L-functions and non-abelian class field theory, from Artin to Langlands

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Introduction

Emil Artin spent the first 15 years of his career in Hamburg. André Weil characterized this period of Artin's career as a "love affair with the zeta function" [77]. Claude Chevalley, in his obituary of Artin [14], pointed out that Artin's use of zeta functions was to discover exact algebraic facts as opposed to estimates or approximate evaluations. In particular, it seems clear to me that during this period Artin was quite interested in using the Artin L-functions as a tool for finding a *non-abelian class field theory*, expressed as the desire to extend results from relative abelian extensions to general extensions of number fields.

Artin introduced his L-functions attached to characters of the Galois group in 1923 in hopes of developing a non-abelian class field theory. Instead, through them he was led to formulate and prove the Artin Reciprocity Law - the crowning achievement of abelian class field theory. But Artin never lost interest in pursuing a non-abelian class field theory. At the Princeton University Bicentennial Conference on the Problems of Mathematics held in 1946 "Artin stated that 'My own belief is that we know it already, though no one will believe me – that whatever can be said about non-Abelian class field theory follows from what we know now, since it depends on the behavior of the broad field over the intermediate fields – and there are sufficiently many Abelian cases.' The critical thing is learning how to pass from a prime in an intermediate field to a prime in a large field. 'Our difficulty is not in the proofs, but in learning what to prove.'." (cf.[29]. The report from the Princeton Conference is on pages 309–329, the Artin quote on page 312.)

In this article I would like to follow the thread of Artin's L-functions through the search for a non-abelian class field theory. We will begin with Artin's work, but will then follow the course of Artin's L-functions, and the parallel L-functions of Hecke, to their role in the formulation of Langlands' program and the eventual proof of what could be called the local non-abelian reciprocity law. I will also

*This paper grew out of a talk given at the conference on "Emil Artin – His Work and His Life" held at the Universität Wien in January of 2006. I would like to thank the Universität Wien and the Erwin Schrödinger Institute for their hospitality during my stay in Vienna in 2006 and again the ESI for my stay in 2011-2012 when the manuscript was expanded. I would like to thank Warren Sinnott for many interesting conversations during the preparation of this manuscript. I also would like to thank Della Dumbaugh, Matthias Künzer and Joachim Schwermer for reading and commenting on earlier versions of this paper. During the preparation of this paper I was partially supported by NSA grant H98230-05-1-0059, NSF grant DMS-0968505, and the Erwin Schrödinger Institute.

discuss the role of Artin's L-functions in Langlands' functoriality conjecture – an avatar of a global non-abelian reciprocity law.

I am not a historian. I have been asked to write about Artin's L-functions and I will address them in terms of the bits of history that I know but I will primarily address them in the way in which they have figured in my own area of research. I am sure that my biases and the gaps in my knowledge will be apparent to many readers. The list of references contains both the sources cited in the text as well as other sources I read in preparing the manuscript.

1. *L*–functions Before Artin

The use of L-functions in number theory goes back at least to Leonhard Euler. Euler introduced what we now call the Riemann zeta function

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

as a function of an integer variable k. The application to arithmetic came from factoring the series into a product over primes, what we now call an Euler product

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = \prod_p (1-p^{-k})^{-1}.$$

Setting $k=1$ Euler obtains the identity

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_p \frac{1}{1-1/p}$$

and from the divergence of the harmonic series he deduces the infinitude of primes. He also stated the functional equation and used it to investigate its special values [9].

It was Peter G. Lejeune Dirichlet who introduced L-functions as we recognize them and use them today [25]. He did this by introducing the series

$$L(\sigma,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^\sigma} = \prod_p (1-\chi(p)p^{-\sigma})^{-1}$$

attached to a character $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$, but now with σ a real variable. Both this type of series and these characters now bear Dirichlet's name. He then showed that the convergence and non-vanishing of these series at $\sigma = 1$ for (non-principal) characters mod m implied the infinitude of primes in arithmetic progressions mod m.

It was Bernhard Riemann that replaced the real variable σ with a complex variable $s = \sigma + it$ in $\zeta(s)$ and discussed the zeta function as a function of a complex variable [64]. He was the first to show that $\zeta(s)$ had a continuation to a meromorphic function of order one with a simple pole at $s = 1$ and satisfying a functional equation

$$Z(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = Z(1-s).$$

He did this by writing $Z(s)$ as the Mellin transform of Jacobi's theta function and using the modular transformation of the theta function. This process remains the paradigm, as developed by Erich Hecke, for all such results. He also discussed the Euler product expansion and its relation to the prime number theorem. This program was carried out by Jacques Hadamard and Charles-Jean de la Valée-Poussin, giving the analytic proof of the prime number theorem.

In his memoir [53] Ernst Kummer introduced the zeta function of a cyclotomic field in order to investigate the class number of these fields following Dirichlet [26]. Richard Dedekind extended the zeta function to the context of arbitrary number fields by introducing what is now known as the Dedekind zeta function of an algebraic extension $K/\mathbb{Q}$ [22]:

$$\zeta_K(s) = \sum_{\mathfrak{a}}^{\infty} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}.$$

It had to wait for Hecke to prove that this function enjoyed a meromorphic continuation and a functional equation analogous to that of the Riemann zeta function [38]. The analogue of Dirichlet L-functions attached to finite order Dirichlet characters then becomes the L-series of Heinrich M. Weber attached to finite order ray-class characters of a number field k:

$$L(s,\chi) = L(s,\chi;k) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1}.$$

It was again Hecke who finally proved that these L-functions also possessed an analytic continuation and functional equation [39]. Finally, Hecke introduced his Größencharakter, now called Hecke characters, and attached to them a generalization of both the L-series of Dirichlet and Weber defined by a similar Dirichlet series with Euler product [40].

The L-functions of Weber were used extensively in the early development of abelian class field theory. In particular, the proof of the first fundamental inequality (now called the second) of class field theory was purely analytic.

Theorem: *Let K be a finite abelian extension of k with $n = (K : k)$. Let $\mathfrak{m}$ be a modulus for k. Then K and $\mathfrak{m}$ determine a subgroup $H_{\mathfrak{m}}$ of the group $I_k(\mathfrak{m})$ of fractional ideals of k relatively prime to $\mathfrak{m}$, namely $H_{\mathfrak{m}} = N_{K/k}(I_K(\mathfrak{m}))P_k(\mathfrak{m})$ where $P_k(\mathfrak{m})$ is the group of principal ideals (α) of k generated by $\alpha \equiv 1 \pmod{\mathfrak{m}}$. If $h_{\mathfrak{m}} = (I_k(\mathfrak{m}) : H_{\mathfrak{m}})$ then $h_{\mathfrak{m}} \leq n$.*

The proof was based on the Euler product expansion of the Dedekind zeta function and its pole at one and the computation of the Dirichlet density of the primes that split completely in an abelian extension [74, 68]. The only proof of this was analytic until Chevalley gave an algebraic proof in 1940 [14].

The Weber L-functions also figured into the factorization of the Dedekind zeta function of a relative abelian extension K/k through characters of the generalized class group $I_k(\mathfrak{f})/H_{\mathfrak{f}}$ of k associated to K/k, which was first obtained by Weber up to a finite number of Euler factors and then shown by Hecke to hold for the completed L-functions.

Theorem:

$$\zeta_K(s) = \prod_{\chi \in \widehat{I_k(\mathfrak{f})/H_{\mathfrak{f}}}} L(s,\chi) = \left(\prod_{\chi \neq \chi_0} L(s,\chi) \right) \zeta_k(s).$$

As a consequence of this one observes that the ratio $\zeta_K(s)/\zeta_k(s)$ is an entire function of s. Also, from this factorization and the functional equation of the Hecke L-functions Helmut Hasse gave his first proof of the conductor–discriminant formula

$$D_{K/k} = \prod_{\chi} f(\chi)$$

where $f(\chi)$ is the conductor of χ [34, 44]. This factorization will play an important role in Artin's development of the Artin L-functions.

All of these zeta-functions or L-series begin life as convergent Dirichlet series for $Re(s) > 1$. They are then shown to enjoy the following properties.

- They all have Euler product factorizations.
- They have meromorphic or analytic continuation.
- The continuations are entire or meromorphic functions of finite order.
- They all satisfy a functional equation of the form

$$\Lambda(s,\chi) = \Gamma_\infty(s,\chi)L(s,\chi) = \varepsilon(s,\chi)\Lambda(1-s,\overline{\chi}).$$

The applications to class field theory all seem to come through computing Dirichlet densities of various sets of primes through the pole of the Dedekind zeta function at $s = 1$ and the factorization. Note that as a consequence of the Existence Theorem of class field theory these same considerations let you prove the non-vanishing of the $L(s,\chi)$ at $s = 1$ and hence deduce the generalization of Dirichlet's Theorem on primes in arithmetic progressions. The point to be made is that the the analytic techniques embodied in the Dedekind zeta function and the $L(s,\chi)$ for abelian characters were intimately intertwined with the class field theory until Chevalley's work.

