# ON TANGLES AND MATROIDS

STEPHEN HUGGETT
*School of Mathematics and Statistics*
*University of Plymouth, Plymouth PL4 8AA, Devon*

ABSTRACT

Given matroids $M$ and $N$ there are two operations $M \oplus_2 N$ and $M \otimes N$. When $M$ and $N$ are the cycle matroids of planar graphs these operations have interesting interpretations on the corresponding link diagrams. In fact, given a planar graph there are *two* well-established methods of generating an alternating link diagram, and in each case the Tutte polynomial of the graph is related to polynomial invariant (Jones or HOMFLY) of the link. Switching from one of these methods to the other corresponds in knot theory to tangle insertion in the link diagrams, and in combinatorics to the tensor product of the cycle matroids of the graphs.

*Keywords*: tangles, knots, Jones polynomials, Tutte polynomials, matroids

## 1. Introduction

Two binary operations on matroids, the two-sum and the tensor product, were introduced in [13] and [4]. In the case when the matroids are the cycle matroids of graphs these give rise to corresponding operations on graphs. Strictly speaking, however, these operations on graphs are not in general well defined, being sensitive to the Whitney twist. Consider the two-sum for a moment: when it is well defined it can be seen simply in terms of the replacement of an edge by a new subgraph: a special case of this is the operation leading to the homeomorphic graphs studied in [11].

Planar graphs define alternating link diagrams, and so the two-sum and tensor product operations are defined on these too: they correspond to insertions of tangles. The Whitney twist referred to above is precisely what is known in knot theory as "mutation" of a knot. Two graphs related by a Whitney twist have the same cycle matroid, which means that any matroid invariant will inevitably be unable to detect mutations in the corresponding knots.

In particular, of course, the Tutte polynomial is a matroid invariant, and is therefore unchanged when a Whitney twist is performed on a graph. On transition to the corresponding link diagram the Tutte polynomial becomes the Jones polynomial, and hence the inability of the latter to see mutations in knots. In addition to providing a neat framework in which to discuss mutations, we will see that the two-sum can be used to describe the operation of inversion on tangles (defined in [7]).

For its part, the tensor product operation provides us with an elegant description of another (less well known) feature of the relationship between planar graphs and link diagrams. Given a planar graph $G$ one can construct its medial graph $\mathcal{G}$ which then defines an alternating link diagram $D_1(G)$ as follows: each vertex of $\mathcal{G}$, being of degree four, corresponds to a single crossing in $D_1(G)$. (This was first discovered in [14], but see [1] for a recent account, whose conventions we will use.) However, one could instead associate a pair of crossings, forming a twist, with each vertex in $\mathcal{G}$, obtaining a completely different link diagram $D_2(G)$. This was the approach taken in [5].

In either case there is a relationship between the Tutte polynomial of $G$ and a polynomial invariant (Jones and HOMFLY, respectively) of the link diagram. Now suppose that each edge of $G$ had a new vertex (of degree two) placed half way along it, giving the graph $H$. Clearly $D_1(H) = D_2(G)$, which prompts us to find the relationship between the Tutte polynomial of $G$ and that of $H$.

Our graph $G$ has a cycle matroid $M(G)$ and the triangle graph $C_3$ has cycle matroid the uniform matroid $U_{2,3}$. The matroid $M \otimes U_{2,3}$ is the cycle matroid of the graph $H$. Furthermore, the Tutte polynomial of $M \otimes U_{2,3}$ can be expressed in terms of those for $M$ and $U_{2,3}$.

This enables us to show how the relationship given in [14] between the Tutte polynomial of $G$ and the Jones polynomial of $D_1(G)$ can be reconciled with the relationship given in [5] between the Tutte polynomial of $G$ and the HOMFLY polynomial of $D_2(G)$ : the details are in section 4.

Here is a brief plan of the article. Section 2 reviews the definition of a matroid, the cycle matroid of a graph, and the Tutte polynomials of matroids and graphs. In section 3 the two-sum of two matroids is defined (by giving its rank function) and then the tensor product is defined in terms of the two-sum. Graphical interpretations of both of these operations are given, and then we compute some simple examples. Our main result appears in section 4, as we have already noted, but in section 5 we include a discussion setting this result into a wider context of tangle replacements.

