with k components, and two orders ϕ, ϕ' differing by a transposition. If the edges whose order changes lie in different components, then we have $W_0(G,c,\phi) = W_0(G,c,\phi')$. If they lie in the same component, then the result follows from (3.12) and Theorem 2.

For necessity we consider the same graphs as in Section 3.3. With (3.12) and Theorem 2, we see that for $\overline{W}_0$ not to depend on the order on these graphs, we must have $\alpha_1 I_0 \subseteq I$. Considering the same graphs with isolated vertices added, we deduce that $\alpha_i I_0 \subseteq I$, completing the proof. □

As in the connected case, given an ideal $I \subseteq \mathbb{Z}_{\Lambda,\alpha_i}$ containing I_0', so that $\overline{W}_0(G,c,\phi)$ does not depend on ϕ, we shall write W for the map $W : \mathscr{G}_c \to \mathbb{Z}_{\Lambda,\alpha_i}/I$ defined by $W(G,c) = \overline{W}_0(G,c,\phi)$ for any order ϕ. Again, when I is not specified, we take $I = I_0'$. Note that in this case, the map W we obtain is the most general coloured graph invariant with a spanning forest expansion of the form described above, allowing arbitrary normalization on the empty graphs E_n.

We now turn to the second definition of the Tutte polynomial given in the Introduction, *i.e.*, that via recurrence relations.

3.5. Recurrence relations

In this section we have two apparently distinct aims. On the one hand, we would like to investigate the properties of the invariant $W(G,c)$ defined above and, in particular, to show that this invariant satisfies certain recurrence relations, analogous to the relations (1.1) obeyed by the Tutte polynomial. On the other hand, we would like to answer the second question raised in the Introduction, by finding the most general coloured graph invariant satisfying such relations. It turns out that these aims coincide: with the aid of Corollary 4, we shall show that $W(G,c)$ is exactly the invariant we are looking for.

To state recurrence relations for coloured graphs, given $(G,c) \in \mathscr{G}_c$, and an edge $e \in E(G)$, we consider G/e and $G-e$ to be coloured with the restrictions of c to $E(G/e)$ and $E(G-e)$ (which, as noted earlier, are naturally identified). The basic idea will be to use the relations (stated precisely below) to define a polynomial ω, and then show that ω is equal to W. The problem is that ω need not exist, since $\omega(G,c)$ must obey several relations defining it in terms of the values of ω on graphs (G',c') with fewer edges. We can get round this by re-introducing an order ϕ, and using this to choose one relation for each $(G,c,\phi) \in \mathscr{G}_{c,o}$.

Lemma 5. *For any ideal $I \subseteq \mathbb{Z}_{\Lambda,\alpha_i}$, the map $\overline{W}_0 : \mathscr{G}_{c,o} \to \mathbb{Z}_{\Lambda,\alpha_i}/I$ is the unique map $\omega_0 : \mathscr{G}_{c,o} \to \mathbb{Z}_{\Lambda,\alpha_i}/I$ satisfying the equations*

$$\omega_0(E_n,c,\phi) = \alpha_n$$

and

$$\omega_0(G,c,\phi) = \begin{cases} X_\lambda \omega_0(G/e,c,\phi) & \text{if } e \text{ is a bridge,} \\ Y_\lambda \omega_0(G-e,c,\phi) & \text{if } e \text{ is a loop,} \\ x_\lambda \omega_0(G/e,c,\phi) + y_\lambda \omega_0(G-e,c,\phi) & \text{if } e \text{ is neither,} \end{cases} \tag{3.13}$$

where (G,c) is any non-empty coloured graph, ϕ is any order on $E(G)$, e is the last edge in the order ϕ, and e has colour λ.

Proof. It is clear that the map ω_0, if it exists, is unique. (In fact, the relations above can be considered as an inductive definition of ω_0, so ω_0 will always exist.) Since we have that $\overline{W}_0(E_n,c,\phi) = \alpha_n$, it only remains to check that $\overline{W}_0$ satisfies (3.13). Since these relations only modify one component of G at a time, equation (3.12) shows that it is sufficient to check that $\overline{W}_0$ satisfies (3.13) for connected graphs.

Let $(G,c,\phi) \in \mathscr{G}^*_{c,o}$, e, and λ be as above. Suppose first that e is a bridge. Then every spanning tree T of G contains e, and the spanning trees of G are in one to one correspondence with the spanning trees of G/e, by removing the edge e. Also, as e is always the only edge in cut$(T-e)$, it is always active. Furthermore, the cuts $T-f$ in G and $T-e-f$ in G/e are exactly the same, as are the cycles of $T \cup g$ in G and $T-e\cup g$ in G'. Thus the edges other than e have the same activity with respect to T in G as with respect to $T-e$ in G/e. This shows that $w(G,c,\phi,T) = X_\lambda w(G/e,c,\phi,T-e)$. Summing over trees, we see that $W_0(G,c,\phi) = X_\lambda W_0(G/e,c,\phi)$, from which we deduce that $\overline{W}_0$ satisfies (3.13) in the case that e is a bridge.

In the case when e is a loop the argument is similar. This time no tree contains e, and e is the only edge in cyc$(T \cup e)$, and so it is always externally active, and contributes a factor of Y_λ.

In proving the third relation, we use the fact that e is the last edge in the order ϕ, and hence is never active. Now the spanning trees T containing e are in bijection with the spanning trees of G/e. Also, for such a tree T and another edge $f \in T$, the cuts $T-f$ in G and $T-e-f$ in G/e consist of exactly the same edges. For $f \notin T$, the cycles in $T \cup f \subseteq G$ and in $T-e\cup f \subseteq G/e$ agree, or differ only in that the first contains the edge e. Since e comes last in the order, f has the same activity in either case. Similarly, the spanning trees not containing e are in bijection with the spanning trees of $G-e$, with the activities of other edges remaining the same. Combining these observations gives the third case above. □

We have now done all the work necessary to prove the following result, concerning the relations obeyed by $W(G,c)$ when it is well defined.

Theorem 6. *Let $I \subseteq \mathbb{Z}_{\Lambda,\alpha_i}$ be any ideal. If $I'_0 \subseteq I$, then the map $W : \mathscr{G}_c \to \mathbb{Z}_{\Lambda,\alpha_i}/I$ is the unique map $\omega : \mathscr{G}_c \to \mathbb{Z}_{\Lambda,\alpha_i}/I$ satisfying the equations*

$$\omega(E_n,c) = \alpha_n$$

and

$$\omega(G,c) = \begin{cases} X_\lambda \omega(G/e,c) & \textit{if } e \textit{ is a bridge,} \\ Y_\lambda \omega(G-e,c) & \textit{if } e \textit{ is a loop,} \\ x_\lambda \omega(G/e,c) + y_\lambda \omega(G-e,c) & \textit{if } e \textit{ is neither,} \end{cases} \tag{3.14}$$

for all non-empty coloured graphs (G,c) and edges $e \in E(G)$ with $c(e) = \lambda$. Furthermore, if $I_0' \not\subseteq I$, then there is no such map ω.

Proof. Clearly the solution ω is unique, if it exists. If $I_0' \subseteq I$ then, from Corollary 4, $\overline{W}_0(G,c,\phi)$ does not depend on ϕ. Combined with Lemma 5, this implies that $W(G,c)$ solves (3.14). Indeed, when we consider a particular (G,c) and $e \in E(G)$, we pick an order ϕ in which e comes last, and use the fact that $\overline{W}_0(G,c,\phi)$ solves (3.13).

Conversely, suppose that $\omega'(G,c)$ is a solution of (3.14), and define $\omega_0'(G,c,\phi) = \omega'(G,c)$ for all orders ϕ. Then ω_0' is a solution of (3.13), so $\omega_0' = \overline{W}_0$. However, by definition $\omega_0'(G,c,\phi)$ does not depend on ϕ, so in this case $\overline{W}_0(G,c,\phi)$ does not depend on ϕ. Hence, by Corollary 4, we must have $I_0' \subseteq I$. This completes the proof of Theorem 6. □

The main force of this result is that the map $W : \mathcal{G}_c \longrightarrow \mathbb{Z}_{\Lambda,\alpha_i}/I_0'$ is the most general coloured graph invariant satisfying recurrence relations of the above form. Indeed, if W' is another such map, we may write X_λ, Y_λ, *etc.* for the coefficients in the relations obeyed by W'. We may then apply Theorem 6, which makes no assumptions about the relationship between X_λ, Y_λ, *etc.*, to deduce that W' can be obtained from W by composing with a ring homomorphism.

Remark 1. Theorem 6 generalizes Theorem 1, by showing that the relations (3.14), which generalize (1.1), have a solution under certain conditions. This result could be proved directly, by adapting the proof of Theorem 1 given in the Introduction. However, this proof would be significantly longer, due to cases like that of two parallel edges of different colours, which corresponds to a trivial case in the proof of Theorem 1.

Remark 2. In the proof of the necessity part of Theorem 2 we only used the fact that W was defined on graphs with up to three edges. We thus have that Theorem 6 holds not only for invariants defined on all coloured graphs, but for invariants defined on any class of coloured graphs closed under contraction and deletion, and containing all coloured graphs with up to three edges. This will be important in Section 6, where we consider the set of signed planar graphs.

Remark 3. All our results for connected graphs can be extended to matroids. To see this, note that W is defined in terms of spanning trees, cuts and cycles. All these concepts can be expressed in terms of which sets of edges are *dependent*, that is, contain a cycle, and which are *independent*. Thus, like the usual Tutte polynomial, $W(G,c)$ can be calculated in terms of the cycle matroid of (G,c). Furthermore, all the properties of independent sets of edges we use hold for arbitrary matroids. We can thus extend W to arbitrary coloured matroids, obtaining results like Theorem 2 and Theorem 6, with all the α_i replaced by 1, and I_0' by I_0. We have thus found the universal coloured matroid invariant W with a 'spanning tree' (*i.e.*, basis) expansion as described in Section 3.1. We have also shown that W is the universal coloured matroid invariant satisfying recurrence relations of the form (3.14).

Remark 4. Zaslavsky [40] classifies all coloured matroid invariants satisfying recurrence relations of a form equivalent to (3.14), but with coefficients in a field K. This appears to be very close to the question we have answered. Indeed, since all fields are rings, Theorem 6 implies that all such invariants can be obtained from W. The reverse need not be the case, however. In fact, whether we take fields or rings seems to make rather a significant difference. For the uncoloured case (coefficients $X_e, \ldots$ depending on e), Zaslavsky obtains *seven* different classes of invariant, and in the coloured case, four. In contrast, we obtain a single universal invariant W in each case. Zaslavsky's results correspond to solving (3.3)–(3.5) in a field K, first with λ, μ, ν distinct, and then with λ, μ, ν arbitrary.