2. Artin L-functions

Artin introduced his L-functions in a paper of 1923 [2] and completed his work on them in the papers of 1930 and 1931 [6, 7]

2.1. Artin L-functions – The 1923 Paper. Artin must have been interested in class field theory; it was one of the great advances in number theory of the period. The piece that really seems to have piqued his interest was the factorization of zeta functions and L-functions as embodied in the theorem of Weber and Hecke above. Artin wrote one paper on such factorizations in 1923, before he turned to the Artin L-functions. It seems to me that a motivating question for Artin was the following.

Question: *How does one extend the factorization of Weber (and possibly also class field theory) to non-abelian extensions K/k?*

He would no longer have the ray class groups $G = I_k(\mathfrak{m})/N_{K/k}(I_K(\mathfrak{m}))P_k(\mathfrak{m})$ and its Dirichlet L-functions to work with, but by the Isomorphism Theorem, these are also the characters of $\mathrm{Gal}(K/k)$, and he still had $\mathrm{Gal}(K/k)$ to work with even in the non-abelian situation.

In the same period (late 19^{th} and early 20^{th} centuries) we find the beginnings of representation theory of finite groups and their characters. Artin seems to have been particularly inspired by the work of F. Georg Frobenius. There were two results that were particularly inspiring for Artin.

(a) **Non-abelian characters** [32]: If

$$\rho : G \to GL_n(\mathbb{C})$$

is a representation of a (possibly non-abelian) group G, then its character is

$$\chi_\rho(g) = tr(\rho(g))$$

which is a class function on G. In fact, the characters χ_ρ for irreducible ρ span the algebra of class functions on G.

(b) **The Frobenius substitution** [31]: Suppose K/k is an extension of number fields. Let $\mathfrak{p} \subset \mathfrak{o}_k$ be a prime ideal of k and $\mathfrak{P} \subset \mathfrak{o}_K$ a prime of K above $\mathfrak{p}$. Let

$$D_{\mathfrak{P}} = \{\sigma \in \mathrm{Gal}(K/k) \mid \sigma\mathfrak{P} = \mathfrak{P}\}$$

be the decomposition group for $\mathfrak{P}$. Then we have a short exact sequence

$$1 \longrightarrow I_{\mathfrak{P}} \longrightarrow D_{\mathfrak{P}} \longrightarrow \mathrm{Gal}(\kappa_{\mathfrak{P}}/\kappa_{\mathfrak{p}}) \longrightarrow 1$$

where we use $\kappa_{\mathfrak{P}}$ and $\kappa_{\mathfrak{p}}$ for the residue fields and $I_{\mathfrak{P}}$ is the inertia subgroup. Then $\mathrm{Gal}(\kappa_{\mathfrak{P}}/\kappa_{\mathfrak{p}})$ is a cyclic group generated by $\varphi_{\mathfrak{P}}$ defined by $\varphi_{\mathfrak{P}}(x) = x^{N(\mathfrak{p})}$. Then a *Frobenius substitution* $\sigma_{\mathfrak{P}}$ for $\mathfrak{P}$ is any inverse image of $\varphi_{\mathfrak{P}}$ in $D_{\mathfrak{P}}$. Note that

(i) $\sigma_{\mathfrak{P}}$ is well defined up to the inertia subgroup $I_{\mathfrak{P}}$, and if $\mathfrak{p} \not| \mathfrak{D}_{K/k}$ then $I_{\mathfrak{P}} = \{e\}$.

(ii) If $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are any two primes over $\mathfrak{p}$, say for $\mathfrak{p} \not| \mathfrak{D}_{K/k}$, then $\sigma_{\mathfrak{P}_1}$ and $\sigma_{\mathfrak{P}_2}$ are conjugate in $\mathrm{Gal}(K/k)$, so for any class function χ of $\mathrm{Gal}(K/k)$ we have

$$\chi(\sigma_{\mathfrak{P}_1}) = \chi(\sigma_{\mathfrak{P}_2}) = \chi(\sigma_{\mathfrak{p}})$$

where we can let $\sigma_{\mathfrak{p}}$ denote the conjugacy class of the $\sigma_{\mathfrak{P}}$ in $\mathrm{Gal}(K/k)$.

Now let us return to the formulas of Weber. First, if we look at the Euler product expansion of Weber's L-function

$$L(s,\chi)=\prod_{\mathfrak{p}}(1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1}$$

for χ a (abelian) ray class character, and look at the log of one Euler factor, we find the expression

$$\log L_{\mathfrak{p}}(s,\chi)=\log(1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s})^{-1}=-\sum_{\ell=1}^{\infty}\frac{\chi(\mathfrak{p})^{\ell}}{\ell N(\mathfrak{p})^{\ell s}}=-\sum_{\ell}\frac{\chi(\mathfrak{p}^{\ell})}{\ell N(\mathfrak{p})^{\ell s}}.$$

If we consider the right most expression here, then this makes perfect sense if we replace the abelain character χ by a non-abelian character χ_ρ associated to a representation ρ of $\mathrm{Gal}(k/k)$ and the prime $\mathfrak{p}$ by its Frobenius substitution $\sigma_{\mathfrak{p}}$, at least for the primes where $I_{\mathfrak{P}}$ is trivial, that is, for $\mathfrak{p}\not|\mathfrak{D}_{K/k}$. So this gave Artin his definition of the Artin Euler factors for $\mathfrak{p}\not|\mathfrak{D}_{K/k}$, namely

$$\log L_{\mathfrak{p}}(s,\chi_\rho,K/k)=-\sum_{\ell=1}^{\infty}\frac{\chi_\rho(\sigma_{\mathfrak{p}}^{\ell})}{\ell N(\mathfrak{p})^{\ell s}}$$

which then exponentiates to

$$L_{\mathfrak{p}}(s,\chi_\rho,K/k)=\det(I_n-N(\mathfrak{p})^{-s}\rho(\sigma_{\mathfrak{p}}))^{-1}$$

a very pleasing generalization of Weber's Euler factor, giving an Euler factor of degree n to a n-dimensional representation of the Galois group. And so Artin made the following first definition of his L-functions.

Definition: *Let $\rho:\mathrm{Gal}(K/k)\to GL_n(\mathbb{C})$ be a n-dimensional representation of K/k. Then set*

$$L(s,\rho)=L(s,\chi_\rho,K/k)=\prod_{\mathfrak{p}\not|\mathfrak{D}_{K/k}}\det(I_n-N(\mathfrak{p})^{-s}\rho(\sigma_{\mathfrak{p}}))^{-1}.$$

Artin himself used the notation $L(s,\chi_\rho,K/k)$. The first thing that Artin did was to make sure that this Euler product converged for $Re(s)>1$.

Given this definition, what were the important properties of these L-functions that Artin would have been interested in? Well, if we return to the formula of Weber and Hecke on factoring the zeta function, namely

$$\zeta_L(s)=\prod_{\chi\in\widehat{G}}L(s,\chi)\quad\text{for}\quad G=I_K(\mathfrak{m})/H$$

then conceptually this seems linked to the decomposition of the regular representation r_G of a group G coming from the group acting on itself by translation, namely

$$r_G=\bigoplus_{\rho\in\widehat{G}}(\dim\rho)\rho\qquad\text{or}\qquad\chi_{r_G}=\sum_{\rho\in\widehat{G}}\chi_\rho(1)\chi_\rho$$

where for a non-abelian group G, $\widehat{G}$ is the set of all irreducible representations of G. If he really expected these to correspond, what properties of his L-functions would he need?

