# DERIVATIVES AND L-FUNCTIONS FOR $GL_n$

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*To the memory of Boris Moishezon*

## Introduction

Let $K$ denote a field. The group $GL_n$ has an amazingly useful subgroup $P_n$, called the mirabolic subgroup. Geometrically, $P_n(K)$ is the subgroup which stabilizes an $n-1$ dimensional subspace of $K^n$ and acts trivially on the quotient line. In terms of matrices, if we fix a standard basis $\{e_1, \ldots, e_n\}$ of $K^n$ and think of $P_n$ as stabilizing the span of the first $n-1$ of these vectors, then

$$P_n(K) = \left\{ p = \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} \mid g \in GL_{n-1}(K),\ u \in K^{n-1} \right\}$$

As a group, $P_n$ has the structure of a semidirect product $P_n \cong GL_{n-1} \ltimes U_n$ where $U_n$ is the unipotent radical of $P_n$, i.e.,

$$U_n(K) = \left\{ \begin{pmatrix} I_{n-1} & u \\ 0 & 1 \end{pmatrix} \mid u \in K^{n-1} \right\} \cong K^{n-1}.$$

Now let $K$ be a non-archimedean local field. Bernstein and Zelevinsky [2, 3, 12], following the lead of Gelfand and Kazhdan [7], analyze the structure of admissible representations of $GL_n(K)$ by restricting the representations to $P_n(K)$ and analyzing them using the representation theory of $P_n(K)$. Using the restriction to $P_n(K)$, Bernstein and Zelevinsky

(i) analyze the irreducibility of representations of $GL_n(K)$ which are induced from supercuspidal representations
(ii) classify the quasi-square-integrable representations in terms of supercuspidal representations
(iii) classify the generic representations in terms of the quasi-square-integrable representations.

In the theory of L-functions, Jacquet, Piatetski-Shapiro, and Shalika [9] use the restriction of admissible representations of $GL_n(K)$ to $P_n(K)$ to obtain

(iv) the existence of the local functional equation for $GL_n \times GL_m$.

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Moreover, if one wants to prove the rationality of the local Rankin–Selberg integrals for $GL_n \times GL_m$ using the method of Bernstein [1] as outlined in Gelbart–Piatetski-Shapiro [6], then the restriction to $P_n(K)$ is used to obtain the uniqueness statement needed for this method. So we add to our list

(v) the proof of rationality of the local Rankin–Selberg integrals for $GL_n \times GL_m$ via Bernstein's method.

We will come back to this in Section 3 of this paper when we discuss the "rationality in parameters" of these integrals using Bernstein's method.

The purpose of this paper is to give a new application of the technique of restriction to $P_n$, namely

(vi) explicitly compute the $GL_n \times GL_m$ local L-factor for generic representations in terms of L-functions for supercuspidal representations.

The expressions we obtain are not new; they can be found in the later sections of [9]. However, we have found that this method seems (at least to us) easier to generalize to other L-functions for $GL_n$. In particular, this method will let us compute the local exterior-square L-function for generic representations of $GL_n$ in the non-archimedean case. We will return to this in the future. We also expect that this method will be adaptable to the analogous archimedean calculation.

Let us briefly describe the contents of this paper. Section 1 begins with a review of the theory of derivatives of Bernstein and Zelevinsky [2, 3, 12]. Then we investigate how these derivatives manifest themselves in terms of restrictions and asymptotics in the Whittaker models of generic representations, and more generally, representations of Whittaker type. In Section 2, we apply these results to the computation of the local L-functions $L(s, \pi \times \sigma)$ for representations $\pi$ of $GL_n$ and $\sigma$ of $GL_m$ in terms of the (exceptional) L-functions of their derivatives. This allows us to compute the L-functions in the cases where both $\pi$ and $\sigma$ are supercuspidal, as in Gelbart–Jacquet [5], and where both are square–integrable in terms of the L-functions of supercuspidals, as in [9]. These techniques are not sufficient for computing the L-functions for generic representations. In Section 3, we discuss deformations of representations. We show that the Rankin–Selberg integrals defining the L-functions $L(s, \pi_u \times \sigma_w)$ for deformed representations are "rational in parameters" by using Bernstein's theorem [1, 6]. We also investigate how the derivatives behave under deformation. In Section 4 we return to the computation of the L-function. By combining the method of Section 2, the deformations of Section 3, and Hartog's theorem we are able to prove a weak version of Theorem 3.1 of [9]: the multiplicativity of $\gamma$ and the divisibility of L. We then follow the methods of [9] to complete the computation of $L(s, \pi \times \sigma)$ for $\pi$ and $\sigma$ generic, and more generally, irreducible.

## 1. Derivatives and Asymptotics

Let $K$ be a non-archimedean local field, $\mathfrak{o}$ its ring of integers, $\varpi$ a uniformizing element, and $q$ the order of its residue field. Throughout the paper, we will abuse notation by letting $GL_n = GL_n(K)$, $P_n = P_n(K)$, etc.

1.1. **The representation theory of $P_n$.** Let us first recall the basic facts about the representation theory of $P_n$, following Bernstein and Zelevinsky [2, 3, 12]. We have noted that $P_n \cong GL_{n-1} \ltimes U_n$ with $GL_{n-1}$ embedded in $P_n$ in the upper left hand block and $U_n$ the unipotent radical. Let $\mathrm{Rep}(P_n)$ denote the category of smooth (algebraic) representations of $P_n$, $\mathrm{Rep}(GL_n)$ the category of smooth representations of $GL_n$, etc.

The representations of $P_n$ are analyzed by the use of four functors

$$\mathrm{Rep}(P_{n-1}) \quad \underset{\Phi^-}{\overset{\Phi^+}{\rightleftarrows}} \quad \mathrm{Rep}(P_n) \quad \underset{\Psi^-}{\overset{\Psi^+}{\leftrightarrows}} \quad \mathrm{Rep}(GL_{n-1}).$$

$\Phi^+$ and $\Psi^+$ are induction functors, while $\Phi^-$ and $\Psi^-$ are localization functors or Jacquet functors. All are normalized. They are defined as follows:

(a) $\mathrm{Rep}(P_n) \quad \underset{\Psi^-}{\overset{\Psi^+}{\leftrightarrows}} \quad \mathrm{Rep}(GL_{n-1})$.

To define $\Psi^-$ we consider the space of $U_n$ covariants. We let $(\tau, V_\tau)$ be a smooth representation of $P_n$ and let

$$V_\tau(U_n, \mathbf{1}) = \langle \tau(u)v - v \mid v \in V_\tau,\ u \in U_n \rangle.$$

Then the space of $\Psi^-(\tau)$ is $V_\tau / V_\tau(U_n, \mathbf{1})$, the largest quotient of $V_\tau$ on which $U_n$ acts trivially. Since $GL_{n-1}$ preserves $U_n$, $GL_{n-1}$ will stabilize $V_\tau(U_n, \mathbf{1})$ and we have the natural action of $GL_{n-1}$ on $V_\tau / V_\tau(U_n, \mathbf{1})$. Letting $\sigma$ denote $\Psi^-(\tau)$, then $\sigma$ is the normalized action of $GL_{n-1}$ on $V_\tau / V_\tau(U_n, \mathbf{1})$ given by

$$\sigma(g)(v + V_\tau(U_n, \mathbf{1})) = |\det(g)|^{-1/2}(\tau(g)v + V_\tau(U_n, \mathbf{1})).$$

The functor $\Psi^+$ is just induction, or in this case, normalized extension by the trivial representation. Given a smooth representation $(\sigma, V_\sigma)$ of $GL_{n-1}$ we let $\tau = \Psi^+(\sigma)$ be the representation of $P_n$ on $V_\sigma$ such that $U_n$ acts trivially and $GL_{n-1}$ acts by $\tau(g) = |\det(g)|^{1/2}\sigma(g)$.

(b) $\mathrm{Rep}(P_{n-1}) \quad \underset{\Phi^-}{\overset{\Phi^+}{\rightleftarrows}} \quad \mathrm{Rep}(P_n)$.

Here we consider $P_{n-1} \hookrightarrow GL_{n-1} \hookrightarrow P_n$. If we fix a non-trivial additive character $\psi$ of $K$, then $\psi$ defines a character of $U_n$, which by abuse of notation we again denote by $\psi$, defined by $\psi(u) = \psi(u_{n-1,n})$. $GL_{n-1}$ is the stabilizer of $U_n$ and the stabilizer of this character in $GL_{n-1}$ is exactly$P_{n-1}$.

To construct $\Phi^-$, let $(\tau, V_\tau)$ be a smooth representation of $P_n$. We form the space of $(U_n, \psi)$-covariants by taking

$$V_\tau(U_n, \psi) = \langle \tau(u)v - \psi(u)v \mid u \in U_n,\ v \in V_\tau \rangle$$

and forming the quotient vector space $V_\tau / V_\tau(U_n, \psi)$. This is the largest quotient on which $U_n$ acts by the character $\psi$. Then $\sigma = \Phi^-(\tau)$ is the normalized representation of $P_{n-1}$ on $V_\tau / V_\tau(U_n, \psi)$ given by

$$\sigma(p)(v + V_\tau(U_n, \psi)) = |\det(p)|^{-1/2}(\tau(p)v + V_\tau(U_n, \psi)).$$

$\Phi^+$ is the functor of normalized compactly supported induction. If $(\sigma, V_\sigma)$ is a smooth representation of $P_{n-1}$ we extend it to a representation of $P_{n-1}U_n$ by letting $U_n$ act by the character $\psi$. Then

$$\tau = \Phi^+(\sigma) = \mathrm{ind}_{P_{n-1}U_n}^{P_n}(|\det|^{1/2}\sigma \otimes \psi)$$

where the induction ind is non-normalized using smooth functions of compact support modulo $P_{n-1}U_n$.

Using fairly elementary geometry, realizing these representations in sections of sheaves over $\widehat{U_n}$, Bernstein and Zelevinsky establish the following basic properties of these functors:

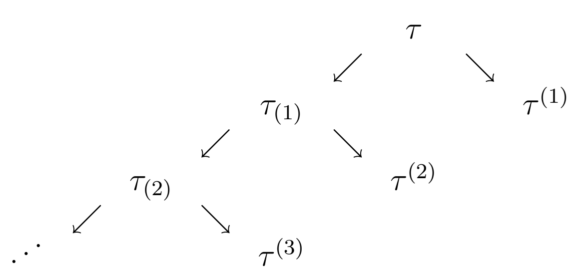
(1) $\Phi^\pm$ and $\Psi^\pm$ are all exact.
(2) $\Psi^-$ is left adjoint to $\Psi^+$.
(3) $\Phi^+$ is left adjoint to $\Phi^-$.
(4) $\Phi^-\Phi^+ \simeq \mathrm{id}$ and $\Psi^-\Psi^+ \simeq \mathrm{id}$.
(5) $\Phi^-\Psi^+ = 0$ and $\Psi^-\Phi^+ = 0$.
(6) $0 \to \Phi^+\Phi^- \to \mathrm{id} \to \Psi^+\Psi^- \to 0$ is exact.

From these basic properties, they derive the following consequences:

(1) $\Phi^+$ and $\Psi^+$ carry irreducible representations to irreducible representations.
(2) Any irreducible representation of $\tau$ of $P_n$ is of the form $\tau \simeq (\Phi^+)^{k-1}\Psi^+(\rho)$ with $\rho$ an irreducible representation of $GL_{n-k}$. The index $k$ and the representation $\rho$ are completely determined by $\tau$.
(3) The derivatives: Let $\tau \in \mathrm{Rep}(P_n)$. For each $k = 1, 2, \ldots, n$ there are representations $\tau_{(k)} \in \mathrm{Rep}(P_{n-k})$ and $\tau^{(k)} \in \mathrm{Rep}(GL_{n-k})$ associated to $\tau$ by

$$\tau_{(k)} = (\Phi^-)^k(\tau) \qquad \text{and} \qquad \tau^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau).$$

Diagrammatically:



where all leftward arrows represent an application of $\Phi^-$ and the rightward arrows an application of $\Psi^-$. $\tau^{(k)}$ is called the $k^{th}$ derivative of $\tau$.

(4) The filtration by derivatives: By successive application of the sixth basic property listed above, any $\tau \in \mathrm{Rep}(P_n)$ has a natural filtration by $P_n$ submodules

$$0 \subset \tau_n \subset \tau_{n-1} \subset \cdots \subset \tau_2 \subset \tau_1 = \tau$$

such that $\tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\tau)$. The successive quotients are completely determined by the derivatives of $\tau$ since

$$\tau_k/\tau_{k+1} = (\Phi^+)^{k-1}\Psi^+(\tau^{(k)}).$$

The proofs of these statements can be found in the work of Bernstein and Zelevinsky [2, 3].

1.2. **Derivatives for $\mathbf{GL_n}$.** Let $\pi \in \mathrm{Rep}(GL_n)$. Then the derivatives of $\pi$ can be defined by using the restriction of $\pi$ to $P_n$. The $0^{th}$ derivative is $\pi$ itself, i.e., $\pi^{(0)} = \pi$. The higher derivatives are defined through the restriction of $\pi$ to $P_n$, which we denote by $\pi_{(0)}$ in keeping with the previous section. If we set $\tau = \pi_{(0)} = \pi|_{P_n}$ then $\pi_{(k)} = \tau_{(k)}$ and $\pi^{(k)} = \tau^{(k)}$ for $k = 1, \dots, n$.

These derivatives are quite useful in discussing the representation theory of $GL_n$. For example, the work of Gelfand and Kazhdan [7] can be interpreted as the statement that $\pi$ is quasi-cuspidal iff $\pi^{(k)} = 0$ for $0 < k < n$ and $\pi^{(n)} \neq 0$, and in this situation $\pi$ is irreducible iff $\pi^{(n)} = \mathbf{1}$.

The derivatives for $GL_n$ also satisfy a type of Leibniz rule. This is easiest stated using (essentially) the notation introduced by Bernstein and Zelevinsky. If $\pi \in \mathrm{Rep}(GL_n)$ and $\sigma \in \mathrm{Rep}(GL_m)$ let $\rho \tilde{\times} \sigma$ denote the induced representation

$$\pi \tilde{\times} \sigma = \mathrm{Ind}_{Q_{n,m}}^{GL_{n+m}}(\pi \otimes \sigma)$$

where $Q_{n,m}$ is the standard parabolic with Levi factor $GL_n \times GL_m$, and the induction is normalized parabolic induction on the space of smooth functions. Then the derivatives of this induced representation $(\pi \tilde{\times} \sigma)^{(k)}$ are glued from the $\pi^{(i)} \tilde{\times} \sigma^{(k-i)}$ for $0 \leq i \leq k$, i.e., $(\pi \tilde{\times} \sigma)^{(k)}$ has a filtration whose successive quotients are the $\pi^{(i)} \tilde{\times} \sigma^{(k-i)}$ [3].

1.3. **Connections with Whittaker models.** Let $(\pi, V_\pi) \in \mathrm{Rep}(GL_n)$. Then one can explicitly compute the $n^{th}$ derivative $\pi^{(n)}$. One finds that $\pi^{(n)}$ is a representation of $GL_0$, i.e., just a vector space. In fact, the space of $\pi^{(n)}$ is $V_\pi/V_\pi(N_n, \psi)$ where $V_\pi(N_n, \psi) = \langle \pi(n)v - \psi(n)v \mid v \in V_\pi,\ n \in N_n \rangle$, $N_n$ is the maximal unipotent subgroup of upper triangular unipotent matrices, and for $n \in N_n$ we set $\psi(n) = \psi(n_{1,2} + \cdots + n_{n-1,n})$. This is the maximal quotient of $V_\pi$ on which $N_n$ acts via the non-degenerate character $\psi$. Thus the dual linear space $(V_\pi/V_\pi(N_n, \psi))^*$ is the space of $\psi$-Whittaker functionals on $\pi$ and we have $\dim(\pi^{(n)})$ is precisely the number of (independent) Whittaker functionals on $\pi$. If $\pi$ is irreducible then $\dim(\pi^{(n)}) \leq 1$ [2, 7].

$\pi$ is called *generic* if it is irreducible and $\dim(\pi^{(n)}) = 1$. In this case, let $\lambda$ be a non-trivial $\psi$-Whittaker functional. Then the space of functions

$$\mathcal{W}(\pi,\psi) = \{W_v(g) = \lambda(\pi(g)v) \mid v \in V_\pi,\ g \in GL_n\}$$

with the natural action of $GL_n$ by right translation is called the *Whittaker model* of $\pi$.

The main purpose of this section is to analyze the restriction of $\pi$ to $P_n$ and the functors $\Phi^-$ and $\Psi^-$ for a generic representation in terms of its Whittaker model. One of the initial goals of the Bernstein–Zelevinsky series was to provide a model for $\pi$ in terms of the restriction of the functions in the Whittaker model $\mathcal{W}(\pi,\psi)$ to $P_n$. This is the so–called Kirillov model of $\pi$. The result is:

**Theorem** [3] *Let $(\pi, V_\pi)$ be generic. Then the map*

$$v \mapsto W_v(p) \text{ for } v \in V_\pi \text{ and } p \in P_n$$

*is injective, i.e., if $W_v \in \mathcal{W}(\pi,\psi)$ then the restriction of $W_v$ to $P_n$ as a function cannot vanish identically.*

**Corollary.** *The space of functions on $P_n$ given by $\{W_v(p) \mid v \in V_\pi,\ p \in P_n\}$ is a model for the restriction $\pi_{(0)}$ of $\pi$ to $P_n$, the action being by right translation.*

Let us make the following simple observation. Let $W_v \in \mathcal{W}(\pi,\psi)$ and let $p = \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in P_n$ with $g \in GL_{n-1}$. Then

$$W_v(p) = W_v\left(\begin{pmatrix} I_{n-1} & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) = \psi(u_{n-1,n}) W_v\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence we have a model of the restriction of $\pi$ to $P_n$ on the space of functions

$$\left\{ W_v\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \mid v \in V_\pi,\ g \in GL_{n-1} \right\}$$

with action given by

$$W_{\pi(p)v}\begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} = W_v\left(\begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} p\right) = \psi(g'u) W_v\begin{pmatrix} g'g & 0 \\ 0 & 1 \end{pmatrix}.$$

We will call this model a Whittaker model of $\pi_{(0)}$ and denote it by $\mathcal{W}(\pi_{(0)},\psi)$.

Let us now consider the functor $\Phi^-$ in this context. To simplify the notation, let $\tau$ denote the restriction $\pi_{(0)}$ of $\pi$ to $P_n$. Recall that for $(\tau, V_\tau)$ a smooth representation of $P_n$, $\Phi^-(\tau)$ is the normalized representation of $P_{n-1}$ on the space $V_\tau / V_\tau(U_n,\psi)$ with $V_\tau(U_n,\psi) = \langle \tau(u)v - \psi(u)v \mid u \in U_n,\ v \in V_\tau \rangle$. There is another characterization of the subspace $V_\tau(U_n,\psi)$ due to Jacquet and Langlands [2], namely $v \in V_\tau(U_n,\psi)$ iff there exists a compact open subgroup $Y \subset U_n$ such that

$$\int_Y \psi^{-1}(y)\tau(y)v\, dy = 0.$$

We would now like to give a third characterization of the space $V_\tau(U_n,\psi)$ in terms of the model of $\tau=\pi_{(0)}$ on the space of functions $\mathcal{W}(\tau,\psi)$.

**Proposition 1.1.** *$V_\tau(U_n,\psi)=\left\{W_v\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix} \mid W_v\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}\equiv 0 \text{ for } p\in P_{n-1}\right\}.$*

*Proof:* Let

$$A=\left\{W_v\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix} \mid W_v\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}\equiv 0 \text{ for } p\in P_{n-1}\right\}.$$

We first claim that $V_\tau(U_n,\psi)\subset A$. To see this, let $p'\in P_{n-1}$. Then if $v'=\tau(u)v-\psi(u)v\in V_\tau(U_n,\psi)$ we see that

$$\begin{aligned} W_{v'}\begin{pmatrix} p' & 0\\ 0 & 1\end{pmatrix} &= W_v\left(\begin{pmatrix} p' & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} I_{n-1} & u\\ 0 & 1\end{pmatrix}\right)-\psi(u)W_v\begin{pmatrix} p' & 0\\ 0 & 1\end{pmatrix}\\ &=(\psi(p'u)-\psi(u))W_v\begin{pmatrix} p' & 0\\ 0 & 1\end{pmatrix}. \end{aligned}$$

But $P_{n-1}$ is the stabilizer of the character $\psi$ of $U_n$. Therefore we see that $\psi(p'u)=\psi(u)$ and $W_{v'}\begin{pmatrix} p' & 0\\ 0 & 1\end{pmatrix}\equiv 0$. Thus $W_{v'}\in A$.

We next claim that $A\subset V_\tau(U_n,\psi)$. We now use the equivalent characterization of $V_\tau(U_n,\psi)$ due to Jacquet and Langlands, namely, $v\in V_\tau(U_n,\psi)$ iff there is a compact open subgroup $Y\subset U_n$ such that

$$\int_{\mathrm{Y}}\psi^{-1}(y)\tau(y)v\,dy=0.$$

In terms of the Whittaker model of $\tau$, this characterization becomes $W_v\in\mathcal{W}(\tau,\psi)$ iff there is a compact open subgroup $Y\subset U_n$ such that

$$\int_Y W_v(py)\psi^{-1}(y)\,dy\equiv 0.$$

Now write $p\in P_n$ as $p=gu$ with $g\in GL_{n-1}\hookrightarrow P_n$ and $u\in U_n$. Then

$$\int_Y W_v(guy)\psi^{-1}(y)\,dy=W_v(gu)\int_Y\psi(gyg^{-1})\psi^{-1}(y)\,dy.$$

For $Y$ sufficiently large, $\int_Y\psi(gyg^{-1})\psi^{-1}(y)\,dy=0$ unless $g\in\mathrm{Stab}_{GL_{n-1}}(\psi)=P_{n-1}$. In this case, setting $g=p'\in P_{n-1}$ we have

$$\begin{aligned} \int_Y W_v(p'uy)\psi^{-1}(y)\,dy &= W_v(p'u)\int_{\mathrm{Y}}\psi(y)\psi^{-1}(y)\,dy\\ &=\psi(p'up'^{-1})W_v\begin{pmatrix} p' & 0\\ 0 & 1\end{pmatrix}\mathrm{vol}(Y). \end{aligned}$$

Hence if $W_v \in A$ we see that this is again identically equal to 0. Therefore $A \subset V_\tau(U_n, \psi)$. $\square$

**Corollary.** *If $\tau = \pi_{(0)}$, then $\Phi^-(\tau) = \pi_{(1)}$ has as a model the space of functions*

$$\mathcal{W}(\pi_{(1)}, \psi) = \left\{ |\det(g)|^{-1/2} W_v \begin{pmatrix} g & \\ & I_2 \end{pmatrix} \,\middle|\, v \in \mathrm{V}_\pi,\ g \in GL_{n-2} \right\}$$

*with the natural action by right translation.*

If we now proceed by induction, we find natural models for all of the $\pi_{(k)}$.

**Proposition 1.2.** *The representation $\pi_{(k-1)}$ of $P_{n-k+1}$ has as a model the space of functions*

$$\mathcal{W}(\pi_{(k-1)}, \psi) = \left\{ |\det(g)|^{-(k-1)/2} W_v \begin{pmatrix} g & \\ & I_k \end{pmatrix} \,\middle|\, v \in \mathrm{V}_\pi,\ g \in GL_{n-k} \right\}$$

*with the natural action by right translation.*

We will refer to these models as Whittaker models for the $\pi_{(k-1)}$.

Now turn to the functor $\Psi^-$ leading to the derivative. Given $\tau = \pi_{(0)} = \pi|_{P_n}$, the representation $\Psi^-(\tau)$ is the normalized representation of $GL_{n-1}$ on the quotient $V_\tau / V_\tau(U_n, \mathbf{1})$. As before, we have two characterizations of $V_\tau(U_n, \mathbf{1})$. By definition,

$$V_\tau(U_n, \mathbf{1}) = \langle \tau(u)v - v \mid u \in U_n,\ v \in V_\tau \rangle$$

whereas the Jacquet–Langlands characterization [2] is that $v \in V_\tau(U_n, \mathbf{1})$ iff there exists a compact open subgroup $Y \subset U_n$ such that

$$\int_Y \tau(y) v \; dy \equiv 0.$$

We now give a third characterization in terms of the Whittaker model for $\tau = \pi_{(0)}$ given above.

