L-FUNCTIONS AND FUNCTORIALITY

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1. Prelude: Arithmetic L-functions

Let M be an arithmetic/geometric object over $\mathbb{Q}$.

To M is associated a very interesting complex analytic invariant: its L-function:

$$M \quad \mapsto \quad L(M,s) = L_\infty(M,s)\prod_p L_p(M,s) \qquad Re(s) >> 0$$

Examples:

M	$L_\infty(M,s)$	typical $L_p(M,s)$		degree
$\mathbb{Q}$	$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$	$(1-p^{-s})^{-1}$		1
E	$(2\pi)^{-s}\Gamma(s)$	$(1-a_pp^{-s}+pp^{-2s})^{-1}$	$a_p = p+1-\lvert\overline{E}(\mathbb{F}_p)\rvert$	2
K	$\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}$	$\prod_{\mathfrak{p}\mid p}(1-N(\mathfrak{p})^{-s})^{-1}$	$\mathfrak{p}$ primes of K	$(K:\mathbb{Q})$
ρ		$\det(1-\rho(Fr_p)p^{-s})^{-1}$	$Fr_p =$ Frobenius at p	$\dim(\rho)$
M	$\Gamma_M(s)$	$Q_{M,p}(p^{-s})$	local (mod p) information	$\deg(Q_{M,p}(X))$

Here E is an elliptic curve defined over $\mathbb{Q}$, K is an algebraic number field, so a finite extension of $\mathbb{Q}$, and $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{C})$ is a n-dimensional Galois representation.

These are all conjectured to be NICE:

(1) $L(M,s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ (entire if M irreducible and $\dim(M) > 1$);
(2) $L(M,s)$ is bounded in vertical strips (BVS);
(3) $L(M,s)$ satisfies a standard functional equation

$$L(M,s) = \varepsilon(M,s)L(M^\vee, 1-s)$$

These complex analytic invariants are built as a convergent Euler product in $Re(s) >> 0$ out of local information. However they (conjecturally) carry interesting global information after analytic continuation, and particularly in the critical strip $0 \leq Re(s) \leq 1$ about the line of symmetry $Re(s) = 1/2$.

M	$L(M,s)$	location	Conjecture/Fact
$\mathbb{Q}$	$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$	$Re(s) = 1$ $Re(s) = 1/2$	Prime Number Theorem Riemann Hypothesis
E	$L(E,s)$	$s = 1/2$	Birch and Swinnerton–Dyer
K	$\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s)$	$s = 1$	Analytic Class Number Formula
ρ	$L(\rho,s)$	$\mathbb{C}$	Artin Conjecture

2. Automorphic L-functions

2.1. **Classical – Hecke. Modular forms:** $f : \mathfrak{H} \to \mathbb{C}$ is a modular form of weight k for $\Gamma \subset SL_2(\mathbb{Z})$ if

(1) f is holomorphic;
(2) for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z);$$

(3) f is holomorphic at the cusps of Γ.

Examples:

(1) $\theta_q(z)$ the theta series attached to a quadratic form $q(x)$;
(2) $\Delta(z)$ the discriminant function from the theory of elliptic modular functions.

We will restrict to $\Gamma = SL_2(\mathbb{Z})$ for simplicity of exposition.

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$ then $f(z+1) = f(z)$ and we have the Fourier expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

Then $f(z)$ is cuspidal, or a cusp form, if $a_0 = \int_0^1 f(z+x)\, dx = 0$, i.e.,

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

These Fourier coefficients often carry interesting arithmetic information:

(1) If $f(z) = \theta_q(z)$, then $a_n = r(n, q)$ counts the number of times n is represented by the quadratic form q.
(2) If $f(z) = \Delta(z)$, then $a_n = \tau(n)$ is Ramanujan's τ–function.

Hecke attached to each cusp form a complex analytic invariant – its L-function:

$$\begin{aligned} L(s,f) &= \int_0^{\infty} f(iy) y^s d^{\times} y = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s} \\ &= (2\pi)^{-s}\Gamma(s) \prod_p (1 - a_p p^{-s} + p^{k-1} p^{-2s})^{-1} \end{aligned}$$

where the last equality is valid if $f(z)$ is "arithmetic", i.e., an eigen-function of the Hecke operators. Due to the relations of the analytic invariant $L(s,f)$ and the analytic object $f(z)$ through the Mellin transform, Hecke could prove the following.