Remark 5. Schwärzler and Welsh [26] also consider coloured matroid invariants satisfying recurrence relations, in the special case $\Lambda = \{+, -\}$. Although their relations have a slightly different form, their Proposition 2.1 is equivalent to the assertion that W gives a well-defined signed matroid invariant with values in $\mathbb{Z}_\Lambda/I$ if and only if $I \supseteq I'$, the ideal generated by r and s, with

$$r = X_- y_+ - y_- X_+ - x_- Y_+ + Y_- x_+$$

and

$$s = X_- y_+ - y_- X_+ - x_- y_+ + y_- x_+.$$

Note that this contradicts Theorem 6, since $I' \neq I_0$, the ideal generated by the set $\{r, X_+ s, X_- s, Y_+ s, Y_- s\}$. To see that these ideals are different, note that substituting $X_+ = X_- = Y_+ = Y_- = 0$ maps every element of I_0 to 0, but not the element s of I'. What Schwärzler and Welsh actually prove is that W is well defined when $I \supseteq I'$, and that if it is well defined, then $I \supseteq I''$, where I'' is generated by r and $X_+ s$. Since $I'' \subset I_0 \subset I'$, this is consistent with Theorem 6, as it should be!

Remark 6. We already know that $W(G, c)$ is given by its values on the components of G. The relations above allow us to go slightly further. Since whether an edge $e \in E$ is a loop or a bridge or neither can be determined by looking only at the block of G containing e, we can apply Theorem 6 in each block in turn, to deduce that $W(G, c)$ is given by

$$W(G, c) = \alpha_{k(G)} \prod_{i=1}^{b} \frac{W(B_i, c)}{\alpha_1}, \tag{3.15}$$

where $B_1, \ldots B_b$ are the blocks of G.

Note that the freedom to specify the values of W on each empty graph independently is not surprising, since the relations (3.14) only involve graphs with the same number of components.

In this section we have shown that the recurrence relations (3.14) are essentially equivalent to the spanning tree expansion as a definition of a coloured graph polynomial. We have then used Corollary 4 of Theorem 2 to give necessary and sufficient conditions for these relations to have a solution.

3.6. Sample calculations

In this section we shall use Theorem 6 to calculate W on some slightly more complicated graphs than those considered in Section 3.3. For simplicity, and because the results will be useful later, we take the set of colours Λ to have just two elements, $+$ and $-$. Also, as the examples we consider will all be connected, their polynomials will all have a factor of α_1, which we set to 1 for simplicity.

Example 3. Any tree $T_{r,s}$ with r positive edges and s negative edges. In this case we can calculate W directly from the definition; there is only one spanning tree, and every edge is internally active, so

$$W(T_{r,s}) = X_+^r X_-^s. \tag{3.16}$$

In particular, for a path $P_{r,s}$ with r positive edges and s negative edges, we have the same expression for W. Note that (3.16) also follows immediately from (3.15) above.

Example 4. A cycle $C_{r,0}$, consisting of r positive edges. For $r = 1$, we take $C_{1,0}$ to be just a positive loop, and we have $W(C_{1,0}) = Y_+$. For $r \geqslant 1$, by Theorem 6,

$$\begin{aligned} W(C_{r,0}) &= x_+ W(C_{r-1,0}) + y_+ W(P_{r-1,0}) \\ &= x_+ W(C_{r-1,0}) + y_+ X_+^{r-1}. \end{aligned}$$

Hence, by induction,

$$W(C_{r,0}) = \sum_{i=0}^{r-2} x_+^i y_+ X_+^{r-i-1} + x_+^{r-1} Y_+, \tag{3.17}$$

for all $r \geqslant 1$. Similarly, for a cycle $C_{0,s}$ consisting of s negative edges, we have

$$W(C_{0,s}) = \sum_{i=0}^{s-2} x_-^i y_- X_-^{s-i-1} + x_-^{s-1} Y_-.$$

Example 5. A cycle $C_{r,s}$ with r positive edges and s negative ones. We have just calculated the case $s = 0$ above. If $s \geqslant 1$, then applying Theorem 6 to one of the negative edges gives

$$\begin{aligned} W(C_{r,s}) &= x_- W(C_{r,s-1}) + y_- W(P_{r,s-1}) \\ &= x_- W(C_{r,s-1}) + y_- X_+^r X_-^{s-1}. \end{aligned}$$

Hence, by induction,

$$W(C_{r,s}) = \sum_{i=0}^{s-1} x_-^i y_- X_-^{s-i-1} X_+^r + x_-^s W(C_{r,0}),$$

for all $s \geqslant 0$. If $r \geqslant 1$, combining this with (3.17) gives

$$W(C_{r,s}) = \sum_{i=0}^{s-1} x_-^i y_- X_-^{s-i-1} X_+^r + \sum_{i=0}^{r-2} x_-^s x_+^i y_+ X_+^{r-i-1} + x_-^s x_+^{r-1} Y_+. \tag{3.18}$$

We could also obtain this formula directly from the definition of W, as follows. We take the order ϕ such that all positive edges precede all negative edges. The spanning trees of $C_{r,s}$ are obtained by omitting one edge e, and this edge will be externally active if and only if it is this first edge in the whole cycle. Thus we have a factor of Y_+ in the last term, corresponding to omitting the first edge, and we have $r-1$ other terms containing a factor of y_+, corresponding to omitting some other positive edge, and s terms containing a factor of y_-, corresponding to omitting a negative edge. An edge f in the tree is internally active if and only if it precedes e in the order. In (3.18) we have written each term as a product of the weights of the edges taken in the reverse order to ϕ. Thus we have factors of X_+ or X_- to the right of the factor y_+ or y_-, and factors of x_+ or x_- to the left.

Note that we could calculate $W(C_{r,s})$ differently, by applying Theorem 6 to the edges in some other order, or by taking a different order ϕ above. For example, applying Theorem 6 to the positive edges first, we would obtain

$$W(C_{r,s}) = \sum_{i=0}^{r-1} x_+^i y_+ X_+^{r-i-1} X_-^s + \sum_{i=0}^{s-2} x_+^r x_-^i y_- X_-^{s-i-1} + x_+^r x_-^{s-1} Y_-. \tag{3.19}$$

At first sight, equations (3.18) and (3.19) appear to be inconsistent. However, we are always working in a quotient ring satisfying the conditions of Corollary 4. In this case, Corollary 4 guarantees that the two forms of $W(C_{r,s})$ calculated are equal.

3.7. Adding colours

When considering weighted graphs, it is natural to consider the operation of replacing two parallel edges with weights w_1 and w_2 by a single edge of weight $w_1 + w_2$. Here we have colours rather than weights, but we can use the recurrence relations established in Section 3.5 to show that, for a suitable choice of colour, replacing two edges by one does not affect the polynomial W.

Theorem 7. *Let (G, c) be a coloured graph with two parallel edges, e_1 and e_2, of colours λ and μ respectively. Let (G', c') be the graph formed from G by deleting these edges and replacing them by a single edge e of colour v. Then if $I \subseteq \mathbb{Z}_{\Lambda,\alpha_i}$ is an ideal with $I_0' \subseteq I$ such that the equation of 4-tuples*

$$(X_v, Y_v, x_v, y_v) = (X_\lambda y_\mu + Y_\lambda x_\mu, Y_\lambda Y_\mu, x_\lambda y_\mu + Y_\lambda x_\mu, y_\lambda y_\mu) \tag{3.20}$$

holds in $\mathbb{Z}_{\Lambda,\alpha_i}/I$, we have $W(G, c) = W(G', c')$ in $\mathbb{Z}_{\Lambda,\alpha_i}/I$.

Proof. When e is a loop, that is, e_1 and e_2 are both loops, we have from Theorem 6 that $W(G) = Y_\lambda Y_\mu W(G - e_1 - e_2) = Y_v W(G' - e) = W(G')$, where we have suppressed the dependence on c or c' for conciseness. When e is a bridge, we have that neither e_1 nor e_2 is a bridge, but that when one of them is deleted, the other becomes a bridge.

Applying Theorem 6 to G, first to e_2 and then to e_1, shows that $W(G) = x_\mu Y_\lambda W(G/e_2 - e_1) + y_\mu X_\lambda W(G - e_2/e_1)$. However, since $G/e_2 - e_1 = G - e_2/e_1 = G'/e$, this reduces to $W(G) = (x_\mu Y_\lambda + y_\mu X_\lambda)W(G'/e)$. However, from (3.20), this is just $X_\nu W(G'/e) = W(G')$. If e is neither a bridge nor a loop the argument is similar – applying Theorem 6 to e_1 and e_2, we express $W(G)$ in terms of $W(G'/e)$ and $W(G' - e)$, and the coefficients x_ν, y_ν are such that applying Theorem 6 again shows that $W(G) = W(G')$. □

We call a colour ν whose variables satisfy (3.20) the *parallel sum* of the colours λ and μ. In the special case when the variables are of the form $X_c = t + x_c$, $Y_c = 1 + zx_c$, $y_c = 1$, the sum formula was given by Traldi [32].

Having shown how colours can be 'added', it is natural to ask whether there is a 'zero' colour, such that the presence or absence of edges of this colour does not affect $W(G)$. As it stands the answer to this question is 'no'. For example, consider the graph (G, c) with n vertices and one edge, of colour 0. This graph has $W(G, c) = X_0\alpha_{n-1}$. Since the α_i are independent, there is no choice of X_0 for which this will always be equal to $W(E_n) = \alpha_n$. This is an example of a situation where we can obtain some additional property of W by insisting that the α_n are of the form $C\alpha^{n-1}$.

Theorem 8. *Let $f : \mathbb{Z}_{\Lambda,\alpha_i}/I_0' \to R$ be a ring homomorphism, with $f(\alpha_n) = C\alpha^{n-1}$ for some C, $\alpha \in R$. Suppose in addition that $f(X_0) = \alpha$, $f(Y_0) = 1$, $f(x_0) = 0$ and $f(y_0) = 1$. Then, for any coloured graph (G, c) and edge $e \in E$ of colour 0, we have*

$$f(W(G, c)) = f(W(G - e, c)).$$

Proof. As before, we split into cases according to whether e is a bridge, a loop, or neither. If e is a bridge, then we have that $f(W(G, c)) = \alpha f(W(G/e, c))$. Now G/e and $G - e$ have the same blocks, but $G - e$ has one more component. The result thus follows from (3.15). In the remaining cases, the result follows immediately from Theorem 6. □

Note that the zero colour is indeed a zero for the addition of colours described in Theorem 7. Note also that under the conditions of Theorem 8, we can add edges of colour 0 between all vertices of a graph G not already connected, without changing $W(G)$. We can then use Theorem 7 to replace any parallel edges by single edges of a new colour. This gives a complete graph which may have some loops attached to some of its vertices. Since the effect of a loop of colour λ is to multiply W by Y_λ, we can replace these by a single loop of an appropriate colour. Thus, if we like, any graph can be considered as a complete graph with one loop, for the purpose of calculating $W(G)$.