**Proposition 1.3.** *Let $\tau = \pi_{(0)}$ realized in its model on*

$$\mathcal{W}(\tau, \psi) = \left\{ W_v \begin{pmatrix} g & \\ & 1 \end{pmatrix} \mid v \in V_\pi \right\}.$$

*Then $V_\tau(U_n, \mathbf{1})$ consists of those $W_v$ for which there exists an $N > 0$, depending on $v$, such that $W_v \begin{pmatrix} g & \\ & 1 \end{pmatrix} \equiv 0$ whenever the last row of $g$ satisfies the estimate $\max_i\{|g_{n-1,i}|\} < q^{-N}$.*

*Proof:* As before, let

$$A = \left\{ v \in V_\pi \mid \text{there exists } N > 0 \text{ such that } \max_i \{|g_{n-1,i}|\} < q^{-N} \right.$$
$$\left. \text{implies } W_v \begin{pmatrix} g & \\ & 1 \end{pmatrix} \equiv 0 \right\}.$$

Let $v_0 \in V_\tau(U_n, \mathbf{1})$. Using the characterization of $V_\tau(U_n, \mathbf{1})$ as the span $\langle \tau(u)v - v | v \in V_\tau,\ u \in U_n\rangle$, we see that if $v_0 = \tau(u)v - v$ where

$$u = \begin{pmatrix} 1 & & & u_1 \\ & \ddots & & \vdots \\ & & 1 & u_{n-1} \\ & & & 1 \end{pmatrix}$$

then

$$W_{v_0}\begin{pmatrix} g & \\ & 1 \end{pmatrix} = \left(\prod_i \psi(g_{n-i,i}u_i) - 1\right) W_v \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

So, assuming $\psi$ is normalized, if we take $N$ such that $q^{-N} \le \min_i\{|u_i|^{-1}\}$, we see that $v_0 \in A$. Hence $V_\tau(U_n, \mathbf{1}) \subset A$.

On the other hand, using the characterization of $V_\tau(U_n, \mathbf{1})$ by Jacquet and Langlands, if $v \in A$ with associated $N > 0$ let

$$v_0 = \int_{Y_N} \tau(y)v \; dy$$

where

$$Y_N = \left\{ y = \begin{pmatrix} 1 & & & y_1 \\ & \ddots & & \vdots \\ & & 1 & y_{n-1} \\ & & & 1 \end{pmatrix} \;\middle|\; |y_i| \le q^{-N} \text{ for } i = 1, \dots, n-1 \right\}.$$

then we see

$$W_{v_0}\begin{pmatrix} g & \\ & 1 \end{pmatrix} = \left(\prod_{i=1}^{n-1} \int_{\{|y_i| \le q^{-N}\}} \psi(g_{n-i,i}y_i)\; dy_i\right) W_v \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

But, again assuming that $\psi$ is normalized,

$$\int_{\{|y_i| \le q^{-N}\}} \psi(g_{n-1,i}y_i)\; dy_i = \begin{cases} 0 & |g_{n-i}| > q^{-N} \\ \operatorname{meas}(\{|y_i| \le q^{-N}\}) & |g_{n-i}| \le q^{-N} \end{cases}.$$

Hence $W_{v_0}\begin{pmatrix} g & \\ & 1 \end{pmatrix} \equiv 0$. But the map $v \mapsto W_v \begin{pmatrix} g & \\ & 1 \end{pmatrix}$ is injective. Hence $v_0 = 0$ and $v \in V_\tau(U_n, \mathbf{1})$ by the Jacquet–Langlands criterion. □

Applying this argument inductively, we obtain a similar result for $(\Phi^-)^{k-1}(\pi|_{P_n}) = \pi_{(k-1)}$.

**Proposition 1.4.** *labelp1.4 Let $\tau = \pi_{(k-1)}$. Then in terms of the Whittaker model of $\tau$*

$$V_\tau(U_{n-k+1}, \mathbf{1}) = \left\{ |\det(g)|^{-(k-1)/2} W_v \begin{pmatrix} g & \\ & I_k \end{pmatrix} \;\middle|\; v \in V_\pi, g \in GL_{n-k}, \text{ and there exists } N > 0 \text{ such that } W_v \begin{pmatrix} g & \\ & I_k \end{pmatrix} \equiv 0 \text{ whenever } \max_i\{|g_{n-k,i}|\} < q^{-N} \right\}.$$

Again, let us let $\tau = \pi_{(0)} = \pi|_{P_n}$. Recall that we have the decomposition of $\tau$ via

$$0 \to \Phi^+\Phi^-(\tau) \to \tau \to \Psi^+\Psi^-(\tau) \to 0.$$

The representation $\Psi^-(\tau)$ is the normalized representation of $GL_{n-1}$ on the space of covariants $V_\tau/V_\tau(U_n, \mathbf{1})$. Both $V_\tau(U_n, \mathbf{1})$ and the quotient are naturally $P_n$–modules. In fact, $V_\tau/V_\tau(U_n, \mathbf{1})$ is the maximal quotient on which $U_n \subset P_n$ acts trivially. Since $\Psi^+(\rho)$ is always the normalized extension of a $GL_{n-1}$ module to $P_n$ by letting $U_n$ act trivially, we see that $\Psi^+\Psi^-(\tau) \cong V_\tau/V_\tau(U_n, \mathbf{1})$ as a $P_n$–module. Hence $\Phi^+\Phi^-(\tau) \cong V_\tau(U_n, \mathbf{1})$. The same argument works for the representations $(\Phi^-)^{k-1}(\tau) = \pi_{(k-1)}$ and we arrive at the following result.

**Proposition 1.5.** *As $P_{n-k+1}$ modules we have*

$$\Phi^+(\pi_{(k)}) \simeq \left\{ |\det(g)|^{-(k-1)/2} W_v \begin{pmatrix} g & \\ & I_k \end{pmatrix} \;\middle|\; v \in V_\pi, g \in GL_{n-k}, \text{ and there exists } N > 0 \text{ such that } W_v \begin{pmatrix} g & \\ & I_k \end{pmatrix} \equiv 0 \text{ whenever } \max_i\{|g_{n-k,i}|\} < q^{-N} \right\}.$$

1.4. **Asymptotics and Derivatives.** We have seen that we have a model for the representation of $P_{k+1}$ given by $\pi_{(n-k-1)}$, which we have called the Whittaker model, on the space of functions

$$\mathcal{W}(\pi_{(n-k-1)}, \psi) = \left\{ |\det(g)|^{-(n-k-1)/2} W \begin{pmatrix} g & \\ & I_{n-k} \end{pmatrix} \;\middle|\; W \in \mathcal{W}(\pi, \psi), g \in GL_k \right\}$$

and that in this model

$$V_{\pi_{(n-k-1)}}(U_{k+1}, \mathbf{1}) = \left\{ |\det(g)|^{-(n-k-1)/2} W \begin{pmatrix} g & \\ & I_{n-k} \end{pmatrix} \;\middle|\; \text{there exists } N > 0 \text{ such that if } \max_i\{|g_{k,i}|\} < q^{-N} \text{ then } W \begin{pmatrix} g & \\ & I_{n-k} \end{pmatrix} \equiv 0 \right\}$$

Let $\tau = \pi_{(n-k-1)}$ and $V_\tau$ the space for $\tau$. For each $v \in V_\tau$ we have a function $F_v(g)$ on $GL_k$ defined as follows. If $v' \in V_\pi$ which projects onto $v$ then

$$F_v(g) = |\det(g)|^{-(n-k-1)/2} W_{v'} \begin{pmatrix} g & \\ & I_{n-k} \end{pmatrix}.$$

The space of functions $\{F_v(g)\}$ is essentially the Whittaker model for $\tau$, which we will denote again by $\mathcal{W}(\tau, \psi)$. The functions $F_v \in \mathcal{W}(\tau, \psi)$ satisfy the following properties:

(i) $F_v(ng) = \psi(n) F_v(g)$ for $n \in N_k$.
(ii) $F_{\tau(p)v}(g') = \psi(g'u) F_v(g'g)$ if $p = \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} \in P_{k+1}$.

where, in the first condition, $\psi(n) = \psi(n_{1,2} + \cdots + n_{k-1,k})$ and in the second condition, $\psi(g'u) = \psi(g'_{k,1}u_1 + \cdots + g'_{k,k}u_k)$.

To better describe the space $V_\tau(U_{k+1},\mathbf{1})$, let us introduce the following definition of a *stable limit.* If $f(a)$ is a function on $K^\times$ we say $\lim_{a\to 0} f(a)=c$ iff there exists $N>0$ such that $f(a)\equiv c$ for $|a|<q^{-N}$. In $m$ variables, if $F$ is a function on $K^m\setminus\{0\}$ we say $\lim_{v\to 0} F(v)=c$ iff there exists $N>0$ such that $F(v)\equiv c$ for $\max_i\{|v_i|\}<q^{-N}$. Then we have the characterization

$$V_\tau(U_{k+1},\mathbf{1})=\{F(g)\in\mathcal{W}(\tau,\psi)\mid \lim_{g_k\to 0}F(g)\equiv 0\}$$

with $g_k$ denoting the last row of $g\in GL_k$.

If we use the Iwasawa decomposition in $GL_k$ and write $g=nzak$ with $n\in N_k$, $z=\mathrm{diag}(z,\dots,z)\in Z_k$, $a=\mathrm{diag}(a_1,\dots,a_{k-1},1)\in A_k$, and $k\in GL_k(\mathfrak{o})$ then $|z|$ is well defined and $g_k\to 0$ if and only if $z\to 0$.

Heuristically, the derivative $\pi^{(n-k)}$ which is the normalized representation of $GL_k$ on $V_\tau/V_\tau(U_{k+1},\mathbf{1})$ now becomes the *asymptotics along the center* in these coordinates. Writing $g=nak$ and using the center and the simple roots as coordinates on $A$, the limit is uniform in the other coordinates, i.e., depends only on the function $F_v(g)$. Let us now make this more precise.

Now let $\pi_0^{(n-k)}\subset\pi^{(n-k)}$ be an irreducible subrepresentation of $\pi^{(n-k)}$, the normalized quotient representation on $V_{\pi^{(n-k)}}=V_\tau/V_\tau(U_{k+1},\mathbf{1})$. Let $p:V_\tau\to V_{\pi^{(n-k)}}$ be the normalized projection map and let $V_{\tau_0}=p^{-1}(V_{\pi_0^{(n-k)}})$. Then $\tau_0$ is a subrepresentation of $\tau$ and $V_{\tau_0}\supset V_\tau(U_{k+1},\mathbf{1})$. Let $\omega_0$ denote the central character of $\pi_0^{(n-k)}$.

**Proposition 1.6.** $\lim\limits_{a\to 0}\omega_0(a)^{-1}|a|^{-k/2}F(aI_k)$ *exists for* $F\in W(\tau_0,\psi)$.

*Proof:* For each $v\in V_{\tau_0}\cong\mathcal{W}(\tau_0,\psi)$ define a function of one variable $f_v(a)=F_v(aI_k)$. This is a smooth function of $a$. Let $\mathcal{F}(\tau_0)=\{f_v(a)\mid v\in V_{\tau_0}\}$ be the space of all such functions as $v$ runs over the space of $\tau_0$ and $\mathcal{F}_0(\tau_0)=\mathcal{F}(\tau_0)\cap\mathcal{S}(K^\times)$ those functions which vanish as $a$ approaches 0 in the stable sense. The image of $V_\tau(U_{k+1},\mathbf{1})$ lies in $\mathcal{F}_0(\tau_0)$.

If $v\in V_{\tau_0}(U_{k+1},\mathbf{1})$ then there is an $N>0$ such that if $|a|<q^{-N}$ then $f_v(a)=0$. So

$$\lim_{a\to 0}\omega_0^{-1}(a)|a|^{-k/2}F_v(aI_k)=\lim_{a\to 0}\omega_0^{-1}(a)|a|^{-k/2}f_v(a)=0$$

for $v\in V_{\tau_0}(U_{k+1},\mathbf{1})$.

Next, suppose that $v\in V_{\tau_0}$ but not in $V_\tau(U_{k+1},\mathbf{1})$. Let $p(v)$ be the image of $v$ in $V_{\pi_0^{(n-k)}}$. Since the projection $p$ is a $GL_k$ intertwining map, we have

$$|a|^{-k/2}p(\tau_0(aI_k)v)=\pi_0^{(n-k)}(aI_k)p(v)=\omega_0(a)p(v).$$

Thus $\tau_0(aI_k)v-\omega_0(a)|a|^{k/2}v\in\ker(p)=V_\tau(U_{k+1},\mathbf{1})$. So, given $v\in V_{\tau_0}$ and $a\in K^\times$ there exists $N=N_{v,a}>0$ such that $f_{\tau_0(aI_k)v-\omega_0(a)|a|^{k/2}v}(\lambda)=0$ for $|\lambda|<q^{-N}$, i.e., $f_v(\lambda a)=\omega_0(a)|a|^{k/2}f_v(\lambda)$ for $|\lambda|<q^{-N}$.

We next consider a general property of the support of $f_v$. Let us assume, without loss of generality, that $\psi$ is normalized so that $\psi$ is trivial on $\mathfrak{o}$ but $\psi(\varpi^{-1}) \neq 1$. Then we have that for $u \in U_{k+1}$

$$f_{\tau_0(u)v}(a) = F_{\tau_0(u)v}(aI_k) = \psi(au_k)F_v(aI_k) = \psi(au_k)f_v(a).$$

On the other hand, we have, by the smoothness of $\tau_0$, a constant $M = M_v > 0$ such that $\tau_0(u)v = v$ if $\max_i |u_i| < q^{-M}$, i.e., $u \in Y_M$. Hence, for $u \in Y_{M_v}$, $f_v(a) = \psi(au_k)f_v(a)$. For $|a| \geq q^{M_v}$ there is a choice of $u$ for which this character is non-trivial. Hence $f_v(a) = 0$ for $|a| \geq q^{M_v}$.

**Claim.** *There exists $N = N_v > 0$ such that $f_v(\lambda a) = \omega_0(a)|a|^{k/2}f_v(\lambda)$ whenever $|\lambda| < q^{-N}$ uniformly for $a \in \mathfrak{o}$.*

*Proof:* [Proof of the claim] Since both $f_v$ and $\omega_0$ are smooth on $K^\times$ there exists an open subset $\mathfrak{u}$ of $\mathfrak{o}^\times$ such that $f_v(\lambda u) = f_v(\lambda)$ and $\omega_0(au) = \omega_0(a)$ for $u \in \mathfrak{u}$. Let $u_1, \ldots, u_\ell$ be a set of coset representatives for $\mathfrak{o}^\times/\mathfrak{u}$. Set $N_v = \max\{N_{v,u_1}, \ldots, N_{v,u_\ell}, N_{v,\varpi}\}$.

Let $a \in \mathfrak{o}$ , $a \neq 0$, with $|a| < q^{-N_v}$ and $\lambda \in K^\times$ with $|\lambda| < q^{-N_v}$. Write $a = \varpi^j u_i u$ with $u \in \mathfrak{u}$ and $j \geq 0$. we have

$$f_v(\lambda a) = f_v(\lambda \varpi^j u_i u) = f_v(\lambda \varpi^j u_i).$$

Since $j \geq 0$ we have $|\lambda\varpi^j| \leq |\lambda| < q^{-N_v}$ so that

$$f_v(\lambda\varpi^j u_i) = \omega_0(u_i)f_v(\lambda\varpi^j).$$

Since $|\lambda\varpi^i| \leq |\lambda|$ for all $0 \leq i \leq j$ repeatedly using the above argument gives

$$f_v(\lambda\varpi^j u_i) = \omega_0(\varpi^j u_i)|\varpi^j|^{k/2}f_v(\lambda).$$

Since $\omega(u) = 1$ for $u \in \mathfrak{u}$ we finally arrive at

$$f_v(\lambda a) = \omega_0(a)|a|^{k/2}f_v(\lambda)$$

for all $a \in \mathfrak{o}$, $a \neq 0$, as long as $|\lambda| < q^{-N_v}$. □

Once we have this claim, we see that for any $\lambda$ with $|\lambda| < q^{-N_v}$ we have

$$\lim_{a\to 0} \omega_0^{-1}(a)|a|^{-k/2}f_v(a) = \omega_0^{-1}(\lambda)|\lambda|^{-k/2}f_v(\lambda).$$

In order to see this, consider $|a| < q^{-N_v}$. Then we can write $a = \lambda a_0$ with $a_0 \in \mathfrak{o}$ and $|\lambda| < q^{-N_v}$. Then

$$\begin{aligned}\omega_0^{-1}(a)|a|^{-k/2}f_v(a) &= \omega_0^{-1}(\lambda a_0)|\lambda a_0|^{-k/2}f_v(\lambda a_0)\\ &= \omega_0^{-1}(\lambda a_0)|\lambda a_0|^{-k/2}\omega_0(a_0)|a_0|^{k/2}f_v(\lambda)\\ &= \omega_0^{-1}(\lambda)|\lambda|^{-k/2}f_v(\lambda).\end{aligned}$$

Hence

$$\lim_{a\to 0} \omega_0^{-1}(a)|a|^{-k/2}f_v(a) = \omega_0^{-1}(\lambda)|\lambda|^{-k/2}f_v(\lambda)$$

for any $\lambda$ with $|\lambda| < q^{-N_v}$. □

**Corollary.** *If $\lambda_1$ and $\lambda_2$ both satisfy $|\lambda_i| < q^{-N_v}$ then*

$$\omega_0^{-1}(\lambda_1)|\lambda_1|^{-k/2} f_v(\lambda_1) = \omega_0^{-1}(\lambda_2)|\lambda_2|^{-k/2} f_v(\lambda_2).$$

We continue taking $\tau = \pi_{(n-k-1)}$ and are working under the assumption that $\pi_0^{(n-k)}$ is irreducible non-zero with central character $\omega_0$. We have seen that for every $a \in K^\times$ and every $v \in V_{\tau_0}$ we have $\tau_0(aI_k)v - \omega_0(a)|a|^{k/2}v$ lies in $V_\tau(U_{k+1}, \mathbf{1})$ since it vanishes under the normalized projection to $V_{\pi_0^{(n-k)}} = V_{\tau_0}/V_\tau(U_{k+1}, \mathbf{1})$. Here we view $aI_k \in GL_k \hookrightarrow P_{k+1}$. Hence there is a positive integer $N = N_{v,a}$ such that for $g \in GL_k$ and

$$F_{\tau_0(aI_k)v-\omega_0(a)|a|^{k/2}v}(g) = 0 \text{ whenever } |g_k| < q^{-N_{v,a}}$$

that is

$$F_v(ag) = \omega_0(a)|a|^{k/2} F_v(g) \text{ whenever } |g_k| < q^{-N_{v,a}}.$$

We now claim that we can choose $N = N_v$ uniformly for $a \in \mathfrak{o}$ as before. The argument is essentially the same as above. From the smoothness of $F_v$ and $\omega_0$ there is an open set $\mathfrak{u} \subset \mathfrak{o}^\times$ such that $F_v(gu) = F_v(g)$ and $\omega_0(u) = 1$. Let $u_1, \ldots, u_m$ be a set of coset representatives of $\mathfrak{o}^\times/\mathfrak{u}$ and let $N_v = \max\{N_{v,u_1}, \ldots, N_{v,u_m}, N_{v,\varpi}\}$. Now let $a \in \mathfrak{o} \cap K^\times$ and $g \in GL_k$ with $|g_k| < q^{-N_v}$. Write $a = \varpi^j u_1 \tilde{u}$ with $\tilde{u} \in \mathfrak{u}$ with $j \geq 0$. Then

$$\begin{aligned} F_v(ag) &= F_v(\varpi^j u_i g) \\ &= \omega_0(u_i) F_v(\varpi^j g) \text{ since } |\varpi^j g_k| \leq q^{-N_v - j} \leq q^{-N_{v,u_i}} \\ &= \omega_0(\varpi^j u_i)|\varpi^j|^{k/2} F_v(g) \text{ since } |\varpi^i g_k| \leq q^{-N_v - i} \leq q^{-N_{v,\varpi}} \text{ for all } i \\ &= \omega_0(a)|a|^{k/2} F_v(g) \end{aligned}$$

We next show that there exists a $v \in V_{\tau_0}$ such that $\lim_{a\to 0} \omega_0^{-1}(a)|a|^{-k/2} F_v(a) \neq 0$. First, select $v_0 \in V_{\tau_0}$, $v_0 \notin V_\tau(U_{k+1}, \mathbf{1})$, so that $p(v_0) \neq 0$. Now, there is $N = N_{v_0}$ such that $F_{v_0}(ag) = \omega_0(a)|a|^{k/2} F_{v_0}(g)$ whenever $a \in \mathfrak{o}$ and $|g_k| < q^{-N}$. Since $v_0 \notin V_\tau(U_{k+1}, \mathbf{1})$ there must be a $g' \in GL_k$ with $|g'_k| < q^{-N}$ for which $F_{v_0}(g') \neq 0$. Then, if we let $v = \tau_0 \begin{pmatrix} g' & \\ & 1 \end{pmatrix} v_0$ we have

(i) $F_v(I_k) = F_{v_0}(g') \neq 0$,
(ii) $F_v(a) = \omega_0(a)|a|^{k/2} F_v(I_k)$ whenever $|a| \leq 1$, $a \neq 0$,

and hence

(iii) $\lim_{a\to 0} \omega_0^{-1}(a)|a|^{-k/2} F_v(a) = F_v(I_k) \neq 0$

as desired.

**Proposition 1.7.** *The linear functional*

$$\lambda(v) = \lim_{a\to 0} \omega_0^{-1}(a)|a|^{-k/2} F_v(a)$$

*defines a non-trivial Whittaker functional on* $\pi_0^{(n-k)}$.

*Proof:* It is easy to see that $\lambda$ is linear. We have seen that $\lambda$ vanishes on $V_\tau(U_{k+1}, \mathbf{1})$, so that $\lambda$ factors through to a functional on $V_{\tau_0}/V_\tau(U_{k+1}, \mathbf{1}) = V_{\pi_0^{(n-k)}}$, and that there exists $v \in V_{\tau_0}$ such that $\lambda(v) \neq 0$.

To see that $\lambda$ is a Whittaker functional, we compute for $n \in N_k$ that

$$\begin{aligned}\lambda(\pi^{(n-k)}(n)v) &= \lambda\left(\tau_0\begin{pmatrix} n & \\ & 1\end{pmatrix} v\right) = \lim_{a\to 0} \omega_0^{-1}(a)|a|^{-k/2} F_{\tau_0\begin{pmatrix} n & \\ & 1\end{pmatrix} v}(a) \\ &= \lim_{a\to 0} \omega_0^{-1}(a)|a|^{-k/2} F_v(an) = \psi(n) \lim_{a\to 0} \omega_0^{-1}(a)|a|^{-k/2} F_v(a) \\ &= \psi(n)\lambda(v).\end{aligned}$$

□

Before we end these preliminaries and turn to the Rankin–Selberg integrals, let us note the following connection between the Whittaker model of $\pi_0^{(n-k)}$ determined by $\lambda$ and the Whittaker model of $\pi$. If $v' \in V_{\pi_0^{(n-k)}}$ define $W'_{v'}(g) = \lambda(\pi^{(n-k)}(g)v')$ for $g \in GL_k$. If $v \in V_{\tau_0}$ which projects to $v'$ then we have $W'_{v'}(g) = |\det(g)|^{-1/2} \lim_{a\to 0} \omega_0^{-1}(a)|a|^{-k/2} F_v(ag)$. But if $|g_k| < q^{-N_v}$ and $|a| \leq 1$ we have $F_v(ag) = \omega(a)|a|^{k/2} F_v(g)$. So, if $|g_k| < q^{-N_v}$ we have $W'_{v'}(g) = |\det(g)|^{-1/2} F_v(g)$. Therefore if we let $\Phi_v(x) \in \mathcal{S}(K^k)$ be the characteristic function of the lattice $(\varpi^{N_v}\mathfrak{o})^k$ and remember the definition of $F_v$ we have

$$\begin{aligned}W'_{v'}(g)\Phi_v(e_k g) &= |\det(g)|^{-1/2} F_v(g)\Phi_v(e_k g) \\ &= |\det(g)|^{-(n-k)/2} W_v\begin{pmatrix} g & \\ & I_{n-k}\end{pmatrix} \Phi_v(e_k g)\end{aligned}$$

where, as usual, $e_k = (0, \ldots, 0, 1) \in K^k$. For future reference, we record this as a Corollary to Proposition 1.7.