(a) **Additivity**: The most obvious thing is that the L-functions should be additive with respect to direct sums of representations or sums of characters, that is

$$L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1)L(s, \rho_2)$$

or

$$L(s, \chi_1 + \chi_2, K/k) = L(s, \chi_1, K/k)L(s, \chi_2, K/k).$$

This would then give a factorization of the form

$$L(s, \chi_{r_G}, K/k) = \prod_{\rho \in \widehat{G}} L(s, \chi_\rho, K/k)^{\chi_\rho(1)}.$$

(b) **Inflation**: In the formula of Weber, on the right hand side the zeta function of K, ζ_K, corresponds to the factor coming from the trivial character $\mathbf{1}$ of G. In the decomposition of the regular representation, this factor would correspond to the trivial representation $\mathbf{1_G}$ of G. But the completed ζ_K really corresponds to the trivial representation $\mathbf{1}$ of $\mathrm{Gal}(k/k)$. So we expect that

$$L(s, \mathbf{1}_{\mathrm{Gal}(K/k)}) = \zeta_K(s) = L(s, \mathbf{1}_{\mathrm{Gal}(k/k)})$$

or

$$L(s, \chi_{\mathbf{1}}, K/k) = \zeta_K(s) = L(s, \chi_{\mathbf{1}}, k/k)$$

at least up to a finite number of Euler factors. These two trivial representations are related by the following diagram

$$\begin{tikzcd} \mathrm{Gal}(K/k) \arrow[r] \arrow[dr, "\mathbf{1}"] & \mathrm{Gal}(k/k) \arrow[r] \arrow[d, "\mathbf{1}"] & 1 . \\ & \mathbb{C}^\times & \end{tikzcd}$$

Then in general Artin expected his L-functions to behave well under *inflation*, that is, if we have extensions $k \subset M \subset K$, all Galois over k and $\rho : \mathrm{Gal}(M/k) \to GL_n(\mathbb{C})$, then it inflates to a representation of $\mathrm{Gal}(K/k)$ by the diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Gal}(K/M) \arrow[r] & Gal(K/k) \arrow[r] \arrow[dr, "\rho"] & \mathrm{Gal}(M/k) \arrow[r] \arrow[d, "\rho"] & 1 \\ & & & GL_n(\mathbb{C}) & \end{tikzcd}$$

and Artin wanted

$$L(s, \chi_\rho, K/k) = L(s, \chi_\rho, M/k) = L(s, \rho)$$

to be independent of which ever extension it factors through. This would then give

$$L(s, \chi_{r_G}, K/k) = \zeta_K(s) \prod_{\chi \neq \chi_{\mathbf{1}}} L(s, \chi_\rho, K/k)^{\chi_\rho(1)}.$$

(c) **Inductivity**: If we now think about the left hand side of of this equation in the same way and compare it with Weber, then we would want

$$L(s, \chi_{r_G}, K/k) = \zeta_L(s) = L(s, \chi_{\mathbf{1}}, K/K).$$

In terms of representation theory, the regular representation is induced from the trivial representation of the identity, that is

$$r_G = \mathrm{Ind}_{\{e\}}^G(\mathbf{1}_{\{e\}})$$

or for Galois groups

$$r_{\mathrm{Gal}(K/k)} = \mathrm{Ind}_{\mathrm{Gal}(K/K)}^{\mathrm{Gal}(K/k)}(\mathbf{1}_{\mathrm{Gal}(K/K)})$$

so Artin would want

$$L(s, r_G, K/k) = L(s, \mathrm{Ind}(\mathbf{1}), K/k) = L(s, \mathbf{1}, K/K).$$

In general Artin expected his L-functions to be *inductive*, that is, if we have extensions $k \subset M \subset K$ and $\rho_0 : \mathrm{Gal}(K/M) \to GL_m(\mathbb{C})$, then we can induce it to obtain $\rho : \mathrm{Gal}(K/k) \to GL_{m(M:k)}(\mathbb{C})$ by

$$\rho = \mathrm{Ind}_{\mathrm{Gal}(K/M)}^{\mathrm{Gal}(K/k)}(\rho_0),$$

with a diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(K/M) \arrow[r] \arrow[d, "\rho_0"] & \mathrm{Gal}(K/k) \arrow[d, "\rho=\mathrm{Ind}(\rho_0)"] \\
& GL_m(\mathbb{C}) & GL_{m(M;k)}(\mathbb{C})
\end{tikzcd} ,$$

and then Artin wanted

$$L(s, \rho, K/k) = L(s, \mathrm{Ind}(\rho_0), K/k) = L(s, \rho_0, K/M).$$

This would finally give him a factorization

$$\zeta_L(s) = \zeta_K(s) \prod_{\rho \neq \mathbf{1}} L(s, \chi_\rho, K/k)^{\chi_\rho(1)}$$

as he desired.

So in his 1923 paper Artin proved the following Theorem.

Theorem: (a) *Let ρ be a n-dimensional representation of* $\mathrm{Gal}(K/k)$*. Let*

$$L(s, \chi_\rho, K/k) = \prod_{\mathfrak{p} \nmid \mathfrak{D}_{K/k}} \det(I_n - N(\mathfrak{p})^{-s}\rho(\sigma_\mathfrak{p}))^{-1}.$$

Then $L(s, \chi_\rho, K/k)$ converges for $Re(s) > 1$.

(b) *The collection of $L(s, \chi_\rho, K/k)$, as L and ρ vary, are additive, satisfy inflation, and are inductive.*

Once you have the formulation, the theorem itself is not that hard to prove. It is a pleasing combination of number theory (splitting of primes in extensions) and representation theory (Frobenius reciprocity for induction and restriction of representations). Then as an immediate corollary, he obtains his factorization.

Corollary: *Let K/k be Galois with Galois group G. Then, up to a finite number of Euler factors,*

$$\zeta_L(s) = \zeta_K(s) \prod_{\substack{\rho \in \widehat{G} \\ \rho \neq \mathbf{1}}} L(s, \chi_\rho, K/k)^{\chi_\rho(1)}.$$

At this point, Artin returned to the case K/k abelian, so L is a class field for K, and he compared his factorization of $\zeta_L(s)$ to that of Weber. One way to phrase this question is:

Question: *Let K/k be a relatively abelian extension of number fields, with K/k the class field associated to the generalized ideal class group $I_k(\mathfrak{f})/H_\mathfrak{f}$. Is there is a bijection*

$$\{\chi : Gal(K/k) \to \mathbb{C}^\times\} \leftrightarrow \{\tilde{\chi} : I_k(\mathfrak{f})/H_\mathfrak{f} \to \mathbb{C}^\times\}$$

such that $L(s, \chi) = L(s, \tilde{\chi})$?

I state this in this way because it is parallel to the current statements of the Langlands conjectures (see below). Of course, by class field theory Artin knew that $Gal(K/k) \simeq I^\mathfrak{f}/H_\mathfrak{f}$ and so the character groups are the same. On the other hand, given the equality of the character groups and then looking at the *definition* of the individual Euler factors in the L-functions, Artin's definition evaluating the characters at the Frobenius elements while that of Weber and Hecke evaluating their characters at ideal classes, led Artin to formulate the General Reciprocity Law (or Artin Reciprocity Law), his "Satz 2".

"Theorem 2": *Suppose K/k is abelian and $\mathfrak{p} \nmid \mathfrak{D}_{K/k}$. Then*

(a) *The substitutetion $\sigma_\mathfrak{p}$ in* $\mathrm{Gal}(K/k)$ *depends only on the ideal class in the ray class group $I_K k\mathfrak{m})/N_{K/k}(I_K(\mathfrak{m}))P_k(\mathfrak{m})$ in which $\mathfrak{p}$ lies.*

(b) *The correspondence*

$$I_k(\mathfrak{m})/N_{K/k}(I_K(\mathfrak{m}))P_K(\mathfrak{m}) \to Gal(K/k)$$

sending $\mathfrak{p} \mapsto \sigma_\mathfrak{p}$ is bijective and gives an isomorphism of groups.

This is the statement of the general (Artin) reciprocity law. Artin cannot prove this at this time, except in cases where a reciprocity law was already known (such as cyclotomic extensions of $\mathbb{Q}$). However, he proceeds with the rest of the paper by *assuming* that this is true as stated.

At this point, Artin is interested in establishing continuation and functional equation for his new L-functions. To do this, he needs to use the various properties they satisfy, plus "Theorem 2", to relate them to abelian L-functions, where continuation and functional equations were known thanks to Hecke. He accomplished this by proving the following very beautiful result in group theory.

Proposition: *Let G be a finite group and let χ be a rational class function on G, that is a rational linear combination of irreducible characters. Then there exist cyclic subgroups $H_i \subset G$ and (necessarily) abelian characters ψ_i of H_i such that*

$$\chi = \sum_i a_i \mathrm{Ind}_{H_i}^G(\psi_i)$$

with $a_i \in \mathbb{Q}$.

Combining this with his reciprocity law ("Theorem 2") he arrived at the following corollary.

Corollary: *Let K/k be a Galois extension of number fields and $\rho : \mathrm{Gal}(K/k) \to GL_n(\mathbb{C})$. Then there exist cyclic subgroups $H_i \subset \mathrm{Gal}(K/k)$ and abelian characters ψ_i of H_i and rational numbers a_i such that*

$$L(s, \chi_\rho, K/k) = \prod_i L(s, \psi_i, K/K^{H_i})^{a_i}. \tag{1}$$

Remark: In 1947 Richard Brauer improved Artin's results on rational characters by showing that any such character can be written as

$$\chi = \sum_i n_i \mathrm{Ind}_{H_i}^G(\psi_i)$$

with the $n_i \in \mathbb{Z}$, a great improvement, but now where the H_i are *elementary* subgroups (not necessarily abelian), but the ψ_i are still one dimensional characters, hence factor through an abelian quotient. This would then give a factorization

$$L(s, \chi_\rho, K/k) = \prod_i L(s, \psi_i, K/K^{H_i})^{n_i}. \tag{2}$$

but now with integral exponents.