In this section we have so far considered merging two edges in parallel. In the planar case this operation has a dual – merging two edges e_1 and e_2 having a common end vertex v of degree exactly 2. We say such edges are *in series*. This operation can be applied to any graph, and we have results corresponding to Theorem 7 and Theorem 8. In the planar case we can appeal to (3.2) for their proofs, but in any case the proofs are similar to those of Theorems 7 and 8, so we shall omit them.

Theorem 9. *Let (G, c) be a coloured graph with two edges e_1 and e_2 in series, where $c(e_1) = \lambda$ and $c(e_2) = \mu$, and e_1, e_2 are not loops. Let (G', c') be the graph formed from G by replacing these edges with a single edge of colour v. Then, if $I \subseteq \mathbb{Z}_{\Lambda,\alpha_i}$ is an ideal with $I'_o \subseteq I$ such that the equation of 4-tuples*

$$(X_v, Y_v, x_v, y_v) = (X_\lambda X_\mu, Y_\lambda x_\mu + X_\lambda y_\mu, x_\lambda x_\mu, y_\lambda x_\mu + X_\lambda y_\mu)$$

holds in $\mathbb{Z}_{\Lambda,\alpha_i}/I$, we have $W(G, c) = W(G', c')$ in $\mathbb{Z}_{\Lambda,\alpha_i}/I$.

Also, if (G, c) is any coloured graph, $I \subseteq \mathbb{Z}_{\Lambda,\alpha_i}$ is any ideal containing I_0 and e is an edge of G with colour 1, where

$$(X_1, Y_1, x_1, y_1) = (1, 1, 1, 0),$$

then $W(G, c) = W(G/e, c)$ in $\mathbb{Z}_{\Lambda,\alpha_i}/I$.

We call a colour v as above the *series sum* of the colours λ and μ, and note that a colour 1 as above is a zero for series addition.

3.8. A comment on the noncommutative case

It is natural to ask why, throughout this section, we have considered only maps into *commutative* rings, rather than into quotients of the free polynomial algebra generated over $\mathbb{Z}$ by $\{X_\lambda, Y_\lambda, x_\lambda, y_\lambda : \lambda \in \Lambda\}$. The answer is that, in the noncommutative case, there is a difficulty in choosing an order in which to multiply the edge weights to obtain the weight of a spanning tree. One possibility is to use the same order ϕ used for deciding the edge activities. The problem with this is that, in our proof of the order independence of W, we assume that the weight of a tree does not change when we change the order of two consecutive, nonrelated edges e and f. Since no edge activities or weights change, this is true provided that the weights of e and f commute. These edges could have any possible weights, so we can obtain nothing new in the noncommutative case without substantially modifying this proof.

Another possibility is to use a fixed order ψ, independent of ϕ, with the aim of defining a function $f(G, c, \psi, \phi)$, depending on ψ but not ϕ. In this case, the proof in Section 3.2 will go through with minor modifications, though the conditions required will be more complicated. For example, condition (3.3) will be replaced by

$$X_\lambda w y_\mu - y_\lambda w X_\mu - x_\lambda w Y_\mu + Y_\lambda w x_\mu \in I,$$

where w is any product of edge weights, in any order. Unfortunately, in this case we have a problem with the recurrence relations. For an edge e not the first or last in the order ψ, calculating $f(G)$ from $f(G/e)$ and $f(G - e)$ involves the rather peculiar operation of inserting a factor of $x_{c(e)}$ or $y_{c(e)}$ into a particular position in all the products involved. Thus, while we could use a spanning tree expansion to define a function $f(G, c, \psi)$ taking values in a noncommutative ring, it would not obey simple recurrence relations, and we shall restrict our attention to the commutative case. Nevertheless, we would not like to rule out the possibility of making use of a noncommutative generalization of the Tutte polynomial at some point in the future.

This concludes our description of a Tutte polynomial for coloured graphs. In the next two sections we shall describe two weighted-graph polynomials that have been defined in different ways, and show that they can be obtained from the polynomial W defined here.

4. Random subgraph definition

4.1. The random cluster model of Fortuin and Kasteleyn

In [7], Fortuin and Kasteleyn define a polynomial on weighted graphs as follows. Given a graph G, a real number p_e for each edge e of G, and one extra real number κ,

$$Z(G,(p_e),\kappa) = \sum_{F\subseteq E(G)} \left(\prod_{e\in F} p_e\right)\left(\prod_{e\in E-F} q_e\right)\kappa^{k(F)},$$

where $q_e = 1 - p_e$, and $k(F)$ is the number of components of F, considered as a spanning subgraph of G. In the case where each p_e lies between 0 and 1, we can consider F as a random subgraph of G, where the edges are selected independently, with e selected with probability p_e. In this case $Z(G,(p_e),\kappa)$ is just the expectation of $\kappa^{k(F)}$. Note that in the above definition we may take κ and p_e to be elements of any ring, and not necessarily real numbers. We shall show that the resulting polynomial Z can be obtained from the coloured Tutte polynomial W (defined in the previous section) by suitable substitutions.

In [7], Fortuin and Kasteleyn show that Z satisfies

$$Z(E_n) = \kappa^n, \tag{4.1}$$

and the relation

$$Z(G) = p_e Z(G/e) + q_e Z(G-e), \tag{4.2}$$

for any edge e of G. Clearly there can be only one polynomial satisfying (4.1) and (4.2), so if we can find some choice of variables for which W satisfies these equations, this evaluation of W will be equal to Z. Now taking $\alpha_n = \kappa^n$ gives $W(E_n) = \kappa^n$, as required. Considering the weights p_e as colours and taking $x_\lambda = \lambda$, $y_\lambda = 1 - \lambda$, we have from Theorem 6 that W satisfies (4.2) when e is neither a bridge nor a loop. Note that for a loop $G/e = G-e$ and for a bridge $W(G-e) = \alpha_{k(G)+1} W(G/e)/\alpha_{k(G)}$. Hence, taking $Y_\lambda = 1$ and $X_\lambda = \lambda + \kappa(1-\lambda)$, we have from Theorem 6 that W satisfies (4.2) in all cases. It is straightforward to check that these choices satisfy (3.6), so that there is a well-defined evaluation of W satisfying (4.1) and (4.2). As noted above, this instance of W is thus precisely Z.

In fact it is easy to see that, in certain cases, W can be obtained from Z. On the one hand, if we restrict our attention to graphs where all the edges have the same colour or weight, that is, to unweighted graphs, then W is just the usual Tutte polynomial, while Z is equivalent to the rank generating function. As mentioned in the Introduction, these can be obtained from each other in a simple way. On the other hand, if we consider graphs where all the edges have different colours, then, as we can set each p_e to 0 or 1 independently, the polynomial Z contains all the information about how many components G has after certain edges are contracted and the rest deleted. It is straightforward to check that this information allows the calculation of $W(G)$ from its recurrence relations.

4.2. A slight extension of Z

We return to the interpretation of $Z(G,(p_e),\kappa)$ as the expectation of $\kappa^{k(F)}$, where F is a random subgraph of G, with edges selected independently with probabilities p_e. Setting $\kappa = 0$ in the polynomial $\kappa^{-1}Z$, which has no negative powers of κ, gives the probability that F is connected. Can we get the probability that F has exactly r components? The answer is 'yes': we can define $Z'(G,(p_e),(\kappa_s)_{s=1}^{\infty})$ by replacing $\kappa^{k(F)}$ by $\kappa_{k(F)}$ in the definition of Z. Now, setting $\kappa_r = 1$ and $\kappa_s = 0$ for $s \neq r$ gives a polynomial in (p_e) which gives the probability that F has exactly r components. There is a cost, however. While Z' still obeys (4.2), it no longer has a spanning tree expansion. This is because the variable X_λ in the expansion of Z is equal to $\lambda + \kappa(1-\lambda)$, and multiplying by κ makes no sense for Z'. On the other hand, all the information contained in Z' is given by Z. In particular, we can calculate Z' by evaluating Z with κ as a free variable, using its spanning tree expansion if we wish, and then replacing κ^r by κ_r.

5. Rank generating function definition

As mentioned in the Introduction, the Tutte polynomial $T(G;x,y)$ is simply related to the dichromatic polynomial $Q(G;t,z)$, which is essentially the rank generating function of G. In [32], Traldi defines a dichromatic polynomial for graphs G with weights w_e on the edges as follows:

$$Q(G;t,z)(w_e)_{e\in E(G)} = \sum_{F\subseteq E}\left(\prod_{e\in F} w_e\right) t^{k(F)} z^{n(F)}, \tag{5.1}$$

where $k(F)$ and $n(F)$ are the number of components and the nullity of F, as before. This can also be considered as generalizing the rank generating function $R(G;w,z)$ for unweighted graphs, defined by (2.3). As in the unweighted case, whether we take $t^{k(F)}$ or $w^{r(F)}$ in (5.1) makes no difference: since $r(F) = |V(G)| - k(F)$, one polynomial can be obtained from the other by substituting $t = w^{-1}$ and multiplying by $w^{|V(G)|}$.

Now, in the unweighted case there is a difference between the rank generating function, which is a polynomial in two variables, and the random cluster model, which is a polynomial in only one variable. However, in the weighted case this difference disappears. In particular, since $n(F) = |F| - |V(G)| + k(F)$,

$$\begin{aligned} Q(G;t,z)(w_e)_{e\in E} &= z^{-|V(G)|}\sum_{F\subseteq E}\left(\prod_{e\in F} z w_e\right)(tz)^{k(F)} \\ &= z^{-|V(G)|} Q(G;tz,1)(z w_e)_{e\in E}. \end{aligned}$$

This shows that the polynomial Q can be recovered from its evaluation with $z = 1$.

Finally, the substitutions $w_e = \frac{p_e}{1-p_e}$ and $p_e = \frac{w_e}{1+w_e}$ convert between $Q(G;t,1)$ and Z as follows:

$$\begin{aligned} Q(G;t,1)(w_e)_{e\in E} &= \sum_{F\subseteq E}\left(\prod_{e\in F} w_e\right) t^{k(F)} \\ &= \left(\prod_{e\in E} 1+w_e\right)\sum_{F\subseteq E}\left(\prod_{e\in F}\frac{w_e}{1+w_e}\right)\left(\prod_{e\in E-F}\frac{1}{1+w_e}\right) t^{k(E)} \\ &= \left(\prod_{e\in E} 1+w_e\right) Z(G,(p_e),t). \end{aligned}$$

In summary, we have shown that the random cluster model of Fortuin and Kasteleyn [7] is equivalent to the generalization of the rank generating function introduced by Traldi [32], and that both can be obtained from the polynomial W, defined by generalizing the spanning tree expansion of the Tutte polynomial as far as possible.