**Corollary.** *Let* $\pi_0^{(n-k)}$ *be an irreducible submodule of* $\pi^{(n-k)}$ *and* $\tau_0$ *the corresponding submodule of* $\pi_{(n-k-1)}$ *which projects to* $\pi_0^{(n-k)}$ *under the canonical projection. For every* $W_\circ \in \mathcal{W}(\pi_0^{(n-k)}, \psi)$ *there is a* $W \in \mathcal{W}(\tau_0, \psi)$ *and* $\Phi_\circ \in \mathcal{S}(K^k)$ *with* $\Phi_\circ(0) \neq 0$ *such that*

$$W_\circ(g)\Phi_\circ(e_k g) = |\det(g)|^{-(n-k)/2} W\begin{pmatrix} g & \\ & I_{n-k}\end{pmatrix} \Phi_\circ(e_k g).$$

*Moreover, for every* $W \in \mathcal{W}(\tau_0, \psi)$ *and every* $\Phi_\circ \in \mathcal{S}(K^k)$ *locally constant and supported in a sufficiently small neighborhood of* $0$ *there is a* $W_\circ \in \mathcal{W}(\pi_0^{(n-k)}, \psi)$ *such that*

$$W\begin{pmatrix} g & \\ & I_{n-k}\end{pmatrix} \Phi_\circ(e_k g) = |\det(g)|^{(n-k)/2} W_\circ(g)\Phi_\circ(e_k g).$$

1.5. **Representations of Whittaker type.** The results of Sections 1.3 and 1.4 are valid for a wider class of representations than just generic representations. In this section we would like to discuss these extensions.

For our purposes, a representation $\pi$ of $GL_n$ will be called of *Whittaker type* if $\pi$ is an induced representation of the form

$$\pi = \mathrm{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$$

where each $\Delta_i$ is an irreducible quasi–square–integrable representation of $GL_{n_i}$ and the induction is normalized from the standard parabolic $Q$ associated to the partition $(n_1, \ldots, n_t)$ of $n$. These are included in the representations of Whittaker type of [9] and in fact the only representations of Whittaker type needed for the applications therein. Such $\pi$ need not be irreducible, but they have a unique (up to scalars) non-trivial $\psi$-Whittaker functional $\lambda$ and hence a (not necessarily injective) Whittaker model $\mathcal{W}(\pi, \psi)$ defined by $v \in V_\pi \mapsto W_v(g) = \lambda(\pi(g)v) \in \mathcal{W}(\pi, \psi)$. The space $\mathcal{W}(\pi, \psi)$ is not really a model for $\pi$ but rather for a non–degenerate quotient of $\pi$. For these representations, it is not known in general whether the restriction of the functions $W_v \in \mathcal{W}(\pi, \psi)$ to $p \in P_n$ can vanish identically, that is, whether a version of the Kirillov model holds for these Whittaker quotients. However, there is a natural representation of $P_n$ for which the analysis of Sections 1.3 and 1.4 hold. Namely, if in this context we set

$$\mathcal{W}_{(0)}(\pi, \psi) = \{W_v(p) \mid v \in V_\pi,\ p \in P_n\}$$

and let $\tau$ be the representation of $P_n$ in this space by right translation, then we may analyze $\tau$ and its derivatives as above. In particular, Propositions 1.1 through 1.7 and their Corollaries remain valid if we replace the representation $\pi^{(0)}$ of $GL_n$ by the Whittaker quotient $\mathcal{W}(\pi, \psi)$, the representation $\pi_{(0)}$ of $P_n$ by the space of restricted Whittaker functions $\mathcal{W}_{(0)}(\pi, \psi)$, and then define $\pi^{(k)}$ and $\pi_{(k)}$ through the representation $\mathcal{W}_{(0)}(\pi, \psi)$ of $P_n$.

There is one case where the representation $\pi$ is a possibly reducible representation of Whittaker type and still the Propositions 1.1 through 1.7 and their Corollaries hold without modification. These are the *induced representations of Langlands type*. Let $\nu(g) = |\det(g)|$ denote the unramified determinantal character of any $GL_r$. An induced representation of Langlands type is a representation of the form

$$\pi = \mathrm{Ind}(\Delta_1 \nu^{u_1} \otimes \cdots \otimes \Delta_t \nu^{u_t})$$

where each $\Delta_i$ is now an irreducible (unitary) square–integrable representation of $GL_{n_i}$, each $u_i$ is real and they are ordered so that $u_1 \geq u_2 \geq \cdots \geq u_t$. The induction is normalized from the standard (upper) parabolic $Q$ associated to the partition $(n_1, \ldots, n_t)$ of $n$. Note that these are representations of Whittaker type.

From the work of Jacquet and Shalika [10] we know that for these induced representations of Langlands type the map $v \mapsto W_v(g)$ is actually an isomorphism of $\pi$ with its Whittaker model $\mathcal{W}(\pi, \psi)$ and moreover the restriction of these functions to $P_n$ can never vanish identically, so the set of functions $\{W_v(p) | v \in V_\pi,\ p \in P_n\}$ does give a model for the restriction

$\pi_{(0)}$ of $\pi$ to $P_n$, that is, $\mathcal{W}_{(0)}(\pi,\psi) = \mathcal{W}(\pi_{(0)},\psi)$. Hence, for these induced representations of Langlands type, the Propositions 1.1 through 1.7 and their Corollaries hold as stated.

## 2. Derivatives and Local Factors

We want to use the theory of derivatives to compute the L-function for a non-archimedean local field $K$ for the case of $GL_n \times GL_m$ with $m \leq n$.

2.1. **The basic existence theorem.** We first recall the basic definitions and the basic existence theorem from the paper of Jacquet, Piatetski-Shapiro, and Shalika [9].

Let

$$\pi = \mathrm{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$$

be a representation of Whittaker type on $GL_n$ as in Section 1.5. Let $\pi^\iota$ denote the representation of $GL_n$ on the same space $V_\pi$ but with action $\pi^\iota(g) = \pi({}^t g^{-1})$. If $\pi$ is irreducible, then $\pi^\iota = \tilde{\pi}$, the contragredient representation. $\pi^\iota = \mathrm{Ind}(\tilde{\Delta}_t \otimes \cdots \otimes \tilde{\Delta}_1)$ is again of Whittaker type. Let $w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ denote the long Weyl element in $GL_n$. If $W \in \mathcal{W}(\pi,\psi)$ then the function $\tilde{W}(g) = W(w_n\, {}^t g^{-1}) \in \mathcal{W}(\pi^\iota, \psi^{-1})$.

Now let $\pi$ be a representation of $GL_n$ of Whittaker type and $\sigma$ a representation of $GL_m$ of Whittaker type. We always assume $m \leq n$. For each $W \in \mathcal{W}(\pi,\psi)$ and $W' \in \mathcal{W}(\sigma,\psi^{-1})$ we associate an integral

$$I(s : W, W', \Phi) = \int_{N_n \backslash GL_n} W(g) W'(g) \Phi(e_n g) |\det(g)|^s \, dg$$

for each $\Phi \in \mathcal{S}(K^n)$ in the case $m = n$, or

$$I_j(s; W, W') = \int_{N_m \backslash GL_m} \int_{M_{j,m}} W \begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-m-j} \end{pmatrix} W'(g) |\det(g)|^{s-(n-m)/2} \, dx \, dg$$

for each $0 \leq j \leq n-m-1$ in the case $m < n$. In the case $m < n$, we will let $I(s; W, W') = I_0(s; W, W')$ when appropriate to ease the notation.

The basic existence theorem of Jacquet, Piatetski-Shapiro, and Shalika [9] is the following.

**Theorem.** [9] *(i) Each of the integrals $I(s : W, W', \Phi)$, in the case $m = n$, and the integrals $I_j(s : W, W')$, in the case $m < n$, is absolutely convergent for* $\mathrm{Re}(s)$ *large.*
*(ii) They are rational functions of $q^{-s}$. More precisely, if $m = n$ the integrals $I(s; W, W', \Phi)$ form a fractional ideal $\mathcal{I}(\pi,\sigma)$ of the ring $\mathbb{C}[q^s, q^{-s}]$ of the form $L(s, \pi \times \sigma)\mathbb{C}[q^s, q^{-s}]$; the factor $L(s, \pi \times \sigma)$ is of the form $P(q^{-s})^{-1}$ where $P(X) \in \mathbb{C}[X]$ and $P(0) = 1$. If $m < n$, there is a similar factor $L(s, \pi \times \sigma)$, independent of $j$, generating the ideal $\mathcal{I}(\pi,\sigma)$ spanned by the integrals $I_j(s; W, W')$. The same results are true of the pair $(\pi^\iota, \sigma^\iota)$.*

*(iii) Suppose $m = n$. Then there is a monomial factor $\varepsilon(\pi \times \sigma, s, \psi)$ of the form $\alpha q^{-as}$, $a \in \mathbb{Z}$, such that if*

$$\gamma(s, \pi \times \sigma, \psi) = \frac{\varepsilon(s, \pi \times \sigma, \psi) L(1 - s, \pi^\iota \times \sigma^\iota)}{L(s, \pi \times \sigma)}$$

*then*

$$I(1 - s; \tilde{W}, \tilde{W}', \hat{\Phi}) = \omega_\sigma(-1)^{n-1} \gamma(s, \pi \times \sigma, \psi) I(s; W, W').$$

*Similarly, if $m < n$ there is a monomial factor $\varepsilon(s, \pi \times \sigma, \psi)$, independent of $j$, such that if*

$$\gamma(s, \pi \times \sigma, \psi) = \frac{\varepsilon(s, \pi \times \sigma, \psi) L(1 - s, \pi^\iota \times \sigma^\iota)}{L(s, \pi \times \sigma)}$$

*then*

$$I_{n-m-j-1}(1 - s; \pi^\iota(w_{n,m}) \tilde{W}, \tilde{W}') = \omega_\sigma(-1)^{n-1} \gamma(s, \pi \times \sigma, \psi) I_j(s; W, W')$$

*where* $w_{n,m} = \begin{pmatrix} I_{n-m} & \\ & w_m \end{pmatrix}$.

*Proof:* [Remarks on the proof] Statements (i) and (ii) are proved by analyzing the asymptotics of the Whittaker functions involved on the torus $T_n$ or $T_m$ as the entries go to 0. The rationality of the integrals can also be proven using Bernstein's Theorem, as presented in Section 3 here. Statement (iii), the local functional equation, is an application of the theory of derivatives and is quite in keeping with the spirit of this paper. □

In order to compute $L(s, \pi \times \sigma)$ we can use the following elementary characterization of the polynomial $P(q^{-s})$, namely $P(q^{-s})$ is the minimal polynomial (in the sense of divisibility) in $q^{-s}$ such that $P(q^{-s}) I(s; W, W', \Phi)$, in the case $m = n$, or $P(q^{-s}) I_j(s; W, W')$, in the case $m < n$, is an entire function of $s$ for all $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\sigma, \psi^{-1})$, and $\Phi \in \mathcal{S}(K^n)$ (if necessary). Once we normalize so that $P(0) = 1$, this characterizes $P(q^{-s})$ and hence $L(s, \pi \times \sigma)$ uniquely.

The cases which are of most interest for us are when $\pi$ and $\sigma$ are either generic or induced of Langlands type, so that the results of Sections 1.3 and 1.4 hold as stated. These cases also best illustrate the ideas involved. However, when we deform our representations in Section 3 we will of necessity leave the realm of generic representations and even of induced representations of Langlands type, although when the parameters are in "general position" the representations in question will be generic. So we must work with representations of Whittaker type. Throughout this section we will use the notation of Sections 1.3 and 1.4. However, in the case when $\pi$ and $\sigma$ are only of Whittaker type and not the more special generic or induced of Langlands type, then we invoke the conventions of Section 1.5, that is, we take $\pi^{(0)}$ to be $\mathcal{W}(\pi, \psi)$, $\pi_{(0)}$ to be $\mathcal{W}_{(0)}(\pi, \psi)$, $\sigma^{(0)}$ to be $\mathcal{W}(\sigma, \psi^{-1})$, and $\sigma_{(0)}$ to be $\mathcal{W}_{(0)}(\sigma, \psi^{-1})$. Since our computations involve the manipulation of integrals of Whittaker functions, this is most natural. We hope this does not cause too much confusion.

2.2. **The case $m = n$.** Let us now begin to analyze the locations of the poles of the rational functions in $\mathcal{I}(\pi, \sigma)$, since these poles and their orders will determine $P(q^{-s}) = L(s, \pi \times \sigma)^{-1}$.

Since this ideal is linearly spanned by the $I(s; W, W', \Phi)$ it will suffice to understand the poles of these integrals.

Suppose there is a function in $\mathcal{I}(\pi, \sigma)$ having a pole of order $d$ at $s = s_0$ and that this is the highest order pole of the family at $s = s_0$. Consider a rational function defined by an individual integral $I(s; W, W', \Phi)$. Then the Laurent, or partial fraction, expansion about $s = s_0$ will have the form

$$I(s; W, W', \Phi) = \frac{B_{s_0}(W, W', \Phi)}{(q^s - q^{s_0})^d} + \text{ higher order terms.}$$

The coefficient of the leading term, $B(W, W', \Phi)$, will define a non-trivial trilinear form on $\mathcal{W}(\pi, \psi) \times \mathcal{W}(\sigma, \psi^{-1}) \times \mathcal{S}(K^n)$ satisfying the quasi–invariance

$$B_{s_0}(\pi(g)W, \sigma(g)W', \rho(g)\Phi; s) = |\det(g)|^{-s_0} B_{s_0}(W, W', \Phi)$$

where $\rho$ denotes the representation of $GL_n$ on $\mathcal{S}(K^n)$ by right translation.

The Schwartz functions have a natural $GL_n$–stable filtration $\mathcal{S}(K^n) \supset \mathcal{S}_0(K^n) \supset \{0\}$, where $\mathcal{S}_0(K^n) = \{\Phi \in \mathcal{S}(K^n) \mid \Phi(0) = 0\}$.

**Definition.** *The pole at $s = s_0$ of the family $\mathcal{I}(\pi, \sigma)$ is called* exceptional *if the associated trilinear form $B_{s_0}(W, W', \Phi)$ vanishes identically on $\mathcal{S}_0(K^n)$.*

If $s_0$ is an exceptional pole of $I(\pi, \sigma)$ then the trilinear form $B_{s_0}$ factors to a non–zero trilinear form on $\mathcal{W}(\pi, \psi) \times \mathcal{W}(\sigma, \psi^{-1}) \times (\mathcal{S}(K^n)/\mathcal{S}_0(K^n))$. The quotient $\mathcal{S}(K^n)/\mathcal{S}_0(K^n)$ is isomorphic to $\mathbb{C}$ via the map $\Phi \mapsto \Phi(0)$. Hence if $s_0$ is an exceptional pole, then the form $B_{s_0}$ can be written as $B_{s_0}(W, W', \Phi) = B^\circ_{s_0}(W, W')\Phi(0)$ with $B^\circ_{s_0}$ a quasi–invariant bilinear form on $\mathcal{W}(\pi, \psi) \times \mathcal{W}(\sigma, \psi^{-1})$ satisfying $B^\circ_{s_0}(\pi(g)W, \sigma(g)W') = |\det(g)|^{-s_0} B^\circ_{s_0}(W, W')$. Note that in the case that $\pi$ and $\sigma$ are irreducible, which will be true for $\pi$ and $\sigma$ in "general position", such a pairing implies an isomorphism between $\tilde{\pi}$ and $\sigma\nu^{s_0}$, where we let $\nu(g)$ denote the determinantal character $\nu(g) = |\det(g)|$ for any size $GL_n$, and puts a severe restriction on the possible exceptional poles $s_0$. In general, we would have to have such an isomorphism between constituents of $\pi$ and $\sigma$. Let us emphasize this fact.

**Proposition 2.1.** *If $\pi$ and $\sigma$ are irreducible, then the exceptional poles $s_0$ of the family $\mathcal{I}(\pi, \sigma)$ can only occur among those $s$ for which $\tilde{\pi} \cong \sigma\nu^s$.*

If the ideal $\mathcal{I}(\pi, \sigma)$ has an exceptional pole of order $d$ at $s = s_0$, then this pole contributes a factor of $(1 - q^{s_0}q^{-s})^d$ to $L(s, \pi \times \sigma)^{-1}$. For distinct exceptional poles (as elements of $\mathbb{C}/\frac{2\pi i}{\log(q)}\mathbb{Z}$), these factors will be independent (that is, relatively prime in $\mathbb{C}[q^s, q^{-s}]$).

**Definition.** *Let $L_{ex}(s, \pi \times \sigma)^{-1}$ denote the product of these factors $(1 - q^{s_0}q^{-s})^d$ as $s_0$ runs over the exceptional poles of $\mathcal{I}(\pi, \sigma)$, with $d$ the (maximal) order of the pole.*

Then $L_{ex}(s, \pi \times \sigma)^{-1}$ divides $L(s, \pi \times \sigma)^{-1}$. We will refer to $L_{ex}(s, \pi \times \sigma)$ as the exceptional contribution to $L(s, \pi \times \sigma)$.

Next, consider a pole at $s = s_0$ of the family $\mathcal{I}(\pi,\sigma)$ which is not exceptional. Let $d$ be the maximal order with which it occurs in $\mathcal{I}(\pi,\sigma)$. Then each integral $I(s;W,W',\Phi)$ will still have an expansion

$$I(s;W,W',\Phi) = \frac{B_{s_0}(W,W',\Phi)}{(q^s - q^{s_0})^d} + \text{ higher order terms.} \tag{2.1}$$

about $s = s_0$ and the coefficient of the leading term, $B_{s_0}(W,W',\Phi)$, still defines a non-trivial trilinear form on $\mathcal{W}(\pi,\psi) \times \mathcal{W}(\sigma,\psi^{-1}) \times \mathcal{S}(K^n)$ satisfying the quasi–invariance

$$B_{s_0}(\pi(g)W, \sigma(g)W', \rho(g)\Phi) = |\det(g)|^{-s_0} B_{s_0}(W,W',\Phi),$$

but now this form restricts non–trivially to $\mathcal{S}_0(K^n)$. Let $\Phi^\circ \in \mathcal{S}_0(K^n)$ for which there exist $W \in \mathcal{W}(\pi,\sigma)$ and $W' \in \mathcal{W}(\sigma,\psi^{-1})$ with $B_{s_0}(W,W',\Phi^\circ) \neq 0$.

Let $K_n = GL_n(\mathfrak{o})$ be the maximal compact subgroup of $GL_n$. Then we may decompose our integral as

$$\begin{aligned} I(s;W,&W',\Phi^\circ) = \\ &= \int_{K_n}\int_{N_n\backslash P_n} W(pk)W'(pk)|\det(p)|^{s-1}\int_{K^\times}\omega_\pi(a)\omega_\sigma(a)|a|^{ns}\Phi^\circ(e_n ak)\ d^\times a\ dp\ dk. \end{aligned}$$

Take $K_n^\circ \subset K_n$ a compact open subgroup which stabilizes $W$, $W'$, and $\Phi^\circ$. Write $K_n = \cup_i k_i K_n^\circ$ and let $\Phi_i^\circ = \rho(k_i)\Phi^\circ$, $W_i = \pi(k_i)W$, and $W_i' = \sigma(k_i)W'$. Then each $\Phi_i^\circ(0) = 0$ so that each $\Phi_i^\circ(e_n a)$ has compact support on $K^\times$. Let $U^\circ$ be an open compact subgroup of $K^\times$ such that each $\Phi_i^\circ$ as well as $\omega_\pi$ and $\omega_\sigma$ are invariant under $U^\circ$. Let $S = \cup\,\mathrm{Supp}(\Phi_i^\circ(e_n a))$ and write $S = \cup a_j U^\circ$. Then, the integral $I(s;W,W',\Phi^\circ)$ can be decomposed as a finite sum of the form

$$\begin{aligned} I(s;W,&W',\Phi^\circ) = \\ &= c\sum_{i,j}\omega_\pi(a_j)\omega_\sigma(a_j)|a_j|^{ns}\Phi_i^\circ(e_n a_j)\int_{N_n\backslash P_n} W_i(p)W_i'(p)|\det(p)|^{s-1}\ dp \end{aligned} \tag{2.2}$$

with $c > 0$ a volume term. We still have the expansion (2.1) with $B_{s_0}(W,W',\Phi^\circ) \neq 0$ for some choice of $W$ and $W'$. Hence, for such a choice of $W$ and $W'$ we must have that at least one of the rational functions defined by the integrals

$$\int_{N_n\backslash P_n} W_i(p)W_i'(p)|\det(p)|^{s-1}\ dp \tag{2.3}$$

must have a pole of order $d$ at $s = s_0$. Moreover, as is apparent by looking at (2.2), for suitable choice of $\Phi^\circ$ each of these integrals will individually occur in $\mathcal{I}(\pi,\sigma)$ and hence completely account for the pole at $s = s_0$ with its maximal order. Let us denote the integral occurring in (2.3) by $I_{(0)}(s;W,W')$. As is suggested, the integrals only depend on the functions $W$ and $W'$ through their restriction to $P_n$, that is, on $\pi_{(0)}$ and $\sigma_{(0)}$. Moreover, the integral over $N_n\backslash P_n$ can be reduced to an integral over $N_{n-1}\backslash GL_{n-1}$. We summarize this in the following statement.

**Proposition 2.2.** *The poles of the family $\mathcal{I}(\pi,\sigma)$ which are not exceptional are precisely the poles of maximal order of the family of rational functions $\mathcal{I}_{(0)}(\pi,\sigma)$ spanned by the integrals*

$$I_{(0)}(s;W,W') = \int_{N_{n-1}\backslash GL_{n-1}} W\begin{pmatrix} g & \\ & 1\end{pmatrix} W'\begin{pmatrix} g & \\ & 1\end{pmatrix} |\det(g)|^{s-1}\, dg$$

*with $W \in \mathcal{W}(\pi,\psi)$ and $W' \in \mathcal{W}(\sigma,\psi^{-1})$.*

Let us now analyze the rational functions $I_{(0)}(s;W,W')$. We again take a pole $s = s_0$ of the family $\mathcal{I}_{(0)}(\pi,\sigma)$ and let $d$ be its maximal order in the family. Then each rational function will have an expansion of the usual shape, which in this case we write

$$I_{(0)}(s;W,W') = \frac{B_{(0),s_0}(W,W')}{(q^s - q^{s_0})^d} + \text{ higher order terms}$$

where now we may view $B_{(0),s_0}$ as a non-trivial bilinear form on $\mathcal{W}(\pi_{(0)},\psi) \times \mathcal{W}(\sigma_{(0)},\psi^{-1})$ which satisfies $B_{(0),s_0}(\pi_{(0)}(p)W, \sigma_{(0)}(p)W') = |\det(p)|^{-s_0+1} B_{(0),s_0}(W,W')$.

Since we are now dealing with representations of $P_n$, we can use the filtration of $\pi_{(0)}$ and $\sigma_{(0)}$ by derivatives. To ease notation, let us denote $\pi_{(0)}$ by $\tau$ for the moment. So $\tau$ has the filtration $\tau = \tau_1 \supset \tau_2 \supset \cdots \supset \tau_n \supset 0$ with $\tau_i = (\Phi^+)^{i-1}(\Phi^-)^{i-1}(\tau) = (\Phi^+)^{i-1}(\pi_{(i-1)})$. We know that the bilinear form $B_{(0),s_0}$ is nontrivial on $\tau = \tau_1$. It must be trivial on $\tau_n$, for the functions $W$ which come from $\mathcal{W}(\tau_n,\psi)$ are compactly supported on $P_n$ modulo $N_n$. Hence the integral defining $I_{(0)}(s;W,W')$ for such functions will become a finite sum, resulting in a Laurent polynomial which is entire. So there will be a smallest $k$ such that $B_{(0),s_0}$ restricts trivially on $\mathcal{W}(\tau_{n-k+1},\psi)$ but is non–zero on $\mathcal{W}(\tau_{n-k},\psi)$.