## 2. Matroids, graphs, and Tutte polynomials

In this section we briefly review the basic definitions of matroids, cycle matroids of graphs, and Tutte polynomials. We refer to [6], an excellent article, for the details.

A matroid $M$ is a pair $(S, \rho)$, where $S$ is the ground set and $\rho$ the rank function, satisfying:

1. $0 \leq \rho(A) \leq |A|$, $A \subseteq S$
2. $\rho(X) \leq \rho(Y)$, $X \subseteq Y \subseteq S$
3. $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$, $X, Y \subseteq S$

A subset $A$ of $S$ is called independent if $\rho(A) = |A|$, and in fact the matroid could have been defined by specifying (subject to some constraints) the independent subsets of $S$. In that case the rank function would be given, for any subset $A$ of $S$, by $\rho(A) = |B|$, where $B$ is the largest independent subset of $A$.

Given the matroid $M$ and an element $e \in S$ we can define two new matroids as follows. The deletion of $e$ from $M$ is the matroid $M \backslash e = (S \backslash \{e\}, \rho_\backslash)$, where $\rho_\backslash(A) = \rho(A)$ for every $A \subseteq S \backslash \{e\}$. The contraction of $e$ from $M$ is the matroid $M/e = (S \backslash \{e\}, \rho_/)$, where $\rho_/(A) = \rho(A \cup e) - \rho(e)$ for every $A \subseteq S \backslash \{e\}$.

Each matroid $M = (S, \rho)$ has an associated dual matroid $M^* = (S, \rho^*)$, where for any $A \subseteq S$

$$\rho^*(A) = |A| + \rho(S \backslash A) - \rho(S). \tag{2.1}$$

An element $e \in S$ is called a loop of $M$ if $\rho(\{e\}) = 0$, and a bridge of $M$ if it is a loop of $M^*$.

Given a graph $G$ with edge set $E$ we define its cycle matroid as follows. The ground set is $E$, and the independent subets of $E$ are those containing no cycle. So for $A \subseteq E$ $\rho(A)$ is the number of edges in the largest forest in $A$. Deletion and contraction in cycle matroids correspond to the usual deletion and contraction in their graphs, and similarly for bridges and loops.

The Tutte polynomial of a matroid $M = (S, \rho)$ is defined by

$$T(M; x, y) = \Sigma_{A \subseteq S} (x-1)^{\rho(S)-\rho(A)} (y-1)^{|A|-\rho(A)}. \tag{2.2}$$

When $M$ is the cycle matroid of the graph $G$ then this gives the usual Tutte polynomial of $G$. Whether thought of as a polynomial invariant of matroids or graphs, it satisfies the deletion-contraction relations

$$T(G; x, y) = \begin{cases} xT(G/e; x, y) & e \text{ a bridge} \\ yT(G \backslash e; x, y) & e \text{ a loop} \\ T(G \backslash e; x, y) + T(G/e; x, y) & \text{otherwise.} \end{cases} \tag{2.3}$$

These relations, together with $T$ having the value one on any edgeless graph, may be taken as an alternative definition of the polynomial.

## 3. Two-sums and tensor products

In this section we describe the two-sum and tensor product of two matroids, interpret them in the case where these matroids are cycle matroids of graphs, and give the relationship between the Tutte polynomial of the tensor product and those of the factors. Again we refer to [6] for the details.

Given matroids $M = (S, \rho)$ and $N = (T, \lambda)$ we define the two-sum as follows. We first have to pick $e \in S$ and $f \in T$. For any $A \subseteq S \backslash \{e\}$ and $B \subseteq T \backslash \{f\}$ define

$$\delta(A, B) = \begin{cases} 1 & \rho(A) = \rho(A \cup \{e\}) \text{ and } \lambda(B) = \lambda(B \cup \{f\}) \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Then $M \oplus_2 N$ along $e$ and $f$ is the matroid with ground set $(S \cup T) \backslash \{e, f\}$ for which the rank of $A \cup B$ is

$$\rho(A) + \lambda(B) - \delta(A, B) + \delta(\emptyset, \emptyset). \tag{3.5}$$

If $M$ and $N$ are the cycle matroids of graphs $G$ and $H$ containing edges $e$ and $f$ respectively, then $M \oplus_2 N$ is the cycle matroid of the graph obtained from $G$ and $H$ by identifying the edges $e$ and $f$ into a new edge, which is then deleted. There will in general be two ways of doing this, depending on which way round $e$ and $f$ are identified: switching from one way to the other is known as the Whitney twist (see [12] and [18]). In fact, any two 2-connected[a] graphs having the same matroid are related by a sequence of Whitney twists [19].