6. Applications to links

In the mid-1980s, Jones [11] discovered a new and very powerful polynomial invariant of knots and links. Related polynomial invariants were later discovered by Kauffman [15] and by four groups working independently [8]. The second of these, the Homfly polynomial, is the universal link invariant satisfying certain linear relations, called *skein* relations, with arbitrary coefficients (see [8] or [6]). For relations of different forms, much effort has been spent on finding the most general polynomial invariants satisfying such conditions (for example, [13], [17], [18], [23]). Soon it was shown that the Jones polynomial and the Kauffman bracket are closely related to the Tutte polynomial [29]; this relationship is frequently expressed in terms of a 'state model', or a rank generating function. A more general link invariant, the two-variable Kauffman polynomial, is also related to the Tutte polynomial. These connections were used by Kauffman [13], Murasugi [19, 20], and Thistlethwaite [29, 30] to answer several questions about links, including long-standing conjectures of Tait [28]. For a review of many of these results, see [16].

Our aim in this section is to find the most general link invariant under different conditions from all those previously considered. We demand that our link invariant have a spanning tree expansion, that is, we look for the most general link invariant that can be obtained from the coloured Tutte polynomial W. For this reason, we wish to use graphs to represent links. A different connection between graphs and links has been used by Jaeger [9] and by Traldi [32], to relate the Tutte polynomial and the Homfly polynomial. However, this works the other way round, relating the Tutte polynomial of any planar graph to the Homfly polynomials of certain special links.

6.1. The relationship between links and signed graphs

In this section we shall describe a (non-unique) way of associating a signed plane graph to a link diagram, such that the link diagram can be recovered from the graph. We shall then give precise 'if and only if' conditions for two graphs to represent the same link.

A *link diagram D* is a union of finitely many smooth closed curves in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ which has finitely many transverse double points, called *crossings*, and no other multiple

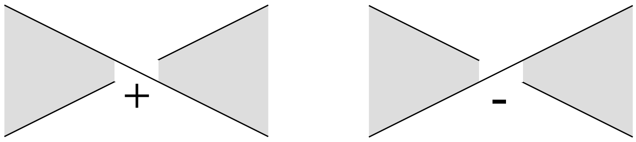


Figure 1 The sign of an edge associated with a crossing.

points, together with information specifying which arc is on top at each crossing. In other words, a link diagram is a 4-regular graph drawn in S^2, possibly with some extra simple closed curves, together with crossing information.

By a *shaded link diagram* we mean a link diagram D with alternate regions shaded; there are two shaded link diagrams for each D.

Given a connected link diagram D, which we may as well assume to lie in $\mathbb{R}^2$, we can associate a connected 2-coloured plane graph to each shading of D as follows. Take a vertex for each shaded region and an edge for each crossing, joining the two shaded regions which meet there. Colour each edge $+$ or $-$ according to the sense of the crossing, as shown in Figure 1. For disconnected diagrams, there is a complication. For example, shading a diagram with n components and no crossings will give anywhere from 1 to n shaded regions, depending on the arrangement of the components. However, we would like each component with no crossings to be represented by a vertex, to avoid losing information. Thus, for a disconnected diagram, we first shade the diagram, and then take the disjoint union of the graphs corresponding to each component, with the induced shading. We shall say that two shaded diagrams are *equivalent* if we can pair off their components such that corresponding components are related by an isotopy of S^2. Note that we require corresponding components to have the same shading. Note also that there is only one shading of a loop with no crossings, as we are working in S^2. Similarly, two signed graphs drawn in $\mathbb{R}^2 = S^2 - \{\infty\}$ are equivalent if their components are related by isotopies of S^2. We shall write $\mathscr{P}_s$ for the set of equivalence classes of signed plane graphs. Note that we have a one-to-one correspondence between equivalence classes of shaded diagrams and elements of $\mathscr{P}_s$. From now on we shall regard equivalent graphs or diagrams as being the same.

Now that we have a way of associating graphs to link diagrams, we would like to know when two graphs represent the same link, so that we can give conditions under which a signed plane graph invariant gives rise to a link invariant.

From [24], or [5], we know that two link diagrams represent the same link if and only if they are related by a series of Reidemeister moves. Now Reidemeister moves have a natural interpretation on shaded diagrams, and hence on signed plane graphs. We shall refer to these operations on graphs as *graph Reidemeister moves*, or *GR-moves*. Since these operations are of central importance to defining link polynomials via graphs, we first list them precisely, and then show the correspondence with Reidemeister moves. In describing these moves, we shall call two parallel edges of a plane graph *adjacent* if they bound a face.

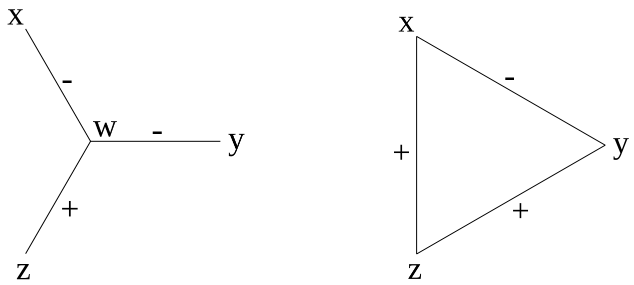


Figure 2 The star-delta transformation.

Two signed plane graphs, that is, elements of $\mathscr{P}_s$, are *related by a GR-move* if one can be obtained from the other by one of the following operations:

1a. replacing G by G/e, where e is a pendant edge, that is, an edge having an end-vertex of degree 1, and so being a bridge
1b. replacing G by $G-e$, where e is a loop
2a. replacing G by $G-e_1-e_2$, where e_1 and e_2 are adjacent parallel edges, with opposite sign, including the case where e_1 and e_2 are loops
2b. replacing G by $G/e_1/e_2$, where e_1 and e_2 are edges of opposite sign, which are in series (*i.e.*, share a vertex of degree 2), but are not parallel
2c. replacing G by the disjoint union of G_1 and G_2, where G has edges e_1 and e_2 of opposite sign between distinct vertices x and y, where x has degree 2, and the closed curve $e_1 \cup e_2$ splits $G-x$ into subgraphs G_1 and G_2 meeting at y
3a. the star-delta transformation of replacing the first configuration shown in Figure 2 by the second, or
3b. as 3a, but with the signs of all edges changed.

Note that in the star-delta transformation of 3a and 3b, the vertices x, y and z need not be distinct. However, in the second configuration the three (distinct) edges shown must form the boundary of a face.

In each of the moves 1a and 1b we shall distinguish two cases, according to the sign of e, so we shall consider GR-moves $1a^+$, $1a^-$, $1b^+$ and $1b^-$.

We can now state precisely the relationship between Reidemeister moves and GR-moves.

Lemma 10. *Two shaded link diagrams L_1 and L_2 are related by a shading-respecting Reidemeister move if and only if the corresponding graphs G_1 and G_2 are related by a GR-move.*

Proof. We consider each Reidemeister move in turn, showing that two shaded link diagrams are related by such a move (in either direction) if and only if the corresponding graphs are related by one of the corresponding GR-moves (in either direction).

1. The first Reidemeister move: removing a kink. There are four cases, depending on the sign of the kink, and the shading of the graph. We start with a positive kink, shaded as

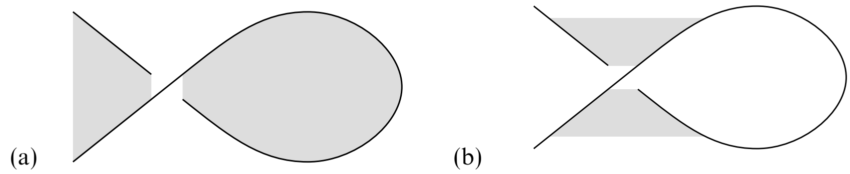


Figure 3 The two shadings of a positive kink.

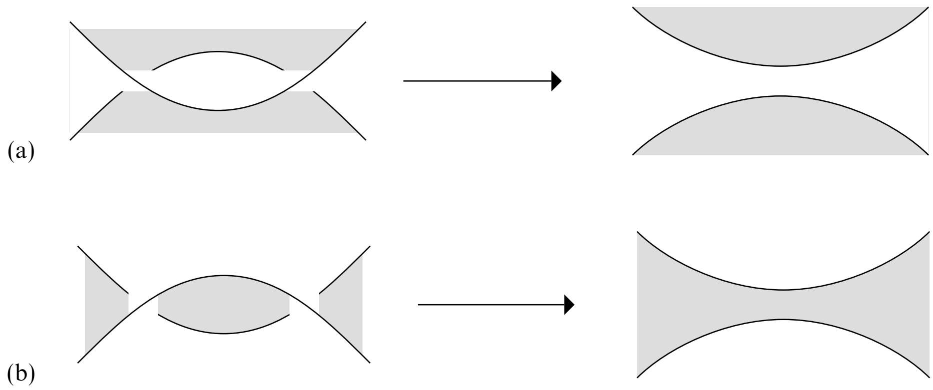


Figure 4 The two shadings of the second Reidemeister move.

in Figure 3(a). Now the shaded diagrams L containing such a kink are precisely those corresponding to graphs G with a negative pendant edge e. Also, removing the kink gives the diagram L' corresponding to the graph $G' = G/e$. Thus two shaded diagrams are related by this case of Reidemeister 1 if and only if the corresponding graphs are related by GR-move $1a^-$, *i.e.*, contracting a negative pendant edge.

Next we consider the other shading of a positive kink, which is shown in Figure 3(b). This time, removing the kink corresponds to GR-move $1b^+$, that is, deleting a positive loop. Similarly, removing a negative kink corresponds to GR-move $1a^+$, that is, contracting a negative pendant edge, or to GR-move $1b^-$, deleting a negative loop. Thus two shaded diagrams are related by Reidemeister move 1 if and only if the corresponding graphs are related by GR-move 1a or 1b.

2. The second Reidemeister move. In this case we must be careful, as the number of components of the link diagram may change. We start by considering the shading shown in Figure 4(a). This corresponds to removing a pair of adjacent parallel edges e_1 and e_2 (which may be loops) with opposite sign, which is exactly GR-move 2a. Note that a component of the link diagram becomes disconnected under this move if and only if the two shaded regions involved are distinct, and are not connected at other crossings via a sequence of other shaded regions. This is the same condition as e_1 and e_2 having distinct end points, and their removal disconnecting these vertices. Thus the link diagram and the

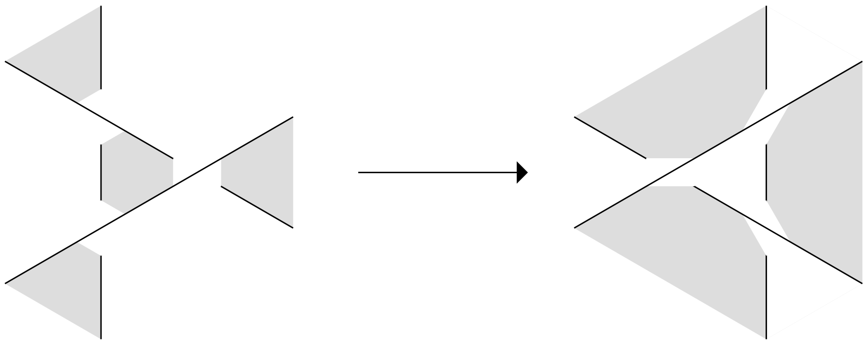


Figure 5 A shaded version of the third Reidemeister move.

graph gain a component in the same circumstances, and the association between possibly disconnected diagrams and possibly disconnected graphs described above is preserved.

