Tutte polynomial

Chromatic polynomial $C_\Gamma(q)$.

A *coloring* of Γ with q colors is a map $c : V(\Gamma) \to \{1, \dots, q\}$. A coloring c is *proper* if for any edge e: $c(v_1) \neq c(v_2)$, where v_1 and v_2 are the endpoints of e.

Definition 1. *$C_\Gamma(q) :=$ # of proper colorings of Γ in q colors.*

Properties (Definition 2).

$C_\Gamma = C_{\Gamma - e} - C_{\Gamma/e}$;
$C_{\Gamma_1 \sqcup \Gamma_2} = C_{\Gamma_1} \cdot C_{\Gamma_2}$, for a disjoint union $\Gamma_1 \sqcup \Gamma_2$;
$C_\bullet = 1$.

Tutte polynomial $T_\Gamma(x, y)$.

Definition 1.

$T_\Gamma = T_{\Gamma - e} + T_{\Gamma/e}$ *if e is neither a bridge nor a loop ;*
$T_\Gamma = x T_{\Gamma/e}$ *if e is a bridge ;*
$T_\Gamma = y T_{\Gamma - e}$ *if e is a loop ;*
$T_{\Gamma_1 \sqcup \Gamma_2} = T_{\Gamma_1 \cdot \Gamma_2} = T_{\Gamma_1} \cdot T_{\Gamma_2}$ *for a disjoint union $\Gamma_1 \sqcup \Gamma_2$ and a one-point join $\Gamma_1 \cdot \Gamma_2$;*
$T_\bullet = 1$.

Properties.

$T_\Gamma(1,1)$ is the number of spanning trees of Γ ;
$T_\Gamma(2,1)$ is the number of spanning forests of Γ ;
$T_\Gamma(1,2)$ is the number of spanning connected subgraphs of Γ ;
$T_\Gamma(2,2) = 2^{|E(\Gamma)|}$ is the number of spanning subgraphs of Γ .
$C_\Gamma(q) = q^{k(\Gamma)}(-1)^{r(\Gamma)} T_\Gamma(1-q, 0)$.

Definition 2.

Let
- F be a graph;
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;

$$T_\Gamma(x,y) \; := \sum_{F \subseteq E(\Gamma)} (x-1)^{r(\Gamma) - r(F)} (y-1)^{n(F)}$$

Dichromatic polynomial $Z_\Gamma(q, v)$ **(Definition 3).**

Let $Col(\Gamma)$ denote the set of colorings of Γ with q colors, and let $D : E(\Gamma) \times Col(\Gamma) \to \{0, 1\}$ be defined by the formula $D(e, c) = 1$ is and only if $c(v_1) = c(v_2)$, where v_1 and v_2 are the endpoints of e.

$$Z_\Gamma(q, v) \; := \sum_{c \in Col(\Gamma)} \prod_{e \in E(\Gamma)} (1 + vD(e, c))$$

Properties .

$Z_\Gamma = Z_{\Gamma - e} + v Z_{\Gamma/e}$;

$Z_{\Gamma_1 \sqcup \Gamma_2} = Z_{\Gamma_1} \cdot Z_{\Gamma_2}$, for a disjoint union $\Gamma_1 \sqcup \Gamma_2$;

$Z_\bullet = q$;

$$Z_\Gamma(q,v) = \sum_{F \subseteq E(\Gamma)} q^{k(F)} v^{e(F)} \ ;$$

$C_\Gamma(q) = Z_\Gamma(q,-1)$;

$Z_\Gamma(q,v) = q^{k(\Gamma)} v^{r(\Gamma)} T_\Gamma(1 + qv^{-1}, 1+v)$;

$T_\Gamma(x,y) = (x-1)^{-k(\Gamma)}(y-1)^{-v(\Gamma)} Z_\Gamma((x-1)(y-1), y-1)$.

Definition 4.

For a connected graph Γ fix an order of its edges: $e_1, e_2, \ldots, e_m$. Let T be a spanning tree.

An edge $e_i \in E(T)$ is called *internally active (live)* if $i < j$ for any edge e_j connecting the two components of $T - e_i$

An edge $e_j \notin E(T)$ is called *externally active (live)* if $j < i$ for any edge e_i in the unique cycle of $T \cup e_j$.

Let $i(T)$ and $j(T)$ be the numbers of internally and externally active edges correspondingly.

$$\boxed{T_\Gamma(x,y) \ := \sum_T \ x^{i(T)} y^{j(T)}}$$