Of course there are many graphs $H$ for which $G \oplus_2 H$ is unchanged under a Whitney twist: a simple but important case is $H = C_p$. Observe that $G \oplus_2 C_p$ is homeomorphic to $G$; in fact we could generate a whole homeomorphism class by starting with a $G$ having no vertex of degree two and successively taking two-sums between $G$ and $C_p$ for various $p$. The behaviour of the Tutte polynomials of these graphs will follow from the formula for the Tutte polynomial of $M \oplus_2 N$ given in [4], in which "pointed" matroids are used to distinguish the edges $e$ and $f$. An alternative method uses a colouring of the edges of the graph and then the coloured Tutte polynomial of [2]; we will introduce this in the final section. (In recent work even more general results have been obtained: Tutte polynomials of two-sums of "weighted" matroids in [16], and of $k$-sums of matroids in [3].)

The tensor product $M \otimes N$ is defined to be the matroid obtained by taking the two-sum of $M$ with $N$ at *each* point of $M$ and the distinguished point $f$ of $N$. Again, if we let $M$ and $N$ be the cycle matroids of graphs $G$ and $H$ we can imagine an operation $G \otimes H$, yielding a graph obtained from $G$ by replacing each of its edges by $H \backslash \{f\}$. As before, this is not necessarily well defined as a *graph* operation, unless for example there is an isomorphism from $H$ to itself which exchanges the two end vertices of $f$.

Again we consider the simple case $H = C_p$, here with $p = 3$. $M \otimes N$ is the cycle matroid of the graph obtained from $G$ by replacing each edge by a path of length two; figure 1 below illustrates this. If instead $H$ is the graph dual to $C_3$, having two vertices with three edges between them, then $M \otimes N$ is the cycle matroid of the graph obtained from $G$ by doubling each edge.

[a]We use Tutte's definition of $n$-connected [17].

Figure 1.

Our next move is to consider the behaviour of the Tutte polynomial under the tensor product of the two matroids $M$ and $N$. It is calculated in [6] as follows (but see also [20]):

$$T(M \otimes N; x, y) = T_C(N; x, y)^{|S|-\rho(S)} T_L(N; x, y)^{\rho(S)} T(M; X, Y) \tag{3.6}$$

where

$$\begin{aligned} X &= \frac{(x-1)T_C(N; x, y) + T_L(N; x, y)}{T_L(N; x, y)}, \\ Y &= \frac{T_C(N; x, y) + (y-1)T_L(N; x, y)}{T_C(N; x, y)}, \end{aligned} \tag{3.7}$$

and where $T_C$ and $T_L$ are polynomials determined by the equations

$$\begin{aligned} (x-1)T_C(N; x, y) + T_L(N; x, y) &= T(N \backslash d; x, y) \\ T_C(N; x, y) + (y-1)T_L(N; x, y) &= T(N/d; x, y). \end{aligned} \tag{3.8}$$

Uniform matroids form an especially simple class. The uniform matroid $U_{r,n}$ has a ground set of size $n$ and all subsets of it of size $r$ are maximal independent subsets.

The simple example referred to above had $H = C_3$. Then $N = U_{2,3}$ and it follows quickly from equations (3.6–3.8) that

$$T(M \otimes N, x, y) = (x+1)^{|S|-\rho(S)} T(M, x^2, \frac{x+y}{x+1}). \tag{3.9}$$

More generally, if $H$ is a cycle of length $p$, then $N = U_{p-1,p}$ and

$$T(M \otimes N, x, y) = (x^p + \ldots + x + 1)^{|S|-\rho(S)} T(M, x^p, \frac{x^p + \ldots + x + y}{x^p + \ldots + x + 1}). \tag{3.10}$$

In our dual example $H$ was the graph on two vertices with three edges between them. Then $N = U_{1,3}$ and

$$T(M \otimes N, x, y) = (1+y)^{\rho(S)} T(M, \frac{x+y}{1+y}, y^2). \tag{3.11}$$

