REMARKS ON RANKIN-SELBERG CONVOLUTIONS

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Dedicated to Joe Shalika

In this paper we would like to present two types of results on the theory of Rankin-Selberg convolution L-functions for $GL_n \times GL_m$. Both families of results are based on the foundational work of Shalika with Jacquet and the second author of this paper [10, 11, 12, 13] on the analysis of these L-functions via the theory of integral representations.

In the first section we present results on the local archimedean Rankin-Selberg convolutions. This section was written in response to a question of D. Ramakrishnan as to whether the local L-function as defined by Jacquet and Shalika in [13] was indeed the "correct" factor in the sense that it is precisely the standard archimedean Euler factor which is determined by the poles of the family of local integrals using either K-finite data or smooth data (i.e., without passing to the Casselman-Wallach completion). In Section 1 we answer this affirmatively as a consequence of showing that the ratio of the local integral divided by the L-function is continuous in the appropriate topology, uniformly on compact subsets of $\mathbb{C}$. As a consequence we establish a non-vanishing result for this ratio which is necessary for the completion of the global theory of Rankin-Selberg convolutions.

In the second section we complete global theory of Rankin-Selberg convolutions from the point of view of integral representations. This section was motivated by the comment of Jacquet that, although known to the experts, this completion had never appeared in print. Most of the necessary results can be found in the paper [12] by Jacquet and Shalika, though not always explicitly stated. One missing ingredient was the non-vanishing result for the archimedean Rankin-Selberg integrals alluded to above. With this in hand, in Section 2 we combine the global results of [10, 12] with the local results of [11, 13] and Section 1 of this paper to give a proof of the fact that the global L-functions $L(s, \pi \times \pi')$ are *nice*, in the sense that they have meromorphic continuation, are bounded in vertical strips, and satisfy a global functional equation, within the context of integral representations. Actually, we are only able to establish the boundedness in vertical strips within the method for $m = n$ and $m = n - 1$. Outside of these cases we must rely on the results of Gelbart and Shahidi [6]. In addition we establish the location of poles for these L-functions, giving the proof of Jacquet, Piatetski-Shapiro, and Shalika of these results alluded to in the appendix of [14]. If one combines these results with the strong multiplicity one results of [12] and the converse theorems [2, 3] we can consider the basic global theory of Rankin-Selberg convolutions via integral representations to now be essentially complete, with the exception of the cases of boundedness in vertical strips alluded to above.

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1. Archimedean Rankin–Selberg Convolutions

This section complements the material in the paper of Jacquet and Shalika [13] and is meant to show that indeed the results there are enough for most applications. Unless otherwise noted, the notation is as in [13].

1.1. **An extension of Dixmier–Malliavin.** Let E be a Fréchet space, G a real Lie group, $\mathfrak{g}$ its complexified Lie algebra, and π a continuous representation of G on E. Let $\{p_j\}$ be a set of semi-norms on E defining the topology on E.

Let E^∞ be the smooth vectors of E. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $\{u_i\}$ be a basis of $U(\mathfrak{g})$. The the topology on E^∞ is defined by the seminorms $q_{i,j}(\xi) = p_j(\pi(u_i)\xi)$ for $\xi \in E^\infty$. With this topology, E^∞ is again a Fréchet space [1]. For convenience, reindex the family $\{q_{i,j}\}$ by a single index $\{q_i\}$.

Let $\xi_k \to \xi_0$ be a convergent sequence in E^∞. The purpose of this section is to prove the following extension of Theorem 3.3 of [4]. Our proof is a variation of that in [4] which we follow.

Proposition 1.1. *There exists a finite set of functions $f_j \in C_c^\infty(G)$ and a collection of vectors $\xi_{k,j} \in E^\infty$ such that $\xi_k = \sum \pi(f_j)\xi_{k,j}$ for all $k \geq 0$ and such that for each j, $\xi_{k,j}$ converge to $\xi_{0,j}$ in E^∞.*

Proof: Since E^∞ is linear, it suffices to consider the case $\xi_0 = 0$.

Let $\{X_1, \dots, X_m\}$ be a basis of $\mathfrak{g}$ with the property that under the map

$$(t_1, \dots, t_m) \mapsto e^{t_1 X_1} \cdots e^{t_m X_m}$$

from $\mathbb{R}^m$ to G the open set $(-1,1)^m$ is mapped diffeomorphically onto an open set Ω of G.

Lemma 1.1. *For each choice of seminorm q_i and non-negative integer n the set of real numbers $\{q_i(\pi(X_1)^{2n}\xi_k)\}$ is bounded.*

Proof: Since $\pi(X_1)^{2n}$ acts continuously and the seminorm q_i is continuous, the sequence $q_i(\pi(X_1)^{2n}\xi_k)$ converges to $q_i(\pi(X_1)^{2n}0) = 0$. Hence the sequence of real numbers $q_i(\pi(X_1)^{2n}\xi_k)$ is bounded. □

Let $M_{n,i}$ be an upper bound for $\{q_i(\pi(X_1)^{2n}\xi_k)\}$.

Lemma 1.2. *There exist positive real numbers β_n such that the sum $\sum_n \beta_n M_{n,i}$ is convergent for all i.*

Proof: For each i there are positive numbers $\beta_{n,i}$ such that $\sum\limits_n \beta_{n,i} M_{n,i}$ converges. Let $\beta_n^{(k)} = \min\limits_{1 \le i \le k} \beta_{n,i}$ and set $\beta_n = \beta_n^{(n)}$. Then

$$\sum_n \beta_n M_{n,i} = \sum_{n \le i} \beta_n M_{n,i} + \sum_{n > i} \beta_n M_{n,i}.$$

For $n > i$, $\beta_n = \min\limits_{1 \le j \le n} \beta_{n,j} \le \beta_{n,i}$. So

$$\sum_{n>i} \beta_n M_{n,i} \le \sum_{n>i} \beta_{n,i} M_{n,i} < \infty.$$

□

Now let $\epsilon \in (0, \frac{1}{2}]$. Then by Lemma 2.5 and Remark 2.6 of [4] there is a sequence of positive numbers α_n and functions $g(t)$ and $h(t)$ in $C_c^\infty(\mathbb{R})$, supported in $(-\epsilon, \epsilon)$ such that

$$\sum_n \alpha_n M_{n,i} < \infty \text{ for all } i$$

and

$$\sum_{n=0}^{p} (-1)^n \alpha_n \delta_0^{(2n)} * g \to \delta_0 + h$$

in the space $\mathcal{E}'(\mathbb{R})$ of compactly supported distributions on $\mathbb{R}$. δ_0 is the Dirac measure supported at the origin of $\mathbb{R}$.

The measures $g(t)dt$ and $h(t)dt$ induce measures μ_1 and ν_1 on G under the map $\mathbb{R} \to G$ given by $t \mapsto e^{tX_1}$. Then

$$\mu_1 * \sum_{n=0}^{p} (-1)^n \alpha_n X_1^{2n} = \sum_{n=0}^{p} (-1)^n \alpha_n X_1^{2n} * \mu_1 \to \delta_e + \nu_1$$

in the space $\mathcal{E}'(G)$ of compactly supported distributions on G and

$$\pi(\mu_1) \sum_{n=0}^{p} (-1)^n \alpha_n \pi(X_1)^{2n} \xi_k \to \xi_k + \pi(\nu_1)\xi_k$$

in the weak topology on E.

However, by our choice of α_n, $\sum\limits_{n=0}^{\infty} q_i(\alpha_n \pi(X_1)^{2n} \xi_k) < \infty$ for each seminorm q_i. Therefore $\sum\limits_{n=0}^{p} (-1)^n \alpha_n \pi(X_1)^{2n} \xi_k$ converges to a vector η_k in E^∞. Therefore we have

$$\xi_k = \pi(\mu_1)\eta_k - \pi(\nu_1)\xi_k$$

for each ξ_k.

Lemma 1.3. *The sequence η_k converges to 0 in E^∞.*

Proof: By continuity of the seminorms,

$$q_i(\eta_k) \leq \lim_{p\to\infty} \sum_{n=0}^{p} \alpha_n q_i(\pi(X_1)^{2n}\xi_k) \leq \sum_{n=0}^{\infty} \alpha_n q_i(\pi(X_1)^{2n}\xi_k).$$

Since the sum $\sum \alpha_n q_i(\pi(X_1)^{2n}\xi_k)$ is absolutely convergent, we can interchange limit and summation to obtain

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} \alpha_n q_i(\pi(X_1)^{2n}\xi_k) = 0.$$

Therefore $\lim_{k\to\infty} q_i(\eta_k) = 0$ for all q_i. Hence $\eta_k \to 0$ in E^∞. □

Now apply the same process for X_2 through X_m. In this way we obtain a finite collection of measures $\{\mu_{i,j}\}$, where each $\mu_{i,j}$ is the image of a measure $g_{i,j}(t_i)dt_i$ under the map $t_i \mapsto e^{t_i X_i}$ as above, and sequences $\xi_{k,j}$ such that

$$\xi_k = \sum_j \pi(\mu_{1,j} * \cdots * \mu_{m,j})\xi_{j,k}$$

for each k with $\lim_{k\to\infty} \xi_{j,k} = 0$ for each j.