Returning to the Artin L-functions, since the continuation and functional equation of the abelian L-functions was known, after completing the L-function at the remaining primes, including the archimedean ones, Artin deduced the following result from the group theoretic result above, his "Theorem 2", and the work of Hecke.

Theorem: *Let* $\rho : \mathrm{Gal}(K/k) \to GL_n(\mathbb{C})$. *Then (some power of)* $L(s, \chi_\rho, K/k)$ *extends to a meromorphic functions of* s *and (after filling in the remaining Euler factors) satisfies a functional equation*

$$\Lambda(s, \rho, K/k) = \varepsilon(s, \rho, K/k)\Lambda(1 - s, \rho^\vee, K/k). \tag{3}$$

Note that the extra Euler factors that one fills in are those coming from the functional equations of Hecke for the abelian L-factors in the above expression (1) or (2). Artin then made the following conjecture about the analytic behavior of his L-functions

Conjecture: *If* ρ *is irreducible, but* $\rho \neq \mathbf{1}$, *then* $L(s, \rho, K/k)$ *is entire.*

Note that this would imply that the ratio $\zeta_K(s)/\zeta_k(s)$ would again be entire as in the abelian case! Artin then finishes out the paper by investigating and comparing some of the factorizations of zeta functions that he now has.

In 1927 Artin proved his reciprocity law, his "Theorem 2", in general. His L-functions played no role in the proof, but they were instrumental in the formulation as we have seen. Instead Artin proved the theorem by "crossing with a sufficient number of cyclotomic extensions", an idea which he credits to Chebotarev from his proof of the Chebotarev density theorem. With this in hand the conditional results in this first paper on L-functions become firmly established.

2.2. Artin *L*-functions – The 1930 Paper. Artin was not completely satisfied with his L-functions for non-abelian characters. At the primes $\mathfrak{p}$ that divided the discriminant of K/k, his formula (1) in terms of induced characters let him fill in the Euler factors for these primes from those for abelian L-factors. The same was true of the archimedean Γ-factors that one needed for a functional equation. So Artin took the task of finding an intrinsic definition of these factors, and this he did in this paper.

If he were to define an Euler factor for the primes dividing the discriminant and archimedean primes and have them agree with the factors that appear in the functional equation coming from the abelian Euler factors (3), then these factors would also need to behave formally as his original ones did. So his guiding principles remained

(1) additivity,

(2) inflation,

(3) inductivity,

(4) agreement with the abelian case.

Let us first look at $\mathfrak{p}$ that divide the discriminant $\mathfrak{D}_{K/k}$. The prime is no longer unramified in the extension L, for each prime $\mathfrak{P}$ above $\mathfrak{p}$ we have a non-trivial inertia group $I_{\mathfrak{P}}$. One thing that Artin realized was that prime-by-prime, the Euler factor he had previously defined only depended on $\rho_{\mathfrak{p}} = \rho|_{D_{\mathfrak{P}}}$, the restriction of the representation to the decomposition group at $\mathfrak{P}$. Once he does this, then $I_{\mathfrak{P}}$ is a normal subgroup of $D_{\mathfrak{P}}$. As a first step, consider representations ρ of $\mathrm{Gal}(K/k)$ which are *unramified* at $\mathfrak{p}$, that is, so that $\rho|_{I_{\mathfrak{P}}}$ is trivial. Then the local representation $\rho_{\mathfrak{P}}$ of $D_{\mathfrak{P}}$ should factor through the fixed points of $I_{\mathfrak{P}}$. Recalling that $D_{\mathfrak{P}} \simeq \mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}})$ then we would have

$$\begin{tikzcd}
1 \arrow[r] & I_{\mathfrak{P}} \arrow[r] & \mathrm{Gal}(K_{\mathfrak{P}}/k_{\mathfrak{p}}) \arrow[r] \arrow[d, "\rho_{\mathfrak{P}}"] & \mathrm{Gal}(K_{\mathfrak{P}}^{I_{\mathfrak{P}}}/k_{\mathfrak{p}}) \arrow[r] \arrow[dl, "\rho_{\mathfrak{P}}"] & 1 \\
& & GL_n(\mathbb{C}) & &
\end{tikzcd}$$

but now the extension $K_{\mathfrak{P}}^{I_{\mathfrak{P}}}/k_{\mathfrak{p}}$ is unramified, and so Artin has a definition of an Euler factor here, $L(s, \rho_{\mathfrak{P}}, K_{\mathfrak{P}}^{I_{\mathfrak{P}}}/k_{\mathfrak{p}})$ and by inflation this should also agree with $L(s, \rho_{\mathfrak{P}}, K_{\mathfrak{P}}/k_{\mathfrak{p}})$. So he knows what the Euler factor at a prime for which ρ is unramified should be, whether or not the prime ramifies in K.

More generally, Artin looked at representations $\rho : \mathrm{Gal}(K/k) \to GL_n(\mathbb{C})$ that were trivial on a normal subgroup H of $\mathrm{Gal}(K/k)$. The we have

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] & Gal(K/k) \arrow[r] \arrow[d, "\rho"] & \mathrm{Gal}(K^H/k) \arrow[r] \arrow[dl, "\rho"] & 1 \\
& & GL_n(\mathbb{C}) & &
\end{tikzcd}$$

and so $L(s, \chi_\rho, K/k) = L(s, \chi_\rho, K^H/k)$. At this point, Artin analyses what this formula says from the point of view of the original representation and, after doing a bit of representation theory and number theory, sees that

$$L_{\mathfrak{p}}(s, \chi_\rho, K/k) = \det(I - N(\mathfrak{p})^{-s}\rho(\sigma_{\mathfrak{p}})|V_\rho^H)^{-1}$$

so that the L-function agrees with the inverse of the characteristic polynomial of Frobenius on the fixed points of H.

Returning to the Euler factor at a prime $\mathfrak{p}|\mathfrak{D}_{K/k}$, Artin now *defines* the Euler factor by

$$L_{\mathfrak{p}}(s, \rho, K/k) = L(s, \rho_{\mathfrak{P}}, K_{\mathfrak{P}}/k_{\mathfrak{p}}) = \det(I - N(\mathfrak{p})^{-s}\rho(\sigma_{\mathfrak{p}})|V_\rho^{I_{\mathfrak{P}}})^{-1},$$

whether the representation is unramified at $\mathfrak{p}$ or not, and shows that with this formula the L-factor at $\mathfrak{p}$ is additive, satisfies inflation, is inductive, and agrees with the abelian case, and in fact is consistent with his former definition at the primes which are unramified in K/k. Hence he can now define

$$L(s, \chi_\rho, K/k) = \prod_{\mathfrak{p}} (I - N(\mathfrak{p})^{-s}\rho(\sigma_{\mathfrak{p}})|V_\rho^{I_{\mathfrak{P}}})^{-1}$$

and know that this is consistent with the Euler factors at the $\mathfrak{p}|\mathfrak{D}_{K/k}$ which were forced on him by his relation with abelian L-functions (1) or (2) and the functional equation (3).

Artin now applies a similar reasoning for the archimedean places. If v is an archimedean place of k, he knows the archimedean Γ-factors should only depend on the restriction of ρ to the decomposition group D_w for w a place of K above v. But the archimedean decomposition groups are quite simple:

$$D_w = \begin{cases} \{1\} & K_w = k_v \\ \{1, \sigma\} & k_v = \mathbb{R}, K_w = \mathbb{C} \end{cases}$$

where σ is complex conjugation, which Artin takes to be the Frobenius substitution for $\mathbb{C}/\mathbb{R}$. They are all abelian. So if $\rho : \mathrm{Gal}(K/k) \to GL_n(\mathbb{C})$ then $\rho|_{D_w}$ would decompose into a sum of one dimensional representations. Additivity would then let him focus on these one dimensional abelian representations. And enforcing compatibility with the abelian situation Artin then defines

$$\Gamma_v(s, \chi, K/k) = \begin{cases} \left(\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\right)^{\chi(1)} & v \text{ complex} \\ \left(\Gamma\left(\frac{s}{2}\right)\right)^{\frac{\chi(1)+\chi(\sigma)}{2}} \left(\Gamma\left(\frac{s+1}{2}\right)\right)^{\frac{\chi(1)-\chi(\sigma)}{2}} & v \text{ real} \end{cases}$$

and then proceeds to show that these factors are additive, satisfy inflation and are inductive. Since they were constructed to agree with the abelian factors, he has that they satisfy all the formal properties to see that they give the same factors for the functional equation as those that were previously forced on him by his relations with abelian L-functions (1) or (2) and the functional equation (3).