Theorem 2.1. *$L(s,f)$ is* NICE*: entire, BVS, and satisfies a functional equation.*

The functional equation comes from the modular transformation law under $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ sending $z \mapsto -1/z$.

Since the Mellin transform has an inverse integral transform, Hecke was able to prove the CONVERSE to this THEOREM.

Theorem 2.2. *If $D(s) = (2\pi)^{-s}\Gamma(s) \sum a_n/n^s$ is* NICE *with the correct functional equation then $f(z) = \sum a_n e^{2\pi i n z}$ is a cusp form of weight k for $SL_2(\mathbb{Z})$ and $D(s) = L(s,f)$.*

The modularity of $f(z)$ essentially comes from the Fourier expansion and the functional equation.

Note that Weil proved a corresponding Converse Theorem for $\Gamma_0(N)$ by using the functional equation not just for $L(s,f)$ but also for

$$L(s,f,\chi) = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{\chi(n) a_n}{n^s}$$

with Dirichlet characters χ of conductor prime to the level N.

2.2. GL_n**.** In the modern analytic theory of automorphic forms, the modular form f of Hecke is replaced by the automorphic representation π (Gelfand, Piatetski-Shapiro, Jacquet, Langlands, Shalika, ...)

The object of study becomes the space of cuspidal automorphic forms

$$\mathcal{A}_0(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})).$$

Here

$$\mathbb{A} = \mathbb{R}\prod_p{}' \mathbb{Q}_p$$

is the ring of adeles of $\mathbb{Q}$ and we have

$$\mathbb{Q} \hookrightarrow \mathbb{A} \text{ discrete ;} \quad \mathbb{Q}\backslash\mathbb{A} \text{ compact.}$$

Then analogously

$$GL_n(\mathbb{A}) = GL_n(\mathbb{R})\prod_p{}' GL_n(\mathbb{Q}_p)$$

and again

$$GL_n(\mathbb{Q}) \hookrightarrow GL_n(\mathbb{A}) \text{ discrete ;} \quad GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}) \text{ finite volume mod center.}$$

The functions $\varphi \in \mathcal{A}_0(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ are analogues of classical modular forms. They satisfy

(1) modularity: $\varphi(\gamma g) = \varphi(g)$ for $\gamma \in GL_n(\mathbb{Q})$ and $g \in GL_n(\mathbb{A})$;
(2) regularity: smooth, satisfying a system of differential equations (analogue of holomorphy);
(3) uniform moderate growth (analogue of holomorphy at the cusps);
(4) cuspidality: analogous constant term integrals vanish.

The space $\mathcal{A}_0(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$ has a natural action of $GL_n(\mathbb{A})$ by right translation. A theorem of Gelfand and Piatetski-Shapiro tells us we have a discrete decomposition

$$\mathcal{A}_0(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})) = \bigoplus m(\pi) V_\pi$$

with finite multiplicities $m(\pi)$ (in fact equal to 0 or 1). The constituents (π, V_π) are the *cuspidal automorphic representations* of $GL_n(\mathbb{A})$. Be warned – they are infinite dimensional.

Just as

$$GL_n(\mathbb{A}) = GL_n(\mathbb{R})\prod_p{}' GL_n(\mathbb{Q}_p)$$

each cuspidal representation π decomposes

$$\pi = \pi_\infty \otimes_p' \pi_p = \otimes_v' \pi_v$$

with π_∞ a representation of $GL_n(\mathbb{R})$ and π_p a representation of $GL_n(\mathbb{Q}_p)$, all infinite dimensional.

Following Hecke's theory of integral representations, Jacquet and Langlands ($n = 2$) and then Jacquet, Piatetski-Shapiro, and Shalika associated to these representations an L-function

$$\begin{aligned} \pi_\infty &\longrightarrow L(s,\pi_\infty) \longleftrightarrow \Gamma(s) \\ \pi_p &\longrightarrow L(s,\pi_p) = Q_p(p^{-s})^{-1} \text{ with } Q_p(X) \in \mathbb{C}[X] \text{ of degree } \le n \\ \pi &\longrightarrow L(s,\pi) = L(s,\pi_\infty)\prod_p L(s,\pi_p) \quad Re(s) >> 0. \end{aligned}$$

As with Hecke, they were able to show that these complex analytic invariants were indeed nice:

Theorem 2.3 (J,P-S,S)**.** *$L(s,\pi)$ is NICE: entire, BVS and satisfies a functional equation*

$$L(s,\pi) = \varepsilon(s,\pi)L(1-s,\widetilde{\pi}).$$

In fact, they were able to construct and analyze the *twisted* L-functions $L(s,\pi\times\pi')$ for π' a cuspidal representation of some $GL_m(\mathbb{A})$ and show that if $m < n$ that these L-functions were also nice.