We now consider the other shading of Reidemeister 2, shown in Figure 4(b), corresponding to contracting a positive edge e_1 and a negative edge e_2 in series, that is, sharing a vertex of degree 2. We have a correspondence between this Reidemeister move, *restricted to the case where* e_1 *and* e_2 *are not parallel*, and GR-move 2b. Note that under this restriction, which corresponds to the two outer shaded regions in the picture being distinct, the number of components of both the diagram and the graph remains the same. If e_1 and e_2 are also parallel, then the diagram becomes disconnected. The vertex representing the outer region must thus be replaced by two vertices, one for each of the two new components. This is precisely GR-move 2c. Thus, two shaded link diagrams are related by this shading of Reidemeister move 2, if and only if the corresponding graphs are related by GR-move 2b or 2c.

3. The third Reidemeister move. This time it turns out that there are fewer cases than we expect. We start by considering the first version of Reidemeister 3, moving a string underneath a crossing, from one B-region to the other. One shading is shown in Figure 5: this corresponds to GR-move 3a. Next we consider the other shading of the same Reidemeister move. Since this is just Figure 5 backwards, it also corresponds to GR-move 3a. Finally, the two shadings of the other version of Reidemeister move 3, that is, Figure 5 with all crossings reversed, correspond to GR-move 3b.

This completes the proof of the lemma. □

From [24], or [5], two graphs G_1 and G_2 represent the same link if and only if the corresponding *unshaded* link diagrams L_1 and L_2 are related by a sequence of Reidemeister moves. However, to make use of the lemma above, we must consider shaded diagrams. Now, if L_1 and L_2 are related as unshaded diagrams, and we perform the corresponding sequence of GR-moves on G_1, we may not obtain the graph G_2. This is because there is another graph G_2' corresponding to the other shading of L_2. However, as we shall show below, the two graphs obtained from the two shadings of any diagram are always related

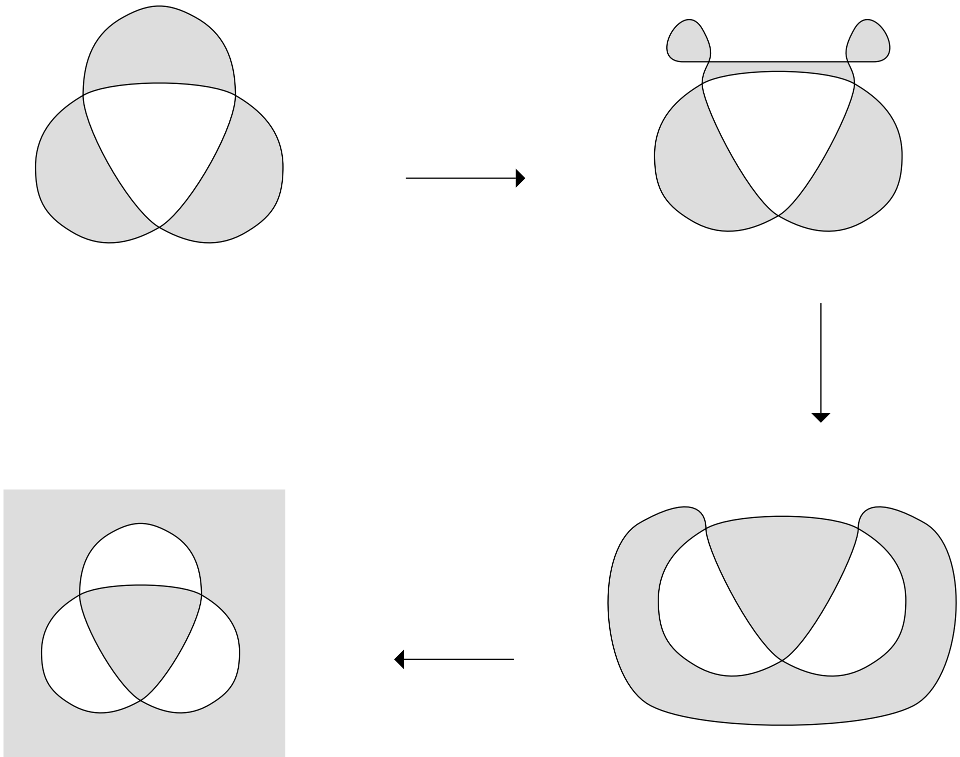


Figure 6 Inverting the shading using Reidemeister moves.

by a sequence of GR-moves. Combined with Lemma 10 above and the result from [5], this will show that two graphs represent the same link if and only if they are related by a sequence of GR-moves.

Lemma 11. *Let L be any link diagram, and let L_1 and L_2 be the two shaded link diagrams corresponding to L. Then the corresponding graphs, G_1 and G_2, are related by a sequence of GR-moves.*

Proof. Given Lemma 10, it suffices to show that L_1 and L_2 are related by a sequence of *shading-respecting* Reidemeister moves. This is true because we can take any string in L_1, and use Reidemeister moves to pull part of this string across the whole diagram, past the point at infinity, and back to where it started. This procedure is illustrated for a trefoil by Figure 6.

Note that at the end of this procedure we have the same link diagram, as we have put the string back where it started. On the other hand, as each crossing has had a string pulled over it using Reidemeister move 3, the shading near each crossing has changed. As there are only two ways of shading a diagram, the shading of the whole diagram must have been inverted. □

As noted above, combining Lemmas 10 and 11 with the result from [24], or [5], that link diagrams represent the same link if and only if they are related by a sequence of Reidemeister moves, we have the following corollary.

Corollary 12. *Two signed plane graphs $G_1, G_2 \in \mathscr{P}_s$ represent the same link if and only if they are related by a sequence of GR-moves.* □

With the aid of this corollary, we can now establish under which conditions the coloured Tutte polynomial W is an invariant of links.

6.2. A link polynomial from the coloured Tutte polynomial

The association between link diagrams and graphs described in the previous section provides a non-unique way of evaluating the polynomial W on a link diagram, and hence on a link. Since we shall have to relate graphs with different numbers of components, we shall use the form of W satisfying $W(E_n) = C\alpha^{n-1}$.

In fact, this choice is forced on us as described in Section 3.7 – writing 0 for the parallel sum of the colours $+$ and $-$, GR-move 2a invariance says that the polynomial is unchanged by deleting edges of colour 0, so considering the graph with n vertices and one edge shows that we must have $\alpha_n = X_0\alpha_{n-1}$.

We wish to establish the conditions on $C, \alpha, X_\pm$, $Y_\pm$, $x_\pm$, $y_\pm$ necessary to make this evaluation on links well defined. Recalling that the Kauffman bracket itself is only invariant under Reidemeister moves 2 and 3, but can be normalized to be fully Reidemeister invariant, we shall consider instead a suitably normalized version of W. To do this we need the concept of the *writhe* of an oriented link diagram.

It is easily seen that in an oriented diagram the crossings can be classified into two types, depending on the relationship between the orientation of the 'overpass' to that of the 'underpass'. The writhe simply counts the number of one type minus the number of the other (see [15] for the appropriate diagrams). The relevant properties of the writhe are that it is unchanged by Reidemeister moves 2 and 3, and that it decreases by one when a positive kink is removed, and increases by one when a negative kink is removed. It is sometimes stated that, since the writhe is only defined for oriented links, polynomials such as the Kauffman bracket can only be normalized to give invariants of oriented links. In fact, as Kauffman himself was aware [15], the *self-writhe* works just as well, and makes sense for unoriented links. The self-writhe of an oriented link diagram is defined in the same way as the writhe, but counting only the crossings involving only one component of the link. The self-writhe changes in exactly the same way as the writhe under Reidemeister moves, but does not actually depend on the orientation, and can thus be defined for unoriented diagrams. For our purposes, there is no reason to introduce orientation, so we shall work with the self-writhe.

Setting $a(G)$ to be the number of positive edges of G, $b(G)$ the number of negative edges, $c(G)$ the number of vertices, $d(G)$ the number of components, and $w(G)$ the self-writhe of the link diagram associated to G, our aim is to find the most general map S of the form

$$S : \mathscr{P}_s \longrightarrow \mathbb{Z}[a,b,c,d,w,w^{-1},C,\alpha,X_+,Y_+,x_+,y_+,X_-,Y_-,x_-,y_-]/I,$$

for some ideal I, where

$$S(G) = a^{a(G)}b^{b(G)}c^{c(G)}d^{d(G)}w^{w(G)}W(G),$$

and $S(G)$ depends only on the link represented by G. In fact, we would like a map S such that every other such map S' is of the form $S' = qS$, where q is a quotient map $A/I \longrightarrow A/J = (A/I)/(J/I)$. However, we do not wish to keep 'dummy' variables, namely, variables whose presence does not give us a more general polynomial.

Before continuing, we remark that if our only aim is to obtain a link invariant from $S(G)$, then we do not have to quotient by an ideal I. In general, it suffices to find some equivalence relation $\sim$ such that $S(G_1) \sim S(G_2)$ whenever G_1 and G_2 represent the same link. However, for the resulting invariant to be of any use, the equivalence relation $\sim$ should be reasonably simple. Here we have chosen to consider only relations corresponding to quotienting by an ideal, but some other choice may be possible.

As we shall see, the ideal I we shall find is unchanged under the simultaneous substitutions $a \mapsto 1$, $X_+ \mapsto aX_+$, $Y_+ \mapsto aY_+$, $x_+ \mapsto ax_+$ and $y_+ \mapsto ay_+$. Also, the definition of W implies that $S(G)$ can be recovered from its evaluation at $a = 1$:

$$\begin{aligned} & S(G)(a,b,c,d,w,w^{-1},C,\alpha,X_+,Y_+,x_+,y_+,X_-,Y_-,x_-,y_-) \\ = \;& S(G)(1,b,c,d,w,w^{-1},C,\alpha,aX_+,aY_+,ax_+,ay_+,X_-,Y_-,x_-,y_-). \end{aligned}$$

This holds since each positive edge contributes one factor of either X_+, Y_+, x_+ or y_+ to the weight of each spanning tree. Similar arguments show that $S(G)(a,b,c,d,\ldots)$ can be recovered from its evaluation at $a = b = c = d = 1$, using the substitutions $b \mapsto 1$, $X_- \mapsto bX_-$, $Y_- \mapsto bY_-$, $x_- \mapsto bx_-$, $y_- \mapsto by_-$, followed by $d \mapsto 1$, $\alpha \mapsto d\alpha$, $C \mapsto dC$, and $c \mapsto 1$, $X_+ \mapsto cX_+$, $X_- \mapsto cX_-$, $x_+ \mapsto cx_+$, $x_- \mapsto cx_-$, $C \mapsto cC$. Finally, to fix normalization, we take $S(E_1) = 1$, that is, $C = 1$. Again, this is just eliminating a dummy variable, as $S(G)$ has a factor of C for every graph $G \in \mathscr{P}_s$.