The measure $\mu_{1,j} * \cdots * \mu_{m,j}$ on G is then the image of the measure on $\mathbb{R}^m$ given by $g_{1,j}(t_1)\cdots g_{m,j}(t_m)dt_1 \cdots dt_m$. If $g_j(t_1, \cdots, t_m) = g_{1,j}(t_1)\cdots g_{m,j}(t_m)$ then g_j is smooth with compact support in $(-\epsilon, \epsilon)^m$. Hence by our choice of basis on $\mathfrak{g}$ the image of the measure $g_j(t)dt$ on $\mathbb{R}^m$ will be of the form $f_j(g)dg$ on G with $f_j \in C_c^\infty(G)$.

Hence we now have a finite collection of $f_j \in C_c^\infty(G)$ and $\xi_{k,j} \in E^\infty$ such that

$$\xi_k = \sum_j \pi(f_j)\xi_{k,j}$$

with the sequence $\xi_{k,j}$ now converging to 0 in E^∞ for each j.

This completes the proof of the proposition. □

1.2. **Continuity of the archimedean local integral.** Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let ψ be a non-trivial additive character of F. Let $GL_r = GL_r(F)$. Let (π, V) be a finitely generated admissible smooth representation of moderate growth of GL_n, as in [1, 13]. Let V_o denote the space of K_n–finite vectors, i.e., the underlying Harish-Chandra module. Similarly, let (σ, E) be a finitely generated admissible smooth representation of moderate growth of GL_m, and E_o its underlying Harish-Chandra module. Note that both V and E are Fréchet spaces and equal to their spaces of smooth vectors.

We further assume that π and σ are of Whittaker type as in [13], with continuous Whittaker functionals λ_π with respect to ψ and λ_σ with respect to ψ^{-1}.

We will let $(\pi \otimes \sigma, V \otimes E)$ denote the algebraic tensor product of (π, V) and (σ, E). We let $(\pi\hat{\otimes}\sigma, V\hat{\otimes}E)$ denote the (projective) topological tensor product. Then $(\pi\hat{\otimes}\sigma, V\hat{\otimes}E)$ is the again an admissible smooth representation of moderate growth of $GL_n \times GL_m$ and is in fact the Casselman-Wallach completion of the algebraic tensor product [1, 13]. (Note: This notation is slightly different from that of [13] where they use $\otimes$ for the topological tensor product.)

The linear functional $\mu = \lambda_\pi \otimes \lambda_\sigma$ is a continuous Whittaker functional on $V \otimes E$ and extends to a Whittaker functional on $V \hat{\otimes} E$ [13]. For each $v \in V \hat{\otimes} E$ let

$$W_v(g, g') = \mu(\pi(g) \hat{\otimes} \sigma(g')v)$$

and let $\mathcal{W}(\pi \hat{\otimes} \sigma, \psi)$ be the spaced spanned by all such functions. Then $\mathcal{W}(\pi \hat{\otimes} \sigma, \psi) \supset \mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\sigma, \psi^{-1})$.

As in [13], define for $W \in \mathcal{W}(\pi \hat{\otimes} \sigma, \psi)$ and $\Phi \in \mathcal{S}(F^n)$

$$\Psi(s; W, \Phi) = \int_{N_n \backslash GL_n} W(g, g)\Phi(e_n g)|\det(g)|^s \, dg \quad \text{if } n = m$$

$$\Psi(s; W) = \int_{N_m \backslash GL_m} W\left(\begin{pmatrix} g & \\ & I_{n-m} \end{pmatrix}, g\right) |\det(g)|^{s-(n-m)/2} \, dg \quad \text{if } n > m$$

$$\Psi(s; W, j) = \int_{N_m \backslash GL_m} \int_X W\left(\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{k+1} \end{pmatrix}, g\right) |\det(g)|^{s-(n-m)/2} dx dg \quad \text{if } n > m$$

where $j + k = n - m - 1$. These are all absolutely convergent for $Re(s) >> 0$.

Define $\tilde{W}(g, g') = W(w_n g^\iota, w_m g'^\iota)$, where w_r is the long Weyl element $\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ and ι is the outer automorphism of GL_r, namely $g \mapsto g^\iota = {}^t g^{-1}$. Then $\tilde{W}$ is in the Whittaker model of $V^\iota \hat{\otimes} E^\iota = (V \hat{\otimes} E)^\iota$. Then we have the functional equation:

$$\Psi(1 - s; \tilde{W}, \hat{\Phi}) = \omega_\sigma(-1)^{n-1} \gamma(s, \pi \times \sigma, \psi) \Psi(s; W, \Phi) \quad \text{if } n = m$$
$$\Psi(1 - s; \rho(w_{n,m})\tilde{W}, j) = \omega_\sigma(-1)^{n-1} \gamma(s, \pi \times \sigma, \psi) \Psi(s; W, k) \quad \text{if } n > m$$

where $j + k = n - m - 1$ and

$$\gamma(s, \pi \times \sigma, \psi) = \frac{\varepsilon(s, \pi \times \sigma, \psi) L(1 - s, \pi^\iota \times \sigma^\iota)}{L(s, \pi \times \sigma)}.$$

Note that here $L(s, \pi \times \sigma)$ is as in [13], i.e., it is the factor attached to the pair (π, σ) by the (arithmetic) Langlands classification.

The purpose of this section is to prove the following result.

Theorem 1.1. *Let Λ_s, respectively $\Lambda_{s,\Phi}$, be the linear functional on $V\hat{\otimes}E$ defined by*

$$\Lambda_s(v) = \frac{\Psi(s;W_v)}{L(s,\pi\times\sigma)} \qquad \text{if } n>m$$
$$\Lambda_{s,\Phi}(v) = \frac{\Psi(s;W_v,\Phi)}{L(s,\pi\times\sigma)} \qquad \text{if } n=m$$

for $v \in V\hat{\otimes}E$. Then Λ_s, respectively $\Lambda_{s,\Phi}$, is continuous on $V\hat{\otimes}E$, uniformly for s in a compact set.

Note that we claim the continuity for all s, not just for those s for which the local integral is absolutely convergent.

We begin by recalling the following result of [13].

Lemma 1.4. *Let $f \in C_c^\infty(GL_n \times GL_m)$. Then there exists a semi-norm β on $V\hat{\otimes}E$ and a gauge ξ on $GL_n\times GL_m$ depending only on f such that*

$$|\rho(f)W_v(g,g')| \le \beta(v)\xi(g,g')$$

for all $v \in V\hat{\otimes}E$.

Proof: The proof is word for word the same as the proof of Proposition 2.1 in [13]. □

We will prove the theorem in the case $n>m$. The proof in the case $n=m$ is the same, with the obvious modifications.

Proposition 1.2. *For s in the half plane of absolute convergence, the functional $v\mapsto \Psi(s;W_v)$ is continuous on $V\hat{\otimes}E$, uniformly for s in a compact set.*

Proof: Since the functional is evidently linear, it is enough to show that the sequence $\Psi(s;W_{v_k})$ converges to 0 whenever $v_k\to 0$ in $V\hat{\otimes}E$, uniformly for s in a compact set.

By Proposition 1.1, there exists a finite collection of functions $f_j \in C_c^\infty(GL_n\times GL_m)$ and sequences $v_{k,j}$ in $V\hat{\otimes}E$ such that $v_k = \sum_j \pi\hat{\otimes}\sigma(f_j)v_{k,j}$ for each k and $v_{k,j}\to 0$ for each j. Then we have

$$W_{v_k}(g,g') = \sum_j \rho(f_j)W_{v_{k,j}}(g,g')$$

so that by Lemma 1.4

$$|W_{v_k}(g,g')| \le \sum_j \beta_j(v_{k,j})\xi_j(g,g')$$

for seminorms β_j and gauges ξ_j depending only on f_j. Then

$$\begin{aligned}|\Psi(s;W_{v_k})| &= \Big|\int W_{v_k}\left(\begin{pmatrix} g & I_{n-m}\end{pmatrix}, g\right)|\det(g)|^{s-(n-m)/2}\,dg\Big| \\ &\leq \int \Big|W_{v_k}\left(\begin{pmatrix} g & \\ & I_{n-m}\end{pmatrix}, g\right)\Big||\det(g)|^{Re(s)-(n-m)/2}\,dg \\ &\leq \sum_j \beta_j(v_{k,j})\int \xi_j\left(\begin{pmatrix} g & \\ & I_{n-m}\end{pmatrix}, g\right)|\det(g)|^{Re(s)-(n-m)/2}\,dg.\end{aligned}$$

In this last expression, each integral involving a gauge ξ_j is absolutely convergent for $Re(s) >> 0$, uniformly for s in compact sets. Since the seminorms β_j are continuous on $V\hat{\otimes}E$ and since each sequence $v_{k,j} \to 0$ as $k \to \infty$ we have $|\Psi(s;W_{v_k})|$ converges to 0 as $k \to \infty$ uniformly for s in a compact set. □

Corollary . *For s in the realm of absolute convergence of the local integrals, the functional $\Lambda_s(v) = \Psi(s;W_v)/L(s,\pi\times\sigma)$ is continuous on $V\hat{\otimes}E$, uniformly for s in a compact set.*

Repeating the proof we also obtain the following.