In the modern formulation, we define the archimedean L-factors a bit differently. To be more compatible with the modern theory of automorphic forms, we set

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \quad \text{and} \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Then if we decompose

$$\rho_w = \rho|_{D_w} = n_+\mathbf{1} \oplus n_- sgn$$

then up to the factors of π and 2 that we have inserted, Artin's factors are essentially

$$L(s, \rho_v, K/k) = \begin{cases} \Gamma_{\mathbb{R}}(s)^{n_+}\Gamma_{\mathbb{R}}(s+1)^{n_-} & k_v = \mathbb{R} \\ \Gamma_{\mathbb{C}}(s)^{\dim(\rho)} & k_v = \mathbb{C} \end{cases}.$$

Now that he has defined Euler factors for all places, including the archimedean ones, Artin now has a completed L-function

$$\Lambda(s, \rho, K/k) = \prod_v L(s, \rho_v, K/k) = \prod_v L_v(s, \chi_\rho, K/k)$$

and since each local factor satisfies the formalism of additivity, inflation, inductivity, and is compatible with the abelian case, then Artin knows that this completed

L-function is the same as the one he had in his 1923 paper, defined purely from the relation with abelian L-functions (1) and the functional equation (3). In particular, he retains the functional equation

$$\Lambda(s,\rho,K/k)=\varepsilon(s,\rho,K/k)\Lambda(1-s,\rho^\vee,K/k)$$

from before.

Looking at this functional equation, Artin now has an intrinsic definition of each Euler factor coming from the structure of ρ and the arithmetic of K/k. However, the exponential factor in the functional equation, the $\varepsilon(s,\rho,K/k)$ is still defined indirectly through the relation with the abelain L-functions and their functional equation. Artin turns to the analysis of this factor in his 1931 paper.

2.3. The Artin Conductor – The 1931 Paper. One can infer from the submission dates that the 1930 and the 1931 paper were written simultaneously. The submission date for the 1930 paper is October 1930 and for the 1931 paper it is November 1930. In fact, the 1931 paper was probably finished first, since it is referred to in detail in the discussion of the functional equation in the 1930 paper. In many ways, the 1931 paper is most interesting from the point of view of number theory of local fields.

In this paper, for each representation $\rho:\mathrm{Gal}(K/k)\to GL_n(\mathbb{C})$, with character $\chi=\chi_\rho$, and each finite prime $\mathfrak{p}_v$ of K, Artin defines the local Artin conductor,

$$\mathfrak{f}_v(\chi,K/k)=\mathfrak{p}_v^{f_v(\chi,K/k)},$$

which is an integral ideal in $\mathfrak{o}$. The definition of $f_v(\chi,K/k)$ is quite intricate and depends on the structure of the restriction of ρ to all of the higher ramification groups of $\mathrm{Gal}(L_\mathfrak{P}/K_\mathfrak{p})$. It is not at all clear that what he initially defines is an integer, so $\mathfrak{f}_v(\chi,K/k)$ integral ideal, and he works quite hard to show this. The integer $f_v(\chi,K/k)$ is zero if $\mathfrak{p}_v$ is unramified in L. Artin then defines the global Artin conductor as the finite product

$$\mathfrak{f}(\chi,K/k)=\prod_{v<\infty}\mathfrak{f}_v(\chi,K/k)\subset\mathfrak{o}.$$

Throughout this investigation, he is guided by his usual principles: that the formula be consistent with additivity, inflation, inductivity, and the abelian conductor. One of the things that makes this challenging is that the term that appears in the abelian functional equation is not just the conductor $\mathfrak{f}_\chi$ of χ, as defined above, but rather it is the product

$$|\mathfrak{D}_{k/\mathbb{Q}}|N_{k/\mathbb{Q}}(\mathfrak{f}_\chi).$$

It is this factor, or its non-abelian analogue, that needs to be additive, satisfy inflation, and be inductive. So for the conductor, these properties become

(i) Additivity: $\mathfrak{f}(\chi_1+\chi_2,K/k)=\mathfrak{f}(\chi_1,K/k)\mathfrak{f}(\chi_2,K/k)$

(ii) Inflation: If $k \subset M \subset K$ and ρ is inflated from a representation of $\mathrm{Gal}(M/k)$ then $\mathfrak{f}(\chi_\rho, K/k) = \mathfrak{f}(\chi_\rho, M/k)$

(iii) Inductivity: If $k \subset M \subset K$ and the representation ρ of $\mathrm{Gal}(K/k)$ is induced from a representation ρ_0 of $\mathrm{Gal}(K/M)$ then

$$\mathfrak{f}(\chi_\rho, K/k) = \mathfrak{D}_{M/k}^{\chi_{\rho_0}(1)} N_{M/k}(\mathfrak{f}(\chi_{\rho_0}, K/M)).$$

and indeed Artin's seemingly complicated definition of the conductor is constructed so that these properties hold. While I have not found time checked the details, it seems fairly certain to me that if one considers the inductivity relation, or more probably the local version for the exponent, then the seemingly complicated formula that Artin gives for the local conductor is accounted for by the fact that as one induces from various subfields, the conductor must vary in parallel with the discriminant, and hence it must register the changes in ramification within the extension.

There are two interesting observations that Artin makes here. First, in the formula for inductivity, if one starts with $\rho_0 = \mathbf{1}$ the trivial representation of $\mathrm{Gal}(K/M)$, then $\mathfrak{f}(\chi_{\mathbf{1}}, K/M) = \mathfrak{o}$ and this formula becomes a formula for the discriminant of M/k, namely

$$\mathfrak{D}_{M/k} = \mathfrak{f}(\chi_\rho, K/k) = \prod_i \mathfrak{f}(\chi_i, K/k)^{g_i}$$

if $\chi_\rho = \chi_{\mathrm{Ind}(\mathbf{1})} = \sum g_i \chi_i$ as a sum of irreducible characters. Also, the analogue of the factorization of the zeta function $\zeta_K(s)$ by Weber now becomes the conductor-discriminant relation of Artin and Hasse (see Satz 16 of [34], Teil I), namely

$$\mathfrak{D}_{K/k} = \prod_{\rho \in \widehat{\mathrm{Gal}(K/k)}} \mathfrak{f}(\chi_\rho, K/k)^{\chi_\rho(1)}.$$

Unfortunately, or interestingly, Artin did not quite fulfill his goal of explaining the factor appearing in the functional equation of his L-functions. By its formal properties, Artin's conductor is compatible with the conductor term that comes from the relation with abelian L-series and Hecke's functional equation. But there remained the mysterious sign in the functional equation and this Artin was not able to give an intrinsic explanation for. In the end, his formula for the term in the functional equation is

$$\varepsilon(s, \chi_\rho, K/k) = W(\rho)[|\mathfrak{D}_{k/\mathbb{Q}}|^{\chi_\rho(1)} N_{k/\mathbb{Q}}(\mathfrak{f}(\chi_\rho, K/k))]^{-(s-1/2)}$$

where $W(\rho)$ is a remaining global constant with $|W(\rho)| = 1$, the so-called Artin root number. It is the sole remnant of the functional equation for the abelian L-factors of Hecke without an intrinsic definition in terms of ρ. Its factorization would have to wait for the work of Pierre Deligne in 1971, but even then, Deligne's factorization of $\varepsilon(s, \rho, K/k)$ was an existence and uniqueness statement, and the local factors remain somewhat mysterious.

2.4. Final Comments. Before I leave Artin for a minute, I want to reiterate a few points. First I want to call attention to Artin's keen interest in a *non-abelian class field theory*, stated as the desire extend results from relative abelian extensions to general extensions of number fields. Next, Artin's taste for *group theoretic methods*, as evidenced in his use of the work of Frobenius and his group theoretic description of the "Artin conductor". Finally, *the primacy of the Euler product*. To my knowledge, Artin's L-functions are the first L-functions whose primary description is through the Euler product. We are familiar with this now through the L-functions associated to, say, elliptic curves E and, more generally, motives M. In these cases there is a definition of the Euler factors $L_p(s, M)$ via purely local information which then form a global L-function through the Euler product $L(s, M) = \prod L_p(s, M)$. Continuation and functional equation are only conjectural. Then these L-functions, through their continuation and functional equations interpolate this local input into global information – they are an analytic embodiment of a local/global principle. This is the content of conjectures like those of Birch and Swinnerton-Dyer and Bloch and Kato. This paradigm shift (Dirichlet series $\to$ Euler product) is to my mind profound, yet rarely mentioned.

3. A Hecke Interlude

While Artin was involved with the creation of his L-function, Hecke was involved in thinking about L-series from another perspective. In 1918 he had introduced his Größencharaktere and their L-series $L(s, \chi)$ [40]. Throughout his career he was interested in modular forms and their applications to arithmetic. In the 1930's, and in particular in a 1936 paper, be initiated the study of L-series attached to modular forms [41, 43].

In their simplest guise, modular forms are holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$ on the upper half plane $\mathfrak{H}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \qquad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = SL_2(\mathbb{Z}).$$

(They are also required to satisfy holomorphy at the "cusps" of Γ.) The integer k appearing in the transformation law is called the *weight* of the modular form. These will have a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$$

and are called *cuspidal* if $a_0 = 0$. Via examples (such as theta series attached to a quadratic form) these modular forms were known to carry arithmetic information in their Fourier coefficients – but their general arithmetic nature is still a bit mysterious. (Although later Martin Eichler was to declare that there were five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms.)