For the rest of this section, we thus confine our attention to this restriction of S, which we consider as a map

$$S : \mathscr{P}_s \longrightarrow \mathbb{Z}[w,w^{-1},\alpha,X_+,Y_+,x_+,y_+,X_-,Y_-,x_-,y_-]/I,$$

given by $S(G) = w^{w(G)}W(G)$.

We now wish to establish necessary and sufficient conditions on the ideal I to ensure that $S(G)$ is well defined on links. More precisely, we shall prove the following result.

Theorem 13. *Given an ideal $I \subseteq \mathbb{Z}[w,w^{-1},\alpha,X_\pm,Y_\pm,x_\pm,y_\pm]$, the map*

$$S : \mathscr{P}_s \longrightarrow \mathbb{Z}[w,w^{-1},\alpha,X_\pm,Y_\pm,x_\pm,y_\pm]/I$$

defined by $S(G) = w^{w(G)}W(G)$ gives rise to a well-defined link invariant if and only if the following relations hold in $\mathbb{Z}[w,w^{-1},\alpha,X_\pm,Y_\pm,x_\pm,y_\pm]/I$:

$$\begin{aligned} X_+ &= Y_- = w \\ X_- &= Y_+ = w^{-1}, \end{aligned} \tag{6.1}$$

$$\alpha = X_- y_+ + Y_- x_+, \tag{6.2}$$

$$X_\epsilon = x_\epsilon + \alpha y_\epsilon \tag{6.3}$$

for $\epsilon = +, -$, *and*

$$Y_\epsilon = \alpha x_\epsilon + y_\epsilon \tag{6.4}$$

for $\epsilon = +, -$.

Remark. Note that (6.3) and (6.4) imply that conditions (3.3)–(3.5) of Theorem 2 are satisfied, and hence (by Corollary 4) that $W(G)$ is well defined on graphs.

Before starting the proof, we note a consequence of equations (6.1)–(6.4) which will be used in the proof, and which is of interest in its own right. Combining (6.1) with (6.3) and (6.4), we have

$$\begin{array}{rcccl} x_+ + \alpha y_+ & = & y_- + \alpha x_- & = & w, \\ \alpha x_+ + y_+ & = & \alpha y_- + x_- & = & w^{-1}. \end{array} \tag{6.5}$$

Eliminating variables from these simultaneous equations, we obtain

$$(\alpha^2 - 1)(x_+ - y_-) = (\alpha^2 - 1)(y_+ - x_-) = 0. \tag{6.6}$$

This tells us that, if $\alpha^2 - 1$ is invertible, the 'dual variables' x_+ and y_- must be equal.

Proof of Theorem 13. We know from Corollary 12 that S is well defined on links if and only if it is invariant under GR-moves, so we consider each of these in turn.

1. GR-move $1a^-$. Let G' be obtained from G by GR-move $1a^-$, so $G' = G/e$, where e is a negative pendant edge. Then, from Theorem 6, we know that $W(G) = X_- W(G')$. Also, if L and L' are the corresponding link diagrams, then L' is obtained from L by removing a positive kink. Thus $w(L) = w(L') + 1$, and hence, by definition, $w(G) = w(G') + 1$. Thus $S(G) = wX_- S(G')$, and we have invariance under this GR-move provided $wX_- = 1$, that is, $X_- = w^{-1}$. This condition is also necessary, as can be seen by taking G to consist of two vertices joined by a negative edge.

2. GR-move $1b^+$. This time let G' be obtained from G by deleting a positive loop, so $W(G) = Y_+ W(G')$. The operation on the link diagram is again removing a positive kink, so the writhe again decreases by one, and it is sufficient to have $wY_+ = 1$, or $Y_+ = w^{-1}$. As before, the case when G is just a positive loop shows this condition to be necessary as well.

3. GR-moves $1a^+$ and $1b^-$. These correspond to removing a negative kink, so the writhe increases by one. Arguing as for moves $1a^-$ and $1b^+$ shows that invariance under these moves is equivalent to requiring $X_+ = Y_- = w$.

To summarize so far, the polynomial S is invariant under the graph equivalents of Reidemeister 1 if and only if (6.1) holds. Unfortunately, for the remaining GR-moves the situation is more complicated, as the effect on the polynomial W depends on the rest of the graph.

4. GR-move 2a: removing a pair of adjacent parallel edges with opposite sign. The writhe does not change, so we require

$$W(G) = W(G - e_1 - e_2), \tag{6.7}$$

for all $G \in \mathscr{P}_s$ and pairs of edges $\{e_1, e_2\}$ as above. Now $W(G)$ does not depend on the embedding of G into the plane. Thus any conditions that make W invariant under GR-move 2a will also make W invariant under the more general operation of deleting any two parallel edges of opposite sign. However, since we are looking for necessary and sufficient conditions for GR-move 2a invariance, we must ensure that in the examples we use to show our conditions are necessary, the edges e_1 and e_2 really are adjacent.

Note that if e_1 and e_2 are loops, then (6.7) follows from (6.1). Thus we need only impose (6.7) for graphs in which e_1 and e_2 are not loops.

Our aim now is to replace the rather daunting set of conditions given by (6.7) by an equivalent, but much simpler, set of conditions. Let G, e_1, e_2 be as above, with G having another edge e_3, which is a positive loop. Set $G' = G - e_1 - e_2$. Then we have from Theorem 6 that

$$W(G) = Y_+ W(G - e_3), \text{ and } W(G') = Y_+ W(G' - e_3).$$

This implies that the relation (6.7) for G follows from (6.7) for the smaller graph $G - e_3$, so we do not need to impose (6.7) for G as a separate condition. A similar argument holds for graphs G containing negative loops, or bridges. Thus it suffices to demand (6.7) for graphs G containing neither loops nor bridges. From now on, we consider only such graphs G.

If G has no edges other than e_1 and e_2, then $W(G) = X_- y_+ + Y_- x_+$, while $W(G - e_1 - e_2) = W(E_2) = \alpha$, so (6.7) becomes

$$\alpha = X_- y_+ + Y_- x_+,$$

which is just (6.2).

If G has another edge e_3, the situation is more complicated. From Theorem 6, we have

$$W(G) = x_\epsilon W(G/e_3) + y_\epsilon W(G - e_3), \tag{6.8}$$

where ϵ is the sign of e_3. Also, *provided e_3 is not a bridge in $G' = G - e_1 - e_2$*, we have

$$W(G') = x_\epsilon W(G'/e_3) + y_\epsilon W(G' - e_3). \tag{6.9}$$

Now, given (6.8) and (6.9), the relation (6.7) for G again follows from (6.7) for smaller graphs, as contracting or deleting e_3 leaves e_1 and e_2 as parallel edges, though they may become loops. Hence, it suffices to consider (6.7) for graphs G such that every edge $e_3 \in E(G')$ is a bridge in G'. Thus G' is a forest which becomes 2-connected after the addition of the parallel edges e_1 and e_2. Hence G' has two end-vertices, which are precisely the vertices of e_1 and e_2, so that G consists of our two edges e_1 and e_2 between two distinct vertices x and y, and an x-y path, shown in Figure 7. Let us write $G_{r,s}$ for this graph G if the x-y path is a $P_{r,s}$ path, that is, one with r positive edges and s negative ones. Also, let us write $A_{r,s}$ for a path with one double edge consisting of edges of opposite sign, and $r + s$ single edges, of which r are positive (and s negative). In this notation, the additional

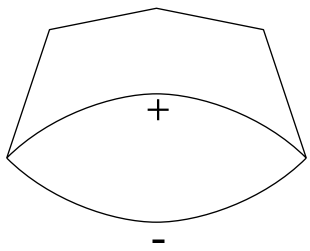


Figure 7 A family of graphs representing the unknot.

conditions we require for GR-move 2a invariance are just

$$W(G_{r,s}) = W(P_{r,s}), \tag{6.10}$$

for all nonnegative r, s with $r + s \geqslant 1$. From the conditions (6.1) and (6.2) we have already imposed, we have that $Y_+Y_- = 1$, and that $W(A_{0,0}) = \alpha$. Thus, when $r + s = 1$, applying Theorem 6 to the edge of $G_{r,s}$ that is not e_1 or e_2, shows that (6.10) and (6.3) are equivalent. Equation (6.3) is thus necessary for the GR-move 2a invariance of S. In fact, with (6.2) it is also sufficient, as can be shown by induction on $r+s$. To see this, note first that

$$W(A_{r,s}) = X_+^r X_-^s \alpha = \alpha W(P_{r,s}). \tag{6.11}$$

Now suppose that $r + s > 1$, and also that $s > 0$. Applying Theorem 6 to a negative edge of $G_{r,s}$ other than e_1 or e_2, we obtain

$$\begin{aligned} W(G_{r,s}) &= x_- W(G_{r,s-1}) + y_- W(A_{r,s-1}) \\ &= x_- W(P_{r,s-1}) + y_- W(A_{r,s-1}) \\ &= (x_- + \alpha y_-) W(P_{r,s-1}) \\ &= X_- W(P_{r,s-1}) = W(P_{r,s}), \end{aligned}$$

with the second equality following from the induction hypothesis, the third from (6.11), and the fourth from (6.3). If $s = 0$, then we can apply Theorem 6 to a positive edge, and argue exactly as above. This completes the induction, showing that (6.10), and hence the invariance of S under GR-move 2a, follows from (6.2) and (6.3).

We have now shown that, given (6.1), the invariance of S under GR-move 2a is equivalent to (6.2) and (6.3).

5. GR-move 2b: contracting a positive edge e_1 and a negative edge e_2 in series. As before, we can assume there are no bridges or loops in the rest of the graph. This time, when we contract e_1 and e_2, another edge e_3 that was not a bridge will still not be a bridge. On the other hand, an edge that was not a loop may become a loop. Thus we need to consider exactly those graphs $G \in \mathscr{P}_s$ with no bridges and loops in which all edges become loops when e_1 and e_2 are contracted, *i.e.*, graphs of the form shown in Figure 8.