Again, more generally we can take $H$ to be the graph on two vertices with $p+1$ edges between them. Then $N = U_{1,p+1}$ and

$$T(M \otimes N, x, y) = (1+y+\ldots+y^p)^{\rho(S)} T(M, \frac{x+y+\ldots+y^p}{1+y+\ldots+y^p}, y^p). \tag{3.12}$$

Suppose the two graphs $G_1$ and $G_2$ have the same Tutte polynomial. Then for any graph $H$, $G_1 \otimes H$ and $G_2 \otimes H$ will also have the same Tutte polynomial. So for any given such pair $G_1$ and $G_2$ there will be infinitely many pairs of graphs with the same Tutte polynomial[b].

## 4. Knot polynomials

The new polynomial invariants of knots and links are defined recursively from skein relations. These, as many authors have noted, bear a striking similarity to deletion-contraction relations. In particular, the Jones polynomial is defined as follows. Let $D$ be a diagram of an oriented link. Then $V(D,t)$ is the Laurent polynomial in $\sqrt{t}$ satisfying two conditions. Firstly,

$$V(\text{unknot}) = 1. \tag{4.13}$$

The second condition is the skein relation. Suppose $D_+$, $D_-$, and $D_0$ are three oriented link diagrams which are identical except near one crossing, which is respectively positive, negative, and absent, as in figure 2:

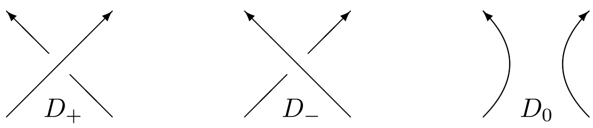


Figure 2.

Then

$$t^{-1}V(D_+) - tV(D_-) + (t^{-1/2} - t^{1/2})V(D_0) = 0. \tag{4.14}$$

In a beautiful paper [5] Jaeger showed how the Tutte polynomial of a planar graph $G$ could be related to the HOMFLY polynomial of an associated oriented link

[b]I am grateful to Ron Read for this observation: he calls these "co-Tutte" pairs.

with diagram $D_2(G)$. In particular, this yielded the following equation between the Jones polynomial and the Tutte polynomial:

$$V(D_2(G),t) = (\frac{1}{\sqrt{t}} - \sqrt{t})^{m(G)-n(G)+1}(1/t)^{m(G)+n(G)-1}T(G,t^2,\frac{t^2+1}{t^2-t}). \quad (4.15)$$

Here $m(G)$ and $n(G)$ are the numbers of edges and vertices of $G$.

On the other hand, given a planar graph $G$ a different association with an oriented link is given in [1], with diagram $D_1(G)$. Here:

$$V(D_1(G),t) = (-1)^w t^{(b-a+3w)/4}T(G,-t,-1/t). \quad (4.16)$$

Here $w$ is the *writhe* of $D_1(G)$, which is obtained by adding the signs of all the crossings in the diagram: a crossing has sign +1 if it is positive and -1 if it is negative. In [1] $a$ and $b$ are the numbers of *A-regions* and *B-regions* in $D_1(G)$, which can here be taken to be the numbers of vertices and faces of $G$, respectively.

In this section, we will describe how equation (3.9) allows us to reconcile these two formulae. We first note that the key relationship between the graph and the link is different in the two cases. In general this relationship is as follows. From a planar graph $G$ we construct the *medial graph* $\mathcal{G}$. The vertices of $\mathcal{G}$ correspond to the edges of $G$, and two vertices in $\mathcal{G}$ are joined when their corresponding edges are adjacent in a face of $G$. Thus $\mathcal{G}$ is a four-valent planar graph, and it is normally face-coloured with two colours, recording where the vertices of $G$ were. To obtain a link diagram we now replace each vertex of $\mathcal{G}$ by a standard tangle (which is just a fragment of a link having precisely four free ends).

It is this tangle which differs in our two cases. Omitting the medial graph step, in (4.15) each edge of $G$ corresponds to a pair of crossings in $D_2(G)$ as follows:

Figure 3.

In (4.16), however, each edge of $G$ corresponds to a single crossing in $D_1(G)$:

Figure 4.

No local rule for assigning orientation can be given in this case, as it can (and is) for $D_2(G)$ above. So we will start from the Jaeger equation (4.15), and eventually the corresponding orientation of $D_1(G)$ will emerge.