Corollary . *For s in the realm of absolute convergence of the local integrals, the functional $\Lambda_{s,j}(v) = \Psi(s;W_v,j)/L(s,\pi\times\sigma)$ is continuous on $V\hat{\otimes}E$, uniformly for s in a compact set.*

From this we obtain:

Corollary . *The functional $\tilde{\Lambda}_{s,j}(v) = \Psi(1-s;\rho(w_{n,m})\tilde{W}_v,j)/L(1-s,\pi^\iota\times\sigma^\iota)$ is continuous on $V\hat{\otimes}E$, uniformly for s in a compact set, in the domain $Re(s) << 0$.*

We are now ready to prove the theorem.

Proof:[Proof of Theorem 1.1] By the first Corollary, we have that the functional $\Lambda_s(v)$ is continuous in a domain $Re(s) > B'$, uniformly for s in a compact set. If we let $\Lambda'_s(v) = \Lambda_s(v)e^{s^2}$ then Λ'_s will also be continuous in this domain with the same uniformity.

Let $B > B'$. Then on the line $Re(s) = B$ we have a uniform estimate $|\Lambda'_s(v)| \leq c_B\Psi(B;|W_v|)$. To see this, write

$$|\Lambda'_s(v)| = |\Psi(s;W_v)|\Big|\frac{e^{s^2}}{L(s,\pi\times\sigma)}\Big|.$$

On the line $Re(s) = B$, the function $e^{(B+it)^2}L(B+it,\pi\times\sigma)^{-1}$ is rapidly decreasing as $|t| \to \infty$. Hence there is a constant c_B so that $|e^{(B+it)^2}L(B+it,\pi\times\sigma)^{-1}| \leq c_B$. On the other hand, it is elementary that $|\Psi(s;W)| \leq \Psi(|W|B;)$. This gives the estimate.

By the functional equation, we have

$$\begin{aligned}\Lambda'_s(v) &= \frac{\Psi(s;W)e^{s^2}}{L(s,\pi\times\sigma)} \\ &= \omega_\sigma(-1)^{n-1}\varepsilon(s,\pi\times\sigma,\psi)^{-1}\frac{\Psi(1-s;\rho(w_{n,m})\tilde{W},n-m-1)e^{s^2}}{L(1-s,\pi^\iota\times\sigma^\iota)} \\ &= \tilde{\Lambda}'_{s,n-m-1}(v).\end{aligned}$$

It follows from the third Corollary that $\Lambda_{s,n-m-1}$ is continuous in a halfplane $Re(s) < A'$, hence so are $\Lambda'_s(v)$ and $\Lambda_s(v) = \Lambda'_s(v)e^{-s^2}$, with uniformity on compact subsets of $Re(s) < A'$.

Arguing as above, if $A < A'$ we have a uniform bound on the line $Re(s) = A$ of the form $|\Lambda'_s(v)| = |\tilde{\Lambda}'_{s,n-m-1}(v)| \leq c_A\Psi(1-A;|\rho(w_{n,m})\tilde{W}|,n-m-1)$.

Consider now the behavior of $\Lambda'_s(v)$ in the strip $A \leq Re(s) \leq B$. The function $\Lambda'_s(v)$, as a function of s, grows sufficiently slowly that we may apply Phragmen–Lindelöf to the strip $A \leq Re(s) \leq B$ and we obtain the estimate

$$|\Lambda'_s(v)| \leq \max(c_B\Psi(B;|W_v|), c_A\Psi(1-A;|\rho(w_{n,m})\tilde{W}|,n-m-1))$$

in this strip. Now suppose that v_k is a sequence converging to 0 in $V\hat{\otimes}E$. Then the proof of Proposition 1.2 shows that both the contributions $\Psi(B;|W_{v_k}|)$ and $\Psi(1-A;|\rho(w_{n,m})\tilde{W}_{v_k}|,n-m-1))$ go to 0 as $k\to\infty$. Hence $\Lambda'_s(v_k)$ converges to 0 in the strip, uniformly for all s. Hence Λ'_s is continuous for s in the strip, and uniformly so. Then $\Lambda_s(v) = \Lambda'_s(v)e^{-s^2}$ will be continuous on this strip, uniformly for s in a compact set.

This completes the proof of the theorem. □

1.3. **Applications.** In this section we would like to present our applications to the analytic properties of the local Rankin-Selberg convolutions, which it turn are needed for the completion of the global theory of Rankin-Selberg convolutions in the following section.

We keep the notation of Section 1.2. Recall that V_o and E_o are the underlying Harish-Chandra modules of V and E. Let $\mathcal{W}_o(\pi,\psi)$ be the subspace of $\mathcal{W}(\pi,\psi)$ spanned by the Whittaker functions associated to vectors in V_o, and similarly for $\mathcal{W}_o(\sigma,\psi^{-1})$.

Theorem 1.2. *(i) For each $W\in\mathcal{W}_o(\pi,\psi)$ and $W'\in\mathcal{W}_o(\sigma,\psi^{-1})$ the ratio*

$$e(s;W,W') = \frac{\Psi(s;W,W')}{L(s,\pi\times\sigma)}$$

is an entire function of s.

(ii) For every $s_0\in\mathbb{C}$ there is a choice of $W_0\in\mathcal{W}_o(\pi,\psi)$ and $W'_0\in\mathcal{W}_o(\sigma,\psi^{-1})$ such that $e(s_0;W_0,W'_0)\neq 0$.

Proof: We have that V_o is dense in V and E_o is dense in E. The Casselman-Wallach completion of the Harish-Chandra module $V_o\otimes E_o$ is $V\hat{\otimes}E$. Hence $V_o\otimes E_o$ is dense in $V\hat{\otimes}E$.

Statement (i) now follows from Theorem 11.1 of [13].

Statement (ii) follows from Theorem 11.1 of [13] and Theorem 2.1 above. By Theorem 11.1 of [13] we know that $L(s, \pi \times \sigma)$ is obtained by $\Psi(s; W)$ for some $W = W_v$ with $v \in V \hat{\otimes} E$. For this v, $\Lambda_s(v) = \Psi(s; W_v)/L(s, \pi \times \sigma) = 1$. Since $V_o \otimes E_o$ is dense, there will be a vector $\tilde{v} \in V_o \otimes E_o$ for which $\Lambda_{s_0}(\tilde{v})$ is close to 1 and in particular is non-zero. Writing $\tilde{v}$ as a sum of decomposable tensors, we find a vector $v_0 \otimes v_0'$ such that $\Lambda_{s_0}(v_0 \otimes v_0') \neq 0$. But

$$\Lambda_{s_0}(v_0 \otimes v_0') = \frac{\Psi(s_0; W_{v_0}, W'_{v_0'})}{L(s_0, \pi \times \sigma)} = e(s_0; W_{v_0}, W'_{v_0'}).$$

Hence (ii). □

The same proof yields the following Corollary.

Corollary . *(i) For each pair $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\sigma, \psi^{-1})$ the ratio*

$$e(s; W, W') = \frac{\Psi(s; W, W')}{L(s, \pi \times \sigma)}$$

is an entire function of s.

(ii) For every $s_0 \in \mathbb{C}$ there is a choice of $W_0 \in \mathcal{W}(\pi, \psi)$ and $W_0' \in \mathcal{W}(\sigma, \psi^{-1})$such that $e(s_0; W_0, W_0') \neq 0$.

These results show that the L-function $L(s, \pi \times \sigma)$ as defined in [13] not only cancels all poles of the local integrals, but also dividing by it introduces no extraneous zeros. Hence this is the minimal standard Euler factor which cancels all poles in the local integrals, even for the K-finite vectors, as in the non-archimedean case [11].

The continuity of the local integrals also plays a role in proving the following result of Stade [16, 17] and Jacquet and Shalika (unpublished).