Hecke associated to such a cusp form a Dirichlet series

$$L(s,f)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

which he showed converged in a half plane $Re(s) >> 0$. Hecke had the advantage of having an analytic relation between $f(\tau)$ and $L(s,f)$, namely

$$\Lambda(s,f)=(2\pi)^{-s}\Gamma(s)L(s,f)=\int_0^{\infty} f(iy)y^s\ d^{\times}y.$$

Through this integral representation, it was relatively easy to derive the analytic properties of $L(s,f)$ from the modular properties of $f(\tau)$. Hecke proved that the $L(s,f)$ were "nice":

(1) $L(s,f)$ continues to an entire function of s,

(2) $L(s,f)$ is of order one as an entire function,

(3) $L(s,f)$ satisfies a functional equation

$$\Lambda(s,f)=i^k\Lambda(k-s,f).$$

Moreover, he turned around and proved his celebrated

Converse theorem: *These analytic properties (particularly the functional equation) characterize the L-series attached to modular forms of weight k for $SL_2(\mathbb{Z})$.*

It was in 1936 at the International Congress of Mathematicians [42, 43] that Hecke introduced an algebra of operators $\mathcal{H}=\{T_n\}$, indexed by the integers, acting on modular forms and proved that if $f(\tau)$ was a simultaneous eigen-function for the operators in $\mathcal{H}$ then $L(s,f)$ possessed an Euler product of degree 2:

$$L(s,f)=\prod_p(1-a_pp^{-s}+p^{-2s})^{-1}.$$

Note that the theories of Artin and Hecke seem to be two sides of the same coin:

- Artin's L-functions $L(s,\chi)$ are
  - defined by an Euler product,
  - their analytic properties are mysterious (conjectural),
  - but they are blatantly arithmetic.
- Hecke's L-series $L(s,f)$ are
  - defined by a Dirichlet series,

– their analytic properties are easy to establish and in fact characterize them,

– but their Euler product expansion and arithmetic meaning are more mysterious.

Even though Artin left Hamburg in 1937, there was sufficient overlap that it would have been quite natural for them to dream of a connection between their two L-functions – what we would now call the *modularity of Galois representations.* As John Tate pointed out [70], results were in place to let them establish such a connection for two dimensional dihedral Galois representations, however, as far as I know, there is no record of them even discussing the subject. (Note that Artin published no papers from 1933 to 1940, which includes this period of Hecke's work.)

4. A Historical Interlude

Before turning to the modern version of Hecke's theory, I want to record some developments on both the Artin and Hecke side of the picture.

In 1934, Chevalley introduced the group of ideles of a number field and gave the first purely algebraic proof of the class field theory using them [12, 13, 14]. For the field of rational numbers, the ideles are a restricted product of the multiplicative groups of all the completions of $\mathbb{Q}$, i.e., $\mathbb{A}^\times = \mathbb{R}^\times \times \prod_p' \mathbb{Q}_p^\times$. Then in 1945, Artin and George Whaples introduced their "valuation vectors" [8], which were essentially the additive version of Chevalley's ideles, now called *adeles* after Weil. As above, for the field $\mathbb{Q}$ these are the restricted product of the completions $\mathbb{A} = \mathbb{R} \times \prod_p' \mathbb{Q}_p = \prod_v' \mathbb{Q}_v$, which can be viewed as vectors indexed by the valuations of $\mathbb{Q}$. The field $\mathbb{Q}$ then embeds into each of its completions and thus embeds diagonally in $\mathbb{A}$; its image is a discrete subgroup with compact quotient.

Brauer proved the meromorphy of Artin's L-functions in 1947 [10]. In essence, Brauer proved the group theoretic fact that one could write

$$\chi = \sum_i n_i \mathrm{Ind}(\chi_i)$$

with the χ_i abelian characters, but now with $n_i \in \mathbb{Z}$. (Compare this with Artin's result above.) Then, following Artin, when one writes

$$L(s,\chi) = \prod_i L(s,\chi_i)^{n_i}$$

one obtains immediately the meromorphy of the Artin L-functions themselves, not powers of them.

In 1950 we have the appearance of Tate's Thesis [69]. Tate was a student of Artin at Princeton. Tate's thesis was essentially to give an adelic theory of Hecke's L-series attached to the Hecke characters. One outcome of Tate's thesis is that each Hecke character χ factors adelically $\chi = \otimes' \chi_v$ into local characters. In fact, this result was proven earlier by one of Artin's previous students, Margaret

Matchett, in her thesis at the University of Indiana, which one can find in this volume [63]. Tate then gives a *purely local* analysis of each individual Euler factor $L_p(s,\chi) = L(s,\chi_p)$ which then bridges one bit of the Artin-Hecke paradigm:

$$L(s,\chi) = \prod_p L(s,\chi_p)$$

can be *defined* as an Euler product! Then the analytic properties (continuation, functional equation, etc.) are proved *globally* via Poisson summation à la Hecke. So, if Chevalley did indeed introduce ideles to banish analysis from class field theory, Artin, through his students Matchett and Tate, turned around and made them the basis for the analytic side of class field theory. Moreover, it is the paradigm of Tate's thesis that we see repeated in the modern formulation of automorphic forms and their L-functions.

As an aside, in 1951 Weil pointed out that the Artin L-functions for one dimensional Galois representations (characters) did not cover all of the L-series attached to Hecke's Größencharaktere [75]. To remedy this, Weil introduced an extension of the (absolute) Galois group (locally and globally), now called the Weil group and denoted by W_k, and extended Artin's L-functions to representations of the Weil group W_k. These were then in a natural 1-1 correspondence with the L-series of Hecke's characters. Here we see the analytic theory expanding the boundaries of the algebraic side.

Finally, in 1967, Weil extended Hecke's converse theorem to modular forms of level N, i.e., for the Hecke congruence groups $\Gamma_0(N)$ [76]. Using this, Weil made precise a conjecture of Taniyama on the *modularity* of the (degree 2) L-function $L(s,E)$ attached to an elliptic curve E over $\mathbb{Q}$.

5. Artin and Hecke Reconciled: The Langlands Program

Beginning in the 1950-60's Hecke's theory of modular forms and their L- series underwent a paradigm shift similar to that brought about by Tate's thesis. Modular forms were adelized (Ichiro Satake, Tsuneo Tamagawa, etc. in Japan) and then converted into a theory of *automorphic representations* by Israel M. Gelfand and Ilya I. Piatetski-Shapiro in Russia, Rogr Godement and Hervé Jacquet in France and Jacquet and Robert P. Langlands in the US. In this new point of view, the modular form $f(\tau)$ of Hecke is replaced by an automorphic representation π of $GL_2(\mathbb{A})$ in a space of functions, called automorphic forms, one realization of which is

$$L^2(Z(\mathbb{A})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})).$$

The locally compact group $GL_2(\mathbb{A})$ acts in this space by right translation and one can then spectrally decompose this space into irreducible representations π. As the adelic group decomposes

$$GL_2(\mathbb{A}) = GL_2(\mathbb{R}){\prod_p}' GL_2(\mathbb{Q}_p)$$

so the representation π decomposes into local representations $\pi = \otimes' \pi_v$ with π_v an irreducible representation of $GL_2(\mathbb{Q}_v)$. Beware: in general these representations are all *infinite dimensional*. Once you have made this paradigm shift, then it is an easy matter to replace the adeles of $\mathbb{Q}$ with the adeles of any number field k and the algebraic group GL_2 by other algebraic groups H, such as GL_n, orthogonal groups, symplectic groups, etc. This group representation point of view probably would have been approved of by Artin, who as we pointed out seemed to be rather fond of group theoretic techniques.

While investigating the theory of Eisenstein series (a special type of modular form or automorphic representations) on general groups $H(\mathbb{A})$, Langlands noticed a similarity between certain Euler products that were arising and the Euler products of Artin (c.f. [56]). This led to his first formulation in 1968 of what is now known of as *The Langlands Program* [57]. There are three main pillars of this program:

I. *The Langlands Correspondence.* This correspondence associated to a reductive algebraic group H, defined over an algebraic number field k, a complex dual group ${}^L H$ and posited a natural correspondence between

$$\left\{\begin{matrix}\text{admissible homomorphisms}\\ \phi : W_k \to {}^L H\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{automorphic representations}\\ \pi \text{ of } H(\mathbb{A})\end{matrix}\right\}$$

and compatible local correspondences between

$$\left\{\begin{matrix}\text{admissible homomorphisms}\\ \phi_v : W_{k_v} \to {}^L H\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{admissible representations}\\ \pi_v \text{ of } H(k_v)\end{matrix}\right\}.$$

One should view the left hand side of this correspondence as special types of Galois representations. For $H = GL_1$ the dual group is ${}^L H = GL_1(\mathbb{C})$ and this becomes a reformulation of abelian class field theory. For $H = GL_n$ the dual group is ${}^L H = GL_n(\mathbb{C})$ and on the left hand side we essentially find the complex n-dimensional Galois representations for which Artin defined his L-functions. Langlands could prove the local correspondence at the finite places when the representation π_v of $H(k_v)$ was *unramified* by reinterpreting results of Satake. One could think of this as giving an *arithmetic parametrization* of automorphic representations, much as Weil had for $n = 1$. Viewed in the other direction, it becomes a statement about the *modularity of certain Galois representations* and it is this view that is connected with a non-abelian class field theory.