Consider now the rational functions defined by the integrals $I_{(0)}(s;W,W')$ with $W \in \mathcal{W}(\pi_{(0),n-k},\psi)$. These rational functions account for the pole at $s = s_0$ with the maximal order $d$. Recall from Section 2 that the Whittaker functions $W \in \mathcal{W}(\pi_{(0),n-k},\psi)$ are characterized by the fact if we view them as functions on $GL_{n-1}$ by $W\begin{pmatrix} g & \\ & 1\end{pmatrix}$ that their support in the last $n-k-1$ rows of $g$ is compact, modulo $N_n$. Hence, if we use a partial Iwasawa decomposition to write the $g \in GL_{n-1}$ as

$$g = \begin{pmatrix} h & & & \\ & a_{k+1} & & \\ & & \ddots & \\ & & & a_{n-1}\end{pmatrix} k \pmod{N_{n-1}}$$

with $h \in GL_k$, $k \in K_{n-1}$, and each $a_i \in K^\times$, the function $W$ will have compact multiplicative support in the $a_i$. Then, as in (2.2), for a fixed $W$ and $W'$ our integral becomes a finite sum of the form

$$I_{(0)}(s;W,W') = \sum_i c_i q^{-\beta_i s} \int_{N_k\backslash GL_k} W_i\begin{pmatrix} h & \\ & I_{n-k}\end{pmatrix} W_i'\begin{pmatrix} h & \\ & I_{n-k}\end{pmatrix} |\det(h)|^{s-(n-k)}\, dh.$$

Set

$$I_{(n-k-1)}(s;W,W') = \\ = \int_{N_k\backslash GL_k} W\begin{pmatrix} h & \\ & I_{n-k}\end{pmatrix} W'\begin{pmatrix} h & \\ & I_{n-k}\end{pmatrix} |\det(h)|^{s-(n-k)}\, dh. \tag{2.4}$$

Then our pole at $s = s_0$ of order $d$ must come from one of the integrals $I_{(n-k-1)}(W, W'; s)$ with $W \in \mathcal{W}(\pi_{(0),n-k}, \psi)$. Moreover, since both $W$ and $W'$ enter into these integrals through their restriction to $GL_k$, they only depend on the images of $W$ and $W'$ in $\mathcal{W}(\pi_{(n-k-1)}, \psi)$ and $\mathcal{W}(\sigma_{(n-k-1)}, \psi^{-1})$ respectively. Moreover, by Lemma 9.2 of [9], it is elementary to see that each $I_{(n-k-1)}(s;W,W')$ actually occurs as a $I_{(0)}(s;W_\circ,W'_\circ)$ for appropriate choice of $W_\circ$ and $W'_\circ$, and hence are elements of $\mathcal{I}_{(0)}(\pi,\sigma) \subset \mathcal{I}(\pi,\sigma)$. Hence these integrals can have at most a pole of order $d$ at $s = s_0$ and hence a pole of order exactly $d$ at $s = s_0$ for appropriate choice of $W$ and $W'$.

Let us denote by $\mathcal{I}_{(n-k-1)}(\pi,\sigma)$ the span of the rational functions defined by the integrals $I_{(n-k-1)}(s;W,W')$. As we have seen, each non–exceptional pole of the family $\mathcal{I}(\pi,\sigma)$ occurs as a pole of the family $\mathcal{I}_{(n-k-1)}(\pi,\sigma)$ for some $k$ and each pole of this family is a pole of the family $\mathcal{I}(\pi,\sigma)$.

Now, analyze our non–exceptional pole at $s = s_0$ in terms of these integrals. Again, look at the expansion about $s = s_0$

$$I_{(n-k-1)}(s;W,W') = \frac{B_{(n-k-1),s_0}(W,W')}{(q^s - q^{s_0})^d} + \text{ higher order terms.}$$

Since the integral involves $W$ and $W'$ restricted to $GL_k$, the integral and the bilinear form $B_{(n-k-1),s_0}(W,W')$ depend only on the functions determined by $W$ and $W'$ in $\mathcal{W}(\pi_{(n-k-1)}, \psi)$ and $\mathcal{W}(\sigma_{(n-k-1)}, \psi^{-1})$ respectively. Recalling the twists involved in the definitions of the derivatives, the quasi–invariance becomes

$$B_{(n-k-1),s_0}(\pi_{(n-k-1)}(p)W, \sigma_{(n-k-1)}(p)W') = |\det(p)|^{-s_0+1} B_{(n-k-1),s_0}(W,W').$$

Furthermore, the index $k$ was chosen so that $B_{(n-k-1),s_0}(W,W')$ is non-trivial, but vanishes for $W$ corresponding to functions in $\mathcal{W}(\pi_{(0),n-k+1}, \psi)$ that is for $W \in \mathcal{W}(\pi_{(n-k-1),2}, \psi) \subset \mathcal{W}(\pi_{(n-k-1)}, \psi)$. As a representation of $P_{k+1}$, $\pi_{(n-k-1)}/\pi_{(n-k-1),2} = \Psi^+(\pi^{(n-k)})$. Hence $B_{(n-k-1),s_0}$ defines a non-trivial bilinear form on $\mathcal{W}(\Psi^+(\pi^{(n-k)}),\psi) \times \mathcal{W}(\sigma_{(n-k-1)}, \psi^{-1})$ which is quasi-invariant with respect to the action of $P_{k+1}$. By Proposition 3.7 of [3], there are no non–trivial quasi–invariant pairings between $\Psi^+(\pi^{(n-k)})$ and $\sigma_{(n-k-1),2} = \Phi^+(\sigma_{(n-k)})$. Hence the pairing defined by $B_{(n-k-1),s_0}$ must actually define a non-zero bilinear form on the space $\mathcal{W}(\Psi^+(\pi^{(n-k)}),\psi) \times \mathcal{W}(\Psi^+(\sigma^{(n-k)}),\psi^{-1})$.

To proceed from this point, we need to make a simplifying assumption. As we shall see in Section 3, this assumption is satisfied for all $\pi$ and $\sigma$ in "general position".

**Assumption.** *Assume the all derivatives $\pi^{(n-k)}$ of $\pi$ and all derivatives $\sigma^{(n-k)}$ of $\sigma$ are completely reducible.*

Continuing our analysis under this assumption, let us write $\pi^{(n-k)} = \oplus \pi_i^{(n-k)}$ and $\sigma^{(n-k)} = \oplus \sigma_j^{(n-k)}$ with each $\pi_i^{(n-k)}$ and $\sigma_j^{(n-k)}$ irreducible. Then the bilinear form $B_{(n-k-1),s_0}(W,W')$ determined by our pole $s_0$ must restrict non-trivially to some pair

$$(W_i, W_j') \in \mathcal{W}(\Psi^+(\pi_i^{(n-k)}),\psi) \times \mathcal{W}(\Psi^+(\sigma_j^{(n-k)}),\psi^{-1}).$$

Recall from the Corollary to Proposition 1.7 that there exist functions $W_\circ \in \mathcal{W}(\pi_i^{(n-k)},\psi)$ and $W_\circ' \in \mathcal{W}(\sigma_j^{(n-k)},\psi^{-1})$ such that for every Schwartz function $\Phi_\circ \in \mathcal{S}(K^k)$ which is the characteristic function of a sufficiently small neighborhood of $0 \in K^k$ we have an equalities

$$W_i \begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} \Phi_\circ(e_k h) = W_\circ(h) |\det(h)|^{(n-k)/2} \Phi_\circ(e_k h)$$
$$W_j' \begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} \Phi_\circ(e_k h) = W_\circ'(h) |\det(h)|^{(n-k)/2} \Phi_\circ(e_k h).$$

We may then decompose the integral $I_{(n-k-1)}(s; W_i, W_j')$ into two parts

$$I_{(n-k-1)}(s; W_i, W_j') = I^0_{(n-k-1)}(s; W_i, W_j') + I^1_{(n-k-1)}(s; W_i, W_j')$$

where

$$I^0_{(n-k-1)}(s; W_i, W_j') = \\ = \int_{N_k \backslash GL_k} W_i \begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} W_j' \begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} \Phi_\circ(e_k h) |\det(h)|^{s-(n-k)}\, dh$$

and

$$I^1_{(n-k-1)}(s; W_i, W_j') = \\ = \int_{N_k \backslash GL_k} W_i \begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} W_j' \begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} (1 - \Phi_\circ(e_k h)) |\det(h)|^{s-(n-k)}\, dh.$$

In the integrals $I^1_{(n-k-1)}(s; W_i, W_j')$, the $(1 - \Phi_\circ(e_k h))$ term will restrict the support of the integrand to being compact in the last row of $h$, modulo $N_k$. Using a partial Iwasawa decomposition as above, we can then write this integral as a finite sum of integrals involving the restrictions of $W_i$ and $W_j'$ to $GL_{k-1}$, that is, depending only on the images of $W_i$ and $W_j'$ in $\mathcal{W}(\pi_{(n-k)},\psi)$ and $\mathcal{W}(\sigma_{(n-k)},\psi^{-1})$. But our bilinear form restricts to zero on these spaces. Hence the integrals of the form $I^1_{(n-k-1)}(s; W_i, W_j')$ cannot contribute to the order $d$ pole at $s = s_0$, and play no role in our analysis.

We can write $I^0_{(n-k-1)}(s; W_i, W_j')$ in terms of $W_\circ$ and $W_\circ'$, namely

$$I^0_{(n-k-1)}(s; W_i, W_j') = \int_{N_k \backslash GL_k} W_\circ(h) W_\circ'(h) \Phi_\circ(e_k h) |\det(h)|^s\, dh.$$

These integrals must contribute the pole at of order $d$ at $s = s_0$. However, as integrals on $GL_k$, these are the standard Rankin–Selberg integrals $I^0_{(n-k-1)}(s; W_i, W_j') = I(s; W_\circ, W_\circ', \Phi_\circ)$ and since $\Phi_\circ(0) = 1 \neq 0$ the pole of order $d$ at $s = s_0$ is an exceptional pole of this integral. Moreover, any integral for $\pi_i^{(n-k)}$ and $\sigma_j^{(n-k)}$ corresponding to an exceptional pole must come from a $I^0_{(n-k-1)}(s;W_i,W_j')$ for some choice of $W_i$ and $W_j$, again by the Corollary to Proposition 1.7, and hence give rise to a pole of the family $\mathcal{I}(\pi,\sigma)$.

Summarizing this analysis, we arrive at the following result.

**Proposition 2.3.** *Under the assumption that all derivatives of $\pi$ and $\sigma$ are completely reducible, any non–exceptional pole $s=s_0$ of order $d$ of the family $\mathcal{I}(\pi,\sigma)$ will correspond to an exceptional pole, again of order d, for some family $\mathcal{I}(\pi_i^{(n-k)},\sigma_j^{(n-k)})$ with $0<k<n$, $\pi_i^{(n-k)}$ an irreducible constituent of $\pi^{(n-k)}$, and $\sigma_j^{(n-k)}$ and irreducible constituent of $\sigma^{(n-k)}$. Furthermore, all exceptional poles of these families occur, with the same order, as poles of the family $\mathcal{I}(\pi,\sigma)$.*

If me combine this with our analysis of the exceptional poles of the family $\mathcal{I}(\pi,\sigma)$, we arrive at the following theorem.

**Theorem 2.1.** *Let $\pi$ and $\sigma$ be representations of $GL_n$ of Whittaker type such that all derivatives of $\pi$ and $\sigma$ are completely reducible. Then*

$$L(s,\pi\times\sigma)^{-1}=l.c.m._{k,i,j}\{L_{ex}(s,\pi_i^{(n-k)}\times\sigma_j^{(n-k)})^{-1}\}$$

*where the least common multiple is with respect to divisibility in $\mathbb{C}[q^s,q^{-s}]$ and is taken over all $k$ with $0<k\leq n$ and for each $k$ all constituents $\pi_i^{(n-k)}$ of $\pi^{(n-k)}$ and all constituents $\sigma_j^{(n-k)}$ of $\sigma^{(n-k)}$.*

2.3. **The case $m<n$.** Now, take $\pi$ a representation of Whittaker type of $GL_n$ and $\sigma$ a representation of Whittaker type of $GL_m$ with $m<n$. Let us analyze the locations of the poles of the rational functions in $\mathcal{I}(\pi,\sigma)$, since these poles and their orders will determine $P(q^{-s})$ and hence $L(s,\pi\times\sigma)$.

Suppose there is a function in $\mathcal{I}(\pi,\sigma)$ having a pole at $s=s_0$ of order $d$ and that this is the highest order with which the pole at $s=s_0$ occurs in the family. Since the ideal $\mathcal{I}(\pi,\sigma)$ is spanned by the rational functions defined by the integrals $I(s;W,W')=I_0(s;W,W')$, this pole must occur with order $d$ for some function $I(s;W,W')$. Consider the rational function determined by an individual integral $I(s;W,W')$ with $W\in\mathcal{W}(\pi,\psi)$ and $W'\in\mathcal{W}(\sigma,\psi^{-1})$. Then this will have a Laurent (or partial fraction) expansion near $s=s_0$ of the form

$$I(s;W,W')=\frac{B_{s_0}(W,W')}{(q^s-q^{s_0})^d}+\text{ higher order terms.}$$

The coefficient $B_{s_0}(W,W')$ of the leading term will be a non-trivial bilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\sigma,\psi^{-1})$ with the quasi-invariance under $GL_m$ given by

$$B_{s_0}\left(\pi\begin{pmatrix}g&\\&I_{n-m}\end{pmatrix}W,\sigma(g)W'\right)=|\det(g)|^{-s_0+(n-m)/2}B_{s_0}(W,W').$$

Since the dependence on $W\in\mathcal{W}(\pi,\psi)$ is through its restriction to $GL_m$, we see that the function $I(s;W,W')$, and hence the form $B_{s_0}(W,W')$, only depends on the restriction of $W$ to $P_n$ and, in fact, only on the image of $W$ in the representation of $P_{m+1}$ on

$\mathcal{W}(\pi_{(n-m-1)}, \psi)$. Thus we see that in fact $B_{s_0}(W, W')$ factors to a $GL_m$ quasi-invariant pairing between $\mathcal{W}(\pi_{(n-m-1)}, \psi)$ and $\mathcal{W}(\sigma, \psi^{-1})$ satisfying

$$B_{s_0}\left(\pi_{(n-m-1)}\begin{pmatrix} g & \\ & 1 \end{pmatrix} W, \sigma(g)W'\right) = |\det(g)|^{-s_0+1/2} B_{s_0}(W, W').$$

Let $\tau = \pi_{(n-m-1)}$ and consider the filtration of $\tau$ by derivatives

$$0 \subset \tau_{m+1} \subset \tau_m \subset \cdots \subset \tau_1 = \tau.$$

The bilinear form $B_{s_0}$ must be trivial on $\tau_{m+1}$, for the functions coming from $\mathcal{W}(\tau_{m+1}, \psi)$ will be compactly supported on $GL_m$ modulo $N_m$, hence the integral defining $I(s; W, W')$ will reduce to a finite sum, resulting in a Laurent polynomial which is entire. Let $k$ be the smallest integer such that $B_{s_0}(W, W')$ is trivial on $\mathcal{W}(\tau_{m+2-k}, \psi)$ but not on $\mathcal{W}(\tau_{m+1-k}, \psi)$.

First, consider the case where $k = m$. In this case, $B_{s_0}(W, W')$ is zero for

$$W \in \mathcal{W}(\pi_{(n-m-1),2}, \psi) \subset \mathcal{W}(\pi_{(n-m-1)}, \psi).$$

The quotient $\pi_{(n-m-1)}/\pi_{(n-m-1),2}$ realizes the representation $\Psi^+(\pi^{(n-m)})$. Hence $B_{s_0}(W, W')$ factors to a non-zero bilinear form on $\mathcal{W}(\Psi^+(\pi^{(n-m)}, \psi) \times \mathcal{W}(\sigma, \psi^{-1})$.

Next, consider the case $k < m$ and consider the rational functions defined by the integrals $I(s; W, W')$ with $W \in \mathcal{W}(\pi_{(n-m-1),m+1-k}, \psi)$. These rational functions account for the pole at $s = s_0$ with the maximal order $d$. Recall from Section 1 that the Whittaker functions $W \in \mathcal{W}(\pi_{(n-m-1),m+1-k}, \psi)$ are characterized by the fact if we view them as functions on $GL_m$ by $W\begin{pmatrix} g & \\ & I_{n-m} \end{pmatrix}$ that their support in the last $m-k$ rows of $g$ is compact, modulo $N_m$. Hence, if we use a partial Iwasawa decomposition to write the $g \in GL_m$ as

$$g = \begin{pmatrix} h & & & \\ & a_{k+1} & & \\ & & \ddots & \\ & & & a_m \end{pmatrix} k \pmod{N_m}$$

with $h \in GL_k$, $k \in K_m$, and each $a_i \in K^\times$, the function $W$ will have compact multiplicative support in the $a_i$. Then, as in (2.2), for a fixed $W$ and $W'$ our integral becomes a finite sum of the form

$$I(s;W, W') = \sum_i c_i q^{-\beta_i s} \int_{N_k\backslash GL_k} W_i\begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} W_i'\begin{pmatrix} h & \\ & I_{m-k} \end{pmatrix} |\det(h)|^{s-(m-k)-(n-m)/2}\, dh.$$

Set

$$I_{(m-k-1)}(s; W, W') = \int_{N_k\backslash GL_k} W\begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} W'\begin{pmatrix} h & \\ & I_{m-k} \end{pmatrix} |\det(h)|^{s-(m-k)-(n-m)/2}\, dh. \tag{2.5}$$

Then our pole at $s = s_0$ of order $d$ must come from one of the integrals $I_{(m-k-1)}(s; W, W')$ with $W \in \mathcal{W}(\pi_{(n-m-1),m+1-k}, \psi)$. Moreover, since both $W$ and $W'$ enter into these integrals through their restriction to $GL_k$, they really only depend on the images of $W$ and $W'$ in $\mathcal{W}(\pi_{(n-k-1)}, \psi)$ and $\mathcal{W}(\sigma_{(m-k-1)}, \psi^{-1})$ respectively. Moreover, by Lemma 9.2 of [9], it is elementary to see that each $I_{(m-k-1)}(s; W, W')$ actually occurs as a $I(s; W_\circ, W'_\circ)$ for appropriate choice of $W_\circ$ and $W'_\circ$, and hence are elements of $\mathcal{I}(\pi, \sigma)$. Hence these integrals can have at most a pole of order $d$ at $s = s_0$ and hence a pole of order exactly $d$ at $s = s_0$ for appropriate choice of $W$ and $W'$.

Let us denote by $\mathcal{I}_{(m-k-1)}(\pi, \sigma)$ the span of the rational functions defined by the integrals $I_{(m-k-1)}(s; W, W')$. As we have seen, each pole of the family $\mathcal{I}(\pi, \sigma)$ occurs as a pole of the family $\mathcal{I}_{(m-k-1)}(\pi, \sigma)$ for some $k$ with $0 < k < m$ and each pole of this family is a pole of the family $\mathcal{I}(\pi, \sigma)$.

Now, analyze our pole at $s = s_0$ in terms of these integrals. Again, look at the expansion about $s = s_0$ which we now write as

$$I_{(m-k-1)}(s; W, W') = \frac{B_{(m-k-1),s_0}(W, W')}{(q^s - q^{s_0})^d} + \text{ higher order terms.}$$

Since the integral involves $W$ and $W'$ restricted to $GL_k$, the integral and the bilinear form $B_{(m-k-1),s_0}(W, W')$ depend only on the functions determined by $W$ and $W'$ in $\mathcal{W}(\pi_{(n-k-1)}, \psi)$ and $\mathcal{W}(\sigma_{(m-k-1)}, \psi^{-1})$ respectively. Recalling the twists involved in the definitions of the derivatives, the quasi–invariance becomes

$$B_{(m-k-1),s_0}(\pi_{(n-k-1)}(p)W, \sigma_{(m-k-1)}(p)W') = |\det(p)|^{-s_0+1} B_{(m-k-1),s_0}(W, W').$$

Furthermore, the index $k$ was chosen so that $B_{(m-k-1),s_0}(W, W')$ is non-trivial, but vanishes for $W$ corresponding to functions in $\mathcal{W}(\pi_{(n-m-1),m+1-k}, \psi)$ that is for functions $W \in \mathcal{W}(\pi_{(n-k-1),2}, \psi) \subset \mathcal{W}(\pi_{(n-k-1)}\psi)$. As a representation of $P_{k+1}$, $\pi_{(n-k-1)}/\pi_{(n-k-1),2} = \Psi^+(\pi^{(n-k)})$. Hence $B_{(n-k-1),s_0}$ defines a non-trivial bilinear form on $\mathcal{W}(\Psi^+(\pi^{(n-k)}), \psi) \times \mathcal{W}(\sigma_{(m-k-1)}, \psi^{-1})$ which is quasi-invariant with respect to the action of $P_{k+1}$. By Proposition 3.7 of [3], there are no non–trivial quasi–invariant pairings between $\Psi^+(\pi^{(n-k)})$ and $\sigma_{(m-k-1),2} = \Phi^+(\sigma_{(m-k)})$. Hence the pairing defined by $B_{(m-k-1),s_0}$ must actually define a non-zero bilinear form on the space $\mathcal{W}(\Psi^+(\pi^{(n-k)}), \psi) \times \mathcal{W}(\Psi^+(\sigma^{(m-k)}), \psi^{-1})$.

To proceed from this point, we need to again make a simplifying assumption. As we shall see in Section 3, this assumption is satisfied for all $\pi$ and $\sigma$ in “general position”.

**Assumption.** *Assume the all derivatives $\pi^{(n-k)}$ of $\pi$ and all derivatives $\sigma^{(m-k)}$ of $\sigma$ are completely reducible.*

Continuing our analysis under this assumption as before, let us write $\pi^{(n-k)} = \oplus \pi_i^{(n-k)}$ and $\sigma^{(m-k)} = \oplus \sigma_j^{(m-k)}$ with each $\pi_i^{(n-k)}$ and $\sigma_j^{(m-k)}$ irreducible. Let us make the convention that, in the case $k = m$, we set $I_{(-1)}(s; W, W') = I(s; W, W')$ and $B_{(-1),s_0}(W, W') = B_{s_0}(W, W')$.

The bilinear form $B_{(m-k-1),s_0}(W, W')$ determined by our pole $s_0$ must restrict non-trivially to some pair $(W_i, W'_j) \in \mathcal{W}(\Psi^+(\pi_i^{(n-k)}), \psi) \times \mathcal{W}(\Psi^+(\sigma_j^{(m-k)}), \psi^{-1})$. From the Corollary to Proposition 1.7 we know there exist functions $W_\circ \in \mathcal{W}(\pi_i^{(n-k)}, \psi)$ and $W'_\circ \in \mathcal{W}(\sigma_j^{(m-k)}, \psi^{-1})$ such that for every Schwartz function $\Phi_\circ \in \mathcal{S}(K^k)$ which is the characteristic function of a sufficiently small neighborhood of $0 \in K^k$ we have an equalities

$$W_i \begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} \Phi_\circ(e_k h) = W_\circ(h)|\det(h)|^{(n-k)/2}\Phi_\circ(e_k h)$$
$$W_j' \begin{pmatrix} h & \\ & I_{m-k} \end{pmatrix} \Phi_\circ(e_k h) = W_\circ'(h)|\det(h)|^{(m-k)/2}\Phi_\circ(e_k h).$$

We may then decompose the integral $I_{(m-k-1)}(W_i, W_j')$ into two parts

$$I_{(m-k-1)}(s; W_i, W_j') = I^0_{(m-k-1)}(s; W_i, W_j') + I^1_{(m-k-1)}(s; W_i, W_j')$$

where

$$I^0_{(m-k-1)}(s; W_i, W_j') = \\ = \int_{N_k\backslash GL_k} W_i \begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} W_j' \begin{pmatrix} h & \\ & I_{m-k} \end{pmatrix} \Phi_\circ(e_k h)|\det(h)|^{s-(m-k)-(n-m)/2}\, dh$$

and

$$I^1_{(m-k-1)}(s; W_i, W_j') = \\ = \int_{N_k\backslash GL_k} W_i \begin{pmatrix} h & \\ & I_{n-k} \end{pmatrix} W_j' \begin{pmatrix} h & \\ & I_{m-k} \end{pmatrix} (1-\Phi_\circ(e_k h))|\det(h)|^{s-(m-k)-(n-m)/2}\, dh.$$

In the integrals $I^1_{(m-k-1)}(W_i, W_j')$, the $(1-\Phi_\circ(e_k h))$ term will restrict the support of the integrand to being compact in the last row of $h$, modulo $N_k$. Using a partial Iwasawa decomposition as above, we can then write this integral as a finite sum of integrals involving the restrictions of $W_i$ and $W_j'$ to $GL_{k-1}$, that is, depending only on the images of $W_i$ and $W_j'$ in $\mathcal{W}(\pi_{(n-k)}, \psi)$ and $\mathcal{W}(\sigma_{(m-k)}, \psi^{-1})$. But our bilinear form restricts to zero on these spaces. Hence the integrals of the form $I^1_{(m-k-1)}(W, W')$ cannot contribute to the order $d$ pole at $s = s_0$, and play no role in our analysis.