Theorem 1.3. *In the cases $m = n$ and $m = n - 1$ there exist a finite collection of K–finite functions $W_i \in \mathcal{W}_o(\pi, \psi)$, $W_i' \in \mathcal{W}_o(\sigma, \psi^{-1})$, and $\Phi_i \in \mathcal{S}(F^n)$ if necessary such that*

$$L(s, \pi \times \sigma) = \sum \Psi(s; W_i, W_i') \quad \text{or} \quad L(s, \pi \times \sigma) = \sum \Psi(s; W_i, W_i', \Phi_i).$$

In the case where both π and σ are unramified, Stade shows that one obtains the L-function exactly with the K–invariant Whittaker functions (and Schwartz function if necessary). Our results are not needed in this case.

In the general case, Jacquet has provided us with a sketch of his argument with Shalika. First one proves that the integrals involving K–finite functions are equal to the product of a polynomial and the L-factor. It suffices to prove this for principal series, since the other representations embed into principal series. For principal series one proceeds by an induction argument on n, however one must prove the $m = n$ and $m = n-1$ cases simultaneously. The (essentially formal) arguments needed are to be found in the published papers of Jacquet and Shalika. The polynomials in question then form an ideal and the point now is to show this ideal is the full polynomial ring. This is then implied by Theorem 1.2 (ii) above.

2. Global Rankin–Selberg Convolutions

It was recently pointed out to us by Jacquet that the global theory of Rankin–Selberg convolutions via integral representations has never appeared in print. We would like to take this opportunity to at least partially correct this situation. All of the necessary global foundational material can be found in [10] and [12] and the necessarily local results are in [11] and [13] with the addition of the material in Section 1 above.

Let k be a global field, $\mathbb{A}$ its ring of adeles, and fix a non-trivial continuous additive character $\psi = \otimes\psi_v$ of $\mathbb{A}$ trivial on k.

Let (π, V_π) be a unitary cuspidal representation of $GL_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ a unitary cuspidal representation of $GL_m(\mathbb{A})$. Since they are irreducible we have restricted tensor product decompositions $\pi \simeq \otimes'\pi_v$ and $\pi' \simeq \otimes'\pi'_v$ with (π_v, V_{π_v}) and $(\pi'_v, V_{\pi'_v})$ irreducible admissible smooth generic unitary representations of $GL_n(k_v)$ and $GL_m(k_v)$ [5, 7, 8]. Let $\omega = \otimes'\omega_v$ and $\omega' = \otimes'\omega'_v$ be their central characters. These are both continuous characters of $k^\times\backslash\mathbb{A}^\times$.

2.1. Global Eulerian Integrals for $GL_n \times GL_m$.

Let us first assume that $m < n$. Then the results we need can be found in Part II of [12]. Let $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ be two cusp forms. The integral representations in this situation are of Hecke type and essentially involve the integration of these cusp forms against a factor of $|\det|^s$, that is, a type of generalized Mellin transform.

In GL_n, let P_n denote the mirabolic subgroup, that is, the stabilizer of the row vector $(0, \ldots, 0, 1)$. Let N_n be the subgroup of upper triangular unipotent matrices, that is, the unipotent radical of the standard Borel subgroup. In the usual way, the additive character ψ defines a non-degenerate character of N_n through its abelianization. Let $Y_{n,m}$ be the unipotent radical of the standard parabolic subgroup attached to the partition $(m+1, 1, \ldots, 1)$. Then ψ defines a character of $Y_{n,m}(\mathbb{A})$ trivial on $Y_{n,m}(k)$ since $Y_{n,m} \subset N_n$. The group $Y_{n,m}$ is normalized by $GL_{m+1} \subset GL_n$ and the mirabolic subgroup $P_{m+1} \subset GL_{m+1}$ is the stabilizer in GL_{m+1} of the character ψ.

Definition . *If $\varphi(g)$ is a cusp form on $GL_n(\mathbb{A})$ define the projection operator $\mathbb{P}^n_m$ from cusp forms on $GL_n(\mathbb{A})$ to cuspidal functions on $P_{m+1}(\mathbb{A})$ by*

$$\mathbb{P}^n_m\varphi(p) = |\det(p)|^{-\left(\frac{n-m-1}{2}\right)} \int_{Y_{n,m}(k)\backslash Y_{n,m}(\mathbb{A})} \varphi\left(y\begin{pmatrix} p & \\ & I_{n-m-1}\end{pmatrix}\right)\psi^{-1}(y)\, dy$$

for $p \in P_{m+1}(\mathbb{A})$.

This function $\mathbb{P}^n_m\varphi$ is essentially the same as the function denoted $V_{\varphi,m}$ in Part II of [12]. As the integration is over a compact domain, the integral is absolutely convergent. We first analyze the behavior on $P_{m+1}(\mathbb{A})$. From Section 3.1 of Part II of [12] we find the proofs of the following Lemmas

Lemma 2.1. *The function $\mathbb{P}^n_m\varphi(p)$ is a cuspidal function on $P_{m+1}(\mathbb{A})$.*

Lemma 2.2. *Let φ be a cusp form on $GL_n(\mathbb{A})$. Then for $h \in GL_m(\mathbb{A})$, $\mathbb{P}^n_m\varphi\begin{pmatrix} h & \\ & 1\end{pmatrix}$ has the Fourier expansion*

$$\mathbb{P}^n_m\varphi\begin{pmatrix} h & \\ & 1\end{pmatrix} = |\det(h)|^{-\left(\frac{n-m-1}{2}\right)} \sum_{\gamma\in N_m(k)\backslash GL_m(k)} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}\right)$$

with convergence absolute and uniform on compact subsets.

We now have the prerequisites for writing down a family of Eulerian integrals for cusp forms φ on GL_n twisted by automorphic forms on GL_m for $m < n$. Let $\varphi \in V_\pi$ be a cusp form on $GL_n(\mathbb{A})$ and $\varphi' \in V_{\pi'}$ a cusp form on $GL_m(\mathbb{A})$. (Actually, we could take φ' to be an arbitrary automorphic form on $GL_m(\mathbb{A})$.) Consider the integrals

$$I(s;\varphi,\varphi') = \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \mathbb{P}^n_m\varphi\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix}\varphi'(h)|\det(h)|^{s-1/2}\, dh.$$

The integral $I(s;\varphi,\varphi')$ is absolutely convergent for all values of the complex parameter s, uniformly in compact subsets, since the cusp forms are rapidly decreasing. Hence it is entire and bounded in any vertical strip.

Let us now investigate the Eulerian properties of these integrals. We first replace $\mathbb{P}^n_m\varphi$ by its Fourier expansion.

$$\begin{aligned} I(s;\varphi,\varphi') &= \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \mathbb{P}^n_m\varphi\begin{pmatrix} h & 0\\ 0 & I_{n-m}\end{pmatrix}\varphi'(h)|\det(h)|^{s-1/2}\, dh \\ &= \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \sum_{\gamma\in N_m(k)\backslash GL_m(k)} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix} h & 0\\ 0 & I_{n-m}\end{pmatrix}\right)\varphi'(h)|\det(h)|^{s-(n-m)/2}\, dh. \end{aligned}$$

Since $\varphi'(h)$ is automorphic on $GL_m(\mathbb{A})$ and $|\det(\gamma)| = 1$ for $\gamma \in GL_m(k)$ we may interchange the order of summation and integration for $Re(s) >> 0$ and then recombine to obtain

$$I(s;\varphi,\varphi') = \int_{N_m(k)\backslash GL_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & 0\\ 0 & I_{n-m}\end{pmatrix}\varphi'(h)|\det(h)|^{s-(n-m)/2}\, dh.$$

This integral is absolutely convergent for $Re(s) >> 0$ by the gauge estimates of [10, Section 13] and this justifies the interchange.

Let us now integrate first over $N_m(k)\backslash N_m(\mathbb{A})$. Recall that for $n \in N_m(\mathbb{A}) \subset N_n(\mathbb{A})$ we have $W_\varphi(ng) = \psi(n)W_\varphi(g)$. Hence we have

$$
\begin{aligned}
I(s;\varphi,\varphi') &= \\
&\int_{N_m(\mathbb{A})\backslash GL_m(\mathbb{A})} \int_{N_m(k)\backslash N_m(\mathbb{A})} W_\varphi\left(\begin{pmatrix} n & 0 \\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix}\right)\varphi'(nh)\ dn\ |\det(h)|^{s-(n-m)/2}\ dh \\
&= \int_{N_m(\mathbb{A})\backslash GL_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix}\int_{N_m(k)\backslash N_m(\mathbb{A})}\psi(n)\varphi'(nh)\ dn\ |\det(h)|^{s-(n-m)/2}\ dh \\
&= \int_{N_m(\mathbb{A})\backslash GL_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_{\varphi'}(h)|\det(h)|^{s-(n-m)/2}\ dh \\
&= \Psi(s;W_\varphi,W'_{\varphi'})
\end{aligned}
$$

where $W'_{\varphi'}(h)$ is the ψ^{-1}-Whittaker function on $GL_m(\mathbb{A})$ associated to φ', i.e.,

$$W'_{\varphi'}(h) = \int_{N_m(k)\backslash N_m(\mathbb{A})}\varphi'(nh)\psi(n)\ dn,$$

and we retain absolute convergence for $Re(s) >> 0$.