**Theorem 1.** For any planar graph $G$, and with $D_1(G)$ and $D_2(G)$ as above, we have

$$V(D_2(G), t) = V(D_1(G \otimes C_3), t). \tag{4.17}$$

**Proof**

$$\begin{aligned} V(D_2(G), t) &= (1/t)^{m+n-1}(1/\sqrt{t})^{m-n+1}(1-t)^{m-n+1}T(G, t^2, \frac{t^2+1}{t^2-t}) \\ &= t^{(1-3m-n)/2}(1-t)^{m-n+1}T(G, t^2, \frac{-t-1/t}{1-t}) \\ &= t^{(1-3m-n)/2}T(G \otimes C_3, -t, -1/t) \\ &= (-1)^{2m}(1/t)^{(n+3m-1)/2}T((G \otimes C_3)^*, -1/t, -t), \end{aligned} \tag{4.18}$$

where we have used (3.9) and then taken the planar dual $(G \otimes C_3)^*$ of $G \otimes C_3$. Now we observe that

$$D_1^*(G \otimes C_3) = D_1((G \otimes C_3)^*), \tag{4.19}$$

where $D_1^*(G \otimes C_3)$ denotes the reflection of $D_1(G \otimes C_3)$. All the crossings in $D_1^*(G \otimes C_3)$ are positive, and so

$$\begin{aligned} w &= m(G \otimes C_3) \\ &= 2m(G). \end{aligned} \tag{4.20}$$

Next, the number of vertices of $(G \otimes C_3)^*$ is the number of faces of $G \otimes C_3$, which we denote $a$, and so by Euler's Theorem

$$a = 2 + m(G) - n(G), \tag{4.21}$$

while the number of B-regions in $(G \otimes C_3)^*$ is

$$\begin{aligned} b &= n(G \otimes C_3) \\ &= m(G) + n(G). \end{aligned} \tag{4.22}$$

Hence, from (4.16) and (4.19),

$$\begin{aligned}(-1)^{2m}(1/t)^{(n+3m-1)/2}T((G\otimes C_3)^*,-1/t,-t) &= V(D_1^*(G\otimes C_3),1/t)\\ &= V(D_1(G\otimes C_3),t), \quad (4.23)\end{aligned}$$

so that finally

$$V(D_2(G),t) = V(D_1(G\otimes C_3),t), \tag{4.24}$$

as required. □

The very interesting preprint [9] contains a version of this result, but expressed in the language of chain polynomials. These are used to generate Jones polynomials of links corresponding to graphs in a given homeomorphism class. See also [11] and [10].

In [15] Traldi observed that the correspondence in figure 3 is one of four, and he generalised Jaeger's result (4.15) to the three other cases. Note that $C_3^*$ is the graph on two vertices with three edges between them. It can be seen from Theorem 1 that Traldi's four cases arise by applying the operation $D_1$ to the graphs $G\otimes C_3$, $G\otimes C_3^*$, and their planar duals.

## 5. Operations on tangles

The edge replacements $G\to G\otimes H$ described earlier correspond to an operation on link diagrams whereby each crossing is replaced by an arbitrary tangle $D_1(H)$. For each such operation, there will be an equation of the type (4.15) given by Jaeger.

However, as we noted in the introduction, even the simpler operation $G\to G\oplus_2 H$ is of interest in knot theory, because its ambiguity in graphs is precisely the concept of mutation in knots. Furthermore, the fact that the Tutte polynomial is actually a matroid invariant, together with (4.16), is a nice way of viewing the inability of the Jones polynomial to detect mutation.

In [9] it is suggested that the operation $G\oplus_2 H\to G\oplus_2 H^*$, where $H^*$ is the planar dual, might repay further study: the tangle equivalent is *inversion* [7], in which the tangle is rotated anti-clockwise through a right angle and then all the crossings are switched. The Tutte polynomial *can* in general distinguish $G\oplus_2 H$ from $G\oplus_2 H^*$. Indeed it treats duality very elegantly: for any matroid $M$ we have

$$T(M^*;x,y) = T(M;y,x). \tag{5.25}$$

But can we find a planar graph $H$ (distinct from its dual) such that for any $G$

$$T(G\oplus_2 H;x,y) = T(G\oplus_2 H^*;x,y)? \tag{5.26}$$

In order to develop a formula for the Tutte polynomial of a two-sum we need a way of distinguishing an edge: one way of doing this is to introduce a colouring on the *edges* of the graph. Let $\Lambda$ be a set (of colours), and given $G$ define a mapping