We can write $I^0_{(m-k-1)}(W_i, W_j')$ in terms of $W_\circ$ and $W_\circ'$, namely

$$I^0_{(m-k-1)}(s; W_i, W_j') = \int_{N_k\backslash GL_k} W_\circ(h)W_\circ'(h)\Phi_\circ(e_k h)|\det(h)|^s\, dh.$$

These integrals must contribute the pole at of order $d$ at $s = s_0$. However, as integrals on $GL_k$, these are the standard Rankin–Selberg integrals $I^0_{(m-k-1)}(W_i, W_j') = I(s; W_\circ, W_\circ', \Phi_\circ)$ and since $\Phi_\circ(0) = 1 \neq 0$ the pole of order $d$ at $s = s_0$ is an exceptional pole of this integral. Moreover, any integral for $\pi_i^{(n-k)}$ and $\sigma_j^{(m-k)}$ corresponding to an exceptional pole must come from a $I^0_{(m-k-1)}(s; W_i, W_j')$ for some choice of $W_i$ and $W_j$, again by the Corollary to Proposition 1.7, and hence give rise to a pole of the family $\mathcal{I}(\pi, \sigma)$.

Summarizing this analysis, we arrive at the following result.

**Proposition 2.4.** *Under the assumption that all derivatives of $\pi$ and $\sigma$ are completely reducible, any pole $s = s_0$ of order $d$ of the family $\mathcal{I}(\pi, \sigma)$ will correspond to an exceptional pole, again of order $d$, for some family $\mathcal{I}(\pi_i^{(n-k)}, \sigma_j^{(m-k)})$ with $0 < k \leq m$, $\pi_i^{(n-k)}$ an irreducible* *constituent of* $\pi^{(n-k)}$, *and* $\sigma_j^{(m-k)}$ *and irreducible constituent of* $\sigma^{(m-k)}$. *Furthermore, all exceptional poles of these families occur, with the same order, as poles of the family* $\mathcal{I}(\pi,\sigma)$.

If we rewrite this in terms of exceptional L-functions, we arrive at the following result.

**Theorem 2.2.** *Let* $\pi$ *and* $\sigma$ *be representations of* $GL_n$ *and* $GL_m$ *respectively of Whittaker type such that all derivatives of* $\pi$ *and* $\sigma$ *are completely reducible. Then*

$$L(s,\pi\times\sigma)^{-1} = l.c.m._{k,i,j}\{L_{ex}(s,\pi_i^{(n-k)}\times\sigma_j^{(m-k)})^{-1}\}$$

*where the least common multiple is with respect to divisibility in* $\mathbb{C}[q^s,q^{-s}]$ *and is taken over all* $k$ *with* $0<k\le m$ *and for each* $k$ *all constituents* $\pi_i^{(n-k)}$ *of* $\pi^{(n-k)}$ *and all constituents* $\sigma_j^{(m-k)}$ *of* $\sigma^{(m-k)}$.

2.4. **The Bernstein–Zelevinsky Classification.** Recall that an admissible representation $(\rho,V_\rho)$ of $GL_r$ is called supercuspidal or simply cuspidal if it is killed by all Jacquet functors for proper standard parabolic subgroups $P=MU$ of $GL_r$, that is, $r_M(\rho)=0$ where $r_M(\rho)$ is the natural representation of $M$ on $V_\rho/V_\rho(U,\mathbf{1})$ [3]. In this terminology, cuspidals need not be unitary.

Let $\nu(g)=|\det(g)|$ be the unramified determinantal character of any $GL_r$.

By a segment $\Delta$ we mean a sequence of cuspidal representations of the form

$$\Delta=[\rho,\rho\nu,\dots,\rho\nu^{\ell-1}].$$

If $\rho$ is a cuspidal representation of $GL_r$ then the segment determines a representation, which by abuse of notation we will again denote by $\Delta$, of $GL_{r\ell}$ by setting $\Delta$ to be the unique irreducible quotient of $\mathrm{Ind}(\rho\otimes\rho\nu\otimes\cdots\otimes\rho\nu^{\ell-1})$ where the induction is normalized parabolic induction from the standard parabolic attached to the partition $(r,r,\dots,r)$ of $r\ell$. (Note: Bernstein and Zelevinsky take $\langle\Delta\rangle$ to be the irreducible submodule of this induced and $\langle\Delta\rangle^t$ to be the irreducible quotient. Since we will not need the irreducible submodule, we will simply use $\Delta$ for the irreducible quotient. We hope this does not cause too much confusion.)

**Theorem.** [3, 12] $\pi$ *is an irreducible quasi-square-integrable representation of* $GL_n$ *if and only if* $\pi=\Delta$ *for some segment* $\Delta$.

Following Bernstein and Zelevinsky we say two segments $\Delta_1$ and $\Delta_2$ are linked if neither one is a subsegment of the other, but never the less their union $\Delta_1\cup\Delta_2$ is again a segment.

**Theorem.** [3, 12] $\pi$ *is an irreducible generic representation of* $GL_n$ *if and only if there exist non-linked segments* $\Delta_1,\dots,\Delta_t$ *such that* $\pi\cong\mathrm{Ind}(\Delta_1\otimes\cdots\otimes\Delta_t)$, *with the induction normalized parabolic induction.*

Note that in this situation the induced representation is irreducible [3, 12].

2.5. **Derivatives.** The derivatives of these representations have all been computed by Bernstein and Zelevinsky [3, 12]. The results are as follows.

(i) Let $\rho$ be a cuspidal representation of $GL_r$. Then $\rho^{(0)} = \rho$, $\rho^{(k)} = 0$ for $1 \leq k \leq r-1$, and $\rho^{(r)} = \mathbf{1}$.

(ii) Let $\pi = \Delta$ with $\Delta = [\rho, \rho\nu, \dots, \rho\nu^{\ell-1}]$ and $\rho$ a cuspidal representation of $GL_r$. Then $\pi^{(k)} = 0$ if $k$ is not a multiple of $r$, $\pi^{(0)} = \Delta$, $\pi^{(kr)} = [\rho\nu^k, \rho\nu^{k+1}, \dots, \rho\nu^{\ell-1}]$ for $1 \leq k \leq \ell-1$, and $\pi^{(\ell r)} = \mathbf{1}$. Note that all non-zero derivatives are irreducible and quasi-square-integrable.

(iii) Let $\pi$ be generic and write $\pi = \mathrm{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$. Then $\pi^{(k)}$ is glued from those representations of the form $\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$ which give representations of $GL_{n-k}$. If $\Delta_i$ is an irreducible representation of $GL_{n_i}$, so that $n = n_1 + \cdots + n_t$, then we have a filtration of $\pi^{(k)}$ whose subquotients are the representations $\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$ with $k = k_1 + \cdots + k_t$. In general $\pi^{(k)}$ need not be completely reducible and the $\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$ need not be irreducible. However, in the case where the segments are in "general position", as in Section 3, then the derivatives $\pi^{(k)}$ will be completely reducible and the subquotients $\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$ will be generic and irreducible.

## 2.6. **Computation of the L-function.**

2.6.1. *Cuspidal representations.* Let us begin with cuspidal representations. Let $\pi$ be an irreducible cuspidal representation of $GL_n$ and $\sigma$ an irreducible cuspidal representation of $GL_m$ with $n \geq m$. The poles of the family $\mathcal{I}(\pi, \sigma)$ and hence of the L-function $L(s, \pi \times \sigma)$ are precisely accounted for by the exceptional poles of the integrals

$$\int_{N_k \backslash GL_k} W(g) W'(g) \Phi(e_k g) |\det(g)|^s \, dg$$

with $W \in \mathcal{W}(\pi^{(n-k)}, \psi)$, $W' \in \mathcal{W}(\sigma^{(m-k)}, \psi^{-1})$, and $\Phi \in \mathcal{S}(K^k)$ with support in a neighborhood of 0.

As we have seen, $\pi^{(n-k)} = 0$ unless $k = 0, n$ and $\sigma^{(m-k)} = 0$ unless $k = 0, m$. The case $k = 0$ gives no poles, so that we have

$$L(s, \pi \times \sigma) \equiv 1 \text{ if } n > m.$$

If $n = m$ the computation is given in Gelbart and Jacquet [5]. As we will use this method of computation in the square-integrable case, we sketch it here. For details, see [5]. The poles of the family $\mathcal{I}(\pi, \sigma)$ are again exactly the exceptional poles of the family of integrals

$$I(s : W, W', \Phi) = \int_{N_n \backslash GL_n} W(g) W'(g) \Phi(e_n g) |\det(g)|^s \, dg$$

with $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\sigma, \psi^{-1})$, and $\Phi \in \mathcal{S}(K^k)$ with support in a neighborhood of 0. Let $s = s_0$ be an exceptional pole of one of these families of integrals. On the one hand, at $s = s_0$ we have the expansion

$$I(s; W, W', \Phi) = \frac{B^\circ_{s_0}(W, W')\Phi(0)}{(q^s - q^{s_0})^d} + \text{ higher order terms}$$

where $d$ is the highest order to which the pole occurs in the family. Since the $\pi$ and $\sigma$ are irreducible, by Proposition 2.1 we know that the exceptional poles can only occur at those $s_0$ where $\tilde{\pi} \cong \sigma\nu^{s_0}$.

On the other hand, in each integral we can separate out the integral over the center. To this end, write $g \in GL_n$ as $g = nz \begin{pmatrix} h & \\ & 1 \end{pmatrix} k$ with $n \in N_n$, $z \in Z_n$, the center, $h \in GL_{n-1}$, and $k \in K_n$, the maximal compact subgroup. Then the integral becomes

$$\int_{K_n} \int_{N_{n-1}\backslash GL_{n-1}} W\left(\begin{pmatrix} h & \\ & 1 \end{pmatrix} k\right) W'\left(\begin{pmatrix} h & \\ & 1 \end{pmatrix} k\right) \left(\int_{K^\times} \omega_\pi(z)\omega_\sigma(z)\Phi((0,\dots,0,z)k)|z|^{ns}\, d^\times z\right) |\det(h)|^{s-1}\, dh\, dk.$$

Since the pole is exceptional, it occurs with its highest order for $\Phi$ with $\Phi(0) \neq 0$. These $\Phi$ are exactly those which contribute the poles of the family of integrals

$$I(s;\omega_\pi\omega_\sigma,\Phi) = \int_{K^\times} \omega_\pi(z)\omega_\sigma(z)\Phi(0,\dots,0,z)|z|^{ns}\, d^\times z.$$

This family is the Tate family computing the abelian L-function $L(ns,\omega_\pi\omega_\sigma)$. This L-function has a simple pole at $s = s_1$ if and only if $\omega_\pi^{-1} = \omega_\sigma\nu^{ns_1}$. At such a point, the Tate integral will have the expansion

$$I(s;\omega_\pi\omega_\sigma,\Phi) = \frac{c\Phi(0)}{q^s - q^{s_1}} + \text{ higher order terms}$$

with $c$ a non-zero constant.

But, since at our exceptional pole $s_0$ we have $\tilde{\pi} \cong \sigma\nu^{s_0}$ which implies $\omega_\pi^{-1} = \omega_\sigma\nu^{ns_0}$, we see that the exceptional poles of our original family occur among the poles of this Tate family. But for these poles, if we replace the Tate integral by its Laurent, or partial fraction, expansion, we see that we have

$$\lim_{s\to s_0}(q^s - q^{s_0})I(s;W,W',\Phi) = c\,\Phi(0)\int_{Z_nN_n\backslash GL_n} W(g)W'(g)|\det(g)|^{s_0}\, dg.$$

The integral on the right hand side realizes the standard pairing between the Whittaker models of $\pi$ and $\sigma\nu^{s_0}$ and is absolutely convergent and non-zero [5, 9].

Thus our original expansion must be of the form

$$I(s;W,W',\Phi) = \frac{B^\circ_{s_0}(W,W')\Phi(0)}{q^s - q^{s_0}} + \text{ higher order terms}$$

with

$$B^\circ_{s_0}(W,W') = c\int_{Z_nN_n\backslash GL_n} W(g)W'(g)|\det(g)|^{s_0}\, dg$$

and $c \neq 0$. Hence the exceptional pole at $s = s_0$ exists, is necessarily simple, and has a non-zero residue if and only if $\tilde{\pi} \cong \sigma\nu^{s_0}$. Therefore, we have

$$L_{ex}(s,\pi^{(n-k)} \times \sigma^{(m-k)}) = \prod(1 - \alpha q^{-s})^{-1}$$

where $\alpha$ runs over all $\alpha = q^{s_0}$ with $\tilde{\pi} \cong \sigma\nu^{s_0}$.

Since these are the only possible poles of the L-function, we have the following.

**Theorem.** [5] *Let $\pi$ be a cuspidal representation of $GL_n$ and $\sigma$ a cuspidal representation of $GL_m$. If $m < n$ we have $L(s, \pi \times \sigma) \equiv 1$. If $m = n$ we have*

$$L(s, \pi \times \sigma) = \prod (1 - \alpha q^{-s})^{-1}$$

*with the product over all $\alpha = q^{s_0}$ such that $\tilde{\pi} \cong \sigma \nu^{s_0}$.*

2.6.2. *Quasi-square-integrable representations.* Our method lets us compute the L-function for quasi-square-integrable representations using the same method that Gelbart and Jacquet did for cuspidal representations. Let us take $\pi = \Delta$ for the segment $\Delta = [\rho, \rho\nu, \ldots, \rho\nu^{\ell-1}]$ and $\sigma = \Delta'$ for the segment $\Delta' = [\rho', \rho'\nu, \ldots, \rho'\nu^{\ell'-1}]$. Then the derivatives are all irreducible and quasi-square-integrable. The poles of the family $\mathcal{I}(\pi, \sigma)$ are again exactly the exceptional poles of the family of integrals

$$I(s; W, W', \Phi) = \int_{N_k \backslash GL_k} W(g) W'(g) \Phi(e_k g) |\det(g)|^s \, dg$$

with $W \in \mathcal{W}(\pi^{(n-k)}, \psi)$, $W' \in \mathcal{W}(\sigma^{(m-k)}, \psi^{-1})$, and $\Phi \in \mathcal{S}(K^k)$ with support in a neighborhood of 0. In each family of integrals, the derivatives $\pi^{(n-k)}$ and $\sigma^{(m-k)}$ are irreducible when non-zero.

Let $s = s_0$ be an exceptional pole of one of these families of integrals. So both derivatives are non-zero. On the one hand, at $s = s_0$ we have the expansion

$$I(s; W, W', \Phi) = \frac{B^\circ_{s_0}(W, W')\Phi(0)}{(q^s - q^{s_0})^d} + \text{ higher order terms}$$

where $d$ is the highest order to which the pole occurs in the family. Since the derivatives $\pi^{(n-k)}$ and $\sigma^{(m-k)}$ are irreducible, by Proposition 2.1 we know that the exceptional poles can only occur at those $s_0$ where $(\pi^{(n-k)})^\sim \cong \sigma^{(m-k)} \nu^{s_0}$.

On the other hand, as in the cuspidal case, in each integral we can pull off the central integrals. We again get

$$\int_{K_k} \int_{N_{k-1} \backslash GL_{k-1}} W\left( \begin{pmatrix} h & \\ & 1 \end{pmatrix} k \right) W'\left( \begin{pmatrix} h & \\ & 1 \end{pmatrix} k \right) \left( \int_{K^\times} \omega_{\pi^{(n-k)}}(z) \omega_{\sigma^{(m-k)}}(z) \Phi((0, \ldots, 0, z)k) |z|^{ks} \, d^\times z \right) |\det(h)|^{s-1} \, dh \, dk.$$

Since the pole is exceptional, it occurs with its highest order for $\Phi$ with $\Phi(0) \neq 0$. These $\Phi$ are exactly those which contribute the poles of the family of integrals

$$I(s; \omega_{\pi^{(n-k)}} \omega_{\sigma^{(m-k)}}, \Phi) = \int_{K^\times} \omega_{\pi^{(n-k)}}(z) \omega_{\sigma^{(m-k)}}(z) \Phi(0, \ldots, 0, z) |z|^{ks} \, d^\times z.$$

This family is the Tate family computing the abelian L-function $L(ks, \omega_{\pi^{(n-k)}} \omega_{\sigma^{(m-k)}})$. This L-function has only simple poles and has a pole at $s = s_1$ if and only if $\omega^{-1}_{\pi^{(n-k)}} = \omega_{\sigma^{(m-k)}} \nu^{ks_1}$.

At such a point, the Tate integral will have the expansion

$$I(s;\omega_{\pi^{(n-k)}}\omega_{\sigma^{(m-k)}},\Phi)=\frac{c\Phi(0)}{q^s-q^{s_1}}+\text{ higher order terms}$$

with $c$ a non-zero constant.

But, since at our exceptional pole $s_0$ we have $(\pi^{(n-k)})^\sim \cong \sigma^{(m-k)}\nu^{s_0}$ which implies $\omega^{-1}_{\pi^{(n-k)}} = \omega_{\sigma^{(m-k)}}\nu^{ks_0}$, we see that the exceptional poles of our original family occur among the poles of this Tate family. But for these poles, if we replace the Tate integral by its Laurent, or partial fraction, expansion, we see that we have

$$\lim_{s\to s_0}(q^s-q^{s_0})I(s;W,W',\Phi)=c\,\Phi(0)\int_{Z_kN_k\backslash GL_k}W(g)W'(g)|\det(g)|^{s_0}\,dg.$$

The integral on the right hand side realizes the standard pairing between the Whittaker models of $\pi$ and $\sigma\nu^{s_0}$ and is absolutely convergent and non-zero [5, 9].

Thus our original expansion must be of the form

$$I(s;W,W',\Phi)=\frac{B^\circ_{s_0}(W,W')\Phi(0)}{q^s-q^{s_0}}+\text{ higher order terms}$$

with

$$B^\circ_{s_0}(W,W')=c\int_{Z_kN_k\backslash GL_k}W(g)W'(g)|\det(g)|^{s_0}\,dg$$

and $c\neq 0$. Hence the exceptional pole at $s=s_0$ exists, is necessarily simple, and has a non-zero residue if and only if $(\pi^{(n-k)})^\sim\cong\sigma^{(m-k)}\nu^{s_0}$. Therefore, we have

$$L_{ex}(s,\pi^{(n-k)}\times\sigma^{(m-k)})=\prod(1-\alpha q^{-s})^{-1}$$

where $\alpha$ runs over all $\alpha=q^{s_0}$ with $(\pi^{(n-k)})^\sim\cong\sigma^{(m-k)}\nu^{s_0}$.

Now, we know from Bernstein and Zelevinsky [3, 12] that when the derivatives are non-zero we have $\pi^{(n-k)}$ is the quasi-square-integrable representation associated to the segment $[\rho\nu^i,\dots,\rho\nu^{\ell-1}]$, $(\pi^{(n-k)})^\sim$ is the quasi-square-integrable representation associated to the segment $[\tilde\rho\nu^{1-\ell},\dots,\tilde\rho\nu^{-i}]$, and $\sigma^{(m-k)}\nu^{s_0}$ is the quasi-square-integrable representation associated to the segment $[\rho'\nu^{j+s_0},\dots,\rho'\nu^{\ell'+s_0-1}]$ for appropriate $i$ and $j$. Hence we see that $(\pi^{(n-k)})^\sim\cong\sigma^{(m-k)}\nu^{s_0}$ if and only if $\ell-i=\ell'-j$ and $\tilde\rho\nu^{1-\ell}\cong\rho'\nu^{j+s_0}$. This last condition is exactly the condition that the L-function $L(\ell-1+j+s,rho\times\rho')$ have a pole at $s=s_0$, and this pole will also be simple. Hence, in this case

$$L_{ex}(s,\pi^{(n-k)}\times\sigma^{(m-k)})=L(\ell-1+j+s,\rho\times\rho').$$

Since these L-functions account for all poles of $L(s,\pi\times\sigma)$ as $j$ runs over all permissible values, namely $0\le j\le\ell'-1$, we arrive at the following result.

**Theorem 2.3.** *Suppose $\pi$ and $\sigma$ are both quasi-square-integrable representations, $\pi$ associated to the segment $\Delta=[\rho,\rho\nu,\dots,\rho\nu^{\ell-1}]$ and $\sigma$ to the segment $\Delta'=[\rho',\rho'\nu,\dots,\rho'\nu^{\ell'-1}]$. Then*

$$L(s,\pi\times\sigma)=\prod_{j=0}^{\ell'-1}L(\ell-1+j+s,\rho\times\rho').$$

This recovers the Theorem 8.2 of [9]. Note that if $\rho$ and $\rho'$ are representation of different sized $GL's$ then $\pi$ and $\sigma$ will never have non-vanishing derivatives of the same size and $L(s,\pi\times\sigma)\equiv 1$.

We also get as a Corollary the following result of [9].

**Corollary.** *If $\pi$ and $\sigma$ are both square-integrable, then $L(s,\pi\times\sigma)$ has no poles in* $\mathrm{Re}(s)>0$*.*

*Proof:* If $\pi$ is a square-integrable representation, its segment can be written as

$$\Delta=[\rho_0\nu^{-(\ell-1)/2},\dots,\rho_0\nu^{(\ell-1)/2)}]$$

with $\rho_0$ a unitary cuspidal representation and similarly the segment for $\sigma$ can be written

$$\Delta'=[\rho_0'\nu^{-(\ell'-1)/2},\dots,\rho_0'\nu^{(\ell'-1)/2)}]$$

with $\rho_0'$ unitary cuspidal. Then we have

$$L(s,\pi\times\sigma)=\prod_{j=0}^{\ell'-1}L((\ell-\ell')/2+j+s,\rho_0\times\rho_0').$$

For this to have poles, we must have that $\rho_0$ and $\rho_0'$ must be representations of the same linear group $GL_r$, and then since $n\geq m$ we must have $\ell\geq\ell'$. Since $\rho_0$ and $\rho_0'$ are unitary, the poles of $L(s,\rho_0\times\rho_0')$ must lie on the line $\mathrm{Re}(s)=0$. Hence the poles of $\mathrm{L}(s,\pi\times\sigma)$ will lie on the lines $\mathrm{Re}(\frac{1}{2}(\ell-\ell')+j+s)=0$ or $\mathrm{Re}(s)=-\frac{1}{2}(\ell-\ell')-j$ for $j=0,\dots,\ell'-1$. □

2.6.3. *Generic representations.* Let us first write $\pi=\mathrm{Ind}(\Delta_1\otimes\cdots\otimes\Delta_t)$ and $\sigma=\mathrm{Ind}(\Delta_1'\otimes\cdots\otimes\Delta_r')$. In computing $L(s,\pi\times\sigma)$ we encounter several difficulties:

(1) The derivatives may not be completely reducible.

(2) The individual constituents $\mathrm{Ind}(\Delta_1^{(k_1)}\otimes\cdots\otimes\Delta_t^{(k_t)})$ of the derivatives may not be irreducible.

(3) If the constituents of the derivatives are not quasi-square-integrable, we do not have a good way of explicitly analyzing the residual integrals as above and concluding whether a given possible pole actually occurs.