From this point, the fact that the integrals are Eulerian is a consequence of the uniqueness of the Whittaker model for GL_n [9, 15]. Take φ a smooth cusp form in a cuspidal representation π of $GL_n(\mathbb{A})$. Assume in addition that φ is factorizable, i.e., in the decomposition $\pi = \otimes'\pi_v$ of π into a restricted tensor product of local representations, $\varphi = \otimes\varphi_v$ is a pure tensor. Then there is a choice of local Whittaker models so that $W_\varphi(g) = \prod W_{\varphi_v}(g_v)$. Similarly for decomposable φ' we have the factorization $W'_{\varphi'}(h) = \prod W'_{\varphi'_v}(h_v)$.

If we substitute these factorizations into our integral expression, then since the domain of integration factors $N_m(\mathbb{A})\backslash GL_m(\mathbb{A}) = \prod N_m(k_v)\backslash GL_m(k_v)$ we see that our integral factors into a product of local integrals

$$\Psi(s;W_\varphi,W'_{\varphi'}) = \prod_v \int_{N_m(k_v)\backslash GL_m(k_v)} W_{\varphi_v}\begin{pmatrix} h_v & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_{\varphi'_v}(h_v)|\det(h_v)|_v^{s-(n-m)/2}\ dh_v.$$

If we denote the local integrals by

$$\Psi_v(s;W_{\varphi_v},W'_{\varphi'_v}) = \int_{N_m(k_v)\backslash GL_m(k_v)} W_{\varphi_v}\begin{pmatrix} h_v & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_{\varphi'_v}(h_v)|\det(h_v)|_v^{s-(n-m)/2}\ dh_v,$$

which converges for $Re(s) >> 0$ by the gauge estimate of [10, Proposition 2.3.6], we see that we now have a family of Eulerian integrals.

Now let us return to the question of a functional equation. The functional equation is essentially a consequence of the existence of the outer automorphism $g \mapsto \iota(g) = g^\iota = {}^tg^{-1}$ of GL_n. If we define the action of this automorphism on automorphic forms by setting $\widetilde{\varphi}(g) = \varphi(g^\iota) = \varphi(w_n g^\iota)$ and let $\widetilde{\mathbb{P}}^n_m = \iota \circ \mathbb{P}^n_m \circ \iota$ then our integrals naturally satisfy the functional equation

$$I(s;\varphi,\varphi') = \widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}')$$

where

$$\widetilde{I}(s;\varphi,\varphi') = \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \widetilde{\mathbb{P}}^n_m \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det(h)|^{s-1/2}\, dh.$$

We have established the following result.

Theorem 2.1. *Let $\varphi \in V_\pi$ be a cusp form on $GL_n(\mathbb{A})$ and $\varphi' \in V_{\pi'}$ a cusp form on $GL_m(\mathbb{A})$ with $m < n$. Then the family of integrals $I(s;\varphi,\varphi')$ define entire functions of s, bounded in vertical strips, and satisfy the functional equation*

$$I(s;\varphi,\varphi') = \widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}').$$

Moreover the integrals are Eulerian and if φ and φ' are factorizable, we have

$$I(s;\varphi,\varphi') = \prod_v \Psi_v(s;W_{\varphi_v},W'_{\varphi'_v})$$

with convergence absolute and uniform for $Re(s) >> 0$.

The integrals occurring in the right hand side of our functional equation are again Eulerian. One can unfold the definitions to find first that

$$\widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'})$$

where the unfolded global integral is

$$\widetilde{\Psi}(s;W,W') = \int\int W \begin{pmatrix} h & & \\ x & I_{n-m-1} & \\ & & 1 \end{pmatrix} dx\; W'(h)|\det(h)|^{s-(n-m)/2}\, dh$$

with the h integral over $N_m(\mathbb{A})\backslash GL_m(\mathbb{A})$ and the x integral over $M_{n-m-1,m}(\mathbb{A})$, the space of $(n-m-1)\times m$ matrices, ρ denoting right translation, and $w_{n,m}$ the Weyl element $w_{n,m} = \begin{pmatrix} I_m & \\ & w_{n-m} \end{pmatrix}$ with $w_{n-m} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ the standard long Weyl element in GL_{n-m}. Also, for $W \in \mathcal{W}(\pi,\psi)$ we set $\widetilde{W}(g) = W(w_n g^\iota) \in \mathcal{W}(\widetilde{\pi},\psi^{-1})$. The extra unipotent integration is the remnant of $\widetilde{\mathbb{P}}^n_m$. As before, $\widetilde{\Psi}(s;W,W')$ is absolutely convergent for $Re(s) >> 0$. For φ and φ' factorizable as before, these integrals $\widetilde{\Psi}(s;W_\varphi,W'_{\varphi'})$ will factor as well. Hence we have

$$\widetilde{\Psi}(s;W_\varphi,W'_{\varphi'}) = \prod_v \widetilde{\Psi}_v(s;W_{\varphi_v},W'_{\varphi'_v})$$

where

$$\widetilde{\Psi}_v(s;W_v,W'_v) = \int\int W_v \begin{pmatrix} h_v & & \\ x_v & I_{n-m-1} & \\ & & 1 \end{pmatrix} dx_v\; W'_v(h_v)|\det(h_v)|^{s-(n-m)/2}\, dh_v$$

where now with the h_v integral is over $N_m(k_v)\backslash GL_m(k_v)$ and the x_v integral is over the matrix space $M_{n-m-1,m}(k_v)$. Thus, coming back to our functional equation, we find that the right hand side is Eulerian and factors as

$$\widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'}) = \prod_v \widetilde{\Psi}_v(1-s;\rho(w_{n,m})\widetilde{W}_{\varphi_v},\widetilde{W}'_{\varphi'_v}).$$

Now consider the case of $m = n$. Then the results we need can essentially be found in Part I of [12]. Let (π, V_π) and $(\pi', V_{\pi'})$ be two unitary cuspidal representations of $GL_n(\mathbb{A})$. Let $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ be two cusp forms. The integral representation in this situation is an honest Rankin–Selberg integral and will involve the integration of the cusp forms φ and φ' against a particular type of Eisenstein series on $GL_n(\mathbb{A})$.

To construct the Eisenstein series as in Part I of [12] we observe that $P_n \setminus GL_n \simeq k^n - \{0\}$. If we let $\mathcal{S}(\mathbb{A}^n)$ denote the Schwartz–Bruhat functions on $\mathbb{A}^n$, then each $\Phi \in \mathcal{S}$ defines a smooth function on $GL_n(\mathbb{A})$, left invariant by $P_n(\mathbb{A})$, by $g \mapsto \Phi((0,\dots,0,1)g) = \Phi(e_n g)$. Let η be a unitary idele class character. (For our application η will be determined by the central characters of π and π'.) Consider the function

$$F(g,\Phi;s,\eta) = |\det(g)|^s \int_{\mathbb{A}^\times} \Phi(ae_ng)|a|^{ns}\eta(a)\ d^\times a.$$

If we let $P_n' = Z_n\, P_n$ be the parabolic of GL_n associated to the partition $(n-1,1)$ then one checks that for $p' = \begin{pmatrix} h & y \\ 0 & d \end{pmatrix} \in P_n'(\mathbb{A})$ with $h \in GL_{n-1}(\mathbb{A})$ and $d \in \mathbb{A}^\times$ we have,

$$F(p'g,\Phi;s,\eta) = |\det(h)|^s|d|^{-(n-1)s}\eta(d)^{-1}F(g,\Phi;s,\eta) = \delta_{P_n'}^s(p')\eta^{-1}(d)F(g,\Phi;s,\eta),$$

with the integral absolutely convergent for $Re(s) > 1/n$, so that if we extend η to a character of P_n' by $\eta(p') = \eta(d)$ in the above notation we have that $F(g,\Phi;s,\eta)$ is a smooth section of the normalized induced representation $Ind_{P_n'(\mathbb{A})}^{GL_n(\mathbb{A})}(\delta_{P_n'}^{s-1/2}\eta^{-1})$. Since the inducing character $\delta_{P_n'}^{s-1/2}\eta^{-1}$ of $P_n'(\mathbb{A})$ is invariant under $P_n'(k)$ we may form Eisenstein series from this family of sections by

$$E(g,\Phi;s,\eta) = \sum_{\gamma \in P_n'(k)\setminus GL_n(k)} F(\gamma g,\Phi;s,\eta).$$

If we replace F in this sum by its definition we can rewrite this Eisenstein series as

$$\begin{aligned} E(g,\Phi;s,\eta) &= |\det(g)|^s \int_{k^\times\setminus\mathbb{A}^\times} \sum_{\xi\in k^n-\{0\}} \Phi(a\xi g)|a|^{ns}\eta(a)\ d^\times a \\ &= |\det(g)|^s \int_{k^\times\setminus\mathbb{A}^\times} \Theta'_\Phi(a,g)|a|^{ns}\eta(a)\ d^\times a \end{aligned}$$

and this first expression is convergent absolutely for $Re(s) > 1$ [12].