If there are no edges other than e_1 and e_2, then invariance under contracting e_1 and e_2 follows from (6.1). Otherwise, the situation is exactly as for GR-move 2a – we need to impose the condition $W(G) = W(G/e_1/e_2)$ only for graphs G with one extra edge, that is,

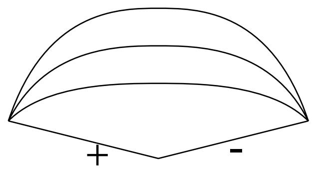


Figure 8 A second family of graphs representing the unknot.

to impose (6.4). Since the argument is exactly analogous to that for GR-move 2a above, we omit it here.

6. GR-move 2c. Let G and G' be related by GR-move 2c, and let e_1, e_2, x, y be as described in the definition of GR-move 2c. Note that y is a cut vertex in $G - e_1 - e_2$, so that G' and $G - e_1 - e_2$ have exactly the same blocks. Since they also have the same number of components, equation (3.15) tells us that $W(G') = W(G - e_1 - e_2)$. However, the argument for GR-move 2a shows that $W(G - e_1 - e_2) = W(G)$, given (6.1)–(6.3). As GR-move 2c does not change the writhe, the invariance of S under this move follows.

7. GR-move 3a: the star-delta transformation of replacing the first configuration in (6.1) by the second. In this case it will turn out that we need impose no extra conditions to ensure the invariance of W, and hence of S. Thus it makes no difference if we drop the constraint that the three edges shown in the second configuration of (6.1) bound a face.

Let $G, G' \in \mathscr{P}_s$ be signed plane graphs, such that G' can be obtained from G by GR-move 3a. As before, the writhe does not change, so for S to be invariant under this operation, we require

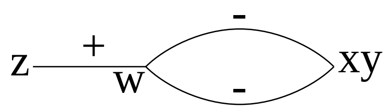
$$W(G) = W(G'). \tag{6.12}$$

If G has another edge e, then e will be a bridge/loop in G if and only if it is a bridge/loop in G', so we can use the recurrence relations (3.14) given by Theorem 6 to simplify (6.12). When we delete e in G and G' we certainly obtain a smaller instance of (6.12). In fact, when we contract e in both graphs this is also the case, as we are allowing some of the vertices x, y and z to be identified. As long as G has edges not shown in (6.1), we may thus use Theorem 6 to express (6.12) for G in terms of the same relation for smaller graphs, so we may assume that G has only the three edges shown. From now on, we consider only such graphs G.

Now, if x, y and z are all distinct, or if z is identified with exactly one of x and y, then G and G' are equivalent under GR-moves 1a–2c, so, assuming invariance of S under these moves, we do not need to impose (6.12) for these graphs. The same applies if x, y and z are all identified. This leaves us with the case where x and y are identified, but are distinct from z. In this case, G is the graph shown in Figure 9(a), and G' the graph in Figure 9(b). Assuming (6.1), we thus require W of a double positive edge to be the same as W of a double negative edge, that is,

$$X_+y_+ + Y_+x_+ = X_-y_- + Y_-x_-. \tag{6.13}$$

(a)



(b)

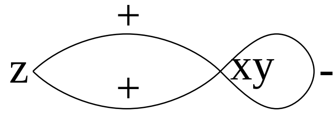


Figure 9 Two different graphs representing the Hopf link.

Given (6.3) and (6.4), this is equivalent to

$$\alpha x_+^2 + 2x_+y_+ + \alpha y_+^2 = \alpha y_-^2 + 2y_-x_- + \alpha x_-^2. \tag{6.14}$$

However, from (6.5) and the fact that $ww^{-1} = 1$, we have

$$\alpha x_+^2 + (\alpha^2+1)x_+y_+ + \alpha y_+^2 = 1 = \alpha y_-^2 + (\alpha^2+1)y_-x_- + \alpha x_-^2. \tag{6.15}$$

Now, subtracting (6.15) and (6.14), we see that the relation we require follows, as long as

$$(\alpha^2-1)x_+y_+ = (\alpha^2-1)y_-x_-.$$

However,

$$(\alpha^2-1)x_+y_+ - (\alpha^2-1)y_-x_- = (\alpha^2-1)(x_+-y_-)y_+ + (\alpha^2-1)y_-(y_+-x_-),$$

which is zero, from (6.6).

Combining the above observations, we see that (6.13), and hence invariance under GR-move 3a, follows from (6.1)–(6.4).

8. GR-move 3b. This operation is just GR-move 3a, but with all the signs changed. Following through the argument above leads to the same condition (6.13) for GR-move 3b invariance as for GR-move 3a invariance.

To conclude, the relations (6.1)–(6.4) are necessary and sufficient conditions for $S(G) = w^{w(G)}W(G)$ to be invariant under all GR-moves, and thus to give a well-defined link polynomial. This completes the proof of Theorem 13. □

We remark that for the three-variable Kauffman bracket, as for S, the conditions for invariance under Reidemeister moves 1 and 2 imply invariance under Reidemeister move 3. In the Kauffman bracket case, there is a short proof, involving resolving a crossing, and applying Reidemeister 2 twice to one of the resulting diagrams. Unfortunately, that proof does not work for S, since the coefficients corresponding to resolving the crossing change from x_+, y_+ say, at the start of the Reidemeister move, to y_-, x_- at the end. As we do not wish to make any unnecessary assumptions, such as $x_+ = y_-$, a different proof of Reidemeister 3 invariance, such as the one given above, is necessary.

Writing J_0 for the ideal of $\mathbb{Z}[w, w^{-1}, \alpha, X_\pm, Y_\pm, x_\pm, y_\pm]$ generated by relations (6.1)–(6.4), Theorem 13 claims that the most general form of a link polynomial arising from the coloured Tutte polynomial is the map $S_0 : \mathscr{P}_s \longrightarrow \mathbb{Z}[w, w^{-1}, \alpha, X_\pm, Y_\pm, x_\pm, y_\pm]/J_0$, given by $S_0(G) = w^{w(G)}W(G)$. In the next section we shall show that S_0 can in fact be obtained from the three-variable Kauffman bracket, considered as a map into a suitable quotient ring.

6.3. A reduction of the link invariant S_0

So far we have attempted to describe the most general link invariant with a spanning tree expansion. As a consequence, the invariant S_0 we have produced has a rather complicated description: it is a map into $\mathbb{Z}[w, w^{-1}, \alpha, X_\pm, Y_\pm, x_\pm, y_\pm]$ quotiented by the relations listed in Theorem 13.

Now, considered as operations on link diagrams, contracting a positive edge and deleting a negative one both have the same effect, that is, resolving a crossing in such a way that the A-regions become joined. It is thus natural to set $x_+ = y_-$: as we shall see later, this loses no information about the link. In other words, we shall consider the composition of S_0 with a ring homomorphism ϕ such that

$$\begin{array}{lllll} \phi(x_+) & = & \phi(y_-) & = & A, \\ \phi(x_-) & = & \phi(y_+) & = & B, \end{array} \tag{6.16}$$

and

$$\phi(\alpha) = d, \tag{6.17}$$

where the last choice is to make our notation consistent with [13]. Since we want Reidemeister invariance, from equations (6.1) and (6.3), we must have

$$\begin{array}{lllllll} \phi(w) & = & \phi(Y_-) & = & \phi(X_+) & = & A + dB, \\ \phi(w^{-1}) & = & \phi(Y_+) & = & \phi(X_-) & = & dA + B, \end{array} \tag{6.18}$$

We thus have all our variables given in terms of A, B, and d. In fact, with these substitutions, the polynomial W becomes the three-variable Kauffman bracket, as can be seen by considering the recurrence relations obeyed by each.

We would like to know under what conditions ϕ makes sense as a map from the quotient ring in which S_0 takes its values. This question is answered by the following result.

Theorem 14. *Let K be an ideal of $\mathbb{Z}[A, B, d]$, and let*

$$\phi : \mathbb{Z}[w, w^{-1}, \alpha, X_\pm, Y_\pm, x_\pm, y_\pm] \longrightarrow \mathbb{Z}[A, B, d]$$

be the ring homomorphism given by (6.16)–(6.18) above. Then ϕ induces a homomorphism

$$\overline{\phi} : \mathbb{Z}[w, w^{-1}, \alpha, X_\pm, Y_\pm, x_\pm, y_\pm]/J_0 \longrightarrow \mathbb{Z}[A, B, d]/K$$

if and only if $K \supseteq K_0$, where K_0 is the ideal generated by

$$(A + dB)(dA + B) = 1, \tag{6.19}$$

and

$$d = A^2 + 2dAB + B^2. \tag{6.20}$$

Proof. The map $\overline{\phi}$ will exist if and only if the relations given by the images of (6.1)–(6.4) hold in $\mathbb{Z}[A,B,d]/K$. Note that the images of (6.3) and (6.4) are automatically satisfied. From (6.1) we must have that $X_+Y_+ = ww^{-1} = 1$, and hence that $\phi(X_+)\phi(Y_+) = 1$, which is just (6.19). With this assumption, the images of all the relations in (6.1) are satisfied. This only leaves (6.2), which becomes (6.20). □

We shall write S_1 for the map $\overline{\phi} \circ S_0 : \mathscr{P}_s \longrightarrow \mathbb{Z}[A,B,d]/K_0$. Note that S_1 is the most general link invariant that can be obtained from the three-variable Kauffman bracket by normalizing using the self-writhe of a diagram. In fact, together with Theorem 13, Theorem 14 can be used to show that the substitution given by (6.16)–(6.18) does not diminish the power of S_0 to distinguish links, as S_0 is determined by S_1.

Theorem 15. *The homomorphism* $\psi : \mathbb{Z}[A,B,d] \longrightarrow \mathbb{Z}[w,w^{-1},\alpha,X_\pm,Y_\pm,x_\pm,y_\pm]$, *defined by* $\psi(A) = x_+$, $\psi(B) = y_+$, *and* $\psi(d) = \alpha$, *induces a homomorphism*

$$\overline{\psi} : \mathbb{Z}[A,B,d]/K_0 \longrightarrow \mathbb{Z}[w,w^{-1},\alpha,X_\pm,Y_\pm,x_\pm,y_\pm]/J_0$$

such that $S_0 = \overline{\psi} \circ S_1$.