We propose to resolve all these difficulties by use of a deformation argument, which we present in detail in the next section. Let us introduce auxiliary complex parameters $u=(u_1,\dots,u_t)\in\mathbb{C}^t$ and $w=(w_1,\dots,w_r)\in\mathbb{C}^r$. Then set

$$\pi_u=\mathrm{Ind}(\Delta_1\nu^{u_1}\otimes\cdots\otimes\Delta_t\nu^{u_t})$$
$$\sigma_w=\mathrm{Ind}(\Delta_1'\nu^{w_1}\otimes\cdots\otimes\Delta_r'\nu^{w_r}).$$

For $u$ and $w$ in general position the derivatives $\pi_u^{(n-k)}$ and $\sigma_w^{(m-k)}$ will be completely reducible and the natural constituents of these derivatives, $\mathrm{Ind}((\Delta_1\nu^{u_1})^{(k_1)}\otimes\cdots\otimes(\Delta_t\nu^{u_t})^{(k_t)})$ and $\mathrm{Ind}((\Delta_1'\nu^{w_1})^{(k_1')}\otimes\cdots\otimes(\Delta_r'\nu^{w_r})^{(k_r')})$, will be irreducible. This will resolve (1) and (2). We will resolve (3) by an argument using Hatrog's Theorem. We will return to the computation of $L(s,\pi\times\sigma)$ in Section 4 after we discuss deformations.

## 3. Deformations of representations

3.1. **Rationality properties of deformations.** In this section we wish to investigate certain rationality properties of deformations of generic representations of $GL_n$.

Let $\pi$ be a representation of $GL_n$ of Whittaker type. We can write

$$\pi = \mathrm{Ind}_Q^{GL_n}(\Delta_1 \otimes \cdots \otimes \Delta_t)$$

where each $\Delta_i$ is a quasi-square-integrable representation of $GL_{n_i}$, $n = n_1 + \cdots + n_t$, and $Q$ is the standard parabolic subgroup of $GL_n$ associated to the partition $(n_1,\dots,n_t)$. Let $M \cong GL_{n_1}\times\cdots\times GL_{n_t}$ denote the Levi subgroup of $Q$.

If $u = (u_1,\dots,u_t)\in\mathbb{C}^t$ then $u$ defines an unramified character of $\nu^u$ of $M$ via $\nu^u(m) = \nu^u(g_1,\dots,g_t) = \nu(g_1)^{u_1}\cdots\nu(g_t)^{u_t}$. Every unramified character of $M$ is of the form $\nu^u$ for some $u$ and we get an isomorphism $\mathfrak{X}_{ur}(M)\cong(\mathbb{C}/\frac{2\pi i}{\log(q)}\mathbb{Z})^t\cong(\mathbb{C}^\times)^t$, where $\mathfrak{X}_{ur}(M)$ denotes the group of unramified characters of $M$. To simplify notation, let $\mathcal{D}=\mathcal{D}_\pi$ denote the complex manifold $(\mathbb{C}/\frac{2\pi i}{\log(q)}\mathbb{Z})^t$. The map $\mathcal{D}\to(\mathbb{C}^\times)^t$ is of course $u\mapsto(q^{u_1},\dots,q^{u_t})$. For convenience, we will let $q^u$ denote $(q^{u_1},\dots,q^{u_t})$.

For each $u\in\mathcal{D}$ we may define the representation $\pi_u$ by

$$\pi_u = \mathrm{Ind}(\Delta_1\nu^{u_1}\otimes\cdots\otimes\Delta_t\nu^{u_t}).$$

This is the family of deformations of $\pi=\pi_0$ we are interested in. Note that each representation $\pi_u$ is of Whittaker type.

This family of representations has the structure of a trivial vector bundle over $\mathcal{D}$ which we would like to describe. Realize each quasi-square-integrable representation $\Delta_i$ in its Whittaker model $\mathcal{W}(\Delta_i,\psi)$. Then we may realize the space $V_\pi$ of $\pi$ as the space of smooth functions

$$f : GL_n \to \mathcal{W}(\Delta_1,\psi)\otimes\cdots\otimes\mathcal{W}(\Delta_t,\psi),$$

which we write as $f(g;m)$ with $g\in GL_n$ and $m\in M$, satisfying

$$f(hg;m) = \delta_Q^{1/2}(m_h)f(g;mm_h)$$

for $h\in Q$, $h = nm_h$ with $m_h\in M$ and $n$ in the unipotent radical $N_Q$ of $Q$. Let $K_n = GL_n(\mathfrak{o})$ denote the maximal compact subgroup of $GL_n$. Since $GL_n = QK_n$ each function $f'$ is determined by its restriction to $K_n$ and we have the so-called compact realization of $\pi$ on the space of smooth functions

$$f : K_n \to \mathcal{W}(\Delta_1,\psi)\otimes\cdots\otimes\mathcal{W}(\Delta_t,\psi),$$

which we again write as $f(k;m)$. The action of $\pi(g)$ is now given by

$$(\pi(g)f)(k;m) = \delta_Q^{1/2}(m')f(k';mm')$$

where $kg = n'm'k'$ with $n' \in N_Q$, $m' \in M$ and $k' \in K_n$. Let us denote this space of functions by $\mathcal{F}_\pi = \mathcal{F}$. Then each $\pi_u$ can also be realized on $\mathcal{F}$ with actions being given by

$$(\pi_u(g)f)(k;m) = \nu^u(m')\delta_Q^{1/2}(m')f(k';mm')$$

where the decomposition of $kg$ is as above. Note that in these realizations, the stabilizer of the function $f \in \mathcal{F}$ under the representation $\pi_u$ is independent of $u$. In the usual model of $\pi_u$ as smooth functions on $GL_n$ having a left transformation law, $f$ determines the function $f_u$ defined by $f_u(g;m) = f_u(n'm'k';m) = \nu^u(m')\delta_Q^{1/2}(m')f(k';mm')$. We may now form a bundle of representations over $\mathcal{D}$ where the fiber over $u \in \mathcal{D}$ is the representation $(\pi_u, \mathcal{F})$. As a vector bundle, this is a trivial bundle $\mathcal{D} \times \mathcal{F}$, with different actions of $GL_n$ in each fibre. Note that the variation of the action from fibre to fibre is actually polynomial in $q^{\pm u}$. (We will use $q^{\pm u}$ as short for $(q^{\pm u_1}, \dots, q^{\pm u_t})$.)

For our purposes we need the Whittaker models of these representations. The unipotent radical $N_Q$ has an exhaustive filtration by compact open subgroups $\{N_i\}$. It is (essentially) a result of Casselman and Shalika [4] that there is a Weyl element $w_Q$ such that for each $f \in \mathcal{F}$ the family of integrals

$$\lambda_u(f) = \int_{N_i} (\pi_u(n)f)(w_Q;e)\psi^{-1}(n)\ dn$$

stabilizes for $i$ large and defines a Whittaker functional on each $\pi_u$. The point of stability depends on $f$, through its stabilizer, and is independent of $u \in \mathcal{D}$. (Casselman and Shalika [4] work with minimal parabolics, but their method carries over. See also Shahidi [11], Section 3.) For each $f \in \mathcal{F}$, let us set

$$W_{f,u}(g) = \lambda_u(\pi_u(g)f).$$

Then for each value of $u \in \mathcal{D}$, $W_{f,u} \in \mathcal{W}(\pi_u, \psi)$. For fixed $g \in GL_n$, we have

$$W_{f,u}(g) = \int_{N_f} (\pi_u(ng)f)(w_Q;e)\psi^{-1}(n)\ dn,$$

with $N_f$ one of the $N_i$ past the stability point for $f$. Since the stabilizer of $f$ in $GL_n$ is independent of $u$ and the integration is compact, this integral will become a finite sum, and we will have

$$W_{f,u}(g) = \sum_{i=1}^{r} (\pi_u(n_i g)f)(w_Q;e)\psi^{-1}(n_i)$$

and if we decompose each $w_Q n_i g = n_i' m_i' k_i'$ we have

$$W_{f,u}(g) = \sum_{i=1}^{r} f(k_i';m_i')\delta_Q^{1/2}(m_i')\nu^u(m_i')\psi^{-1}(n_i)$$

which defines a Laurent polynomial in $\mathbb{C}[q^{\pm u_1}, \dots, q^{\pm u_t}]$. So, for fixed $g$, $W_{f,u}(g)$ is polynomial in $q^{\pm u}$. Note that in terms of the usual realization of the induced representation, this is the standard Whittaker function associated to $f_u$, that is,

$$\begin{aligned} W_{f,u}(g) &= \int_{N_f} (\pi_u(ng)f)(w_Q;e)\psi^{-1}(n)\ dn \\ &= \int_{N_f} f_u(w_Q ng;e)\psi^{-1}(n)\ dn \\ &= W_{f_u}(g). \end{aligned}$$

If, in our bundle of representations, we replace each fibre $(\pi_u, \mathcal{F})$ by its Whittaker model $(\pi_u, \mathcal{W}(\pi_u, \psi))$ defined with respect to the Whittaker functional $\lambda_u$ above, then the functions $W_{f,u}$, as functions of $u \in \mathcal{D}$, correspond to the flat sections above. Let $\mathcal{W}_{\mathcal{F}}$ denote the space of global sections of the form $W_{f,u}$ for $f \in \mathcal{F}$, that is,

$$\mathcal{W}_{\mathcal{F}} = \{W_{f,u}(g) \mid f \in \mathcal{F}\}.$$

This space of sections is not stable under the action of $GL_n$ by right translation, so let $\mathcal{W}_\pi^{(0)} = \mathcal{W}^{(0)}$ denote the representation of $GL_n$ generated by $\mathcal{W}_{\mathcal{F}}$. Since the stabilizer of each $W_{f,u}$ is independent of $u$, this representation is seen to be smooth. Note

$$\mathcal{W}^{(0)} = \langle W_{\pi_u(g')f_u}(g) = W_{f_u}(gg') \mid f \in \mathcal{F},\ g' \in GL_n\rangle.$$

Being a smooth representation of $GL_n$, we could submit $\mathcal{W}^{(0)}$ to a full derivative analysis as in Section 1. However, we will not need this. Instead, we consider only an associated representation of $P_n$ and the bottom piece of the filtration of this representation by derivatives. To this end, by analogy with what we did in Section 1, let us set $\mathcal{W}_{(0)}$ to be the space of restrictions to $P_n$ of the functions in $\mathcal{W}^{(0)}$, that is, $\mathcal{W}_{(0)} = \{W(p) \mid W \in \mathcal{W}^{(0)},\ p \in P_n\}$. This should be a model for the restriction of $\mathcal{W}^{(0)}$ to $P_n$ as a representation, but we did not check this.

If $V$ is any complex vector space, let $\mathcal{S}_\psi(P_n, V)$ denote the space of smooth functions $\varphi : P_n \to V$ which satisfy $\varphi(np) = \psi(n)\varphi(p)$ for $n \in N_n$ and which are compactly supported mod $N_n$. In this notation, the bottom piece of the filtration by derivatives of any irreducible generic representation of $GL_n$ restricted to $P_n$ is $\mathcal{S}_\psi(P_n) = \mathcal{S}_\psi(P_n, \mathbb{C})$. Let $\mathcal{P}_0$ be the vector subspace of $\mathbb{C}[q^{\pm u}]$ consisting of all Laurent polynomials of the form $W(I_n)$ for $W \in \mathcal{W}^{(0)}$. Then the corresponding result on the bottom piece of the filtration by derivatives for $\mathcal{W}_{(0)}$ is the following.

**Proposition 3.1.** $\mathcal{W}_{(0)}$ *contains* $\mathcal{S}_\psi(P_n, \mathcal{P}_0)$.

*Proof:* The proof of this proposition is obtained by repeating the proof of Proposition 2 of Gelfand–Kazhdan [7] in our setting. They treat the case of scalar valued functions, but the method transfers completely. This is also essentially the same argument used to prove Lemma 9.2 of [9] which we have frequently used.

It suffices to prove that for every Laurent polynomial $P \in \mathcal{P}_0$ and every sufficiently small compact open subgroup $H \subset P_n$ there is a function $W_{P,H} \in \mathcal{W}_{(0)}$ such that

(i) $W_{P,H}(I_n) = P$
(ii) $W_{P,H}(ph) = W_{P,H}(p)$ for $p \in P_n$ and $h \in H$
(iii) $\mathrm{Supp}(W_{P,H}) \subset N_n H$.

For each $k = 1, \dots, n$ let us set

$$\mathcal{W}_{(n-k)} = \left\{ W \begin{pmatrix} p & \\ & I_{n-k} \end{pmatrix} \mid W \in \mathcal{W}^{(0)},\ p \in P_k \right\}.$$

Then, as in Section 1, this space is naturally a representation for $P_k$ acting by right translation. When $k = n$, this agrees with our previous definition of $\mathcal{W}_{(0)}$. For $W \in \mathcal{W}_{(n-k)}$ we will write $W(p)$ for $W \begin{pmatrix} p & \\ & I_{n-k} \end{pmatrix}$ to ease notation.

Inductively, we will prove that for each $k$ and every sufficiently small compact open $H \subset P_k$ there is a function $W_{P,H} \in \mathcal{W}_{(n-k)}$ satisfying (i)–(iii) with $n$ replaced by $k$.

If $k = 1$, then $P_1$ is the trivial group $\{I_n\}$ and $\mathcal{W}_{(n-1)} = \{W(I_n) \mid W \in \mathcal{W}^{(0)}\} = \mathcal{P}_0$. So this case is clear.

Assume the statement true for $k$. Take $P \in \mathcal{P}_0$ and a compact open $H_{k+1} \subset P_{k+1}$. Let $H_k = H_{k+1} \cap P_k$. By induction there is a compact open subgroup $H' \subset H_k$ and a function $W_{P,H'} \in \mathcal{W}_{(n-k)}$ satisfying (i)–(iii) relative to $H' \subset P_k$. Choose $W' \in \mathcal{W}_{(n-k-1)}$ whose restriction to $P_k \subset GL_k \subset P_{k+1}$ is $W_{P,H'}$. Let

$$W''(p) = \frac{1}{\mathrm{meas}(H')} \int_{H'} W'(ph)\ dh$$

and let $H''$ be the stabilizer of $W''$ in $P_{k+1}$. $H''$ is open compact and contains $H'$. Let $H = H'' \cap H_{k+1} \subset H_{k+1}$.

Let $\psi_{k+1}$ now denote the standard character on $U_{k+1}$, that is, $\psi_{k+1} \begin{pmatrix} I_k & u \\ & 1 \end{pmatrix} = \psi(u_k)$. Let $\hat{\varphi}_H \in \mathcal{S}(\hat{U}_{k+1})$ be the characteristic function of the $H$ orbit of $\psi_{k+1}$, that is, of $H\psi_{k+1} = \{\psi_{k+1}^h(u) = \psi_{k+1}(huh^{-1})\} \subset \hat{U}_{k+1}$. Let $\varphi_H \in \mathcal{S}(U_{k+1})$ denote its Fourier transform. Let

$$W(p) = \varphi_H * W''(p) = \int_{U_{k+1}} \varphi_H(u) W''(pu)\ du.$$

Then $W \in \mathcal{W}_{(n-k-1)}$ and satisfies

$$W(p) = \hat{\varphi}_H(\psi_{k+1}^{p^{-1}}) W''(p).$$

Thus

(i′) $W(I_{k+1}) = W''(I_{k+1}) = P$
(ii′) $W(p) = 0$ for $p \notin N_{k+1} P_k H$
(iii′) $W(p) = W''(p)$ for $p \in N_{k+1} P_k H$.

But for $p = np'h \in N_{k+1}P_kH$ we have $W''(p) = \psi_{k+1}(n)W''(p') = \psi_{k+1}(n)W_{P,H'}(p')$. So if we take $W_{P,H} = W''$, this satisfies (i)–(iii) relative to $P$ and $H \subset H_{k+1}$. □

The space $\mathcal{S}_\psi(P_n, \mathcal{P}_0)$ should be the bottom piece in the filtration by derivatives of $\mathcal{W}_{(0)}$, although we did not check this.

3.2. **Deformations and local factors.** We now want to consider how the local integrals $I(s; W, W', \Phi)$ and $I(s; W, W')$ behave when we deform the Whittaker functions $W$ and $W'$ in the above manner. If we deform $\pi$ to $\pi_u$ and deform $\sigma$ to $\sigma_w$, and form the corresponding spaces of functions $\mathcal{F}_\pi$, $\mathcal{W}_\pi^{(0)}$, $\mathcal{F}_\sigma$, and $\mathcal{W}_\sigma^{(0)}$ then we will show that both $I(s; W, W', \Phi)$ and $I(s; W, W')$, for $W \in \mathcal{W}_\pi^{(0)}$ and $W' \in \mathcal{W}_\sigma^{(0)}$, are rational functions in $q^{\pm u}$, $q^{\pm w}$, and $q^{\pm s}$.

Our method will be to employ the following theorem of Bernstein. Before the statement, we need some preliminary definitions. Let $V$ denote a vector space over $\mathbb{C}$ having countable dimension and let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ denote its algebraic dual. A system of equations $\Xi$ for a functional $\lambda^* \in V^*$ is a collection of pairs $\Xi = \{(x_r, c_r) \mid r \in R\}$ where $x_r \in V$, $c_r \in \mathbb{C}$, and $R$ is an index set. A solution $\lambda$ of the system $\Xi$ is then a functional $\lambda \in V^*$ such that $\lambda(x_r) = c_r$ for all $r \in R$. We will need to consider polynomial families of such systems. Let $\mathcal{D}$ be an irreducible algebraic variety over $\mathbb{C}$ and suppose that for each $d \in \mathcal{D}$ we are given a system of equations $\Xi_d = \{(x_r(d), c_r(d)) | r \in R\}$ with index set $R$ independent of $d \in \mathcal{D}$. We will say that such a family is a polynomial family of system of equations if, for each $r \in R$, the functions $x_r(d)$ and $c_r(d)$ vary polynomially in $d$, i.e., $x_r(d) \in \mathbb{C}[\mathcal{D}] \otimes_{\mathbb{C}} V$ and $c_r(d) \in \mathbb{C}[\mathcal{D}]$. Let $\mathcal{M} = \mathbb{C}(\mathcal{D})$ denote the field of fractions of $\mathbb{C}[\mathcal{D}]$. Set $V_{\mathcal{M}} = \mathcal{M} \otimes_{\mathbb{C}} V$ and $V_{\mathcal{M}}^* = \mathrm{Hom}_{\mathcal{M}}(V_{\mathcal{M}}, \mathcal{M})$.

**Theorem.** (Bernstein [1]) *With the above notation, suppose that $V$ has countable dimension over $\mathbb{C}$ and suppose that there exists an non-empty subset $\Omega \subset \mathcal{D}$, open in the usual complex topology of $\mathcal{D}$, such that for each $d \in \Omega$ the system $\Xi_d$ has a unique solution $\lambda_d$. Then the system $\Xi = \{(x_r(d), c_r(d)) \mid r \in R\}$ viewed as a system over the field $\mathcal{M} = \mathbb{C}(\mathcal{D})$ has a unique solution $\lambda(d) \in V_{\mathcal{M}}^*$. Moreover, on some subset $\mathcal{D}' \subset \mathcal{D}$, which is the complement of a countable number of hyperplanes, $\lambda(d) = \lambda_d$ is the unique solution of $\Xi_d$.*

We now wish to apply this to our situation.

We take $\pi = \mathrm{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ as above and deform it as above to the family $\pi_u = \mathrm{Ind}(\Delta_1\nu^{u_1} \otimes \cdots \otimes \Delta_t\nu^{u_t})$. We realize each of these representations on the common vector space $\mathcal{F} = \mathcal{F}_\pi$ and form the representation of $GL_n$ on the Whittaker space $\mathcal{W}_\pi^{(0)}$.

Let $\sigma = \mathrm{Ind}(\Delta'_1 \otimes \cdots \otimes \Delta'_r)$ be a representation of $GL_m$ with each $\Delta'_i$ representing a quasi-square-integrable representation of some $GL_{r'_i}$. We may deform $\sigma$ as above by setting $\sigma_w = \mathrm{Ind}(\Delta'_1\nu^{w_1} \otimes \cdots \otimes \Delta'_r\nu^{w_r})$ where $w \in (\mathbb{C}/\frac{2\pi i}{\log(q)})^r = \mathcal{D}_\sigma$. We may realize each of these representations on a common vector space $\mathcal{F}_\sigma$ and form the representation of $GL_m$ on the Whittaker space $\mathcal{W}_\sigma^{(0)}$, of course, with respect to the character $\psi^{-1}$.

Observe that since $\pi$ and $\sigma$ are admissible representations, the vector spaces $\mathcal{F}_\pi$ and $\mathcal{F}_\sigma$ are both countable dimensional over $\mathbb{C}$.

The cases $n = m$ and $n > m$ are slightly different, and we will treat them separately.

3.2.1. *The case $n > m$.* Consider the local integrals for the case $n > m$. For $W \in \mathcal{W}_\pi^{(0)}$ and $W' \in \mathcal{W}_\sigma^{(0)}$ these integrals are

$$I(s; W, W') = \int_{N_m \backslash GL_m} W \begin{pmatrix} g & \\ & I_{n-m} \end{pmatrix} W'(g) |\det(g)^{s-(n-m)/2}\, dg.$$

From the work of Jacquet, Piatetski-Shapiro, and Shalika [8, 9] we know that for fixed $u$ and $w$ these integrals are absolutely convergent for $\mathrm{Re}(s)$ large. If we pay closer attention to their arguments we see that there is in fact a linear form $L_{\pi,\sigma}(s, u, w)$ with real coefficients such that these integrals converge absolutely for $\mathrm{Re}(L_{\pi,\sigma}(s, u, w)) > 0$.

**Proposition 3.2.** *For $W \in \mathcal{W}_\pi^{(0)}$ and $W' \in \mathcal{W}_\sigma^{(0)}$ the integral $I(s; W, W')$ is a rational function of $q^{-u}$, $q^{-w}$ and $q^{-s}$.*

*Proof:* We will need to view these integrals as defining a polynomial system of equations for a functional on $\mathcal{V} = \mathcal{F}_\pi \otimes \mathcal{F}_\sigma$. First note that any $W \in \mathcal{W}_\pi^{(0)}$ is the Whittaker function attached to a finite linear combination of translates of elements of $\mathcal{F}_\pi$, that is, $W = W_{f_u}$ where $f_u = \sum \pi_u(g_i) f_i$ with $g_i \in GL_n$ and $f \in \mathcal{F}_\pi$, and similarly for $W' \in \mathcal{W}_\sigma^{(0)}$. The local integrals have well known quasi-invariance properties, namely

$$I\left(s; \pi_u \begin{pmatrix} g & \\ & I_{n-m} \end{pmatrix} W, \sigma_w(g) W\right) = |\det(g)|^{-s+(n-m)/2} I(s; W, W')$$

for each $g \in GL_m$ and

$$I\left(s; \pi_u \begin{pmatrix} I_m & n_2 \\ & n_1 \end{pmatrix} W, W'\right) = \psi\begin{pmatrix} I_m & n_2 \\ & n_1 \end{pmatrix} I(s; W, W')$$

for $n_2$ an arbitrary $m \times (n-m)$ matrix and $n_1 \in N_{n-m}$.

By Proposition 2.11 of [9], outside of a finite number of hyperplanes in $u, w$, and $s$ there is at most a one dimensional space of functionals on $\mathcal{F}_\pi \otimes \mathcal{F}_\sigma$ having these invariance properties. In fact, the statement in [9] is for irreducible generic $\pi$ and $\sigma$ rather than families. But outside of a finite number of hyperplanes in $u$ and $w$ the $\pi_u$ and $\sigma_w$ will be irreducible and generic. Then, upon analysis of their proof, one finds that the condition for the space of quasi-invariant functionals being of dimension greater than one is a condition of contragredience between constituents of derivatives $\pi_u^{(n-k)}$ and $\sigma_w^{(m-k)} \nu^s$ as representations of $GL_k$. Since these conditions define a finite number of hyperplanes in $u, w$, and $s$, we obtain the extension of Proposition 2.11 of [9] to families.