Inverting the integral representation once again gives a *Converse Theorem*:

Theorem 2.4 (C,P-S)**.** *Let $\pi = \otimes'\pi_v$ be an irreducible admissible representation of $GL_n(\mathbb{A})$. (Think of this as a collection of local data.) Suppose that the formal L-function*

$$L(s,\pi) := \prod_v L(s,\pi_v)$$

converges for some $Re(s) >> 0$ and has a automorphic central character. Suppose that for every $\pi' \in \mathcal{T}_0$, an appropriate cuspidal automorphic twisting set, we have that all $L(s,\pi\times\pi')$ are NICE. Then π is in fact cuspidal automorphic.

Examples of twisting sets are:

- $\mathcal{T}_0 = \mathcal{T}_0(n-1) = \{\pi' \mid \text{ cuspidal automorphic for } GL_m(\mathbb{A}),\ 1 \le m \le n-1\}$
- $\mathcal{T}_0 = \mathcal{T}_0(n-2)$

Moral: *All NICE degree n L-functions are modular, i.e., associated to a cuspidal automorphic representation π of $GL_n(\mathbb{A})$.*

3. Example – Langlands Conjectures

One goal of number theory is:

- understand $\mathcal{G}_\mathbb{Q} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$;
- understand all $\rho : \mathcal{G}_\mathbb{Q} = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C})$;

• understand the associated invariants, i.e., the Artin L-functions $L(\rho, s)$, where for almost all p,

$$L_p(\rho, s) = \det(I - \rho(Fr_p)p^{-s})^{-1}$$

a degree n Euler factor.

In light of our moral, and the expected niceness of the Artin L-functions the following (also known as the *Strong Artin Conjecture*) seem natural.

Global Langlands Conjecture (Naive version): *There exist natural bijections between*

$$Rep_n(\mathcal{G}_{\mathbb{Q}}) = \{\rho : \mathcal{G}_{\mathbb{Q}} \to GL_n(\mathbb{C}); \textit{irreducible}\}$$

and

$$\mathcal{A}_0(n) = \{\pi : \textit{ cuspidal automorphic representations of } GL_n(\mathbb{A})\}$$

such that $L(\rho, s) = L(s, \pi)$ *(among other things).*

And we could then expect local versions:

Local Langlands Conjecture (Naive version): *There exist natural bijections between*

$$Rep_n(\mathcal{G}_{\mathbb{Q}_v}) = \{\rho_v : \mathcal{G}_{\mathbb{Q}_v} = Gal(\overline{\mathbb{Q}_v}/\mathbb{Q}_v) \to GL_n(\mathbb{C})\}$$

and

$$\mathcal{A}_v(n) = \{\pi_v : \textit{ irred. admissible representations of } GL_n(\mathbb{Q}_v)\}$$

such that $L(\rho_v, s) = L(s, \pi_v)$ *(among other things).*

This is naive because of several issues:

1. The difference in the topologies of $\mathcal{G}_{\mathbb{Q}}$ and $GL_n(\mathbb{C})$ is such that one doesn't pick up enough information about the Galois group from complex representations. One needs to use ℓ-adic representations.
2. There are "more" automorphic or admissible representations of GL_n than n-dimensional Galois representations.

Weil dealt with the second issue for $n = 1$ by introducing the local and global Weil groups $W_{\mathbb{Q}}$ or $W_{\mathbb{Q}_v}$ to substitute for $\mathcal{G}_{\mathbb{Q}}$, etc.

Deligne dealt with the first issue and second issue locally for $n \geq 2$ by introducing the local Weil-Deligne group $W'_{\mathbb{Q}_v}$ to replace $W_{\mathbb{Q}_v}$. So

$$\begin{array}{lll} \mathcal{G}_{\mathbb{Q}} \longrightarrow W_{\mathbb{Q}} \longrightarrow ?? & & \text{globally} \\ \mathcal{G}_{\mathbb{Q}_v} \longrightarrow W_{\mathbb{Q}_v} \longrightarrow W'_{\mathbb{Q}_v} & & \text{locally.} \end{array}$$

Which leaves us only with

Local Langlands Conjecture: *There exist natural bijections between*

$$Rep_n(W'_{\mathbb{Q}_v}) = \{\rho_v : W'_{\mathbb{Q}_v} \to GL_n(\mathbb{C}), \textit{ suitably semi-simple } \}$$

and

$$\mathcal{A}_v(n) = \{\pi_v : \text{ irred. admissible representations of } GL_n(\mathbb{Q}_v)\}$$

such that $L(\rho_v, s) = L(s, \pi_v)$ *(among other things).*

This is of course now a **Theorem** due to Harris and Taylor. We might state this as:

Theorem. "*Local Galois representations in characteristic zero are modular.*"

As for the **GLC**, all we can hope for at the moment is a type of Hasse principle – a local/global compatibility.