$c : E(G) \to \Lambda$. Then the coloured Tutte polynomial [2] $\omega(G, c)$ may be defined from the following deletion-contraction rules:

$$\omega(G,c) = \begin{cases} X_\lambda \omega(G/e,c) & e \text{ a bridge} \\ Y_\lambda \omega(G\backslash e,c) & e \text{ a loop} \\ x_\lambda \omega(G/e,c) + y_\lambda \omega(G\backslash e,c) & \text{otherwise} \end{cases} \tag{5.27}$$

for any edge $e$ with $c(e) = \lambda$. There is a rather simple colouring for $G$ which takes us back to the Tutte polynomial. Set $\Lambda = \{\lambda, \mu\}$ and $X_\lambda = x, Y_\lambda = y, x_\lambda = y_\lambda = 1$. Then if $i : E(G) \to \Lambda$ assigns the colour $\lambda$ to all the edges of $G$, we have

$$\omega(G,i) = T(G;x,y). \tag{5.28}$$

Suppose the edge $e$ of $G$ is neither a bridge nor a loop. We choose a colouring $c$ for $G$ which distinguishes the edge $e$, by assigning $\lambda$ to all the edges of $G$ except $e$, and $c(e) = \mu$. We also set $x_\mu = y_\mu = 1$. Then, leaving $e$ until last in the deletions and contractions on $G$, we have

$$\omega(G,c) = \sigma(x,y)X_\mu + \tau(x,y)Y_\mu, \tag{5.29}$$

for some polynomials $\sigma$ and $\tau$. Using the same sequence of deletions and contractions, but now on $G \oplus_2 H$, we obtain

$$T(G \oplus_2 H) = \sigma(x,y)T(H\backslash f) + \tau(x,y)T(H/f). \tag{5.30}$$

Now colour the edges of $H$ just as we did for $G$, this time distinguishing the edge $f$. We obtain

$$\omega(H,c) = \mu(x,y)X_\mu + \nu(x,y)Y_\mu, \tag{5.31}$$

which (after a little work) allows us to express (5.30) in the form

$$T(G \oplus_2 H) = (x-1)\sigma\mu + (y-1)\tau\nu + \sigma\nu + \tau\mu. \tag{5.32}$$

It follows from (5.32) that for $T(G \oplus_2 H)$ to equal $T(G \oplus_2 H^*)$, we seek a graph $H$ such that

$$\begin{aligned} &(x-1)\sigma\mu + (y-1)\tau\nu + \sigma\nu + \tau\mu \\ &= (x-1)\sigma\mu(y,x) + (y-1)\tau\nu(y,x) + \sigma\nu(y,x) + \tau\mu(y,x) \end{aligned} \tag{5.33}$$

(where the argument is $(x, y)$ unless specified). For this to be true for all $G$ we can quickly deduce that

$$\mu(x,y) = \nu(y,x). \tag{5.34}$$

However, (5.34) implies that

$$T(H\backslash f;x,y) = T(H/f;y,x) \tag{5.35}$$

which in turn implies that

$$T(H;x,y) = T(H^*;x,y), \tag{5.36}$$

and yet to avoid triviality we certainly do not want the graph $H$ to be self-dual.

There exist graphs which are not self-dual, but which have the same Tutte polynomials as their duals. To see this we confront something which has been glossed over so far in this article: we have been thinking of our planar graphs as being embedded in the plane, and for 3-connected graphs this is reasonable, since by Steinitz's Theorem [19] such graphs have essentially unique embeddings. However, 2-connected planar graphs clearly have more than one embedding, so that strictly speaking we should be referring to *maps* $(V, E, F)$ rather than simply graphs $(V, E)$. Clearly, map self-duality implies graph self-duality, but it is shown in [12] that for 2-connected graphs the converse is false: there is a 2-connected self-dual graph having no self-dual embedding. So we can find maps $C$ distinct from their map duals $C^*$ but graphically self-dual, so that $T(C) = T(C^*)$. Finding such a $C$ which also satisfies (5.34) is work in progress.

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