Thus, the functional defined by these integrals is determined up to a scalar by the following system of equations. Let us choose a basis $\{f_i\}$ of $\mathcal{F}_\pi$ and a basis $\{f'_j\}$ of $\mathcal{F}_\sigma$. Then the invariance conditions give the system $\Xi'$

$$\begin{aligned}&\Big\{\Big(\pi_u\begin{pmatrix} g & \\ & I_{n-m}\end{pmatrix}\pi_u(g_i)f_i\otimes\sigma_w(g)\sigma_w(g_j)f'_j-|\det(g)|^s\pi_u(g_i)f_i\otimes\sigma_w(g_j)f'_j,0\Big)\Big|\\ &\qquad g\in GL_m,\ g_i\in GL_n,\ g_j\in GL_m\Big\}\\ &\bigcup\Big\{(\pi_u(n)\pi_u(g_i)f_i\otimes\sigma_w(g_j)f'_j-\psi(n)\pi_u(g_i)f_i\otimes\sigma_w(g_j)f'_j,0)\Big|\\ &\qquad n=\begin{pmatrix} I_m & n_2\\ & n_1\end{pmatrix}\in N_n,\ g_i\in GL_n,\ g_j\in GL_m\Big\}\end{aligned}\tag{$\Xi'$}$$

This system is polynomial in $q^{\pm u}$, $q^{\pm w}$, and $q^{\pm s}$. So if we let $\mathcal{D}=\mathcal{D}_\pi\times\mathcal{D}_\sigma\times\mathcal{D}_s$, where $\mathcal{D}_s=(\mathbb{C}/\frac{2\pi i}{\log(q)})\cong\mathbb{C}^\times$ via the map $s\mapsto q^s$, then this system is polynomial over $\mathcal{D}$. Moreover, if we define $\Omega\subset\mathcal{D}$ by the conditions that $\Omega$ is the intersection of the complements of the hyperplanes on which uniqueness fails intersected with the domain $\mathrm{Re}(L_{\pi,\sigma}(s,u,w))>0$ of absolute convergence for our integrals, then the functional $I(s;W,W')$ is the unique solution up to scalars. To be able to apply Bernstein's theorem, we must add one equation to insure uniqueness on $\Omega$. This is a normalization equation. To give it, we may first take any $f'\in\mathcal{F}_\sigma$ such that $W'_{f',w}(I_m)\neq 0$. Let $P'(q^{\pm w})=W'_{f',w}(I_m)$ be the corresponding Laurent polynomial in $q^{\pm w}$. Let $H\subset GL_m$ be the stabilizer of this $W'_{f',w}$. Now, by Proposition 3.1, we can find $W\in\mathcal{W}_\pi^{(0)}$ such that $W(I_n)=P(q^{\pm u})\neq 0$ and the restriction of $W$ to $GL_m\subset GL_n$ is stabilized by $H$ and has support in $Z_mH$. Then we can easily compute that $I(s;W,W'_{f',w})$ is convergent for every $s$ and in fact $I(s;W,W'_{f',w})=\mathrm{vol}(H)P(q^{\pm u})P'(q^{\pm w})$ is independent of $s$. Since $W\in\mathcal{W}_\pi^{(0)}$ it is a linear combination of $GL_n$ translates of functions in $\mathcal{W}_{\mathcal{F}_\pi}$. So we have

$$W=\sum_i\pi_u(g_i)W_{h_i,u}$$

for appropriate $g_i\in GL_n$ and $h_i\in\mathcal{F}_\pi$. Thus to remove the scalar ambiguity in our system of equations we add the single normalization equation

$$\left(\sum_i\pi_u(g_i)h_i\otimes f',\mathrm{vol}(H)P(q^{\pm u})P'(q^{\pm w})\right)\tag{N}$$

This is again a polynomial equation in $\mathcal{D}$.

If $\Xi$ is the system $\Xi'$ with the equation (N) adjoined, we have a system which satisfies the hypotheses of Bernstein's Theorem. Hence we may conclude that each $I(s;W,W')$ defines a rational function in $\mathbb{C}(q^{-u},q^{-w},q^{-s})$. □

In the functional equation for the $GL_n\times GL_m$ local integrals it is not just the $I(s;W,W')$ which occur, but also the integrals $I_j(s;W,W')$ defined by

$$I_j(s;W,W')=\int_{N_m\backslash GL_m}\int_{M_{j,m}}W\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-m-j}\end{pmatrix}W'(g)|\det(g)^{s-(n-m)/2}\,dx\,dg$$

for $0\leq j\leq n-m-1$. For a fixed $j$ these integrals enjoy the same convergence and invariance properties as the $I(s;W,W')$. Hence we see that for $W\in\mathcal{W}_\pi^{(0)}$ and $W'\in\mathcal{W}_\sigma^{(0)}$ these integrals satisfy the same invariance system $\Xi'$ as the $I(s;W,W')$. To show that they are also rational in $q^{-u}$, $q^{-w}$, and $q^{-s}$ we need only a normalization equation for these integrals. This can be easily found in the same manner as above. Hence we have the following Corollary.

**Corollary 1.** *Let $j$ be an integer between $0$ and $n-m-1$. Then for $W \in \mathcal{W}_\pi^{(0)}$ and $W' \in \mathcal{W}_\sigma^{(0)}$ we have $I_j(s;W,W') \in \mathbb{C}(q^{-u},q^{-w},q^{-s})$.*

If we now look at the local functional equation, it reads

$$I_{n-m-1}(1-s;\pi_u(w_{n,m})\tilde{W},\tilde{W}') = \omega_{\sigma_w}(-1)^{n-1}\gamma(s,\pi_u\times\sigma_w,\psi)I_0(s;W,W')$$

for $W\in\mathcal{W}(\pi_u,\psi)$ and $W'\in\mathcal{W}(\sigma_w,\psi^{-1})$. If $W\in\mathcal{W}_\pi^{(0)}$, then $\tilde{W}\in\mathcal{W}_{\pi^\iota}^{(0)}$. Under deformation, $(\pi_u)^\iota=(\pi^\iota)_{u^\iota}$, where if $u=(u_1,\dots,u_t)$ then $u^\iota=(-u_t,\dots,-u_1)$. Note that in the Whittaker model, $\mathcal{W}_{\pi^\iota}^{(0)}$ should be taken with respect to the character $\psi^{-1}$. Hence for $W\in\mathcal{W}_\pi^{(0)}$ and $W'\in\mathcal{W}_\sigma^{(0)}$ the integrals appearing in the left hand side of the functional equation are rational functions of $q^{-u}$, $q^{-w}$, and $q^{-s}$ by Corollary 1. Note that $\omega_{\sigma_w}(-1)=\omega_\sigma(-1)$ is independent of $w$. Hence $\gamma(s,\pi_u\times\sigma_w,\psi)$ must also be rational.

**Corollary 2.** $\gamma(s,\pi_u\times\sigma_w,\psi)\in\mathbb{C}(q^{-u},q^{-w},q^{-s})$.

3.2.2. *The case $n=m$.* The case when $n=m$ runs along the same lines, but the local integrals are different. In this case the local integrals involve not just the Whittaker functions associated to $\pi_u$ and $\sigma_w$ but also a Schwartz–Bruhat function on $K^n$. For $W\in\mathcal{W}_\pi^{(0)}$, $W'\in\mathcal{W}_\sigma^{(0)}$, and $\Phi\in\mathcal{S}(K^n)$ the local integral is

$$I(s;W,W',\Phi)=\int_{N_n\backslash GL_n} W(g)W'(g)\Phi(e_ng)|\det(g)|^s\,dg.$$

Again, from Jacquet,Piatetski-Shapiro, and Shalika [8, 9], there is a linear form $L_{\pi,\sigma}(s,u,w)$ with real coefficients such that the integral is absolutely convergent for $\mathrm{Re}(L_{\pi,\sigma}(s,u,w))>0$.

**Proposition 3.3.** *For every $W\in\mathcal{W}_\pi^{(0)}$, $W'\in\mathcal{W}_\sigma^{(0)}$, and $\Phi\in\mathcal{S}(K^n)$ we have $I(s;W,W',\Phi)\in\mathbb{C}(q^{-u},q^{-w},q^{-s})$.*

*Proof:* We once again must write down a system of equations which are polynomial in $q^{\pm u}$, $q^{\pm w}$, and $q^{\pm s}$ which characterize these functionals. In this case, our underlying vector space is $\mathcal{V}=\mathcal{F}_\pi\otimes\mathcal{F}_\sigma\otimes\mathcal{S}(K^n)$ and is still countable dimensional over $\mathbb{C}$. The invariance properties that this functional satisfies are

$$I(s;\pi_u(g)W,\sigma_w(g)W',\rho(g)\Phi)=|\det(g)|^sI(s;W,W',\Phi)$$

where $\rho$ denotes the action of $GL_n$ on $\mathcal{S}(K^n)$ by right translation. By modifying the proof of Proposition 2.10 of Jacquet, Piatetski-Shapiro, and Shalika [9] to the case of families as before, we know that the space of such functionals is at most one dimensional off of a finite number of hyperplanes in $(s,u,w)$. Taking the bases of $\mathcal{F}_\pi$ and $\mathcal{F}_\sigma$ as above and a basis $\Phi_k$ of $\mathcal{S}(K^n)$, our system of equations $\Xi'$ expressing the invariance of the local integrals is

$$\begin{aligned}&\big\{(\pi_u(g)\pi_u(g_i)f_i\otimes\sigma_w(g)\sigma_w(g_j)f'_j\otimes\rho(g)\Phi_k-|\det(g)|^s\pi_u(g_i)f_i\otimes\sigma_w(g_j)f'_j\otimes\Phi_k,0)\big|\\ (\Xi')\qquad &\quad g\in GL_n,\ g_i\in GL_n,\ g_j\in GL_n\big\}.\end{aligned}$$

This system is polynomial over the complex domain $\mathcal{D} = \mathcal{D}_\pi \times \mathcal{D}_\sigma \times \mathcal{D}_s$. Moreover, if we define $\Omega \subset \mathcal{D}$ by the conditions that $\Omega$ is the intersection of the complements of the hyperplanes on which uniqueness fails intersected with the domain $\mathrm{Re}(L_{\pi,\sigma}(s,u,w)) > 0$ of absolute convergence for our integrals, then the functional $I(s;W,W',\Phi)$ is the unique solution up to scalars. To be able to apply Bernstein's theorem, we must add one equation to insure uniqueness on $\Omega$. This is again a normalization equation, which is slightly more complicated in this situation.

Using the Iwasawa decomposition, we see that for any choice of $W \in \mathcal{W}_\pi^{(0)}$, $W' \in \mathcal{W}_\sigma^{(0)}$, and $\Phi$ we can decompose our local integral as

$$I(s;W,W',\Phi) = \int_{K_n}\int_{N_n\backslash P_n} W(pk)W'(pk)|\det(p)|^{s-1}\int_{K^\times}\omega_{\sigma_w}(a)\omega_{\pi_u}(a)\Phi(e_n ak)|a|^{ns}\,d^\times a\;dp\;dk.$$

For an arbitrarily small compact open subgroup $H \subset P_n$ we can find functions $W$ and $W'$ such that their restrictions to $P_n$ both are invariant under $H$, supported on $N_nH$, and such that $W(I_n) = P(q^{\pm u}) \neq 0$ and $W'(I_n) = P'(q^{\pm w}) \neq 0$. Let $K'$ be a sufficiently small open compact congruence subgroup of $K_n$ such that $K' \cap P_n \subset H$ and $K'$ stabilizes $W$ and $W'$. Now choose $\Phi$ to be the characteristic function $\Phi'$ of $e_nK'$. With these choices the integral reduces to

$$I(s;W,W',\Phi') = c\int_{N_n\backslash P_n} W(p)W'(p)|\det(p)|^{s-1}\,dp$$

with $c > 0$ a volume and for $H$ sufficiently small $|\det(p)| = 1$ and we have $I(s;W,W',\Phi') = c'P(q^{\pm u})P'(q^{\pm w})$ for a positive constant $c'$. Now, as $W \in \mathcal{W}_\pi^{(0)}$ and $W' \in \mathcal{W}_\sigma^{(0)}$ each can be expressed as finite linear combinations

$$W = \sum_i \pi_u(g_i)W_{h_i,u} \qquad \text{and} \qquad W' = \sum_j \sigma_w(g'_j)W_{h'_j,w}$$

for appropriate $g_i,\ g'_j \in GL_n,\ h_i \in \mathcal{F}_\pi$ and $h'_j \in \mathcal{F}_\sigma$. Thus our normalization equation can be written

$$\text{(N)} \qquad \left(\sum_i\sum_j \pi_u(g_i)h_i \otimes \sigma_w(g'_j)h'_j \otimes \Phi', c'P(q^{\pm u})P'(q^{\pm w})\right)$$

This is again a polynomial equation in $\mathcal{D}$.

If $\Xi$ is the system $\Xi'$ with the equation (N) adjoined, we have a system which satisfies the hypotheses of Bernstein's Theorem. Hence we may conclude that each $I(s;W,W',\Phi)$ defines a rational function in $\mathbb{C}(q^{-u},q^{-w},q^{-s})$. □

In the case $n = m$ the local functional equation reads

$$I(1-s;\tilde{W},\tilde{W}',\hat{\Phi}) = \omega_{\sigma_w}(-1)^{n-1}\gamma(s,\pi_u\times\sigma_w,\psi)I(s;W,W',\Phi)$$

for $W \in \mathcal{W}(\pi_u,\psi)$ and $W' \in \mathcal{W}(\sigma_w,\psi^{-1})$. If we take $W \in \mathcal{W}_\pi^{(0)}$ and $W' \in \mathcal{W}_\sigma^{(0)}$, then the local integrals involved are again rational functions of $q^{-u}$, $q^{-w}$, and $q^{-s}$, and hence so must $\gamma(s,\pi_u\times\sigma_w,\psi)$ be.

**Corollary.** $\gamma(s,\pi_u\times\sigma_w,\psi)\in\mathbb{C}(q^{-u},q^{-w},q^{-s})$.

3.3. **Deformations and derivatives.** Let us now consider how the deformation process effects the irreducibility and derivatives of $\pi$.

Take $\pi$ and $\pi_u$ as in Section 3.1. Suppose that the quasi–square–integrable representation $\Delta_i$ corresponds to the segment $[\rho_i,\rho_i\nu,\dots,\rho_i\nu^{\ell_i-1}]$ where $\rho_i$ is a cuspidal representation of $GL_{r_i}$. Then we have $\Delta_i$ is a representation of $GL_{n_i}$ where $n_i=r_i\ell_i$ and $n=\sum n_i=\sum r_i\ell_i$. If we twist $\Delta_i$ with the determinantal character $\nu^{u_i}$ then, setting $\Delta_{i,u_i}=\Delta_i\nu^{u_i}$, we see that $\Delta_{i,u_i}$ is again quasi-square-integrable and associated to the segment $[(\rho_i\nu^{u_i}),(\rho_i\nu^{u_i})\nu,\dots,(\rho_i\nu^{u_i})\nu^{\ell_i-1}]$.

Now consider the reducibility of $\pi_u$. Since

$$\pi_u=\mathrm{Ind}(\Delta_{1,u_1}\otimes\cdots\otimes\Delta_{t,u_t})$$

is still induced from the quasi–square–integrable representations $\Delta_{i,u_i}$, it will be irreducible as long as the segments corresponding to the $\Delta_{i,u_i}$ are unlinked. Since the $\rho_i$ remain constant throughout the deformation, it is not hard to see (but tedious to write down) that the $\Delta_{i,u_i}$ will be unlinked except possibly on a finite number of hyperplanes in $u$. Hence we have the following.

**Proposition 3.4.** *With the possible exception of a finite number of hyperplanes in $u$, the deformed representation $\pi_u$ is irreducible.*

Note that even if we begin with a $\pi$ which is reducible at $u=0$, the deformed representations $\pi_u$ will still be irreducible except on a finite number of hyperplanes in the $u$.

Now consider the derivatives of $\pi_u$. By the results of Bernstein and Zelevinsky, the $k^{th}$ derivative $\pi_u^{(k)}$ will be glued from the representations $\mathrm{Ind}(\Delta_{1,u_1}^{(k_1)}\otimes\cdots\otimes\Delta_{t,u_t}^{(k_t)})$ for all possible partitions $k=k_1+\cdots+k_t$ with $0\le k_i\le n_i$. Let us set

$$\pi_u^{(k_1,\dots,k_t)}=\mathrm{Ind}(\Delta_{1,u_1}^{(k_1)}\otimes\cdots\otimes\Delta_{t,u_t}^{(k_t)}).$$

If we consider a particular $\Delta_{i,u_i}^{(k_i)}$ then since $\Delta_{i,u_i}$ is associated to the segment $[(\rho_i\nu^{u_i}),\dots,(\rho_i\nu^{u_i})\nu^{\ell_i-1}]$ we see that $\Delta_{i,u_i}^{(k_i)}$ is zero unless $k_i=a_ir_i$ with $0\le a_i\le\ell_i$ and $\Delta_{i,u_i}^{(a_ir_i)}$ is the quasi–square–integrable representation attached to the segment $[(\rho_i\nu^{u_i})\nu^{a_i},\dots,(\rho_i\nu^{u_i})\nu^{\ell_i-1}]$. So we have that $\pi_u^{(k_1,\dots,k_t)}=0$ unless $(k_1,\dots,k_t)=(a_1r_1,\dots,a_tr_t)$ with $0\le a_i\le\ell_i$. Moreover, as above, each individual representation $\pi_u^{(a_1r_1,\dots,a_tr_t)}$ will be irreducible, except possibly on a finite number of hyperplanes in the $u$.

Next, consider a fixed derivative $\pi_u^{(k)}$ of $\pi_u$. To be non-zero, $k$ must be of the form $k=a_1r_1+\cdots+a_tr_t$ with the $a_i$ as above. There may be more that one way to write $k$ in this fashion and we have that $\pi_u^{(k)}$ is glued form the $\pi_u^{(a_1r_1,\dots,a_tr_t)}$ with $k=\sum a_ir_i$. It is again easy to see (but tedious to write) that outside of a finite number on hyperplanes in the $u$ the central characters of the $\pi_u^{(a_1r_1,\dots,a_tr_t)}$ will be distinct and hence there can be no non–trivial extensions among these representations. Thus, off these hyperplanes, $\pi_u^{(k)}=\oplus\pi_u^{(a_1r_1,\dots,a_tr_t)}$

where the sum is restricted to those representations where $k = \sum a_i r_i$. Collecting this information, we arrive at the following result.

**Proposition 3.5.** *Outside of a finite number of hyperplanes in the $u$ we have that each non–zero derivative of $\pi_u$ is completely reducible. Moreover, the decomposition is given by*

$$\pi_u^{(k)} = \oplus_{k=a_1r_1+\cdots+a_tr_t} \pi_u^{(a_1r_1,\ldots,a_tr_t)}$$

*with each $\pi_u^{(a_1r_1,\ldots,a_tr_t)}$ irreducible.*

For future reference, we will say that any $u$ for which the proposition is true is in *general position.* The set of $u$ in general position form a Zariski open subset of $\mathcal{D}_\pi$.

## 4. Derivatives and Local Factors, II

Let us now begin with representations $\pi = \mathrm{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)$ of $GL_n$ and $\sigma = \mathrm{Ind}(\Delta'_1 \otimes \cdots \otimes \Delta'_r)$ with the $\Delta_i$ and $\Delta'_j$ irreducible quasi-square-integrable. For now, $\pi$ and $\sigma$ need not be irreducible. Let us take $\Delta_i$ to be associated to the segment $[\rho_i, \ldots, \rho_i \nu^{\ell_i - 1}]$ with $\rho_i$ a cuspidal representation of $GL_{r_i}$ and $\Delta'_j$ to be associated to the segment $[\rho'_j, \ldots, \rho'_j \nu^{\ell'_j - 1}]$ with $\rho'_j$ a cuspidal representation of $GL_{r'_j}$.

We deform each representation to families $\pi_u$ with $u \in \mathcal{D}_\pi$ and $\sigma_w$ with $w \in \mathcal{D}_\sigma$ as in Section 3.

Let us fix points $u$ and $w$ in general position. For $u$ and $w$ in general position, both $\pi_u$ and $\sigma_w$ are irreducible, their derivatives are completely reducible, and they are given by Proposition 3.5. Consider the local integrals for $\pi_u$ and $\sigma_w$, namely the $I_j(s; W_u, W'_w)$ if $n > m$ or the $I(s; W_u, W'_w, \Phi)$ if $n = m$ with $W_u \in \mathcal{W}(\pi_u, \psi)$, $W'_w \in \mathcal{W}(\sigma_w, \psi^{-1})$, and $\Phi \in \mathcal{S}(K^n)$ if necessary, and the fractional ideals $\mathcal{I}(\pi_u, \sigma_w) \subset \mathbb{C}(q^{-s})$ they generate. Then by the results of Section 2, we know that the poles of these families are precisely the poles of the exceptional contributions to the L-functions of the form

$$L_{ex}(s, \pi_u^{(a_1r_1,\ldots,a_tr_t)} \times \sigma_w^{(a'_1r'_1,\ldots,a'_rr'_r)})$$

such that $0 \le a_i \le \ell_i$, $0 \le a'_j \le \ell'_j$, and $n - \sum a_i r_i = m - \sum a'_j r'_j$. In fact,

$$L(s, \pi_u \times \sigma_w)^{-1} = l.c.m. \{ L_{ex}(s, \pi_u^{(a_1r_1,\ldots,a_tr_t)} \times \sigma_w^{(a'_1r'_1,\ldots,a'_rr'_r)})^{-1} \}$$

where the least common multiple is taken in terms of divisibility in $\mathbb{C}[q^s, q^{-s}]$ and normalized to be an standard Euler factor.

The exceptional L-function

$$L_{ex}(s, \pi_u^{(a_1r_1,\ldots,a_tr_t)} \times \sigma_w^{(a'_1r'_1,\ldots,a'_rr'_r)}) \tag{4.1}$$

can have a pole only at those $s$ for which

$$(\pi_u^{(a_1r_1,\ldots,a_tr_t)})^\sim \cong \sigma_w^{(a'_1r'_1,\ldots,a'_rr'_r)} \nu^s$$

or equivalently

$$\mathrm{Ind}(\Delta_{1,u_1}^{(a_1r_1)} \otimes \cdots \otimes \Delta_{t,u_t}^{(a_tr_t)})^\sim \cong \mathrm{Ind}((\Delta'_{1,w_1})^{(a'_1r'_1)} \otimes \cdots \otimes (\Delta'_{r,w_r})^{(a'_rr'_r)}) \nu^s.$$

Since these induced representations are irreducible, the only way that this is possible is there is a bijection between those indices $i$ for which $a_ir_i \neq \ell_i$ and those indices $j$ for which $a'_jr'_j \neq \ell'_j$ and under this bijection we have $(\Delta_i^{(a_ir_i)}\nu^{u_i})^\sim \cong \Delta_j^{\prime(a'_jr'_j)}\nu^{w_j}\nu^s$ or equivalently

$$(\Delta_i^{(a_ir_i)})^\sim \cong \Delta_j^{\prime(a'_jr'_j)}\nu^{u_i+w_j+s}. \tag{4.2}$$

Now, let us see how these conditions vary in $u$ and $w$. Consider now the local integrals $I_j(s;W_u,W'_w)$ or $I(s;W_u,W'_w,\Phi)$ with $W \in \mathcal{W}_\pi^{(0)}$ and $W' \in \mathcal{W}_\sigma^{(0)}$. Then these integrals define rational functions of $q^{\pm u}$, $q^{\pm w}$, and $q^{\pm s}$. For $u$ and $w$ in the Zariski open subset of general position, these rational functions can have poles coming from the exceptional contributions to the L-functions from (4.1). Each such L-function can have poles which lie along the locus defined by the equations (4.2), which define a finite number of hyperplanes, where there is one equation for every pair of indices such that $a_ir_i \neq \ell_i$ and $a'_jr'_j \neq \ell'_j$. If there is more than one pair of such indices then this locus will be defined by 2 or more independent equations and hence will be of codimension greater than or equal to 2. Viewing the local integrals as meromorphic functions of $u$, $w$, and $s$, we know that these integrals can have no isolated singularities of codimension greater than or equal to 2 by Hartog's Theorem. Hence every singularity of our integrals must be accounted for by an exceptional contribution of the form (4.1) where there is exactly one pair of indices $a_ir_i$ and $a'_ir'_i$ with $a_ir_i \neq \ell_i$, $a'_jr'_j \neq \ell'_j$, $n_i - a_ir_i = m_j - a'_jr'_j$, and satisfying (4.2). Hence we have, for $u$ and $w$ in general position

$$L(s,\pi_u \times \sigma_w)^{-1} = l.c.m.\big\{L_{ex}(u_i+w_j+s, \Delta_i^{(a_ir_i)} \times \Delta_j^{\prime(a'_jr'_j)})^{-1}\big\}.$$

Now, still for $u$ and $w$ in general position, the exceptional contributions to the L-functions for different $\Delta_i$ and $\Delta'_j$ will be relatively prime, and collecting the contributions for the derivatives of a fixed $\Delta_i$ and $\Delta'_j$ will give precisely $L(u_i+w_j+s, \Delta_i \times \Delta'_j)$ by the computations of Section 2.6.2. Hence we have the following result.