II. *The Langlands L-functions.* The correspondence then allowed Langlands to *use Artin* (channeled through Weil) to associate to any π and any representation of the dual group

$$r : \; {}^L H \to GL_n(\mathbb{C})$$

an Euler factor, given at the unramified places by

$$L(s, \pi_v, r) = L(s, r \circ \phi_v) = \det(I - r \circ \phi_v(\sigma_{\mathfrak{P}}) N(\mathfrak{p})^{-s})^{-1}$$

and an Euler product, or rather, a partial Euler product ... just as Artin did originally

$$L(s, \pi, r) = \prod_v L(s, \pi_v, r) = \prod_v L(s, r \circ \phi_v) = L(s, r \circ \phi)$$

which is naturally an Artin L-function! This, as in Tate's thesis, gives an Euler product definition of L-functions associated to these general automorphic representations. *However*, in contrast to Tate, there is no global construction of the $L(s, \pi, r)$ to give their analytic properties. So again Langlands was faced with simply *conjecturing* that these had a continuation, functional equation, etc. However, one hopes for an independent, analytic definition of these automorphic L-functions, either through Eisenstein series as in Langlands work or through integral representations as in Hecke. Then we could view this as saying that the correspondence conjectured above, the modularity of the Galois representations, should be mediated by an equality of L-functions, those of Artin and those of Hecke.

III. *The Principle of Functoriality.* Functoriality is an automorphic form phenomenon that is a consequence of thinking about the Langlands conjectures as giving an arithmetic parametrization of automorphic representations. It is associated to what is called an L-homomorphism between L-groups

$$u : \ {}^L H \to {}^L G.$$

This gives you a straightforward mechanism for transferring the arithmetic parameters, or Galois representations, by composition and then the automorphic forms π for $H(\mathbb{A})$ are carried through the Langlands correspondence automorphic forms Π for $G(\mathbb{A})$. Conceptually, the global functoriality diagram is then

$$\begin{tikzcd}
 & & {}^L H \arrow[rr, "u"] & & {}^L G & & \\
\pi \arrow[r, mapsto] & {} & & & & {} \arrow[r, mapsto] & \Pi \\
 & & & W_k \arrow[uul, "\phi"] \arrow[uur, "\Phi"'] & & &
\end{tikzcd} .$$

and it would come with an equality of L-functions for each $r : {}^L G \to \mathbb{C}$ given by

$$L(s, \pi, r \circ u) = L(s, r \circ u \circ \phi) = L(s, r \circ \Phi) = L(s, \Pi, r)$$

along with a list of other compatibility conditions. In particular, let us point out that this is mediated by an equality of Artin L-functions! There is also a local diagram and the local and global formulations should be compatible. As Langlands likes to phrase things, the Langlands correspondence is simply Functoriality where one takes the first group, H in our diagram, to be the trivial group so that ${}^L H = W_k$ and ϕ is the identity. So the Principle of Functoriality encompasses the entire program.

This program has been instrumental in determining the course of research in not only the theory of automorphic forms but throughout a broad swath of number theory. From the formulation one can see that it has part of its roots in both Artin's L-functions and Hecke's L-functions and in turn has impacted our understanding of them.

We have emphasized the local/global nature of Artin's L-functions – the primacy of the Euler product expansion. However there was one part of the theory that had only a global definition, namely the Artin root number $W(\chi)$ occurring in the functional equation. Bernard Dwork addressed the local/global nature of the Artin root number in 1956 [30], but in the 1970's this was revisited by Langlands [58] and Deligne [24].

In 1971 Jacquet and Langlands' book on *Automorphic Forms on* $GL(2)$ appeared [48]. This could be characterized as Tate's thesis for $GL(2)$. It extended Hecke's global theory L-functions attached to modular forms to automorphic representations π of $GL_2(\mathbb{A})$, giving first a local analytic definition of $L(s, \pi_v)$ as in Tate's thesis, then a global analytic definition of $L(s, \pi) = \prod L(s, \pi_v)$ via an integral representation, so the adelic reformulation of Hecke. This let them prove that $L(s, \pi)$ had continuation, functional equation, etc. as in Hecke. They then went on to prove a GL_2–converse theorem as in Hecke characterizing these L-functions. From it they were able to then deduce the modularity of two dimensional dihedral Galois representations.

In 1973, Langlands proved the local Langlands correspondence when the local field was archimedean, i.e., $\mathbb{R}$ or $\mathbb{C}$ [59]. Since it is the local L-function at the archimedean places that corresponds to the "gamma factors" that appeared in the functional equation of the Artin L-functions, this is the analogue for the Langlands program of Artin's explanation of these "gamma factors" in his second paper on the Artin L-functions.

In a series of papers through the 1970-80's, Jacquet, Piatetski-Shapiro, and Joseph Shalika extended the work of Jacquet and Langlands to automorphic representations π of $GL_n(\mathbb{A})$ [49, 50, 51], giving again a local analytic definition of $L(s, \pi_v)$ as in Tate's thesis and a global analytic definition of $L(s, \pi) = \prod L(s, \pi_v)$ via an integral representation, again an adelic reformulation of Hecke, which let them prove that $L(s, \pi)$ had continuation, functional equation, etc. Then they were able to prove a GL_3–converse theorem characterizing these L-functions in terms of their analytic properties and to deduce from it the modularity of three dimensional dihedral Galois representations.

The first serious assault on Functoriality was Langlands' "Base Change for GL_2" which appeared in 1980 [60]. Through the Langlands correspondence and functoriality, any natural construction on Galois representations should have a parallel construction on the level of automorphic forms. On the Galois side you can restrict representations to subgroups and Base Change is the parallel automorphic construction. Langlands used the trace formula, his favorite tool, to establish this in the cyclic and thus solvable cases for the group GL_2. One consequence of this was the modularity of tetrahedral Galois representations by Langlands and the subsequent proof of the modularity of octahedral Galois representations by Gerald Tunnell [71]. From this one can deduce the Artin conjecture for the Artin L-functions, since the associated automorphic forms are cuspidal and hence have entire L-functions. Both proofs begin with the previously established modularity of the three dimensional dihedral Galois representations. Then, given the existence of base change, the proof of modularity amounted to analyzing the structure of these

Galois groups and then finding enough parallel Galois theoretic and automorphic constructions to boot-strap the modularity up to the desired representation. In particular, the formal properties of the Artin L-functions established by Artin played a definite role.

In the 1990's we completed part of Hecke's program for GL_n by proving some converse theorems [19, 20]. These results characterize L-functions for automorphic representations of $GL_n(\mathbb{A})$ in terms of their analytic properties. Morally, what this then lets you say (much as Weil did) is that any nice L-function of degree n (as Artin's are conjectured to be) should be modular!

After this work we once again had both

• an arithmetic theory of L-functions for representations of $Gal(\overline{k}/k)$ or W_k following Artin,

• an analytic theory of L-functions for GL_n following Hecke.

Now we are able to make a more precise, but still not final, formulation of the Artin-Hecke link.

The Global Langlands Conjecture: *There should be compatible bijective correspondences between n-dimensional Galois representations and automorphic representations of $GL_n(\mathbb{A})$*

$$\{\rho : W_k \to GL_n(\mathbb{C})\} \leftrightarrow \{\textit{automorphic representations } \pi \textit{ of } GL_n(\mathbb{A})\}$$

such that $L(s, \chi_\rho) = L(s, \pi)$, among other desirable conditions.

Note the similarity between this formulation of the Langlands correspondence and our first formulation of Artin's reciprocity law in Section 2. For $n = 1$ this follows from Artin Reciprocity. Since now both the Artin and the Hecke side also comes with a compatible local definition of Euler factors, we can formulate a purely local version of the correspondence.

The Local Langlands Conjecture: *There should be compatible local bijective correspondences*

$$\{\rho_v : W_{k_v} \to GL_n(\mathbb{C})\} \leftrightarrow \{\textit{admissible representations } \pi_v \textit{ of } GL_n(k_v)\}$$

such that $L(s, \rho_v) = L(s, \pi_v)$, among other desirable conditions.