If we view the information as flowing

$$\textbf{Automorphic} \longrightarrow \textbf{Galois}$$

which we have emphasized, this is a type of *Class Field Theory*. However, and this is important for us, if one views the information as flowing

$$\textbf{Galois} \longrightarrow \textbf{Automorphic}$$

then this gives an *Arithmetic Parameterization* of automorphic or admissible representations. Then one can ask, as Langlands did, how can we parameterize the representations of other reductive algebraic groups, for example the split $H = SO_n$ or $H = Sp_n$? What replaces the $GL_n(\mathbb{C})$ in the Galois representation?

It was to understand this that Langlands introduced his dual group or L-group. For these split groups, the process is easy: dualize the root data and take the complex points of the resulting group:

H	${}^L H$
GL_n	$GL_n(\mathbb{C})$
SO_{2n+1}	$Sp_{2n}(\mathbb{C})$
Sp_{2n}	$SO_{2n+1}(\mathbb{C})$
SO_{2n}	$SO_{2n}(\mathbb{C})$

Then the Local Langlands Conjecture for H, as an arithmetic parameterization problem, takes the following form

Local Langlands Conjecture for H: *Let*

$$Rep(W'_{\mathbb{Q}_v}, H) = \{\phi_v : W'_{\mathbb{Q}_v} \to^L H(\mathbb{C}), \text{ admissible}\}$$

and

$$\mathcal{A}_v(H) = \{\pi_v : \text{ irred. admissible representations of } H(\mathbb{Q}_v)\}.$$

Then there exists a surjective map

$$\mathcal{A}_v(H) \longrightarrow Rep(W'_{\mathbb{Q}_v}, H)$$

with finite fibres such that

$$L(s, \pi_v) = L(\phi_v, s)$$

(among other things).

This would partition the admissible representations of $H(\mathbb{Q}_v)$ into L-packets, i.e., finite subsets all having the same L-functions.

Known cases:

- $\mathbb{Q}_v = \mathbb{R}$, all H (Langlands).
- $\mathbb{Q}_v = \mathbb{Q}_p$, and π_p unramified (Satake).
- $H = GL_n$ (Harris-Taylor, Henniart).

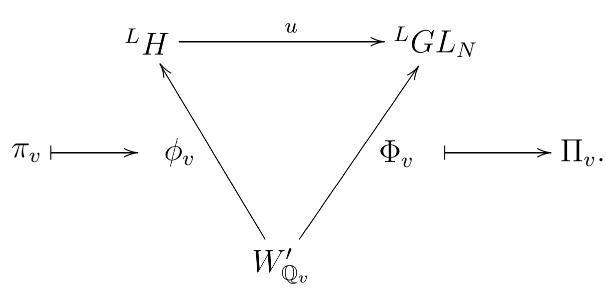
4. Functoriality

Functoriality is a manifestation of viewing either the **GLC** or **LLC** as giving arithmetic parameterizations of automorphic/admissible representations. It involves one extra piece of data, an L**-homomorphism**, which relates the arithmetic parameter spaces. Restricting our attention to functoriality from one of our classical groups H to GL_N this is a complex analytic morphism

$$u : {}^LH \longrightarrow {}^LGL_N = GL_N(\mathbb{C})$$

which we will take as the natural embedding. With this we can formulate functoriality as a way to transfer admissible or automorphic representations from H to GL_N.

Local Functoriality: *If π_v is an irreducible admissible representation of $H(\mathbb{Q}_v)$ then we can obtain an irreducible admissible representation Π_v of $GL_N(\mathbb{Q}_v)$ by following the diagram*



and this should satisfy

$$L(s, \pi_v) = L(\phi_v, s) = L(\Phi_v, s) = L(s, \Pi_v)$$

along with similar equalities for twisted versions and for ε-factors.

In the case of Global Functoriality, since we do not have a global version of the Weil-Deligne group, and so no such global diagram, we rely on local/global compatibility.