**Proposition 4.1.** *For $u$ and $w$ in general position we have*

$$L(s,\pi_u \times \sigma_w) = \prod_{i,j} L(u_i+w_j+s, \Delta_i \times \Delta_j).$$

We would like to specialize this result to $u$ and $w$ not in general position. Knowing that the L-function is given by the above product, which is the inverse of a Laurent polynomial in $\mathbb{C}[q^{\pm u}, q^{\pm w}, q^{\pm s}]$, we know that, for $W_u \in \mathcal{W}_\pi^{(0)}$ and $W'_w \in \mathcal{W}_\sigma^{(0)}$ the rational functions

$$\frac{I_j(s;W_u,W'_w)}{\prod L(u_i+w_j+s, \Delta_i \times \Delta'_j)} \qquad \text{or} \qquad \frac{I(s;W_u,W'_w,\Phi)}{\prod L(u_i+w_j+s, \Delta_i \times \Delta'_j)}$$

have no poles on the Zariski open set of $u$ and $w$ in general position. The removed hyperplanes defining general position are hyperplanes in $u$ and $w$ only and are independent of $s$. However, we know that for each fixed $u$ and $w$ the local integrals $I_j(s;W_u,W'_w)$ or $I(s;W_u,W'_w,\Phi)$ converge absolutely in a half plane $\mathrm{Re}(L_{\pi,\sigma}(u,w,s)) > 0$. Hence no polar locus can lie entirely in these removed hyperplanes. Hence the ratios

$$\frac{I_j(s;W_u,W'_w)}{\prod L(u_i+w_j+s, \Delta_i \times \Delta'_j)} \qquad \text{or} \qquad \frac{I(s;W_u,W'_w,\Phi)}{\prod L(u_i+w_j+s, \Delta_i \times \Delta'_j)}$$

have no poles and hence define an entire rational function of the $q^{-u}$, $q^{-w}$, and $q^{-s}$ and hence lies in $\mathbb{C}[q^{\pm u}, q^{\pm w}, q^{\pm s}]$. If we now specialize to $u = 0$ and $w = 0$, or any other point for that matter, we find that

$$\frac{I_j(s; W, W')}{\prod L(s, \Delta_i \times \Delta'_j)} \qquad \text{or} \qquad \frac{I(s; W, W', \Phi)}{\prod L(s, \Delta_i \times \Delta'_j)}$$

have no poles for all $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\sigma, \psi^{-1})$. From this we cannot conclude that the denominator of these ratios is indeed the local L-function, but only the following.

**Proposition 4.2.** $L(s, \pi \times \sigma) \in \prod_{i,j} L(s, \Delta_i \times \Delta'_j)\mathbb{C}[q^s, q^{-s}]$.

This reproduces a weak version of the second statement of Theorem 3.1 of [9].

To proceed further, we must use the functional equation to recover a weak version of the first part of Theorem 3.1 of [9]. Let us consider again the behavior of the gamma factor for these representations under deformations. We know that $\gamma(s, \pi_u \times \sigma_w, \psi)$ is a rational function of $q^{-u}$, $q^{-w}$, and $q^{-s}$. The local $\varepsilon$-factor satisfies

$$\gamma(s, \pi_u \times \sigma_w, \psi) = \frac{\varepsilon(s, \pi_u \times \sigma_w, \psi) L(1 - s, (\pi_u)^\iota \times (\sigma_w)^\iota)}{L(s, \pi_u \times \sigma_w)}.$$

For fixed $u$ and $w$ we know, by applying the functional equation twice, that $\varepsilon(s, \pi_u \times \sigma_w, \psi)$ is of the form $Aq^{-Bs}$, that is, it is a unit in $\mathbb{C}[q^s, q^{-s}]$ [9].

For $u$ and $w$ in general position, we know that $L(s, \pi_u \times \sigma_w) = \prod_{i,j} L(u_i + w_j + s, \Delta_i \times \Delta_j)$ and $L(1 - s, (\pi_u)^\iota \times (\sigma_w)^\iota) = \prod_{i,j} L(1 - s - u_i - w_j, \tilde{\Delta}_i \times \tilde{\Delta}_j)$. Moreover, all the ratios

$$\frac{I(s; W_u, W'_w)}{\prod L(u_i + w_j + s, \Delta_i \times \Delta'_j)} \qquad \text{and} \qquad \frac{I_{n-m-1}(1 - s; \pi_u(w_{n,m})\tilde{W}_u, \tilde{W}'_w, )}{\prod L(1 - s - u_i - w_j, \tilde{\Delta}_i \times \tilde{\Delta}'_j)}$$

or

$$\frac{I(s; W_u, W'_w, \Phi)}{\prod L(u_i + w_j + s, \Delta_i \times \Delta'_j)} \qquad \text{and} \qquad \frac{I(1 - s; \tilde{W}_u, \tilde{W}'_w, \hat{\Phi})}{\prod L(1 - s - u_i - w_j, \tilde{\Delta}_i \times \tilde{\Delta}'_j)}$$

are Laurent polynomials in $\mathbb{C}[q^{\pm u}, q^{\pm w}, q^{\pm s}]$. If we now define a variant of the $\varepsilon$–factor by

$$\gamma(s, \pi_u \times \sigma_w, \psi) = \frac{\varepsilon^o(s, \pi_u \times \sigma_w, \psi) \prod L(1 - s - u_i - w_j, \tilde{\Delta}_i \times \tilde{\Delta}'_j)}{\prod L s + u_i + w_j, (\Delta_i \times \Delta'_j)}$$

then $\varepsilon^o(s, \pi_u \times \sigma_w, \psi) \in \mathbb{C}(q^{-u}, q^{-w}, q^{-s})$ and for $u$ and $w$ in general position $\varepsilon^o(s, \pi_u \times \sigma_w, \psi) = \varepsilon(s, \pi_u \times \sigma_w, \psi)$. If we apply the functional equation twice as in the usual argument [9] then we find that for $u$ and $w$ in general position

$$\varepsilon^o(s, \pi_u \times \sigma_w, \psi)\varepsilon^o(1 - s, (\pi_u)^\iota \times (\sigma_w)^\iota, \psi^{-1}) = 1.$$

Since both sides of this equality are rational functions in $q^{-u}$, $q^{-w}$, and $q^{-s}$ and agree on the Zariski open subset of $u$ and $w$ in general position, we have that they agree for all $u$, $w$, and $s$. Hence $\varepsilon^o(s, \pi_u \times \sigma_w, \psi)$ must be a unit in $\mathbb{C}[q^{\pm u}, q^{\pm w}, q^{\pm s}]$, that is, a monomial of the form $\varepsilon^o(s, \pi_u \times \sigma_w, \psi) = \alpha q^{-\beta u} q^{-\gamma w} q^{-\delta s}$ where $\alpha = (\alpha_1, \ldots, \alpha_t) \in \mathbb{Z}^t$ with $\alpha u = \sum \alpha_i u_i$ and similarly for $\gamma$ and $\gamma w$.

If we return to our consideration of the behavior of $\gamma(s,\pi_u\times\sigma_w,\psi)$, we now have

$$\gamma(s,\pi_u\times\sigma_w,\psi)=\frac{\varepsilon^o(s,\pi_u\times\sigma_w,\psi)\prod L(1-s-u_i-w_j,\tilde\Delta_i\times\tilde\Delta'_j)}{\prod L(s+u_i+w_j),\Delta_i\times\Delta'_j)}$$

but we also have

$$\gamma(s+u_i+w_j,\Delta_i\times\Delta'_j,\psi)=\frac{\varepsilon(s,\Delta_i\times\Delta'_j,\psi)L(1-s-u_i-w_j,\tilde\Delta_i\times\tilde\Delta'_j)}{L(s+u_i+w_j,\Delta_i\times\Delta'_j)}$$

and together these imply

$$\gamma(s,\pi_u\times\sigma_w,\psi)=\left\{\frac{\varepsilon^o(s,\pi_u\times\sigma_w,\psi)}{\prod\varepsilon(s+u_i+w_j,\Delta_i\times\Delta'_j,\psi)}\right\}\prod\gamma(s+u_i+w_j,\Delta_i\times\Delta'_j,\psi).$$

**Proposition 4.3.** $\gamma(s,\pi_u\times\sigma_w,\psi)$ *and* $\prod\gamma(s+u_i+w_j,\Delta_i\times\Delta'_j,\psi)$ *are equal up to a unit in* $\mathbb{C}[q^{\pm u},q^{\pm w},q^{\pm s}]$.

Hence, under deformation, $\gamma$ is multiplicative up to a monomial factor. If we now specialize to $u=0$ and $w=0$ and introduce the notation for two rational functions $P(q^{-s})$ and $Q(q^{-s})$ that $P\sim Q$ denotes that the ratio is a unit in $\mathbb{C}[q^s,q^{-s}]$, that is, a monomial factor, then we have the following Corollary.

**Corollary.** $\gamma(s,\pi\times\sigma,\psi)\sim\prod\gamma(s,\Delta_i\times\Delta'_j,\psi)$.

This is our version of the first statement in Theorem 3.1 of [9].

From this point on, to compute the L-function $L(s,\pi\times\sigma)$, we have very little to add to the argument of Section 9 of [9]. For completeness, we will sketch the argument here.

We first need the following Proposition, which occurs as Lemma 9.3 of [9].

**Proposition 4.4.** (i) *Suppose* $\pi=\mathrm{Ind}(\pi_1\otimes\pi_2)$ *with each* $\pi_i$ *a representation of Whittaker type and the induction normalized parabolic induction from a standard (upper) maximal parabolic. Then* $L(s,\pi_2\times\sigma)^{-1}$ *divides* $L(s,\pi\times\sigma)^{-1}$*, that is,* $L(s,\pi_2\times\sigma)=Q(q^{-s})L(s,\pi\times\sigma)$ *with* $Q(X)\in\mathbb{C}[X]$.
(ii) *Suppose* $\sigma=\mathrm{Ind}(\sigma_1\otimes\sigma_2)$ *with each* $\sigma_i$ *a representation of Whittaker type and the induction normalized parabolic induction from a standard (upper) maximal parabolic. Then* $L(s,\pi\times\sigma_2)^{-1}$ *divides* $L(s,\pi\times\sigma)^{-1}$*, that is,* $L(s,\pi\times\sigma_2)=Q(q^{-s})L(s,\pi\times\sigma)$ *with* $Q(X)\in\mathbb{C}[X]$.

*Proof:* In Proposition 9.1 of [9] they establish that if $\pi=\mathrm{Ind}(\pi_1\otimes\pi_2)$, with $\pi_2$ a representation of $GL_{n_2}$, then for every $W_2\in\mathcal{W}(\pi_2,\psi)$ and $\Phi\in\mathcal{S}(K^{n_2})$ there is a $W\in\mathcal{W}(\pi,\psi)$ such that

$$W\begin{pmatrix}g&\\&I_{n-n_2}\end{pmatrix}=W_2(g)\Phi(e_{n_2}g)|\det(g)|^{(n-n_2)/2}$$

and similarly for $\sigma$ and $\sigma_2$.

In [9], (i) is established in the case of $m<n$ by using this result to show that in fact every integral occurring in $\mathcal{I}(\pi_2,\sigma)$ is actually an integral in $\mathcal{I}(\pi,\sigma)$, that is, $\mathcal{I}(\pi_2,\sigma)\subset\mathcal{I}(\pi,\sigma)$.

From this the divisibility of the L-functions follows. In the case that $n = m$, (i) is subsumed under (ii) since the statement is symmetric in $\pi$ and $\sigma$.

Jacquet, Piatetski-Shapiro, and Shalika [9] establish (ii) as a consequence of (i) using their Theorem 3.1. Our version of this theorem is not suitable for this purpose, but we can prove this along the lines of their proof of (i). In particular, from either formula (2.4) or (2.5) (which agree in the case $m = n$) and the discussion around them, we know that the ideal $\mathcal{I}(\pi, \sigma)$ contains each of the integrals

$$I_{(m-m_2-1)}(s; W, W') = \\ = \int_{N_{m_2}\backslash GL_{m_2}} W\begin{pmatrix} h & \\ & I_{n-m_2}\end{pmatrix} W'\begin{pmatrix} h & \\ & I_{m-m_2}\end{pmatrix} |\det(h)|^{s-(m-m_2)-(n-m)/2}\, dh$$

with $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\sigma, \psi^{-1})$. If we let $W_2' \in \mathcal{W}(\sigma_2, \psi^{-1})$ and take $W'$ to be an associated element of $\mathcal{W}(\sigma, \psi^{-1})$, we see that this integral becomes

$$\int_{N_{m_2}\backslash GL_{m_2}} W\begin{pmatrix} h & \\ & I_{n-m_2}\end{pmatrix} W_2'(h)\Phi(e_{n_2}h)|\det(h)|^{s-(n-m_2)/2}\, dh$$

and so, for appropriate choice of $\Phi$, becomes

$$I(s; W, W_2') = \int_{N_{m_2}\backslash GL_{m_2}} W\begin{pmatrix} h & \\ & I_{n-m_2}\end{pmatrix} W_2'(h)|\det(h)|^{s-(n-m_2)/2}\, dh.$$

Thus, once again, $\mathcal{I}(\pi, \sigma_2) \subset \mathcal{I}(\pi, \sigma)$ and we obtain the stated divisibility result in (ii). □

To proceed, as in [9], we take both $\pi$ and $\sigma$ as induced representation of Langlands type. We write $\pi = \mathrm{Ind}(\Delta_1\nu^{u_1} \otimes \cdots \otimes \Delta_t\nu^{u_t})$ and $\sigma = \mathrm{Ind}(\Delta_1'\nu^{w_1} \otimes \cdots \otimes \Delta_r'\nu^{w_r})$ with each $\Delta_i$ and $\Delta_j'$ a square-integrable representation, the $u_i$ and $w_j$ real and ordered so that $u_1 \geq \cdots \geq u_t$ and $w_1 \geq \cdots \geq w_r$. Every generic representation can be written this way by the Langlands classification, and in fact every irreducible admissible representation occurs as the unique quotient of such.

**Theorem 4.1.** *Let $\pi$ and $\sigma$ be as above, that is, induced of Langlands type. Then*

$$L(s, \pi \times \sigma) = \prod_{i,j} L(s, \Delta_i \times \Delta_j').$$

*Proof:* By our deformation argument, we know that in general

$$L(\pi \times \sigma) = P(q^{-s}) \prod_{i,j} Ls + u_i + w_j, (\Delta_i \times \Delta_j')$$

and

$$L(1-s, \pi^\iota \times \sigma^\iota) = \tilde{P}(q^{-s}) \prod_{i,j} L(1-s-u_i-w_j, \tilde{\Delta}_i \times \tilde{\Delta}_j')$$

and

$$\gamma(s, \pi \times \sigma, \psi) \sim \prod_{i,j} \gamma(s + u_i + w_j, \Delta_i \times \Delta_j', \psi).$$

Replacing the $\gamma$–factors by their definitions then gives $P(q^{-s}) \sim \tilde{P}(q^{-s})$.

Next, consider the case when $r = 1$, that is, $\sigma$ a quasi-square-integrable representation. We proceed by induction on $t$. If $t = 1$ then $\pi$ is also quasi-square-integrable and it is elementary that $L(s, \pi \times \sigma) = L(s + u_1 + w_1, \Delta_1 \times \Delta_1')$. For $t > 1$, we use transitivity of induction to write $\pi = \mathrm{Ind}(\Delta_1\nu^{u_1} \otimes \pi_2)$ where $\pi_2 = \mathrm{Ind}(\Delta_2\nu^{u_2} \otimes \cdots \otimes \Delta_t\nu^{u_t})$ and $\pi^\iota = \mathrm{Ind}(\tilde{\Delta}_t^{-u_t} \otimes (\pi^\iota)_2)$ where $(\pi^\iota)_2 = \mathrm{Ind}(\tilde{\Delta}_{t-1}\nu^{-u_{t-1}} \otimes \cdots \otimes \tilde{\Delta}_1\nu^{-u_1})$.

By induction we have

$$L(s, \pi_2 \times \sigma) = \prod_{i=2}^{t} L(s + u_i + w_1, \Delta_i \times \Delta_1').$$

By the previous Proposition, there is a polynomial $Q$ such that

$$L(s, \pi_2 \times \sigma) = Q(q^{-s})L(s, \pi \times \sigma)$$

and by the deformation argument there is a polynomial $P$ such that

$$L(s, \pi \times \sigma) = P(q^{-s}) \prod_{i=1}^{t} L(s + u_i + w_1, \Delta_i \times \Delta_1').$$

Combined, these imply that $P(q^{-s})$ divides $L(s+u_1+w_1, \Delta_1 \times \Delta_1')^{-1}$ and hence has its zeros in the half plane $\mathrm{Re}(s) \leq -u_1 - w_1$ by the Corollary to Theorem 2.3.

By the same argument applied to $\pi^\iota$, we find that if $\tilde{P}$ is defined by

$$L(1 - s, \pi^\iota \times \sigma^\iota) = \tilde{P}(q^{-s}) \prod_{i} L1 - s - u_i - w_1, (\tilde{\Delta}_i \times \tilde{\Delta}_1')$$

then $\tilde{P}(q^{-s})$ divides $L(1 - s - u_t - w_1, \tilde{\Delta}_t \times \tilde{\Delta}_1)^{-1}$ and hence has its zeros in the half plane $1 - u_t - w_1 \leq \mathrm{Re}(s)$.

But we know that in general $P(q^{-s}) \sim \tilde{P}(q^{-s})$, and so they must have the same zero set. But $u_1 \geq u_t$. Therefore the halfplanes $\mathrm{Re}(s) \leq -u_1 - w_1$ and $1 - u_t - w_1 \leq \mathrm{Re}(s)$ are disjoint. Hence $P = \tilde{P} \equiv 1$, since they both have no zeros but are of the form of $\prod(1 - \alpha_i q^{-s})$.

This establishes the result in the case $r = 1$ and $t$ arbitrary. The same argument establishes the case $t = 1$ and $r$ arbitrary.

To establish the general case, we proceed by a double induction. We may assume that both $r > 1$ and $t > 1$. We assume we have established the Theorem for the pairs $(t, r - 1)$ and $(t - 1, r)$, and prove the result for the pair $(t, r)$. We decompose $\pi$ and $\pi^\iota$ as above and also decompose $\sigma$ and $\sigma^\iota$. So write $\sigma = \mathrm{Ind}(\Delta_1'\nu^{w_1} \otimes \sigma_2)$ where $\sigma_2 = \mathrm{Ind}(\Delta_2'\nu^{w_2} \otimes \cdots \otimes \Delta_r'\nu^{u_r})$ and $\sigma^\iota = \mathrm{Ind}(\tilde{\Delta}_r'\nu^{-w_r} \otimes (\sigma^\iota)_2)$ where $(\sigma^\iota)_2 = \mathrm{Ind}(\tilde{\Delta}_{r-1}'\nu^{-w_{r-1}} \otimes \cdots \otimes \tilde{\Delta}_1'\nu^{-w_1})$.

By induction we have

$$L(1 - s, (\pi^\iota)_2 \times \sigma^\iota) = \prod_{i=1}^{t-1} \prod_{j=1}^{r} L(1 - s - u_i - w_j, \tilde{\Delta}_i \times \tilde{\Delta}_j') \tag{4.3}$$

and

$$L(s,\pi\times\sigma_2)=\prod_{i=1}^{t}\prod_{j=2}^{r}L(s+u_i+w_j,\Delta_i\times\Delta'_j). \tag{4.4}$$

By the previous Proposition, there are polynomials $\tilde{Q}_1$ and $Q_2$ such that

$$L(1-s,(\pi^\iota)_2\times\sigma^\iota)=\tilde{Q}_1(q^{-s})L(1-s,\pi^\iota\times\sigma^\iota) \tag{4.5}$$

and

$$L(s,\pi\times\sigma_2)=Q_2(q^{-s})L(s,\pi\times\sigma). \tag{4.6}$$

By the deformation argument there are polynomials $P$ and $\tilde{P}$ such that

$$L(s,\pi\times\sigma)=P(q^{-s})\prod_{i=1}^{t}\prod_{j=1}^{r}L(s+u_i+w_j,\Delta_i\times\Delta'_j) \tag{4.7}$$

and

$$L(1-s,\pi^\iota\times\sigma^\iota)=\tilde{P}(q^{-s})\prod_{i=1}^{t}\prod_{j=1}^{r}L(1-s-u_i-w_j,\tilde{\Delta}_i\times\tilde{\Delta}'_j). \tag{4.8}$$

Now, from (4.4), (4.6), and (4.7), we see that the product $P(q^{-s})Q_2(q^{-s})$ divides $\prod_i L(s+u_i+w_1,\Delta_i\times\Delta'_1)^{-1}$. Similarly, from (4.3), (4.5), and (4.8), $\tilde{P}(q^{-s})\tilde{Q}_1(q^{-s})$ divides the product $\prod_j L(1-s-u_t-w_j,\tilde{\Delta}_t\times\tilde{\Delta}'_j)^{-1}$. In general, from the functional equation, we know that $P(q^{-s})\sim\tilde{P}(q^{-s})$ so that these polynomials must have the same zero set. If there is a common zero, then there must be a pair of indices $i$ and $j$ such that $L(s+u_i+w_1,\Delta_i\times\Delta_1)^{-1}$ and $L(1-s-u_t-w_j,\tilde{\Delta}_t\times\tilde{\Delta}_j)^{-1}$ have a common zero. However, the function $L(s+u_i+w_1,\Delta_i\times\Delta_1)^{-1}$ has its zeros in the halfplane $\mathrm{Re}(s)\le -u_i-w_1$ while the function $L(1-s-u_t-w_j,\tilde{\Delta}_t\times\tilde{\Delta}_j)^{-1}$ has its zeros in $1-u_t-w_j\le\mathrm{Re}(s)$. Since $u_i\ge u_t$ and $w_1\ge w_j$, we see that these half planes have no intersection. Hence $P(q^{-s})$ can have no zeros and, as above, $P\equiv 1$.

This completes the proof of the Theorem. □

**Corollary.** *Suppose that $\pi=\mathrm{Ind}(\Delta_1\otimes\cdots\otimes\Delta_t)$ is a generic representation of $GL_n$, so each $\Delta_i$ is quasi–square–integrable, and $\sigma=\mathrm{Ind}(\Delta'_1\otimes\cdots\otimes\Delta'_r)$ is a generic representation of $GL_m$, so each $\Delta'_j$ is quasi–square–integrable. Then*

$$L(s,\pi\times\sigma)=\prod_{i,j}L(s,\Delta_i\times\Delta'_j).$$

*Proof:* Since $\pi$ and $\sigma$ are generic, they are irreducible, and the quasi-square-integrable representations can be rearranged to be in Langlands order without changing $\pi$ or $\sigma$. Then the result is just a restatement of the above Theorem. □

If $\pi$ and $\sigma$ are not generic, then their L-function $L(s, \pi \times \sigma)$ is defined by taking the Langlands induced representations $\Pi = \mathrm{Ind}(\Delta_1 \nu^{u_1} \otimes \cdots \otimes \Delta_t \nu^{u_t})$ and $\Sigma = \mathrm{Ind}(\Delta'_1 \nu^{w_1} \otimes \cdots \otimes \Delta'_r \nu^{w_r})$ such that $\pi$ is the unique irreducible quotient of $\Pi$ and $\sigma$ is the unique irreducible quotient of $\Sigma$ and setting $L(s, \pi \times \sigma) = L(s, \Pi \times \Sigma)$. This later L-function we have computed above. Hence, we have computed $L(s, \pi \times \sigma)$ for all irreducible admissible representations of $GL_n$ and $GL_m$.

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