The second expression essentially gives the Eisenstein series as the Mellin transform of the Theta series

$$\Theta_\Phi(a,g) = \sum_{\xi\in k^n} \Phi(a\xi g),$$

where in the above we have written

$$\Theta'_\Phi(a,g) = \sum_{\xi\in k^n-\{0\}} \Phi(a\xi g) = \Theta_\Phi(a,g) - \Phi(0).$$

This allows us to obtain the analytic properties of the Eisenstein series from the Poisson summation formula for Θ_Φ, namely

$$\begin{aligned}\Theta_\Phi(a,g) &= \sum_{\xi\in k^n}\Phi(a\xi g) = \sum_{\xi\in k^n}\Phi_{a,g}(\xi)\\ &= \sum_{\xi\in k^n}\widehat{\Phi_{a,g}}(\xi) = \sum_{\xi\in k^n}|a|^{-n}|\det(g)|^{-1}\widehat{\Phi}(a^{-1}\xi {}^tg^{-1})\\ &= |a|^{-n}|\det(g)|^{-1}\Theta_{\hat\Phi}(a^{-1},{}^tg^{-1})\end{aligned}$$

where the Fourier transform $\hat\Phi$ on $\mathcal{S}(\mathbb{A}^n)$ is defined by

$$\hat\Phi(x) = \int_{\mathbb{A}^\times}\Phi(y)\psi(y{}^tx)\ dy.$$

This allows us to write the Eisenstein series as

$$\begin{aligned}E(g,\Phi,s,\eta) = |\det(g)|^s &\int_{|a|\geq 1}\Theta'_\Phi(a,g)|a|^{ns}\eta(a)\ d^\times a\\ &+ |\det(g)|^{s-1}\int_{|a|\geq 1}\Theta'_{\hat\Phi}(a,{}^tg^{-1})|a|^{n(1-s)}\eta^{-1}(a)\ d^\times a + \delta(s)\end{aligned}$$

where

$$\delta(s) = \begin{cases} 0 & \text{if } \eta \text{ is ramified}\\ -c\Phi(0)\frac{|\det(g)|^s}{s+i\sigma} + c\hat\Phi(0)\frac{|\det(g)|^{s-1}}{s-1+i\sigma} & \text{if } \eta(a)=|a|^{in\sigma} \text{ with } \sigma\in\mathbb{R}\end{cases}$$

with c a non-zero constant. From this we derive easily the basic properties of our Eisenstein series [12, Part I, Section 4].

Proposition 2.1. *The Eisenstein series $E(g,\Phi;s,\eta)$ has a meromorphic continuation to all of $\mathbb{C}$ with at most simple poles at $s=-i\sigma, 1-i\sigma$ when η is unramified of the form $\eta(a)=|a|^{in\sigma}$. As a function of g it is smooth of moderate growth and as a function of s it is bounded in vertical strips (away from the possible poles), uniformly for g in compact sets. Moreover, we have the functional equation*

$$E(g,\Phi;s,\eta) = E(g^\iota,\hat\Phi;1-s,\eta^{-1})$$

where $g^\iota = {}^tg^{-1}$.

Note that under the center the Eisenstein series transforms by the central character η^{-1}.

Now let us return to our Eulerian integrals. Let π and π' be our irreducible cuspidal representations. Let their central characters be ω and ω'. Set $\eta=\omega\omega'$. Then for each pair of cusp forms $\varphi\in V_\pi$ and $\varphi'\in V_{\pi'}$ and each Schwartz-Bruhat function $\Phi\in\mathcal{S}(\mathbb{A}^n)$ set

$$I(s;\varphi,\varphi',\Phi) = \int_{Z_n(\mathbb{A})\,GL_n(k)\backslash GL_n(\mathbb{A})}\varphi(g)\varphi'(g)E(g,\Phi;s,\eta)\ dg.$$

Since the two cusp forms are rapidly decreasing on $Z_n(\mathbb{A})\,GL_n(k)\backslash GL_n(\mathbb{A})$ and the Eisenstein is only of moderate growth, we see that the integral converges absolutely for all s away from the poles of the Eisenstein series and is hence meromorphic. It will be bounded in vertical strips away from the poles and satisfies the functional equation

$$I(s;\varphi,\varphi',\Phi) = I(1-s;\widetilde\varphi,\widetilde\varphi',\hat\Phi),$$

coming from the functional equation of the Eisenstein series, where we still have $\widetilde{\varphi}(g) = \varphi(g^\iota) = \varphi(w_n g^\iota) \in V_{\widetilde{\pi}}$ and similarly for $\widetilde{\varphi}'$.

These integrals will be entire unless we have $\eta(a) = \omega(a)\omega'(a) = |a|^{in\sigma}$ is unramified. In that case, the residue at $s = -i\sigma$ will be

$$\underset{s=-i\sigma}{Res}\, I(s;\varphi,\varphi',\Phi) = -c\Phi(0)\int_{Z_n(\mathbb{A})\, GL_n(\mathbb{A})\backslash\, GL_n(\mathbb{A})} \varphi(g)\varphi'(g)|\det(g)|^{-i\sigma}\, dg$$

and at $s = 1 - i\sigma$ we can write the residue as

$$\underset{s=1-i\sigma}{Res}\, I(s;\varphi,\varphi',\Phi) = c\hat{\Phi}(0)\int_{Z_n(\mathbb{A})\, GL_n(k)\backslash\, GL_n(\mathbb{A})} \widetilde{\varphi}(g)\widetilde{\varphi}'(g)|\det(g)|^{i\sigma}\, dg.$$

Therefore these residues define $GL_n(\mathbb{A})$ invariant pairings between π and $\pi' \otimes |\det|^{-i\sigma}$ or equivalently between $\widetilde{\pi}$ and $\widetilde{\pi}' \otimes |\det|^{i\sigma}$. Hence a residues can be non-zero only if $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$ and in this case we can find φ, φ', and Φ such that indeed the residue does not vanish.

We have yet to check that our integrals are Eulerian. To this end we take the integral, replace the Eisenstein series by its definition, and unfold:

$$\begin{aligned}
I(s;\varphi,\varphi',\Phi) &= \int_{Z_n(\mathbb{A})\, GL_n(k)\backslash\, GL_n(\mathbb{A})} \varphi(g)\varphi'(g)E(g,\Phi;s,\eta)\, dg \\
&= \int_{Z_n(\mathbb{A})\, P_n'(k)\backslash\, GL_n(\mathbb{A})} \varphi(g)\varphi'(g)F(g,\Phi;s,\eta)\, dg \\
&= \int_{Z_n(\mathbb{A})\, P_n(k)\backslash\, GL_n(\mathbb{A})} \varphi(g)\varphi'(g)|\det(g)|^s \int_{\mathbb{A}^\times} \Phi(ae_ng)|a|^{ns}\eta(a)\, da\, dg \\
&= \int_{P_n(k)\backslash\, GL_n(\mathbb{A})} \varphi(g)\varphi'(g)\Phi(e_ng)|\det(g)|^s\, dg.
\end{aligned}$$

We next replace φ by its Fourier expansion in the form

$$\varphi(g) = \sum_{\gamma\in N_n(k)\backslash\, P_n(k)} W_\varphi(\gamma g)$$

and unfold to find

$$\begin{aligned}
I(s;\varphi,\varphi',\Phi) &= \int_{N_n(k)\backslash\, GL_n(\mathbb{A})} W_\varphi(g)\varphi'(g)\Phi(e_ng)|\det(g)|^s\, dg \\
&= \int_{N_n(\mathbb{A})\backslash\, GL_n(\mathbb{A})} W_\varphi(g)\int_{N_n(k)\backslash\, N_n(\mathbb{A})} \varphi'(ng)\psi(n)\, dn\, \Phi(e_ng)|\det(g)|^s\, dg \\
&= \int_{N_n(\mathbb{A})\backslash\, GL_n(\mathbb{A})} W_\varphi(g)W'_{\varphi'}(g)\Phi(e_ng)|\det(g)|^s\, dg \\
&= \Psi(s;W_\varphi,W'_{\varphi'},\Phi).
\end{aligned}$$

This expression converges for $Re(s) >> 0$ by the gauge estimates as before.