Global Functoriality Conjecture: *If $\pi = \otimes'\pi_v$ is a cuspidal automorphic representation of $H(\mathbb{A})$ then the representation $\Pi = \otimes'\Pi_v$ of $GL_N(\mathbb{A})$ we obtain by following the* *diagram*

$$\begin{array}{ccccccc} & & {}^LH & \xrightarrow{u} & {}^LGL_N & & \\ \pi = \otimes' \pi_v & \pi_v \longmapsto & \phi_v \nwarrow & & \nearrow \Phi_v & \longmapsto \Pi_v & \Pi = \otimes' \Pi_v. \\ & & & W'_{\mathbb{Q}_v} & & & \end{array}$$

should be automorphic and moreover should satisfy

$$L(s,\pi) = \prod_v L(s,\pi_v) = \prod_v L(s,\Pi_v) = L(s,\Pi)$$

along with similar equalities for twisted versions and for ε-factors.

Establishment of instances of Global Functoriality can be considered as evidence of

- The existence of a global version of the Weil-Deligne group, often called the Langlands group, to mediate a global diagram.
- The strong Artin conjecture.
- Specific modularity of orthogonal or symplectic Galois representations.

5. Converse Theorem and Functoriality

The Converse Theorem gives a method for attacking the Global Functoriality Conjecture in the case where the target group is GL_N as above.

To explain this, let H be a split classical group over $\mathbb{Q}$ as above, so $H = SO_{2n}$, $H = SO_{2n+1}$, or $H = Sp_{2n}$ and let

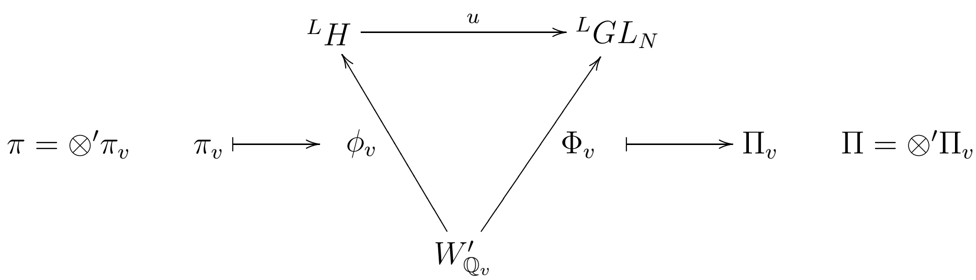
$$u : {}^LH \hookrightarrow GL_N(\mathbb{C}).$$

H	${}^LH$	$u : {}^LH \to {}^LGL_N$	${}^LGL_N$	GL_N
SO_{2n+1}	$Sp_{2n}(\mathbb{C})$	$\hookrightarrow$	$GL_{2n}(\mathbb{C})$	GL_{2n}
SO_{2n}	$SO_{2n}(\mathbb{C})$	$\hookrightarrow$	$GL_{2n}(\mathbb{C})$	GL_{2n}
Sp_{2n}	$SO_{2n+1}(\mathbb{C})$	$\hookrightarrow$	$GL_{2n+1}(\mathbb{C})$	GL_{2n+1}

Let $\pi = \otimes' \pi_v$ be a cuspidal automorphic representation of $H(\mathbb{A})$.

There are three basic steps.

Step 1: Constructing a Candidate Lift: To construct a candidate lift $\Pi = \otimes' \Pi_v$ we just use the diagram from the Global Functoriality Conjecture:



Problem 1: We do not know the **LLC** for $H(\mathbb{Q}_v)$ for all π_v. There will be a finite number of places S at which we do not have the necessary local diagram.

Step 2: Analytic Properties of L-functions: From our global diagram we expect to have the equalities

$$L(s, \pi) = L(s, \Pi) \quad \text{and even} \quad L(s, \pi \times \pi') = L(s, \Pi \times \pi')$$

for all π' in an appropriate cuspidal twisting set $\mathcal{T}_0$. Now, the L-functions $L(s, \pi \times \pi')$ are **automorphic L-functions**, so we have a good chance of controlling them.

In these specific examples, under the additional condition that the cuspidal representation π be *generic*, these can be controlled by the **Langlands-Shahidi method**. This method controls the analytic properties of automorphic L-functions by relating them to both constant terms and non-constant Fourier coefficients of Eisenstein series and then utilizing the analytic properties of the Eisenstein series (the "other" type of automorphic forms).