To be precise, in 1972 Deligne realized that even locally for GL_2 the representations of the Weil group W_{k_v} were not enough to parametrize admissible representations of $GL_2(k_v)$ [23]. He extended W_{k_v} to what is now known as the Weil-Deligne group W'_{k_v}. There is no known global analogue – this would be the conjectural Langlands group L_k. So there remains a well posed Local Langlands Conjecture, but a precise formulation of the Global Conjecture would rely on this conjectural group L_k. On the other hand, the local conjecture is now a theorem.

Theorem (Local Langlands Conjecture): *There exists canonical compatible bijections*

$$\{\rho_v : W'_{k_v} \to GL_n(\mathbb{C})\} \leftrightarrow \{\textit{admissible representations } \pi_v \textit{ of } GL_n(k_v)\}$$

for all n satisfying
– for each n we have $L(s, \chi_{\rho_v}) = L(s, \pi_v)$
– the $n = 1$ correspondence is given by local abelian reciprocity law
plus a list of other natural compatibility conditions.

This was proved by Michael Harris and Richard Taylor using a combination of local and global techniques and the cohomology of Shimura varieties [33]. A simplified proof was later given by Guy Henniart [45], still using geometry and a combination of local and global techniques but with a greater reliance on the analytic properties of the L-functions, that is, the work of Artin. Henniart essentially began with the local version of Brauer's relation

$$\chi = \sum_i n_i \mathrm{Ind}(\chi_i)$$

where $n_i \in \mathbb{Z}$ and the χ_i are abelian characters, and showed that if we view the χ_i as Hecke characters, so GL_1 automorphic forms, through abelian class field theory, then by analyzing parallel Galois/automorphic constructions combined with parallel properties of the Artin/automorphic L-functions he is able to establish that

$$\sigma = \sum_i n_i ss(AI(\chi_i))$$

is a supercuspidal representation of $GL_n(k_v)$. Here ss represents *supercuspidal support* and AI stands for the process of *automorphic induction.* (Automorphic induction is the automorphic construction that parallels simple induction on the Galois representation side, and as we have seen Artin investigated the behavior of his L-functions under induction.) Then if $\chi = \chi_{\rho_v}$ and $\sigma = \pi_v$ then $\rho_v \leftrightarrow \pi_v$ is the basis of the correspondence of the theorem.

This correspondences can be thought of as *local modularity of Galois representations* or a *local non-abelian reciprocity law.* This is a partial answer to Artin's desired non-abelian class field theory. This correspondence is now *mediated* via the equality of Artin and Hecke L-functions, although this could not have been formulated by Artin and Hecke since they did not have a purely local definition of the Hecke Euler factors – this is the impact of Tate's thesis under Artin. If we replace number fields by function fields of curves over finite fields (as Artin considered in his thesis!) then both the local and global correspondences are known, the local by Gérard Laumon, Michael Rapoport, and Ulrich Stuhler [61] and the global by Laurent Lafforgue [54] (after Drinfeld [27]).

6. Functoriality

Although there is no proof of the global modularity of Galois representations, there is a global avatar on the automorphic side. This is Langlands' Global Functoriality Conjecture. The formulation we gave above depended on the global Weil group W_k but, as we have seen, now would need to be formulated in terms of the conjectural group L_k. However, instead one can appeal to a local/global principle, or a type of Hasse principle.

We begin with Local Functoriality. The basic local diagram for the functoriality to GL_n is as follows:

Local Functoriality: *If π_v is an irreducible admissible representation of $H(k_v)$ then we can obtain an irreducible admissible representation Π_v of $GL_N(k_v)$ by following the diagram*

$$\begin{tikzcd}
 & & {}^LH \arrow[rr, "u"] & & {}^LGL_N & & \\
\pi_v \arrow[r, mapsto] & \phi_v & & & & \Phi_v \arrow[r, mapsto] & \Pi_v. \\
 & & & W'_{k_v} \arrow[uul] \arrow[uur] & & &
\end{tikzcd}$$

and this should satisfy

$$L(s, \pi_v, u) = L(s, u \circ \phi_v) = L(s, \Phi_v) = L(s, \Pi_v)$$

along with a list of other compatibility conditions.

Note again that this is mediated by the local version of Artin's L-functions! If we grant the Local Langlands Conjecture, which we know completely for GL_N and at almost all places (either the archimedean places or the finite places where the above π_v is unramified) then we can piece them together to obtain a well formulated version of a Global Functoriality Conjecture.

Global Functoriality Conjecture: *If $\pi = \otimes' \pi_v$ is a cuspidal automorphic representation of $H(\mathbb{A})$ then the representation $\Pi = \otimes' \Pi_v$ of $GL_N(\mathbb{A})$ which we obtain by following the diagram*

$$\begin{tikzcd}
 & & & {}^LH \arrow[rr, "u"] & & {}^LGL_N & & & \\
\pi = \otimes' \pi_v & \pi_v \arrow[r, mapsto] & \phi_v & & & & \Phi_v \arrow[r, mapsto] & \Pi_v & \Pi = \otimes' \Pi_v \\
 & & & & W'_{k_v} \arrow[uul] \arrow[uur] & & & &
\end{tikzcd}$$

should be automorphic and moreover should satisfy

$$L(s, \pi, u) = \prod_v L(s, \pi_v, u) = \prod_v L(s, u \circ \phi_v) = \prod_v L(s, \Phi_v) = \prod_v L(s, \Pi_v) = L(s, \Pi)$$

along with a list of other compatibility conditions.

Again, Artin's L-functions are the mediators. The reason I mention this is that in recent years we have been able to establish many instances of this conjecture by using the converse theorem for GL_N. I would like to conclude with a brief explanation of this process.

For simplicity of explanation let us take H to be a split classical group, so H is the split form of SO_{2n+1}, Sp_{2n}, or SO_{2n}. Then in these cases, the dual group is relatively simple, it is $Sp_{2n}(\mathbb{C})$, $SO_{2n+1}(\mathbb{C})$ or $SO_{2n}(\mathbb{C})$, respectively. In each case we have a natural embedding of ${}^LH$ into an appropriate $GL_N(\mathbb{C}) = {}^LGL_N$. We take this embedding as our L-homomorphism u.

We begin the process with a (globally generic) cuspidal representation $\pi = \otimes'\pi_v$ of $H(k_v)$. We first need to construct our candidate lift Π. At almost all places, namely the archimedean places and the non-archimedean places v where π_v is unramified, we understand the Local Langlands Correspondence and hence the left hand side of the local Functoriality diagram. We then transfer ϕ_v to $\Phi_v = u\circ\phi_v$ and then apply the local Langlands correspondence for GL_N to construct a local lift Π_v as a representation of $GL_N(k_v)$. There remain the finite ramified places to worry about, and we have a way of finessing the lack of a local Langlands correspondence at these places. So this will let us construct our candidate lift $\Pi = \otimes'\Pi_v$ as an irreducible admissible representation of $GL_N(\mathbb{A})$, but not necessarily automorphic.

By piecing together the local functoriality diagrams we have an equality of L-functions $L(s,\pi) = L(s,\Pi)$, mediated through the intermediate local Artin L-functions. In fact, part of the "desirable conditions" is the equality of twisted L-functions $L(s,\pi\times\pi') = L(s,\Pi\times\pi')$ for cuspidal automorphic representations π' of smaller $GL_m(\mathbb{A})$. These are necessary for the application of the converse theorem. Now, following the principle that L-functions associated to automorphic forms are nice (entire continuation, functional equation, etc.), the L-functions $L(s,\pi\times\pi')$ should be nice. But how do we know this? As we mentioned above, Langlands was led to the formulation of his L-functions through the Fourier coefficients of Eisenstein series. One corollary of this is that if one can control the Eisenstein series, one should be able to control the L-functions that appear in their Fourier coefficients. This idea has been resolutely pursued by Shahidi along with his collaborators and students. This is the so called *Langlands-Shahidi method* for analyzing automorphic L-functions (cf [66] and the references therein). So, we can thereby conclude that in fact the L-functions $L(s,\Pi\times\pi')$ have all the nice analytic properties that we expect from automorphic L-functions.

The converse theorem characterizes the L-functions of automorphic forms by their analytic properties. We now know that the L-functions $L(s,\Pi\times\pi')$ behave as they should if Π were automorphic. So the conclusion of the converse theorem in this case is (essentially) that Π is automorphic. This verifies the Langlands Functoriality Conjecture in these cases.

The cases of the split classical groups that we have outlined here can be found in the two papers [17, 18]. By now this process has been applied to many other cases and used to finesse the symmetric cube and fourth power functorialities for GL_2 (cf. [21, 66, 16] for other surveys of these results and references to further cases of functoriality).

So the statement which gave us moral modularity allows us to establish this consequence of the conjectured global modularity of Galois representations, a non-abelian class field theory, which would give us a single global diagram, without actually establishing the modularity of global Galois representations. Note that in keeping with Artin's tradition the L-functions are used as motivation and guidance, not as a tool! It is evidence for a global version of Artin's desired non-abelian class field theory.

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Index

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