To continue, we assume that φ, φ' and Φ are decomposable tensors under the isomorphisms $\pi \simeq \otimes'\pi_v$, $\pi' \simeq \otimes'\pi_v'$, and $\mathcal{S}(\mathbb{A}^n) \simeq \otimes'\mathcal{S}(k_v^n)$ so that we have $W_\varphi(g) = \prod_v W_{\varphi_v}(g_v)$, $W'_{\varphi'}(g) =$

$\prod_v W'_{\varphi'_v}(g_v)$ and $\Phi(g) = \prod_v \Phi_v(g_v)$. Then, since the domain of integration also naturally factors we can decompose this last integral into an Euler product and now write

$$\Psi(s; W_\varphi, W'_{\varphi'}, \Phi) = \prod_v \Psi_v(s; W_{\varphi_v}, W'_{\varphi'_v}, \Phi_v),$$

where

$$\Psi_v(s; W_{\varphi_v}, W'_{\varphi'_v}, \Phi_v) = \int_{N_n(k_v)\backslash GL_n(k_v)} W_{\varphi_v}(g_v) W'_{\varphi'_v}(g_v) \Phi_v(e_n g_v) |\det(g_v)|^s \, dg_v,$$

still with convergence for $Re(s) >> 0$ by the local gauge estimates. We have now established the following result.

Theorem 2.2. *Let $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ cusp forms on $GL_n(\mathbb{A})$ and let $\Phi \in \mathcal{S}(\mathbb{A}^n)$. Then the family of integrals $I(s; \varphi, \varphi', \Phi)$ define meromorphic functions of s, bounded in vertical strips away from the poles. The only possible poles are simple and occur iff $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$ with σ real and are then at $s = -i\sigma$ and $s = 1 - i\sigma$ with residues as above. They satisfy the functional equation*

$$I(s; \varphi, \varphi', \Phi) = I(1 - s; \widetilde{W}_\varphi, \widetilde{W}'_{\varphi'}, \hat{\Phi}).$$

Moreover, for φ, φ', and Φ factorizable we have that the integrals are Eulerian and we have

$$I(s; \varphi, \varphi', \Phi) = \prod_v \Psi_v(s; W_{\varphi_v}, W'_{\varphi'_v}, \Phi_v)$$

with convergence absolute and uniform for $Re(s) >> 0$.

We remark in passing that the right hand side of the functional equation also unfolds as

$$\begin{aligned} I(1 - s; \widetilde{\varphi}, \widetilde{\varphi}', \hat{\Phi}) &= \int_{N_n(\mathbb{A})\backslash GL_n(\mathbb{A})} \widetilde{W}_\varphi(g) \widetilde{W}'_{\varphi'}(g) \hat{\Phi}(e_n g) |\det(g)|^{1-s} \, dg \\ &= \prod_v \Psi_v(1 - s; \widetilde{W}_{\varphi_v}, \widetilde{W}'_{\varphi'_v}, \hat{\Phi}) \end{aligned}$$

with convergence for $Re(s) << 0$.

2.2. **The Global L-function.**

Let S be the finite set of places of k, containing the archimedean places S_∞, such that for all $v \notin S$ we have that π_v, π'_v, and ψ_v are unramified.

For each place v of k local factors $L(s, \pi_v \times \pi'_v)$ and $\varepsilon(s, \pi_v \times \pi'_v, \psi_v)$ have been defined through the local theory of Rankin-Selberg convolutions in [11] for non-archimedean v and in [13] for archimedean v. Then we can at least formally define

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v) \quad \text{and} \quad \varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

We need to discuss convergence of these products. Let us first consider the convergence of $L(s, \pi \times \pi')$. For those $v \notin S$, so π_v, π'_v, and ψ_v are unramified, Jacquet and Shalika have explicitly computed the local factor in [12, Part I, Section 2; Part II, Section 1]. They show

$$L(s, \pi_v \times \pi'_v) = \det(I - q_v^{-s} A_{\pi_v} \otimes A_{\pi'_v})^{-1}$$

where A_{π_v} and $A_{\pi'_v}$ are the associated Satake parameters, and that the eigenvalues of A_{π_v} and $A_{\pi'_v}$ are all of absolute value less than $q_v^{1/2}$ [12, Part I, Corollary 2.5]. Thus, as in [12, Theorem 5.3], the partial (or incomplete) L-function

$$L^S(s,\pi\times\pi') = \prod_{v\notin S} L(s,\pi_v\times\pi'_v) = \prod_{v\notin S} \det(I - q_v^{-s}A_{\pi_v}\otimes A_{\pi'_v})^{-1}$$

is absolutely convergent for $Re(s) >> 0$. Thus the same is true for $L(s,\pi\times\pi')$.

Remark: The local calculation alluded to above is actually the computation of the local integral with the unramified Whittaker functions. For $v\notin S$, in the Whittaker models there will be unique normalized $K = GL(\mathfrak{o}_v)$–fixed Whittaker functions, $W_v^\circ \in \mathcal{W}(\pi_v,\psi_v)$ and $W_v'^\circ \in \mathcal{W}(\pi'_v,\psi_v^{-1})$, normalized by $W_v^\circ(e) = W_v'^\circ(e) = 1$. When when $n=m$ let $\Phi = \Phi_v^\circ$ be the characteristic function of the lattice ${\mathfrak{o}_v}^n \subset k_v^n$. What Jacquet and Shalika show is that

$$\det(I - q_v^{-s}A_{\pi_v}\otimes A_{\pi'_v})^{-1} = \begin{cases} \Psi(s;W_v^\circ,W_v'^\circ) & m<n \\ \Psi(s;W_v^\circ,W_v'^\circ,\Phi_v^\circ) & m=n \end{cases}$$

and hence $\det(I - q_v^{-s}A_{\pi_v}\otimes A_{\pi'_v})$ divides $L(s,\pi_v\times\pi'_v)^{-1}$. To see that this actually calculates the L-function, one needs to combine this calculation with Proposition 9.4 of [11].)

For the ε–factor, it follows from the local calculation cited above and the local functional equation [11, Theorem 2.7 (iii)] that $\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\equiv 1$ for $v\notin S$ so that the product is in fact a finite product and there is no problem with convergence. The fact that $\varepsilon(s,\pi\times\pi')$ is independent of ψ can either be checked by analyzing how the local ε–factors vary as you vary ψ, as is done in [2, Lemma 2.1], or it will follow from the global functional equation presented below.

2.3. **The basic analytic properties.** Our first goal is to show that these L-functions have nice analytic properties.

Theorem 2.3. *The global L–functions $L(s,\pi\times\pi')$ are nice in the sense that*

(1) *$L(s,\pi\times\pi')$ has a meromorphic continuation to all of $\mathbb{C}$,*
(2) *the extended function is bounded in vertical strips (away from its poles),*
(3) *they satisfy the functional equation*

$$L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

To do so, we relate the L-functions to the global integrals.

Let us begin with continuation. In the case $m<n$ for every $\varphi\in V_\pi$ and $\varphi'\in V_{\pi'}$ we know the integral $I(s;\varphi,\varphi')$ converges absolutely for all s. From the unfolding in Section 2.1 and the local calculation mentioned above we know that for $Re(s) >> 0$ and for appropriate choices of φ and φ' we have

$$\begin{aligned}I(s;\varphi,\varphi') &= \prod_v \Psi_v(s;W_{\varphi_v},W_{\varphi'_v})\\ &= \left(\prod_{v\in S}\Psi_v(s;W_{\varphi_v},W_{\varphi'_v})\right) L^S(s,\pi\times\pi')\\ &= \left(\prod_{v\in S}\frac{\Psi_v(s;W_{\varphi_v},W_{\varphi'_v})}{L(s,\pi_v\times\pi'_v)}\right) L(s,\pi\times\pi')\\ &= \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W_{\varphi'_v})\right) L(s,\pi\times\pi')\end{aligned}$$

We know that each $e_v(s;W_v,W'_v)$ is entire. For non-archimedean v this follows from [11, Theorem 2.3] and for archimedean v this follows from Theorem 1.2 above and its Corollary. Hence $L(s,\pi\times\pi')$ has a meromorphic continuation. If $m=n$ then for appropriate $\varphi\in V_\pi$, $\varphi'\in V_{\pi'}$, and $\Phi\in\mathcal{S}(\mathbb{A}^n)$ we again have

$$I(s;\varphi,\varphi',\Phi) = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)\right) L(s,\pi\times\pi').$$

Once again, since each $e_v(s;W_v,W'_v,\Phi_v)$ is entire, $L(s,\pi\times\pi')$ has a meromorphic continuation.