Problem 2: The functorial lift Π of cuspidal representations π of $H(\mathbb{A})$ need not be cuspidal on GL_N. Hence the $L(s, \pi \times \pi')$ may have poles.

Step 3: Apply the Converse Theorem: If Problem 1 and Problem 2 can be solved, then one will have that indeed $L(s, \Pi \times \pi')$ are NICE for all π' in an appropriate twisting set $\mathcal{T}_0$. In fact, both of these problems are finessed using a more flexible variant of the Converse Theorem:

Converse Theorem Variant: *If we fix a finite set of finite places S and a (highly ramified) idele class character η and let $\mathcal{T}_0(S, \eta)$ be the set of cuspidal automorphic representations π' of $GL_m(\mathbb{A})$ for $1 \leq m \leq N-1$ such that*

- $\pi' = \pi'_0 \otimes \eta$;
- $\pi'_0 = \otimes' \pi'_{0,v}$ *with* $\pi'_{0,v}$ *unramified for all* $v \in S$

then if $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}_0(S, \eta)$ then there is an automorphic Π' with $\Pi'_v \simeq \Pi_v$ for all $v \notin S$.

– Taking for S the set of places from Problem 1 and using an η that is highly ramified at those places, we are able to twist away all information at those place (using the stability of the local L– and ε-factors under highly ramified twists) at the cost of losing control at those places. This solves Problem 1.

– An observation of Kim tells us that if we twist by a highly ramified character, then globally the $L(s, \pi \times \pi')$ is entire, thus solving Problem 2.

Theorem 5.1 (C, Kim, Piatetski-Shapiro, Shahidi)**.** *Let H be a split classical group as above and let π be a generic cuspidal representation of $H(\mathbb{A})$. Then there exists an automorphic representation Π of the appropriate $GL_N(\mathbb{A})$ such that Π_v is the local Langlands functorial lift of π_v for all but finitely many places.*

This is our solution of the Global Functoriality Conjecture in these cases.

Applications: Besides providing evidence for a Global Class Field Theory, as we have discussed, one also obtains the following applications of these liftings.

1. Non-trivial bounds towards the Ramanujan Conjecture for these classical groups.

2. Combining this technique with the descent of Ginzburg-Rallis-Soudry, Jiang and Soudry filled in the **LLC** for the places in Problem 1 for $H = SO_{2n+1}$.

3. Various applications to local representation theory for the classical groups (Mœglin's dimension relation for generic discrete series representations, the first analysis of the conductor, holomorphy and non-vanishing of certain local intertwining operators.)

Other Transfers: This general method, and variations thereof, has also been used to establish Functorialities in the following situations.

H	${}^L H$	$u : {}^L H \to {}^L G$	${}^L G$	G	
$GL_2 \times GL_2$	$GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$	$\otimes$	$GL_4(\mathbb{C})$	GL_4	R
$GL_2 \times GL_3$	$GL_2(\mathbb{C}) \times GL_3(\mathbb{C})$	$\otimes$	$GL_6(\mathbb{C})$	GL_6	K & S
GL_4	$GL_4(\mathbb{C})$	$\wedge^2$	$GL_6(\mathbb{C})$	GL_6	K
GL_2/E		Asai		GL_4	R; Kr
$U_{n,n}$		Base Change		GL_{2n}/E	K & Kr
$GSpin_{2n+1}$	$GSp_{2n}(\mathbb{C})$	$\hookrightarrow$	$GL_{2n}(\mathbb{C})$	GL_{2n}	A & S
$GSpin_{2n}$	$GSO_{2n}(\mathbb{C})$	$\hookrightarrow$	$GL_{2n}(\mathbb{C})$	GL_{2n}	A & S

In the attribution column, R = D. Ramakrishnan, K = H. Kim, S = F. Shahidi, Kr = M. Krishnamurthy, and A = M. Asgari.

The tensor product and exterior square functorialities of Kim and Shahidi then led them to the symmetric cube and fourth power liftings from GL_2 to GL_4 and GL_5 respectively and holomorphy results for the symmetric power L-functions for GL_2 up to the ninth power. These results were then applied to:

• improved bounds towards the Ramanujan and Selberg conjectures for GL_2 (Kim and Shahidi, Kim and Sarnak);

• the hyperbolic circle problem (Kim and Shahidi);

• resolution of Hilbert's eleventh problem for positive ternary quadratic forms over a totally real number field (C, Piatetski-Shapiro, and Sarnak).

So, in the end, these seemingly far removed results have had applications to very concrete arithmetic problems.