Proof. From equations (6.1), (6.3) and (6.4) of Theorem 13, we have the following relations in $\mathbb{Z}[w,w^{-1},\alpha,X_\pm,Y_\pm,x_\pm,y_\pm]/J_0$:

$$\begin{array}{lllllll} w & = & X_+ & = & Y_- & = & x_+ + \alpha y_+, \\ w^{-1} & = & X_- & = & Y_+ & = & \alpha x_+ + y_+, \end{array} \tag{6.21}$$

and, since $ww^{-1} = 1$,

$$(x_+ + \alpha y_+)(\alpha x_+ + y_+) = 1. \tag{6.22}$$

Also, from (6.2) and (6.21), we have

$$\alpha = (\alpha x_+ + y_+)y_+ + (x_+ + \alpha y_+)x_+,$$

that is,

$$\alpha = x_+^2 + 2\alpha x_+ y_+ + y_+^2. \tag{6.23}$$

Now equations (6.22) and (6.23) tell us that the homomorphism $\overline{\psi}$ is well defined, as ψ maps the generators of K_0 into J_0. It only remains to show that we can evaluate $S_0(G)$ in terms of the variables x_+, y_+ and α, for all $G \in \mathscr{P}_s$. If so, then the definitions of $\overline{\phi}$ and $\overline{\psi}$ will ensure that $S_0 = \overline{\psi} \circ \overline{\phi} \circ S_0 = \overline{\psi} \circ S_1$. Note that (6.22) implies that $w^{w(G)}$ can always be expressed in terms of x_+, y_+ and α. Thus it only remains to show that $W(G)$ can be expressed in these terms, for all $G \in \mathscr{P}_s$, which we do by induction on the number of edges of G.

Suppose first that G has only bridges and loops. Then $W(G)$ is a product of the variables $X_\pm$, $Y_\pm$ and α. From (6.22), this can be expressed in the required form. Suppose next that G has a positive edge that is neither a bridge nor a loop. Then, from Theorem 6,

$$W(G) = x_+ W(G/e) + y_+ W(G-e).$$

Applying the induction hypothesis to G/e and $G - e$, this can be expressed in terms of x_+, y_+ and α. Otherwise, G has a negative edge e which is neither a bridge nor a loop. Let $\overline{G}$ be the planar dual of G, with all the signs changed. Then $\overline{G}$ and G represent the same link, so $S_0(G) = S_0(\overline{G})$, and $w(G) = w(\overline{G})$, and hence $W(G) = W(\overline{G})$. However, in $\overline{G}$, the edge e is positive, and neither a bridge nor a loop. We thus have that

$$W(G) = W(\overline{G}) = x_+ W(\overline{G}/e) + y_+ W(\overline{G} - e).$$

Since $\overline{G}$ has the same number of edges as G, we can apply the induction hypothesis to $\overline{G}/e$ and $\overline{G} - e$, so $W(G)$ can again be expressed in the required form. This completes the induction, and hence the proof of the theorem. □

Theorems 13, 14 and 15 enable us to give a simple description of the 'most general' link invariant defined in terms of the coloured Tutte polynomial of the underlying graph.

Theorem 16. *Let K_0 be the ideal of $\mathbb{Z}[A, B, d]$ generated by the relations (6.19) and (6.20), and let $S_1 : \mathscr{P}_s \longrightarrow \mathbb{Z}[A, B, d]/K_0$ be given by evaluating $S(G) = w^{w(G)} W(G)$ under the substitutions*

$$\begin{aligned} x_+, y_- &\mapsto A, \\ x_-, y_+ &\mapsto B, \\ w, X_+, Y_- &\mapsto A + dB, \\ w^{-1}, X_-, Y_+ &\mapsto dA + B, \\ \alpha &\mapsto d. \end{aligned}$$

Then S_1 is a link invariant, and any link invariant obtained from W, as described at the start of Section 6.2, is of the form $f \circ S_1$, for some ring homomorphism f.

Proof. Theorems 13 and 14 imply that S_1 is a link invariant. Now Theorem 13 implies that any link invariant S obtained from W is of the form $g \circ S_0$, for some ring homomorphism g. However, from Theorem 15, we have that $S_0 = \overline{\psi} \circ S_1$. Hence $S = (g \circ \overline{\psi}) \circ S_1$, completing the proof. □

Theorem 16 states that the most general link invariant obtained from W is S_1, which is just the three-variable Kauffman bracket $[\cdot](A, B, d)$, multiplied by $(A + dB)^{w(G)}$, evaluated in $\mathbb{Z}[A, B, d]/K_0$, where K_0 is the ideal generated by (6.19) and (6.20). We now turn to more specific examples, in particular the special case of invariants taking values in an integral domain.

6.4. Specific link invariants obtained from the coloured Tutte polynomial

In this section we shall consider special cases of S_1, that is, link invariants of the from $\theta \circ S_1$, where θ is a homomorphism from $\mathbb{Z}[A, B, d]/K_0$ to some other ring R. In the case when R is an integral domain, we can describe all such invariants very easily.

Theorem 17. *Let S be any link invariant obtained from W, as described at the start of Section 6.2, such that S takes values in an integral domain R. Then S can be obtained by composing a ring homomorphism with either the one-variable Jones polynomial, or the invariant $(-1)^{k(L)-1}$, where $k(L)$ is the number of components of a link L.*

Proof. From Theorem 16, it suffices to consider the case $S = \theta \circ S_1$, where θ is a homomorphism from $\mathbb{Z}[A, B, d]/K_0$ to an integral domain R. For notational simplicity, we shall omit θ; thus, for example, we shall write A instead of $\theta(A)$.

Subtracting (6.19) from d times (6.20), we have

$$(d^2 - 1)(AB - 1) = 0.$$

Thus, when our ring R is an integral domain, we must have either $AB = 1$, or $d = \pm 1$.

If $AB = 1$, that is, $B = A^{-1}$, then (6.20) immediately gives $d = -A^2 - A^{-2}$, and, as shown in [13], we obtain the Jones polynomial with the change of variable $t = A^{-4}$.

For the case $d = \pm 1$, we need a little preparation. Let $G \in \mathscr{P}_s$ be any signed plane graph, and $e \in E(G)$ any edge of G. Applying Theorem 6 to all the edges of G other than e, we can express $W(G)$ in the form $fX_\epsilon + gY_\epsilon$, where ϵ is the sign of e, and f and g are polynomials not depending on the sign of e. Thus $W(G) = wf + w^{-1}g$ if e is positive, and $W(G) = w^{-1}f + wg$ if e is negative.

We can now describe $S_1(G)$ in the case that $d = \pm 1$ (strictly speaking, we describe $\theta \circ S_1(G)$ in the case that $\theta(d) = \pm 1$). Suppose first that $d = 1$. Then, from (6.18), $w = A + B$, while $w^{-1} = Y_+ = A + B$. Thus $w = w^{-1}$, and the argument above shows that $W(G)$ is unchanged when the sign of an edge of G is changed. Also, from (6.19) we have that $w^2 = (A + B)^2 = 1$. Since reversing a crossing changes the writhe of a diagram by ± 2, this evaluation of $S_1(G)$ is also invariant under changing the sign of an edge. The same holds if $d = -1$: now $w = A - B = -w^{-1}$, so $W(G)$ changes sign when a crossing is reversed, but $w^2 = -1$, so $w^{w(G)}$ also changes sign, and $S_1(G)$ is unchanged. Now the components of any link can be separated and unknotted if we allow the operation of reversing a crossing. Thus, if G is the graph of any k component link, then $S_1(G) = S_1(E_k) = d^{k-1}$. Since $d = \pm 1$, this completes the proof of the theorem. □

Over more general rings the situation is different. For example, setting $A = 3$, $B = 4$ and $d = -3$ in the ring $\mathbb{Z}/22\mathbb{Z}$ satisfies conditions (6.19) and (6.20), and gives a nontrivial invariant, in that it distinguishes the trefoil from the unknot. At the moment we do not know whether there is a pair of links distinguished by S_1, but not by the Jones polynomial V. Given two different expressions in $\mathbb{Z}[A, B, d]$, to decide whether they are the same requires some calculation, using, for example, the algorithm described in [12]. Having run such calculations using Maple, we know that S_1 does not distinguish the knot 9_{42} from its mirror image, nor 8_8 from 10_{129}. (Our notation for knots is that of [25].)

6.5. Defining a coloured graph polynomial via link diagrams

In [14], Kauffman defines a polynomial $K(G)(A, B, d)$ on signed graphs, *i.e.*, graphs two-coloured with colours + and −. This polynomial, which he calls $Q[G](A, B, d)$, is defined by taking the Kauffman bracket defined on link diagrams and extending it to arbitrary signed graphs. In this section we show that proceeding this way around loses generality, *i.e.*, that the polynomial K is a special case of W, from which W cannot be recovered for signed graphs. In fact, we shall show that no polynomial defined on graphs considered as link diagrams can give as much information about the graph as the polynomial W.

Since, in [14], Kauffman shows that $K(G)$ has a spanning tree expansion, we automatically have that $K(G)$ is a special case of W. In particular, from his expansion we see that $K(G)$ is given by $W(G)$ with

$$\begin{array}{lllll} X_+ & = & Y_- & = & A + dB, \\ X_- & = & Y_+ & = & B + dA, \\ x_+ & = & y_- & = & A, \\ x_- & = & y_+ & = & B, \end{array}$$

and $\alpha_n = d^{n-1}$. Note that $K(G)$ is in fact a polynomial in two variables, since the sum of the degrees of A and B in any term is just the number of edges of G. It is easily seen that $W(G)$ is more general than $K(G)$, in the following sense.

Theorem 18. *The coloured Tutte polynomial $W(G)$ for signed graphs cannot be recovered from $K(G)$. In fact, setting*

$$K'(G)(A, B, a, b, c, d, e) = a^{a(G)} b^{b(G)} c^{c(G)} e^{d(G)} K(G)(A, B, d),$$

where, as before, $a(G)$ is the number of positive edges of G, $b(G)$ the number of negative edges, $c(G)$ the number of vertices, and $d(G)$ the number of components, there are signed graphs G_1 and G_2 with $K'(G_1) = K'(G_2)$ and $W(G_1) \neq W(G_2)$.

Proof. The only property of K we shall use is the fact that K does not distinguish between a signed plane graph G, and $\overline{G}$, its dual with all signs changed. This can be seen from the above and (3.2), or from the fact that K is defined on link diagrams, as these two graphs correspond to the same link diagram.

Let G_1 be the plane graph with four vertices and six edges, given by a four-cycle, with two consecutive edges doubled. Colour three edges $+$, and three $-$, in an arbitrary way. Now, G_1 and $G_2 = \overline{G}_1$ both have three positive edges, three negative edges, four vertices and one component. As we have $K(G_1) = K(G_2)$, we thus have $K'(G_1) = K'(G_2)$. However, the polynomial W does distinguish G_1 and G_2. In particular, taking $C = 1$, $x_\pm = y_\pm = 1$, $X_\pm = 0$, and $Y_\pm = 2$ (which satisfies the conditions of Theorem 2, as the variables for different colours are the same), we have $W(G_1) = 18$, but $W(G_2) = 14$. □

Note that, in defining the extension K' of K, we did not allow a factor of $w^{w(G)}$. This is because K and K' are defined on arbitrary signed graphs, and it is only signed *plane* graphs that have an associated diagram, and hence a writhe.

Since the proof of Theorem 18 only used the fact that K does not distinguish two signed graphs corresponding to the same link, we have actually shown that $W(G)$ as a signed graph polynomial is more general than any polynomial defined on link diagrams.

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