Let us next turn to the functional equation. This will follow from the functional equation for the global integrals given above and the local functional equations [11, Theorem 2.7 (iii)] and [13, Theorem 5.1 (ii)]. We will consider only the case where $m<n$ since the other case is entirely analogous. The functional equation for the global integrals is simply

$$I(s;\varphi,\varphi') = \tilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}').$$

Once again we have for appropriate φ and φ'

$$I(s;\varphi,\varphi') = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v})\right) L(s,\pi\times\pi')$$

while on the other side

$$\tilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \left(\prod_{v\in S} \tilde{e}_v(1-s;\rho(w_{n,m})\widetilde{W}_{\varphi_v},\widetilde{W}'_{\varphi'_v})\right) L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

However, by the local functional equations, for each $v\in S$ we have

$$\begin{aligned}\tilde{e}_v(1-s;\rho(w_{n,m})\widetilde{W}_v,\widetilde{W}'_v) &= \frac{\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_v,\widetilde{W}'_v)}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}\\ &= \omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\frac{\Psi(s;W_v,W'_v)}{L(s,\pi\times\pi')}\\ &= \omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)e_v(s,W_v,W'_v)\end{aligned}$$

Combining these, we have

$$L(s,\pi\times\pi') = \left(\prod_{v\in S}\omega_v'(-1)^{n-1}\varepsilon(s,\pi_v\times\pi_v',\psi_v)\right) L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Now, for $v\notin S$ we know that π_v' is unramified, so $\omega_v'(-1)=1$, and also that $\varepsilon(s,\pi_v\times\pi_v',\psi_v)\equiv 1$. Therefore

$$\begin{aligned}\prod_{v\in S}\omega_v'(-1)^{n-1}\varepsilon(s,\pi_v\times\pi_v',\psi_v) &= \prod_{v}\omega_v'(-1)^{n-1}\varepsilon(s,\pi_v\times\pi_v',\psi_v)\\ &= \omega'(-1)^{n-1}\varepsilon(s,\pi\times\pi')\\ &= \varepsilon(s,\pi\times\pi')\end{aligned}$$

and we indeed have

$$L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Note that this implies that $\varepsilon(s,\pi\times\pi')$ is independent of ψ as well.

Let us now turn to the boundedness in vertical strips. For the global integrals $I(s;\varphi,\varphi')$ or $I(s;\varphi,\varphi,\Phi)$ this simply follows from the absolute convergence. For the L-function itself, the paradigm is the following. For every finite place $v\in S$, by the definition of the local L-function as the generator of the fractional ideal spanned by the local integrals [11, Theorem 2.7 (ii)] we know that there is a choice of finite collections $W_{v,i}$, $W_{v,i}'$, and if necessary $\Phi_{v,i}$ such that

$$L(s,\pi_v\times\pi_v') = \sum\Psi(s;W_{v,i},W_{v'i}') \quad \text{or} \quad L(s,\pi_v\times\pi_v') = \sum\Psi(s;W_{v,i},W_{v'i}',\Phi_{v,i}).$$

If $m=n-1$ or $m=n$ then by the results of Stade [16, 17] or the unpublished work of Jacquet and Shalika presented in Theorem 1.3 above we know that we have similar statements for $v\in S_\infty$. Hence if $m=n-1$ or $m=n$ there are finite global choices φ_i, φ_i', and if necessary Φ_i such that

$$L(s,\pi\times\pi') = \sum I(s;\varphi_i,\varphi_i') \quad \text{or} \quad L(s,\pi\times\pi') = \sum I(s;\varphi_i,\varphi_i',\Phi_i).$$

Then the boundedness in vertical strips for the L-functions follows from that of the global integrals.

However, if $m<n-1$ then all we know at those $v\in S_\infty$ is that there is a function $W_v\in\mathcal{W}(\pi_v\hat{\otimes}\pi_v',\psi_v)=\mathcal{W}(\pi_v,\psi_v)\hat{\otimes}\mathcal{W}(\pi_v',\psi_v^{-1})$ or a finite collection of such functions $W_{v,i}$ and of $\Phi_{v,i}$ such that

$$L(s,\pi_v\times\pi_v') = I(s;W_v) \quad \text{or} \quad L(s,\pi_v\times\pi_v') = \sum I(s;W_{v,i},\Phi_{v,i}).$$

To make the above paradigm work for $m<n-1$ one possibility would be to rework the theory of global Eulerian integrals for cusp forms in $V_\pi\hat{\otimes}V_{\pi'}$. This is naturally the space of smooth vectors in an irreducible unitary cuspidal representation of $GL_n(\mathbb{A})\times GL_m(\mathbb{A})$. So we would need extend the global theory of integrals parallel to Jacquet and Shalika's extension of the local integrals in the archimedean theory. There seems to be no obstruction to carrying this out, and then we would obtain boundedness in vertical strips for $L(s,\pi\times\pi')$ in general within the context of integral representations. However, if one approaches these L-function by the method of constant terms and Fourier coefficients of Eisenstein series, then

Gelbart and Shahidi have shown a wide class of automorphic L-functions, including ours, to be bounded in vertical strips [6]. Thus the boundedness in vertical strips is true, even if we must go "outside the method" for this fact at this point.

2.4. **Poles of L-functions.** Let us determine where the global L-functions can have poles. The poles of the L-functions will be related to the poles of the global integrals. Recall from Section 2.2 that in the case of $m < n$ we have that the global integrals $I(s;\varphi,\varphi')$ are entire and that when $m = n$ then $I(s;\varphi,\varphi',\Phi)$ can have at most simple poles and they occur at $s = -i\sigma$ and $s = 1 - i\sigma$ for σ real when $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$. As we have noted above, the global integrals and global L-functions are related, for appropriate φ, φ', and Φ, by

$$I(s;\varphi,\varphi') = \left(\prod_{v\in S} e_v(s; W_{\varphi_v}, W'_{\varphi'_v})\right) L(s,\pi\times\pi')$$

or

$$I(s;\varphi,\varphi',\Phi) = \left(\prod_{v\in S} e_v(s; W_{\varphi_v}, W'_{\varphi'_v}, \Phi_v)\right) L(s,\pi\times\pi').$$

On the other hand, for any $s_0 \in \mathbb{C}$ and any v there is a choice of local W_v, W'_v, and Φ_v such that the local factors $e_v(s_0; W_v, W'_v) \neq 0$ or $e_v(s_0; W_v, W'_v, \Phi_v) \neq 0$. For archimedean v this is Theorem 1.2 (ii) and its Corollary. For non-archimedean v this follows from the definition of the L-function as the generator of the fractional ideal spanned by the local integrals. As noted above this implies that there are finite collections $W_{v,i}$, $W'_{v,i}$, and $\Phi_{v,i}$ if necessary such that

$$L(s,\pi_v\times\pi'_v) = \sum \Psi(s; W_{v,i}, W'_{v'i}) \quad \text{or} \quad L(s,\pi_v\times\pi'_v) = \sum \Psi(s; W_{v,i}, W'_{v'i}, \Phi_{v,i})$$

which is equivalent to

$$1 = \sum e(s; W_{v,i}, W'_{v'i}) \quad \text{or} \quad 1 = \sum e(s; W_{v,i}, W'_{v'i}, \Phi_{v,i}).$$

Hence for any choice of $s_0 \in \mathbb{C}$ one of the $e(s_0; W_{v,i}, W'_{v'i})$ or $e(s_0; W_{v,i}, W'_{v'i}, \Phi_{v,i})$ must be non-vanishing. So as we vary φ, φ' and Φ at the places $v \in S$ we see that division by these factors can introduce no extraneous poles in $L(s,\pi\times\pi')$, that is, in keeping with the local characterization of the L-factor in terms of poles of local integrals, globally the poles of $L(s,\pi\times\pi')$ are precisely the poles of the family of global integrals $\{I(s;\varphi,\varphi')\}$ or $\{I(s;\varphi,\varphi',\Phi)\}$. Hence from Theorems 2.1 and 2.2 we have.

Theorem 2.4. *If $m < n$ then $L(s,\pi\times\pi')$ is entire. If $m = n$, then $L(s,\pi\times\pi')$ has at most simple poles and they occur iff $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$ with σ real and are then at $s = -i\sigma$ and $s = 1 - i\sigma$.*

If we apply this with $\pi' = \widetilde{\pi}$ we obtain the following corollary.

Corollary . *$L(s,\pi\times\widetilde{\pi})$ has simple poles at $s = 0$ and $s = 1$.*

Since a general, not necessarily unitary, cuspidal representation π is always of the form $\pi = \pi^u \otimes |\det|^r$ with π^u unitary cuspidal, these results extend in a straightforward way to all cuspidal representations. In particular, this gives the proof of Jacquet, Piatetski-Shapiro, and Shalika of these results which was alluded to in the appendix of [14], where these results were proven using the technique of Eisenstein series.

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